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On the general equations of motion of non-holonomic systems

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1. – One knows that the motion without friction of a holonomic or non-holonomic system is characterized by the function $S = \frac{1}{2}\sum m J^2$, where J denotes the acceleration of a point of mass m. If $q_1, q_2, \ldots, q_k$ are the independent parameters whose virtual variations are arbitrary then the function S will be a function of degree two in $q_1'', q_2'', \ldots, q_k''$ that one can suppose to have been reduced to only the terms that contain $q_1'', q_2'', \ldots, q_k''$. The coefficients of that function can depend upon even more parameters whose virtual variations are given functions that are linear and homogeneous in $q_1, q_2, \ldots, q_k$. For a given virtual displacement of the system, the sum of the works done by applied forces will be:

$$Q_1\,\delta q_1 + Q_2\,\delta q_2 + \ldots + Q_k\,\delta q_k .$$

The equations of motion are ([1]):

$$\frac{\partial S}{\partial q''_\alpha} = Q_\alpha \qquad (\alpha = 1, 2, \ldots, k).$$

The function S is called the *energy of acceleration*, by analogy with the name *kinetic energy*, which is given to the *semi-vis viva of system:*

$$T = \tfrac{1}{2}\sum m v^2 ,$$

in which v is the velocity of a point of mass m.

If the system is holonomic then the equations of motion that were given by Lagrange are:

$$\frac{d}{dt}\frac{\partial T}{\partial q'_\alpha} - \frac{\partial T}{\partial q_\alpha} = Q_\alpha \qquad (\alpha = 1, 2, \ldots, k).$$

([1]) Paul APPELL, "Développement, sur une forme nouvelle, des équations de la Dynamique," paper published in J. Math. pures appl. (1900).

Our objective is to write the general equations of motion of a non-holonomic system in a different form, by means of which we will arrive at the differential equations of motion more easily in most cases.

2. – Suppose that a system of material points is subject to constraints that are expressed by finite and differential equations in the parameters that define the position of the system. The left-hand sides of the differential equations are not total differentials and do not have integrating factors.

Let the number of parameters $q_1, q_2, \ldots, q_k, q_{k+1}, \ldots, q_{k+p}$ that fix its position be $k + p$ when one takes into account the finite constraints that are imposed upon the system. Upon supposing that those constraints also depend upon time t, one will have:

$$\text{(1)} \qquad \begin{cases} x = f(t, q_1, q_2, \ldots, q_k, q_{k+1}, \ldots, q_{k+p}), \\ y = \varphi(t, q_1, q_2, \ldots, q_k, q_{k+1}, \ldots, q_{k+p}), \\ z = \omega(t, q_1, q_2, \ldots, q_k, q_{k+1}, \ldots, q_{k+p}) \end{cases}$$

for the coordinates of an arbitrary point of the system.

We obtain a virtual displacement of the system that is compatible *with those constraints* at the moment t by giving arbitrary infinitely-small increments $dq_1, dq_2, \ldots, dq_k, dq_{k+1}, \ldots, dq_{k+p}$ to the parameters $q_1, q_2, \ldots, q_k, q_{k+1}, \ldots, q_{k+p}$, which will give:

$$\text{(2)} \qquad \begin{cases} \delta x = \dfrac{\partial x}{\partial q_1}\delta q_1 + \dfrac{\partial x}{\partial q_2}\delta q_2 + \ldots + \dfrac{\partial x}{\partial q_k}\delta q_k + \dfrac{\partial x}{\partial q_{k+1}}\delta q_{k+1} + \ldots + \dfrac{\partial x}{\partial q_{k+p}} q_{k+p}, \\ \delta y = \dfrac{\partial y}{\partial q_1}\delta q_1 + \dfrac{\partial y}{\partial q_2}\delta q_2 + \ldots + \dfrac{\partial y}{\partial q_k}\delta q_k + \dfrac{\partial y}{\partial q_{k+1}}\delta q_{k+1} + \ldots + \dfrac{\partial y}{\partial q_{k+p}} q_{k+p},, \\ \delta z = \dfrac{\partial z}{\partial q_1}\delta q_1 + \dfrac{\partial z}{\partial q_2}\delta q_2 + \ldots + \dfrac{\partial z}{\partial q_k}\delta q_k + \dfrac{\partial z}{\partial q_{k+1}}\delta q_{k+1} + \ldots + \dfrac{\partial z}{\partial q_{k+p}} q_{k+p}. \end{cases}$$

Now suppose that we add some new constraints to the finite constraints above that depend upon time t, are expressed by p non-integrable differential relations in the parameters $q_1, q_2, \ldots, q_k, q_{k+1}, \ldots, q_{k+p}$, and when those relations are solved for the $dq_{k+1}, dq_{k+2}, \ldots, dq_{k+p}$, they will have the form:

$$\text{(3)} \qquad \begin{cases} dq_{k+1} = \alpha_1\, dq_1 + \alpha_2\, dq_2 + \cdots + \alpha_k\, dq_k + \alpha\, dt, \\ dq_{k+2} = \beta_1\, dq_1 + \beta_2\, dq_2 + \cdots + \beta_k\, dq_k + \beta\, dt, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ dq_{k+p} = \lambda_1\, dq_1 + \lambda_2\, dq_2 + \cdots + \lambda_k\, dq_k + \lambda\, dt, \end{cases}$$

in which the coefficients of $dq_1, dq_2, \ldots, dq_k, dt$ are generally functions of $q_1, q_2, \ldots, q_k, q_{k+1}, \ldots, q_{k+p}$. For a virtual displacement that is compatible with those constraints at the moment t, we have:

$$
(4) \quad \begin{cases}
\delta q_{k+1} = \alpha_1\,\delta q_1 + \alpha_2\,\delta q_2 + \cdots + \alpha_k\,\delta q_k, \\
\delta q_{k+2} = \beta_1\,\delta q_1 + \beta_2\,\delta q_2 + \cdots + \beta_k\,\delta q_k, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\delta q_{k+p} = \lambda_1\,\delta q_1 + \lambda_2\,\delta q_2 + \cdots + \lambda_k\,\delta q_k.
\end{cases}
$$

We will then obtain a virtual displacement of the system that is compatible with two sorts of constraints at the moment t when we introduce the values of δq_{k+1}, δq_{k+2}, …, δq_{k+p} in (4) into (2); hence, we will have:

$$
(5) \quad \begin{cases}
\delta x = \left(\dfrac{\partial x}{\partial q_1} + \alpha_1 \dfrac{\partial x}{\partial q_{k+1}} + \beta_1 \dfrac{\partial x}{\partial q_{k+2}} + \cdots + \lambda_1 \dfrac{\partial x}{\partial q_{k+p}}\right)\delta q_1 \\
\qquad + \left(\dfrac{\partial x}{\partial q_2} + \alpha_2 \dfrac{\partial x}{\partial q_{k+1}} + \beta_2 \dfrac{\partial x}{\partial q_{k+2}} + \cdots + \lambda_2 \dfrac{\partial x}{\partial q_{k+p}}\right)\delta q_2 \\
\qquad + \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\qquad + \left(\dfrac{\partial x}{\partial q_k} + \alpha_k \dfrac{\partial x}{\partial q_{k+1}} + \beta_k \dfrac{\partial x}{\partial q_{k+2}} + \cdots + \lambda_k \dfrac{\partial x}{\partial q_{k+p}}\right)\delta q_k, \\
\delta y = \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\delta z = \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{cases}
$$

for the displacement.

