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On the motion of non-holonomic systems

By **Iv. TZÉNOFF**

Translated by D. H. Delphenich

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1. – Imagine a system that is first subject to constraints that are expressible by relations in finite terms between the coordinates of the various points. While taking those constraints into account, let $k + p$ be the number of independent parameters $q_1, q_2, \ldots, q_k, \ldots, q_{k+p}$ that fix the position of the system.

Now suppose that one adds some new constraints to the preceding ones that depend upon time and are expressible by p differential relations between the $q_1, q_2, \ldots, q_k, \ldots, q_{k+p}$ that have the form:

$$(1) \qquad dq_{k+i} = \sum_{\alpha=1}^{k} a_{i\alpha}\, dq_\alpha + a_i\, dt \qquad \text{or} \qquad q'_{k+i} = \sum_{\alpha=1}^{k} a_{i\alpha}\, q'_\alpha + a_i \qquad (i = 1, 2, \ldots, p).$$

Let T_0 and S_0 denote the semi-*vis viva* and the semi-energy of acceleration of the system, resp., when calculated by taking into account only the finite constraints that were imposed upon the system.

On the other hand, let T and S denote the analogous quantities when one also takes into account the differential constraints that at given by (1).

Finally, let T_1 denote the function T_0, when considered to be a function of only $q'_{k+i}, \ldots, q'_{k+p}$, and let S_1 denote the function S_0, when considered to be a function of only $q''_{k+i}, \ldots, q''_{k+p}$, and of course, one must not forget that $q'_{k+i}, \ldots, q'_{k+p}$ are determined by (1) for T_1, in the same way that $q''_{k+i}, \ldots, q''_{k+p}$ are for S_1.

One can then put the equations of motion of non-holonomic systems into the following form:

$$(2) \qquad \frac{d}{dt}\frac{\partial T}{\partial q'_\alpha} - \frac{\partial T}{\partial q_\alpha} + \frac{\partial T_1}{\partial q_\alpha} - \frac{d}{dt}\frac{\partial T_1}{\partial q'_\alpha} + \frac{\partial S_1}{\partial q''_\alpha} = Q_\alpha \qquad (\alpha = 1, 2, \ldots, k)$$

or

$$(2') \qquad \frac{d}{dt}\frac{\partial T_0}{\partial q'_\alpha} - \frac{\partial T_0}{\partial q_\alpha} + \frac{\partial S_1}{\partial q''_\alpha} = Q_\alpha \qquad (\alpha = 1, 2, \ldots, k),$$

in which Q_α is the coefficient of δq_α in the expression for the sum of the virtual works done by applied forces.

We have obtained those results by following a method that is analogous to that of Lagrange for holonomic systems ([1]).

2. – The functions T_0 and S_0 are constrained by the relation:

$$\frac{\partial S_0}{\partial q''_s} = \frac{d}{dt}\frac{\partial T_0}{\partial q'_s} - \frac{\partial T_0}{\partial q_s} \qquad (s = 1, 2, \ldots, k, \ldots, k + p)\,.$$

Indeed, we have:

$$2\,T_0 = \sum m\,(x_0'^2 + y_0'^2 + z_0'^2)\,, \tag{3}$$

$$2\,S_0 = \sum m\,(x_0''^2 + y_0''^2 + z_0''^2)\,. \tag{4}$$

One has:

$$x'_0 = \frac{\partial x_0}{\partial t} + \sum_{s=1}^{k+r}\frac{\partial x_0}{\partial q_s}q'_s\,, \qquad y'_0 = \ldots, \qquad z'_0 = \ldots, \tag{5}$$

$$x''_0 = \frac{d}{dt}\frac{\partial x_0}{\partial t} + \sum_{s=1}^{k+r}\frac{\partial x_0}{\partial q_s}q''_s + \sum_{s=1}^{k+r}q'_s\frac{d}{dt}\frac{\partial x_0}{\partial q_s}\,, \qquad y''_0 = \ldots, \qquad z''_0 = \ldots \tag{6}$$

When one takes (5) and (6) into account, equation (4) will give:

$$\begin{aligned}\frac{\partial S_0}{\partial q''_s} &= \sum m\left(x''_0\frac{\partial x''_0}{\partial q''_s} + \cdots + \cdots\right) = \sum m\left(x''_0\frac{\partial x'_0}{\partial q'_s} + \cdots + \cdots\right)\\ &= \frac{d}{dt}\sum m\left(x'_0\frac{\partial x'_0}{\partial q'_s} + \cdots + \cdots\right) - \sum m\left(x'_0\frac{d}{dt}\frac{\partial x'_0}{\partial q'_s} + \cdots + \cdots\right)\\ &= \frac{d}{dt}\frac{\partial T_0}{\partial q'_s} - \sum m\left(x'_0\frac{d}{dt}\frac{\partial x_0}{\partial q_s} + \cdots + \cdots\right),\end{aligned}$$

so upon observing that $\frac{d}{dt}\frac{\partial x_0}{\partial q_s} = \frac{\partial x'_0}{\partial q_s}$, …, one will infer from this that:

$$\frac{\partial S_0}{\partial q''_s} = \frac{d}{dt}\frac{\partial T_0}{\partial q'_s} - \frac{\partial T_0}{\partial q_s} \qquad (s = 1, 2, \ldots, k, \ldots, k + p). \tag{7}$$