Upon taking into account all of the constraints that were imposed upon the system, its position will be completely defined at any instant if we know the parameters q_1, q_2, …, q_k at that moment, because the other p parameters q_{k+1}, q_{k+2}, …, q_{k+p} are determined by equations (3); hence, the position of the system depends upon k independent parameters $q_1, q_2, \ldots, q_k$.

The general equation of dynamics, which is deduced from d'Alembert's principle and the principle of virtual work, is:

$$
\sum m\,(x''\,\delta x + y''\,\delta y + z''\,\delta z) = \sum (X\,\delta x + Y\,\delta y + Z\,\delta z)\,.
$$

x'', y'', z'' are the second derivatives of the coordinates with respect to time, and X, Y, Z are the projections of any of the forces that are applied directly.

That equation must be satisfied for all displacements (5) that are compatible with all of the constraints. It therefore decomposes into k equations of the form:

$$
(6) \quad \sum m \Bigg[\; x''\left(\frac{\partial x}{\partial q_1} + \alpha_1 \frac{\partial x}{\partial q_{k+1}} + \beta_1 \frac{\partial x}{\partial q_{k+2}} + \cdots + \lambda_1 \frac{\partial x}{\partial q_{k+p}}\right)
$$
$$
+ y''\left(\frac{\partial y}{\partial q_1} + \alpha_1 \frac{\partial y}{\partial q_{k+1}} + \beta_1 \frac{\partial y}{\partial q_{k+2}} + \cdots + \lambda_1 \frac{\partial y}{\partial q_{k+p}}\right)
$$

$$+ z''\left(\frac{\partial z}{\partial q_1}+\alpha_1\frac{\partial z}{\partial q_{k+1}}+\beta_1\frac{\partial z}{\partial q_{k+2}}+\cdots+\lambda_1\frac{\partial z}{\partial q_{k+p}}\right)\Bigg] = Q_1 ,$$

in which Q_1 is the coefficient of dq_1 in the expression for the sum of the virtual works done by applied forces:

$$\sum(X\,\delta x+Y\,\delta y+Z\,\delta z) = Q_1\,\delta q_1 + Q_2\,\delta q_2 + \ldots + Q_k\,\delta q_k .$$

Transform the left-hand side of equation (6), which we denote by P_1 . We have:

$$\begin{aligned} P_1 = \frac{d}{dt}\sum m\Bigg[\quad & x'\left(\frac{\partial x}{\partial q_1}+\alpha_1\frac{\partial x}{\partial q_{k+1}}+\beta_1\frac{\partial x}{\partial q_{k+2}}+\cdots+\lambda_1\frac{\partial x}{\partial q_{k+p}}\right)\\ &+ y'\left(\frac{\partial y}{\partial q_1}+\alpha_1\frac{\partial y}{\partial q_{k+1}}+\beta_1\frac{\partial y}{\partial q_{k+2}}+\cdots+\lambda_1\frac{\partial y}{\partial q_{k+p}}\right)\\ &+ z'\left(\frac{\partial z}{\partial q_1}+\alpha_1\frac{\partial z}{\partial q_{k+1}}+\beta_1\frac{\partial z}{\partial q_{k+2}}+\cdots+\lambda_1\frac{\partial z}{\partial q_{k+p}}\right)\Bigg]\end{aligned}$$

$$\begin{aligned} -\sum m\Bigg\{ & x'\left[\frac{d}{dt}\left(\frac{\partial x}{\partial q_1}\right)+\frac{d}{dt}\left(\alpha_1\frac{\partial x}{\partial q_{k+1}}\right)+\frac{d}{dt}\left(\beta_1\frac{\partial x}{\partial q_{k+2}}\right)+\cdots+\frac{d}{dt}\left(\lambda_1\frac{\partial x}{\partial q_{k+p}}\right)\right]\\ &+ y'\left[\frac{d}{dt}\left(\frac{\partial y}{\partial q_1}\right)+\frac{d}{dt}\left(\alpha_1\frac{\partial y}{\partial q_{k+1}}\right)+\frac{d}{dt}\left(\beta_1\frac{\partial y}{\partial q_{k+2}}\right)+\cdots+\frac{d}{dt}\left(\lambda_1\frac{\partial y}{\partial q_{k+p}}\right)\right]\\ &+ z'\left[\frac{d}{dt}\left(\frac{\partial z}{\partial q_1}\right)+\frac{d}{dt}\left(\alpha_1\frac{\partial z}{\partial q_{k+1}}\right)+\frac{d}{dt}\left(\beta_1\frac{\partial z}{\partial q_{k+2}}\right)+\cdots+\frac{d}{dt}\left(\lambda_1\frac{\partial z}{\partial q_{k+p}}\right)\right]\Bigg\}.\end{aligned}$$

We get from equations (1) that:

$$(7)\qquad \left\{\begin{aligned} x' &= \frac{\partial x}{\partial t}+\frac{\partial x}{\partial q_1}q_1'+\frac{\partial x}{\partial q_2}q_2'+\cdots+\frac{\partial x}{\partial q_k}q_k'+\frac{\partial x}{\partial q_{k+1}}q_{k+1}'+\cdots+\frac{\partial x}{\partial q_{k+p}}q_{k+p}',\\ y' &= \frac{\partial y}{\partial t}+\frac{\partial y}{\partial q_1}q_1'+\frac{\partial y}{\partial q_2}q_2'+\cdots+\frac{\partial y}{\partial q_k}q_k'+\frac{\partial y}{\partial q_{k+1}}q_{k+1}'+\cdots+\frac{\partial y}{\partial q_{k+p}}q_{k+p}',\\ z' &= \frac{\partial z}{\partial t}+\frac{\partial z}{\partial q_1}q_1'+\frac{\partial z}{\partial q_2}q_2'+\cdots+\frac{\partial z}{\partial q_k}q_k'+\frac{\partial z}{\partial q_{k+1}}q_{k+1}'+\cdots+\frac{\partial z}{\partial q_{k+p}}q_{k+p}'.\end{aligned}\right.$$

Equations (3) give:

(8)
$$\begin{cases} q'_{k+1} = \alpha_1 q'_1 + \alpha_2 q'_2 + \cdots + \alpha_k q'_k + \alpha, \\ q'_{k+2} = \beta_1 q'_1 + \beta_2 q'_2 + \cdots + \beta_k q'_k + \beta, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ q'_{k+p} = \lambda_1 q'_1 + \lambda_2 q'_2 + \cdots + \lambda_k q'_k + \lambda. \end{cases}$$

Hence, by virtue of (7) and (8), we will have:

$$\frac{\partial x'}{\partial q'_1} = \frac{\partial x}{\partial q_1} + \alpha_1 \frac{\partial x}{\partial q_{k+1}} + \beta_1 \frac{\partial x}{\partial q_{k+2}} + \cdots + \lambda_1 \frac{\partial x}{\partial q_{k+p}},$$

$$\frac{\partial y'}{\partial q'_1} = \frac{\partial y}{\partial q_1} + \alpha_1 \frac{\partial y}{\partial q_{k+1}} + \beta_1 \frac{\partial y}{\partial q_{k+2}} + \cdots + \lambda_1 \frac{\partial y}{\partial q_{k+p}},$$

$$\frac{\partial z'}{\partial q'_1} = \frac{\partial z}{\partial q_1} + \alpha_1 \frac{\partial z}{\partial q_{k+1}} + \beta_1 \frac{\partial z}{\partial q_{k+2}} + \cdots + \lambda_1 \frac{\partial z}{\partial q_{k+p}},$$

and the function P_1 will take the form:

$$\begin{aligned} P_1 = {} & \frac{d}{dt} \sum m \left(x' \frac{\partial x'}{\partial q'_1} + y' \frac{\partial y'}{\partial q'_1} + z' \frac{\partial z'}{\partial q'_1} \right) \\ & - \sum m \left[x' \frac{d}{dt} \left(\frac{\partial x}{\partial q'_1} \right) + y' \frac{d}{dt} \left(\frac{\partial y}{\partial q'_1} \right) + z' \frac{d}{dt} \left(\frac{\partial z}{\partial q'_1} \right) \right] \\ & - \sum m \left[x' \frac{d}{dt} \left(\alpha_1 \frac{\partial x}{\partial q_{k+1}} + \beta_1 \frac{\partial x}{\partial q_{k+2}} + \cdots + \lambda_1 \frac{\partial x}{\partial q_{k+p}} \right) \right. \\ & \qquad + x' \frac{d}{dt} \left(\alpha_1 \frac{\partial x}{\partial q_{k+1}} + \beta_1 \frac{\partial x}{\partial q_{k+2}} + \cdots + \lambda_1 \frac{\partial x}{\partial q_{k+p}} \right) \\ & \qquad \left. + z' \frac{d}{dt} \left(\alpha_1 \frac{\partial z}{\partial q_{k+1}} + \beta_1 \frac{\partial z}{\partial q_{k+2}} + \cdots + \lambda_1 \frac{\partial z}{\partial q_{k+p}} \right) \right]. \end{aligned}$$

In equations (1), x, y, z are functions of t, q_1, q_2, …, q_k, q_{k+1}, …, q_{k+p} ; hence, $\frac{\partial x}{\partial q_1}, \frac{\partial y}{\partial q_1}, \frac{\partial z}{\partial q_1}$ will be functions of the same parameters. If we take the derivatives with respect to time of the latter quantities and take equations (8) into account then we will have:

$$\frac{d}{dt} \left(\frac{\partial x}{\partial q_1} \right) = \frac{\partial^2 x}{\partial q_1 \, \partial t} + \frac{\partial^2 x}{\partial q_1^2} q'_1 + \frac{\partial^2 x}{\partial q_1 \, \partial q_2} q'_2 + \cdots + \frac{\partial^2 x}{\partial q_1 \, \partial q_k} q'_k$$

$$+\ \frac{\partial^2 x}{\partial q_1\,\partial q_{k+1}}(\alpha_1 q_1' + \alpha_2 q_2' + \cdots + \alpha_k q_k' + \alpha)$$

$$+\ \frac{\partial^2 x}{\partial q_2\,\partial q_{k+1}}(\beta_1 q_1' + \beta_2 q_2' + \cdots + \beta_k q_k' + \beta)$$

$$+\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+\ \frac{\partial^2 x}{\partial q_1\,\partial q_{k+p}}(\lambda_1 q_1' + \lambda_2 q_2' + \cdots + \lambda_k q_k' + \lambda)\,,$$

$$\frac{d}{dt}\left(\frac{\partial y}{\partial q_1}\right) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$\frac{d}{dt}\left(\frac{\partial z}{\partial q_1}\right) = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.$$

On the other hand, if we take the partial derivatives with respect to q_1 of x', y', z', as given by equations (7), and take equations (8) into account then we will get:

$$\frac{\partial x'}{\partial q_1} = \frac{\partial^2 x}{\partial q_1\,\partial t} + \frac{\partial^2 x}{\partial q_1^2} q_1' + \frac{\partial^2 x}{\partial q_1\,\partial q_2} q_2' + \cdots + \frac{\partial^2 x}{\partial q_1\,\partial q_k} q_k'$$

$$+\ \frac{\partial^2 x}{\partial q_1\,\partial q_{k+1}}(\alpha_1 q_1' + \alpha_2 q_2' + \cdots + \alpha_k q_k' + \alpha)$$

$$+\ \frac{\partial^2 x}{\partial q_2\,\partial q_{k+1}}(\beta_1 q_1' + \beta_2 q_2' + \cdots + \beta_k q_k' + \beta)$$

$$+\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$+\ \frac{\partial^2 x}{\partial q_1\,\partial q_{k+p}}(\lambda_1 q_1' + \lambda_2 q_2' + \cdots + \lambda_k q_k' + \lambda)\,,$$

$$+\frac{\partial x}{\partial q_{k+1}}\frac{\partial q_{k+1}'}{\partial q_1} + \frac{\partial x}{\partial q_{k+2}}\frac{\partial q_{k+2}'}{\partial q_1} + \cdots + \frac{\partial x}{\partial q_{k+p}}\frac{\partial q_{k+p}'}{\partial q_1}\,,$$

$$\frac{\partial y'}{\partial q_1} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

$$\frac{\partial z'}{\partial q_1} = \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots.$$

Hence:

$$\frac{d}{dt}\left(\frac{\partial x}{\partial q_1}\right) = \frac{\partial x'}{\partial q_1} - \left(\frac{\partial x}{\partial q_{k+1}}\frac{\partial q_{k+1}'}{\partial q_1} + \frac{\partial x}{\partial q_{k+2}}\frac{\partial q_{k+2}'}{\partial q_1} + \cdots + \frac{\partial x}{\partial q_{k+p}}\frac{\partial q_{k+p}'}{\partial q_1}\right),$$

$$\frac{d}{dt}\left(\frac{\partial y}{\partial q_1}\right) = \frac{\partial y'}{\partial q_1} - \left(\frac{\partial y}{\partial q_{k+1}}\frac{\partial q_{k+1}'}{\partial q_1} + \frac{\partial y}{\partial q_{k+2}}\frac{\partial q_{k+2}'}{\partial q_1} + \cdots + \frac{\partial y}{\partial q_{k+p}}\frac{\partial q_{k+p}'}{\partial q_1}\right),$$

$$\frac{d}{dt}\left(\frac{\partial z}{\partial q_1}\right) = \frac{\partial z'}{\partial q_1} - \left(\frac{\partial z}{\partial q_{k+1}}\frac{\partial q'_{k+1}}{\partial q_1} + \frac{\partial z}{\partial q_{k+2}}\frac{\partial q'_{k+2}}{\partial q_1} + \cdots + \frac{\partial z}{\partial q_{k+p}}\frac{\partial q'_{k+p}}{\partial q_1}\right).$$

Upon taking these expressions for $\frac{d}{dt}\left(\frac{\partial x}{\partial q_1}\right)$, $\frac{d}{dt}\left(\frac{\partial y}{\partial q_1}\right)$, $\frac{d}{dt}\left(\frac{\partial z}{\partial q_1}\right)$ into account, along with equations (8), we will have:

$$\frac{\partial q'_{k+1}}{\partial q'_1} = \alpha_1\,, \qquad \frac{\partial q'_{k+2}}{\partial q'_1} = \beta_1\,, \qquad \ldots, \qquad \frac{\partial q'_{k+p}}{\partial q'_1} = \lambda_1\,,$$

and from equations (7):

$$\begin{aligned}
&\frac{\partial x'}{\partial q'_{k+1}} = \frac{\partial x}{\partial q_{k+1}}, && \frac{\partial x'}{\partial q'_{k+2}} = \frac{\partial x}{\partial q_{k+2}}, && \ldots, && \frac{\partial x'}{\partial q'_{k+p}} = \frac{\partial x}{\partial q_{k+p}},\\
&\frac{\partial y'}{\partial q'_{k+1}} = \frac{\partial y}{\partial q_{k+1}}, && \frac{\partial y'}{\partial q'_{k+2}} = \frac{\partial y}{\partial q_{k+2}}, && \ldots, && \frac{\partial y'}{\partial q'_{k+p}} = \frac{\partial y}{\partial q_{k+p}},\\
&\frac{\partial z'}{\partial q'_{k+1}} = \frac{\partial z}{\partial q_{k+1}}, && \frac{\partial z'}{\partial q'_{k+2}} = \frac{\partial z}{\partial q_{k+2}}, && \ldots, && \frac{\partial z'}{\partial q'_{k+p}} = \frac{\partial z}{\partial q_{k+p}},
\end{aligned}$$

so the function P will take the form:

$$\begin{aligned}
P_1 = {} & \frac{d}{dt}\sum m\left(x'\frac{\partial x'}{\partial q'_1} + y'\frac{\partial y'}{\partial q'_1} + z'\frac{\partial z'}{\partial q'_1}\right)\\
& - \sum m\left(x'\frac{\partial x'}{\partial q_1} + y'\frac{\partial y'}{\partial q_1} + z'\frac{\partial z'}{\partial q_1}\right)\\
& + \sum m\Bigg[x'\left(\frac{\partial x'}{\partial q'_{k+1}}\frac{\partial q'_{k+1}}{\partial q_1} + \frac{\partial x'}{\partial q'_{k+2}}\frac{\partial q'_{k+2}}{\partial q_1} + \cdots + \frac{\partial x'}{\partial q'_{k+p}}\frac{\partial q'_{k+p}}{\partial q_1}\right)\\
& \qquad + y'\left(\frac{\partial y'}{\partial q'_{k+1}}\frac{\partial q'_{k+1}}{\partial q_1} + \frac{\partial y'}{\partial q'_{k+2}}\frac{\partial q'_{k+2}}{\partial q_1} + \cdots + \frac{\partial y'}{\partial q'_{k+p}}\frac{\partial q'_{k+p}}{\partial q_1}\right)\\
& \qquad + z'\left(\frac{\partial z'}{\partial q'_{k+1}}\frac{\partial q'_{k+1}}{\partial q_1} + \frac{\partial z'}{\partial q'_{k+2}}\frac{\partial q'_{k+2}}{\partial q_1} + \cdots + \frac{\partial z'}{\partial q'_{k+p}}\frac{\partial q'_{k+p}}{\partial q_1}\right)\Bigg]\\
& - \sum m\Bigg[x'\frac{d}{dt}\left(\frac{\partial x'}{\partial q'_{k+1}}\frac{\partial q'_{k+1}}{\partial q'_1} + \frac{\partial x'}{\partial q'_{k+2}}\frac{\partial q'_{k+2}}{\partial q'_1} + \cdots + \frac{\partial x'}{\partial q'_{k+p}}\frac{\partial q'_{k+p}}{\partial q'_1}\right)\\
& \qquad + y'\frac{d}{dt}\left(\frac{\partial y'}{\partial q'_{k+1}}\frac{\partial q'_{k+1}}{\partial q'_1} + \frac{\partial y'}{\partial q'_{k+2}}\frac{\partial q'_{k+2}}{\partial q'_1} + \cdots + \frac{\partial y'}{\partial q'_{k+p}}\frac{\partial q'_{k+p}}{\partial q'_1}\right)
\end{aligned}$$

$$+z'\frac{d}{dt}\left(\frac{\partial z'}{\partial q'_{k+1}}\frac{\partial q'_{k+1}}{\partial q'_1}+\frac{\partial z'}{\partial q'_{k+2}}\frac{\partial q'_{k+2}}{\partial q'_1}+\cdots+\frac{\partial z'}{\partial q'_{k+p}}\frac{\partial q'_{k+p}}{\partial q'_1}\right)\Bigg].$$

Upon denoting the last sum in the function P_1 by M, for the moment, we will have:

$$M=\frac{d}{dt}\sum m\left[x'\left(\frac{\partial x'}{\partial q'_{k+1}}\frac{\partial q'_{k+1}}{\partial q'_1}+\frac{\partial x'}{\partial q'_{k+2}}\frac{\partial q'_{k+2}}{\partial q'_1}+\cdots+\frac{\partial x'}{\partial q'_{k+p}}\frac{\partial q'_{k+p}}{\partial q'_1}\right)\right.$$
$$+\ y'\left(\frac{\partial y'}{\partial q'_{k+1}}\frac{\partial q'_{k+1}}{\partial q'_1}+\frac{\partial y'}{\partial q'_{k+2}}\frac{\partial q'_{k+2}}{\partial q'_1}+\cdots+\frac{\partial y'}{\partial q'_{k+p}}\frac{\partial q'_{k+p}}{\partial q'_1}\right)$$
$$\left.+z'\left(\frac{\partial z'}{\partial q'_{k+1}}\frac{\partial q'_{k+1}}{\partial q'_1}+\frac{\partial z'}{\partial q'_{k+2}}\frac{\partial q'_{k+2}}{\partial q'_1}+\cdots+\frac{\partial z'}{\partial q'_{k+p}}\frac{\partial q'_{k+p}}{\partial q'_1}\right)\right]$$

$$-\sum m\left[x''\left(\frac{\partial x'}{\partial q'_{k+1}}\frac{\partial q'_{k+1}}{\partial q'_1}+\frac{\partial x'}{\partial q'_{k+2}}\frac{\partial q'_{k+2}}{\partial q'_1}+\cdots+\frac{\partial x'}{\partial q'_{k+p}}\frac{\partial q'_{k+p}}{\partial q'_1}\right)\right.$$
$$+\ y''\left(\frac{\partial y'}{\partial q'_{k+1}}\frac{\partial q'_{k+1}}{\partial q'_1}+\frac{\partial y'}{\partial q'_{k+2}}\frac{\partial q'_{k+2}}{\partial q'_1}+\cdots+\frac{\partial y'}{\partial q'_{k+p}}\frac{\partial q'_{k+p}}{\partial q'_1}\right)$$
$$\left.+z''\left(\frac{\partial z'}{\partial q'_{k+1}}\frac{\partial q'_{k+1}}{\partial q'_1}+\frac{\partial z'}{\partial q'_{k+2}}\frac{\partial q'_{k+2}}{\partial q'_1}+\cdots+\frac{\partial z'}{\partial q'_{k+p}}\frac{\partial q'_{k+p}}{\partial q'_1}\right)\right].$$

However, the last sum in that expression for M can take another form. Upon taking equations (7) and (8) into account, we will have:

$$\frac{\partial x'}{\partial q'_{k+1}}=\frac{\partial x''}{\partial q''_{k+1}},\qquad \frac{\partial x'}{\partial q'_{k+2}}=\frac{\partial x''}{\partial q''_{k+2}},\qquad \ldots,\qquad \frac{\partial x'}{\partial q'_{k+p}}=\frac{\partial x''}{\partial q''_{k+p}},$$

$$\frac{\partial y'}{\partial q'_{k+1}}=\frac{\partial y''}{\partial q''_{k+1}},\qquad \frac{\partial y'}{\partial q'_{k+2}}=\frac{\partial y''}{\partial q''_{k+2}},\qquad \ldots,\qquad \frac{\partial y'}{\partial q'_{k+p}}=\frac{\partial y''}{\partial q''_{k+p}},$$

$$\frac{\partial z'}{\partial q'_{k+1}}=\frac{\partial z''}{\partial q''_{k+1}},\qquad \frac{\partial z'}{\partial q'_{k+2}}=\frac{\partial z''}{\partial q''_{k+2}},\qquad \ldots,\qquad \frac{\partial z'}{\partial q'_{k+p}}=\frac{\partial z''}{\partial q''_{k+p}},$$

$$\frac{\partial q'_{k+1}}{\partial q'_1}=\frac{\partial q''_{k+1}}{\partial q''_1},\qquad \frac{\partial q'_{k+2}}{\partial q'_1}=\frac{\partial q''_{k+2}}{\partial q''_1},\qquad \ldots,\qquad \frac{\partial q'_{k+p}}{\partial q'_1}=\frac{\partial q''_{k+p}}{\partial q''_1}.$$

We will then have the following expression for P_1 :

$$
\begin{aligned}
P_1 = {} & \frac{d}{dt}\sum m\left(x'\frac{\partial x'}{\partial q_1'} + y'\frac{\partial y'}{\partial q_1'} + z'\frac{\partial z'}{\partial q_1'}\right) \\
& - \sum m\left(x'\frac{\partial x'}{\partial q_1} + y'\frac{\partial y'}{\partial q_1} + z'\frac{\partial z'}{\partial q_1}\right) \\
& + \sum m\left[x'\left(\frac{\partial x'}{\partial q_{k+1}'}\frac{\partial q_{k+1}'}{\partial q_1} + \frac{\partial x'}{\partial q_{k+2}'}\frac{\partial q_{k+2}'}{\partial q_1} + \cdots + \frac{\partial x'}{\partial q_{k+p}'}\frac{\partial q_{k+p}'}{\partial q_1}\right)\right. \\
& \qquad + y'\left(\frac{\partial y'}{\partial q_{k+1}'}\frac{\partial q_{k+1}'}{\partial q_1} + \frac{\partial y'}{\partial q_{k+2}'}\frac{\partial q_{k+2}'}{\partial q_1} + \cdots + \frac{\partial y'}{\partial q_{k+p}'}\frac{\partial q_{k+p}'}{\partial q_1}\right) \\
& \qquad \left. + z'\left(\frac{\partial z'}{\partial q_{k+1}'}\frac{\partial q_{k+1}'}{\partial q_1} + \frac{\partial z'}{\partial q_{k+2}'}\frac{\partial q_{k+2}'}{\partial q_1} + \cdots + \frac{\partial z'}{\partial q_{k+p}'}\frac{\partial q_{k+p}'}{\partial q_1}\right)\right] \\
& - \frac{d}{dt}\sum m\left[x'\left(\frac{\partial x'}{\partial q_{k+1}'}\frac{\partial q_{k+1}'}{\partial q_1'} + \frac{\partial x'}{\partial q_{k+2}'}\frac{\partial q_{k+2}'}{\partial q_1'} + \cdots + \frac{\partial x'}{\partial q_{k+p}'}\frac{\partial q_{k+p}'}{\partial q_1'}\right)\right. \\
& \qquad + y'\left(\frac{\partial y'}{\partial q_{k+1}'}\frac{\partial q_{k+1}'}{\partial q_1'} + \frac{\partial y'}{\partial q_{k+2}'}\frac{\partial q_{k+2}'}{\partial q_1'} + \cdots + \frac{\partial y'}{\partial q_{k+p}'}\frac{\partial q_{k+p}'}{\partial q_1'}\right) \\
& \qquad \left. + z'\left(\frac{\partial z'}{\partial q_{k+1}'}\frac{\partial q_{k+1}'}{\partial q_1'} + \frac{\partial z'}{\partial q_{k+2}'}\frac{\partial q_{k+2}'}{\partial q_1'} + \cdots + \frac{\partial z'}{\partial q_{k+p}'}\frac{\partial q_{k+p}'}{\partial q_1'}\right)\right] \\
& + \sum m\left[x''\left(\frac{\partial x''}{\partial q_{k+1}''}\frac{\partial q_{k+1}''}{\partial q_1''} + \frac{\partial x''}{\partial q_{k+2}''}\frac{\partial q_{k+2}''}{\partial q_1''} + \cdots + \frac{\partial x''}{\partial q_{k+p}''}\frac{\partial q_{k+p}''}{\partial q_1''}\right)\right. \\
& \qquad + y''\left(\frac{\partial y''}{\partial q_{k+1}''}\frac{\partial q_{k+1}''}{\partial q_1''} + \frac{\partial y''}{\partial q_{k+2}''}\frac{\partial q_{k+2}''}{\partial q_1''} + \cdots + \frac{\partial y''}{\partial q_{k+p}''}\frac{\partial q_{k+p}''}{\partial q_1''}\right) \\
& \qquad \left. + z''\left(\frac{\partial z''}{\partial q_{k+1}''}\frac{\partial q_{k+1}''}{\partial q_1''} + \frac{\partial z''}{\partial q_{k+2}''}\frac{\partial q_{k+2}''}{\partial q_1''} + \cdots + \frac{\partial z''}{\partial q_{k+p}''}\frac{\partial q_{k+p}''}{\partial q_1''}\right)\right],
\end{aligned}
$$

and equation (6) will have the form:

$$(6') \qquad P_1 = Q_1 .$$

Let T denote the semi-*vis viva* of the system when we take into account the finite and differential constraints that are imposed upon it, and let T_0 denote the semi-*vis viva* of the system when we take into account only the finite constraints. The function T is obtained from T_0 by substituting the values that are defined by equations (8) for the q_{k+1}', q_{k+2}', …, q_{k+p}' in it. The function T_0 is composed of two parts: One of them contains terms that depend upon q_{k+1}', q_{k+2}', …, q_{k+p}', and we denote it by T_1 . The other one contains the other terms, and we denote it by T_0'. In that manner, we have:

$$T_0 = T_1 = T_0' .$$

On the other hand, let S_0 denote the semi-energy of acceleration of the system when one takes into account only the finite constraints, and let S_1 be the function that is obtained from S_0 by keeping only the terms that contain the quantities q''_{k+1}, q''_{k+2}, …, q''_{k+p} that are also defined by equations (8) when one differentiates them with respect to t.

The equation of motion (6′) will then take the form:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial q_1'}\right) - \frac{\partial T}{\partial q_1} + \frac{\partial T_1}{\partial q_1} - \frac{d}{dt}\left(\frac{\partial T_1}{\partial q_1'}\right) + \frac{\partial S_1}{\partial q_1''} = Q_1 ,$$

or rather:

$$\frac{d}{dt}\left(\frac{\partial T_0'}{\partial q_1'}\right) - \frac{\partial T_0'}{\partial q_1} + \frac{\partial S_1}{\partial q_1''} = Q_1 .$$

Hence, the equations of motion of the system are:

(9) $$\frac{d}{dt}\left(\frac{\partial T}{\partial q_\alpha'}\right) - \frac{\partial T}{\partial q_\alpha} + \frac{\partial T_1}{\partial q_\alpha} - \frac{d}{dt}\left(\frac{\partial T_1}{\partial q_\alpha'}\right) + \frac{\partial S_1}{\partial q_\alpha''} = Q_\alpha \qquad (\alpha = 1, 2, \ldots, k)$$

or

(10) $$\frac{d}{dt}\left(\frac{\partial T_0'}{\partial q_\alpha'}\right) - \frac{\partial T_0'}{\partial q_\alpha} + \frac{\partial S_1}{\partial q_\alpha''} = Q_\alpha \qquad (\alpha = 1, 2, \ldots, k),$$

in which is a function of only the true independent parameters q_1, q_2, …, q_k , and their derivatives, and S_1 is a function of only the second derivatives of the dependent parameters, which are determined as functions of the second derivatives of the independent parameters by means of equations (8).

In most cases, the functions T_0 and S_1 are easier to determine than the part of the function S that Appell introduced that gives the semi-energy of acceleration by taking into account all of the constraints that are imposed on the system.

We shall explain that with some examples that present themselves in non-holonomic systems.

Upon writing the differential equations of motion in the form (9), we deduce the following corollaries:

1. If any of the independent parameters does not enter into equation (8) then the differential equation for that parameter will be obtained by Lagrange's method.

However, one can also obtain the differential equation for any of the independent parameters by Lagrange's method – for example, the parameter q_s that enters into equations (8) – provided that we have:

$$\frac{\partial T_1}{\partial q_s} - \frac{d}{dt}\left(\frac{\partial T_1}{\partial q'_s}\right) + \frac{\partial S_1}{\partial q''_s} = 0.$$

2. The expression:

$$\frac{\partial T_1}{\partial q_\alpha} - \frac{d}{dt}\left(\frac{\partial T_1}{\partial q'_\alpha}\right) + \frac{\partial S_1}{\partial q''_\alpha}$$

contains terms that must be added to the left-hand side of the equation of motion that one deduces for the parameter q_α by Lagrange's method in order to have the true differential equation of motion that pertains to that parameter.

3. Example 1. – A circle of radius *a* and mass unity (viz., a hoop) rolls without friction or slipping on a fixed horizontal plane ([1]).

Take two fixed axes *Ox* and *Oy* in the horizontal plane *xOy* and draw a third axis *Oz* through the point *O* that is perpendicular to the plane and points upward. Draw three axes $Gx'y'z'$ through the center of gravity *G* of the circle that are parallel to the axes *Oxyz*. Let *GX* be the intersection of the plane of the circle with the plane $x'G'y'$, let *GY* denote the axis that passes through *G* and the point of contact *H* of the circle and the plane *xOy*, and finally let *GZ* be the axis of the hoop. If we let *GD* denote a line that is invariantly coupled to the circle and situated in its plane then the position of the circle around *G* will be defined by angles:

$$\widehat{x'X} = \psi, \qquad \widehat{XD} = \varphi, \qquad \widehat{z'Z} = \theta.$$

The projections of the instantaneous rotation of the rectangular trihedron *GXYZ* onto the axes *GX*, *GY*, *GZ* are:

$$(11) \qquad P = \theta', \qquad Q = \psi' \sin \theta', \qquad R = \psi' \cos \theta',$$

and those of the instantaneous rotation of the solid body for its motion around *G* are:

$$(12) \qquad p = \theta', \qquad q = \psi' \sin \theta', \qquad r = \psi' \cos \theta' + \omega',$$

so:

$$(13) \qquad P = p, \qquad Q = q, \qquad R = q \cos \tan \theta.$$

If *u*, *v*, *w* are the projections of the velocity V_0 of the point *G* onto the axes *GXYZ* then in order to write down that the idea the circle rolls without slipping on the plane *yOx*, one must express the idea that the velocity of the material point *H* (0, – *a*, 0) is equal to zero; one will then have:

$$u + a\,r = 0, \qquad v = 0, \qquad w - a\,p = 0.$$

([1]) Paul APPELL, *Traité de Mécanique rationelle*, t. II, pp. 241, 372, 381.

Those equations are equations (8).

If θ, ψ, φ are independent parameters then the function T_0 in that case will be the semi-*vis viva* for the motion around G :

$$T_0' = \tfrac{1}{2}[A(p^2 + q^2) + C r^2)],$$

and the function S_1 , which is the semi-energy of acceleration when all of the mass of the circle is concentrated at G, will be:

$$S_1 = \tfrac{1}{2}[(u' + Qw)^2 + (w' - Q u)^2] + \ldots$$

The applied force, which is the weight Mg, is derived from the force function:

$$u = - M g a \sin \theta.$$

The equation for φ is then:

$$\frac{d}{dt}(C r) + (u' + Q w)(- a) = 0$$

or

(I) $$(C + a^2) r' - a^2 pq = 0.$$

The equation for θ is:

$$\frac{d}{dt}(A q \sin \theta + C r \cos \theta) + (u' + Q w)(a \cos \theta) = 0,$$

or when one takes (I) into account, one will have:

(II) $$A q' + (A R - C r) p = 0.$$

Finally, the equation for θ is:

$$\frac{d}{dt}(A p) - A g \psi' \cos \theta + C r \psi' \sin \theta + (w' - Q u) a = - a g \cos \theta$$

or

(III) $$(A + a^2) p' - (A R - C r) q + a^2 q r = - a g \cos \theta;$$

equations (I), (II), (III) are the equations of motion.

It is easy to see that the Lagrange method will give equation (III) when it is applied to the parameter θ. Indeed, the function T will be:

$$T = \tfrac{1}{2}[(A + a^2) p^2 + A g^2 + (C + a^2) r^2]$$

in that case, and the function T_1 will be:

$$T_1 = \tfrac{1}{2}(u^2 + w^2)\ ;$$

obviously, we will have:

$$\frac{\partial T_1}{\partial \theta} - \frac{d}{dt}\left(\frac{\partial T_1}{\partial \theta'}\right) + \frac{\partial S_1}{\partial \theta''} = 0.$$

4. Example II. – A heavy, homogeneous, solid body of revolution rolls without slipping on a moving horizontal plane that rotates uniformly around a fixed vertical axis ([1]).

Let Ox and Oy be two perpendicular axes in the moving horizontal plane, and let OZ be a third axis that coincides with the fixed vertical axis and is directed upwards.

The relative motion of the body will be known when we know the motion of its center of gravity G and its motion around G.

The motion of G will be defined when we know the coordinates ξ, η, ζ of that point with respect to the axes $Oxyz$ as a function of time.

The point G of the body is situated on the rotational axis that we denote by GZ.

The relative motion around G – i.e., around the axes $G\,x'y'z'$, which are parallel to the axes $Oxyz$ – is a motion of a body around a fixed point.

Take a horizontal axis GX that is perpendicular to the vertical plane $Z'\,G\,Z$ and another axis GY that is perpendicular to the plane XGZ. In that manner, we will determine a rectangular trihedron $GXYZ$ whose position with respect to the trihedron $G\,x'y'z'$ is determined by the angles $\psi = \widehat{X\,G\,x'}$, $\theta = \widehat{z'G\,Z}$.

The motion relative to the body around G is defined by the angles:

$$\psi,\quad \theta,\quad \varphi = \widehat{X\,G\,D}\,,$$

in which GD is a line that is coupled to the body and situated in the plane XGY.

The condition for the body to touch the moving horizontal plane is expressed by the finite constraint:

$$\zeta = f(\theta),$$

in which f is a given function.

The parameters $q_1, q_2, \ldots, q_k, \ldots, q_{k+p}$ in this case are:

$$\varphi,\ \psi,\ \theta,\ \xi,\ \eta\,.$$

The components of the rotation of the trihedron $GXYZ$ and those of the rotation of the body for its motion around G are given by equations (11) and (12), respectively.

The body rolls on the horizontal plane when its velocity relative to the material point of contact is zero. Let us find the coordinates of that point. The meridian curve, which

([1]) Iv. TSENOFF, "Mouvement sans frottement d'un corps solide pesant der révolution sur un plan horizontal," Annuaire de l'Université de Sofia, 1916, 1917, 1918.

generates the surface of revolution, is situated in the vertical plane *YGZ*. One will obtain it as the envelope of the tangent. The equation of the tangent at *H* is:

$$- y \sin \theta - z \cos \theta = \zeta;$$

from that equation and the equation:

$$- y \cos \theta + z \sin \theta = \xi',$$

one will then get:

$$(14)\qquad \begin{cases} x = 0, \\ y = - f(\theta) \sin\theta - f'(\theta)\cos\theta, \\ z = - f(\theta)\cos\theta + f'(\theta)\sin\theta \end{cases}$$

for the coordinates of the point *M* with respect to the axes *GXYZ*. The relative velocity of that point is the resultant of the velocity of translation of the axes *G x′y′z′* and the velocity that is due to the rotation of the body around *G*. The projections of that velocity onto the axes *GXYZ* are:

$$\begin{aligned} \xi' \cos \psi + \eta' \sin \psi + q\, z - r\, y &= 0, \\ - \xi' \sin \psi \cos \theta + \eta' \cos \psi \cos \theta + f' p \sin \theta - p\, z &= 0, \\ \xi' \sin \psi \sin \theta - \eta' \cos \psi \sin \theta + f' p \cos \theta + p\, y &= 0, \end{aligned}$$

or, upon taking into account the values of *y* and *z*, those three equations will reduce to the following two:

$$\begin{aligned} \xi' &= q \cos \psi (f \cos \theta - f' \sin \theta) - r \cos \psi (f \sin \theta + f' \cos \theta) + p\, f \sin \psi, \\ \eta' &= q \sin \psi (f \cos \theta - f' \sin \theta) - r \sin \psi (f \sin \theta + f' \cos \theta) + p\, f \cos \psi, \end{aligned}$$

which correspond to equations (8).

The independent parameters are φ, θ, ψ and the dependent ones are – ξ, η.

The function T_0' is found immediately in this case; it is:

$$T_0' = \tfrac{1}{2} [A\, p^2 + A\, (q + \mu \sin \theta)^2 + C\, (r + \mu \cos \theta)^2] + M f'^2 p^2,$$

in which μ is the velocity with which the horizontal plane turns around the fixed vertical axis. The function S_1 exhibits the part that comes from the absolute energy of acceleration of G_1, which is a function that is also easy to find:

$$S_1 = \tfrac{1}{2} M\, [\xi''^2 + \eta''^2 - 2\xi''(2\mu\, \eta' + \mu^2\, \xi) + 2\eta''(2\mu\, \xi' - \mu^2\, \eta)] + \ldots$$

The equation for φ will then be:

$$\frac{d}{dt} C\, (r + \mu \cos \theta) + \frac{\partial S_1}{\partial \xi''}\frac{\partial \xi''}{\partial r'}\frac{\partial r'}{\partial \varphi''} + \frac{\partial S_1}{\partial \eta''}\frac{\partial \eta''}{\partial r'}\frac{\partial r'}{\partial \varphi''} = 0$$

or

$$\text{(I)} \qquad C\,(r' - \mu \sin\theta\, p) - M\,(\xi'' - 2\,\mu\,\eta' - \mu^2 \xi) \cos\psi\,(f \sin\theta + f' \cos\theta) \\ - M\,(\eta'' + 2\,\mu\,\xi' - \mu^2 \eta) \sin\psi\,(f \sin\theta + f' \cos\theta) = 0.$$

The equation for ψ will be:

$$\frac{d}{dt}[A\,(q + \mu \sin\theta) \sin\theta + C\,(r + \mu \cos\theta) \cos\theta] \\ + \frac{\partial S_1}{\partial \xi''}\left(\frac{\partial \xi''}{\partial q'}\frac{\partial q'}{\partial \psi''} + \frac{\partial \xi''}{\partial r'}\frac{\partial r'}{\partial \psi''}\right) + \frac{\partial S_1}{\partial \eta''}\left(\frac{\partial \eta''}{\partial q'}\frac{\partial q'}{\partial \psi''} + \frac{\partial \eta''}{\partial r'}\frac{\partial r'}{\partial \psi''}\right) = 0,$$

or, upon taking into account equation (I), we will get:

$$\text{(II)} \qquad A\,q' + (A\,q \cot\theta - C\,r)\,p + (2A - C)\,\mu\,p \cos\theta \\ + (f \cos\theta - f' \sin\theta)\,[M\,(\xi'' - 2\mu\eta' - \mu^2\xi) \cos\psi + M\,(\eta'' + 2\mu\xi' - \mu^2\eta) \sin\psi] \\ = 0.$$

Finally, the equation for q will be:

$$\frac{d}{dt}(A + M f'^2)\,p - A\,(q + \mu \sin\theta)\,(\psi' \cos\theta + \mu \cos\theta) - C\,(r + \mu \cos\theta)(-\psi' \sin\theta - \mu \sin\theta) \\ - M f' f'' p^2 + \frac{\partial S_1}{\partial \xi''}\frac{\partial \xi''}{\partial p'} + \frac{\partial S_1}{\partial \eta''}\frac{\partial \eta''}{\partial p'} = -M\,g\,f'$$

or

$$\text{(III)} \qquad (A + M f^2)\,p' + M f' f'' p^2 - A\,(q + \mu \sin\theta)\,(q \cot\theta + \mu \cos\theta) \\ + C\,(r + \mu \cos\theta)(q + \mu \sin\theta) + M\,(\xi'' - 2\mu\eta' - \mu^2\xi)\,f \sin\psi \\ - M\,(\eta'' + 2\mu\xi' - \mu^2\eta)\,f \cos\psi = -M\,g\,f'.$$

In the case where the plane is fixed, we have the following problem:

A heavy solid body of revolution rolls without slipping on a fixed horizontal plane ([1]).

The equations of motion are obtained from (I), (II), (III) by setting $\mu = 0$ in them. They are:

$$C\,r' - (f \sin\theta + f' \cos\theta)\,M\,(\xi'' \cos\psi + \eta'' \sin\psi) = 0,$$

$$A\,q' + (A\,q \cot\theta - C\,r)\,p + (f \cos\theta - f' \sin\theta)\,M\,(\xi'' \cos\psi + \eta'' \sin\psi) = 0,$$

(1) Paul APPELL, "Développement sur une forme nouvelle des équations de la Dynamique," J. Math. pures appl. (1900), pp. 33.

$$(A + M f'^2)\, p' + M f' f'' p^2 - (A \cot\theta - C r)\, q + M f\, (\xi'' \sin\psi - \eta'' \cos\psi) = - M g f'.$$

However, by virtue of equations (14), we will have:

$$C r' + M y\, (\xi'' \cos\psi + \eta'' \sin\psi) = 0,$$

$$A q' + (A q \cot\theta - C r)\, p - M z\, (\xi'' \cos\psi + \eta'' \sin\psi) = 0$$

for those equations.

Appell gave another form to those two equations by introducing the projections u, v, w or the velocity of the center of gravity G onto the axes GX, GY, GZ.

Let us find those equations. The function T_0' will be:

$$2T_0 = M\,(u^2 + v^2 + w^2) + A\,(p^2 + q^2) + C\, r^2 .$$

The equations that express the condition that the body must roll will be:

$$u + q z - r y = 0, \qquad v - p z = 0, \qquad w + p y = 0.$$

Hence:

$$u = r y - q z, \qquad v = p z, \qquad w = - p y ;$$

those equations correspond to equations (8).

Hence:

$$T_0' = \tfrac{1}{2}\,[A\,(p^2 + q^2) + C\, r^2] + \ldots$$

reduces to just the semi-*vis viva* for the motion around G.

The function S_1 is the semi-energy of acceleration of the entire mass when it is concentrated at G:

$$S_1 = \tfrac{1}{2} M\, [(u' + q w - R v)^2 + (v' + R u - p w)^2 + (w' + p v - q u)^2].$$

The equation for φ is:

$$\frac{d}{dt}(C r) + M\,(u' + q w - R v)\, y = 0$$

or

(IV) $$C r' + M u' + M\,(q y + R z)\, w = 0 .$$

The equation for ψ is:

$$\frac{d}{dt}(A q \sin\theta + C r \cos\theta) + M\,(u' + q w - R v)\,(y \cos\theta - z \sin\theta) = 0$$

or

$A q' \sin\theta + C r' \cos\theta + A p q \cos\theta - C r p \sin\theta$

$$+ M (u' + q\,w - R\,v)\, y \cos\theta - M (u' + q\,w - R\,v)\, y \cos\theta - M (u' + q\,w - R\,v)\, z \sin\theta = 0,$$

or finally, upon taking into account equation (IV) and dividing the two sides by sin θ, we will get:

$$\text{(V)} \qquad A\,q' + (A\,q \cot\theta - C\,r)\,p - M\,u'z + M\,(q\,y + R\,z)\,v = 0.$$

(IV) and (V) are the desired equations.

The equation above for θ can be replaced with the integral of the *vis viva*.

5. Example III. – A sphere rolls without slipping on a horizontal plane that turns with a constant angular velocity μ around a vertical axis ([1]).

That problem is a special case of the preceding one. Here, $\zeta = a$, where a is the radius of the sphere. If we take φ, ψ, θ to be independent parameters then we will proceed as in § **4**. However, we shall take the independent parameters to be ξ, η, ζ.

If we equate to zero the projections onto the axes Gx', y', z' of the velocity relative to the contact point H with coordinates 0, 0, – a then, as with equations (8), we will get the following equations:

$$\xi' - a\,q = 0,$$
$$\eta' + a\,p = 0,$$

in which:

$$p = \theta' \cos\psi + \varphi' \sin\theta \sin\psi,$$
$$q = \theta' \sin\psi - \varphi' \sin\theta \cos\psi.$$

In order to find the functions T_0' and S_1, one must necessarily calculate T_0 and S_0 completely.

The absolute *vis viva* T_0 and the absolute energy of acceleration S_0 of the sphere, when one takes into account only the finite constraint $\zeta = 0$, are very easy to calculate. Those functions are:

$$2\,T_0 = M\,[(\xi' - \mu\,\eta)^2 + (\eta' + \mu\,\xi)^2] + A\,[p^2 + q^2 + (r + \mu)^2],$$

$$2\,S_0 = M\,[(\xi'' - 2\mu\eta - \mu^2\xi)^2 + (\eta'' + 2\,\mu\xi' - \mu\eta)^2] + A\,[p'^2 + q'^2 + r'^2 + 2\mu\,(pq' - qp')] + \ldots,$$

respectively, where ([2]):

$$r = \psi' + \varphi' \cos\theta.$$

We will then have:

([1]) Iv. TSENOFF, "Mouvement sans frottement, etc." Annuaire de l'Université de Sofia, 1916, 1917, 1918.

([2]) The expressions p, q, r are the projections of the relative instantaneous angular velocity of the sphere onto the axes $G\,x'$, y', z'.

$$2T_0' = M\,[(\xi' - \mu\,\eta)^2 + (\eta' + \mu\,\xi)^2] + A\,(r + \mu)^2.$$

We omit the term $A\,(p^2 + q^2)$, because it is equal to $A\,(\theta'^2 + \varphi'^2 \sin^2\theta)$, and:

$$2S_1 = A\,[p'^2 + q'^2 + r'^2 + 2\mu\,(pq' - qp')].$$

The equation for ξ is:

$$\frac{d}{dt} M\,(\xi' - \mu\,\eta) + (\eta' + \mu\,\xi)\,\mu + \frac{A}{a} q' + 2\frac{A}{a}\mu\,p = 0$$

or

(15) $$\frac{7}{5}\xi'' - \frac{12}{15}\mu\eta' - \mu^2\xi = 0.$$

The equation for η is:

$$\frac{d}{dt}(\eta' + \mu\,\xi)\,\mu + M\,(\xi' - \mu\,\eta) - \frac{A}{a} p' + 2\frac{A}{a}\mu\,q = 0$$

or

(16) $$\frac{7}{5}\eta'' + \frac{12}{15}\mu\xi' - \mu^2\eta = 0.$$

Finally, the equation for ψ is:

$$\frac{d}{dt} A\,(r + \mu) = 0$$

or

$$r' = 0.$$

That equation is obtained by the Lagrange method, because ψ' does not enter into $\xi' = a\,q$, $\eta' = -\,a\,p$.

The general integrals of the system of equations (15) and (16) are:

$$\xi = \ \ A\cos(\mu\,t + \alpha) + B\cos\left(\frac{5}{7}\mu\,t + \beta\right),$$

$$\eta = -\,A\sin(\mu\,t + \alpha) - B\sin\left(\frac{5}{7}\mu\,t + \beta\right).$$

Those equations give the law of motion for the center of gravity of the sphere.

The absolute trajectory of the projection of G onto a fixed horizontal plane is a circle.