3. – Upon taking the relation (7) into account, one will have:

([1]) Iv. TZÉNOFF, J. Math. pures et appl., 1920.

$$\frac{\partial T_1}{\partial q_\alpha}-\frac{d}{dt}\frac{\partial T_1}{\partial q'_\alpha}+\frac{\partial S_1}{\partial q''_\alpha}=\sum_{i=1}^{p}\frac{\partial T_1}{\partial q'_{k+i}}\frac{\partial q'_{k+i}}{\partial q_\alpha}-\sum_{i=1}^{p}\frac{\partial T_0}{\partial q'_{k+i}}\frac{\partial q_{k+i}}{\partial q'_\alpha}+\sum_{i=1}^{r}\frac{\partial S_0}{\partial q''_\alpha}\frac{\partial q''_{k+i}}{\partial q''_\alpha}.$$

Upon appealing to the relations (7) for $s = k + 1, \ldots, k + p$, and the relation:

$$\frac{\partial q'_{k+i}}{\partial q'_\alpha}=\frac{\partial q''_{k+i}}{\partial q''_\alpha}\qquad (i = 1, 2, \ldots, p),$$

which one easily deduces from (1), the complementary term will take the form:

$$\frac{\partial T_1}{\partial q_\alpha}-\frac{d}{dt}\frac{\partial T_1}{\partial q'_\alpha}+\frac{\partial S_1}{\partial q''_\alpha}=\sum_{i=1}^{p}\left[\frac{\partial T_0}{\partial q'_{k+i}}\left(\frac{\partial q'_{k+i}}{\partial q_\alpha}-\frac{d}{dt}\frac{\partial q'_{k+i}}{\partial q'_\alpha}\right)-\frac{\partial T_0}{\partial q_{k+i}}\frac{\partial q'_{k+i}}{\partial q'_\alpha}\right]. \tag{8}$$

It then follows that the complementary terms that one must add to the left-hand side of the Lagrange equations in order to obtain the equations of motion of non-holonomic material systems *do not contain second derivatives with respect to time.*

The equations of motion (2) are:

$$\text{(9)}\quad\begin{cases}\dfrac{d}{dt}\dfrac{\partial T}{\partial q'_\alpha}-\dfrac{\partial T}{\partial q_\alpha}+\displaystyle\sum_{i=1}^{p}\left[\frac{\partial T_0}{\partial q'_{k+i}}\left(\frac{\partial q'_{k+i}}{\partial q_\alpha}-\frac{\partial q''_{k+i}}{\partial q'_\alpha}\right)-\frac{\partial T_0}{\partial q_{k+i}}\frac{\partial q'_{k+i}}{\partial q'_\alpha}\right]=Q_\alpha & (\alpha=1,2,\cdots k),\\[2ex] q'_{k+i}=\displaystyle\sum_{\alpha=1}^{k}a_{i\alpha}\,q'_\alpha+a_i & (i=1,2,\ldots,p).\end{cases}$$

That shows us that in order to be able to write the equations of motion of a non-holonomic system, it will not be necessary to know Appel's S function or the function S_1 that we have introduced. That simplifies the calculations greatly, because the function S is difficult to calculate.

4. – We shall now *deduce Appell's equations from our own and conversely.*

We start from the equation:

$$\frac{d}{dt}\frac{\partial T_0}{\partial q'_\alpha}-\frac{\partial T_0}{\partial q_\alpha}+\frac{\partial S_1}{\partial q''_\alpha}=Q_\alpha\qquad(\alpha = 1, 2, \ldots, k).$$

Upon taking equation (7) into account for $\alpha = 1, 2, \ldots, k$, the equation above will become:

$$\frac{\partial S_0}{\partial q''_\alpha}+\frac{\partial S_1}{\partial q''_\alpha}=Q_\alpha, \tag{10}$$

when one takes:

$$\frac{\partial S_1}{\partial q''_\alpha}=\sum_{i=1}^{p}\frac{\partial S_0}{\partial q''_{k+i}}\frac{\partial q''_{k+i}}{\partial q''_\alpha}.$$

Appell's S function represents the energy of acceleration when one takes into account all of the finite and differential constraints. One can obtain S_0 upon taking the relations (1) into account. One will then have:

$$\frac{\partial S}{\partial q''_\alpha} = \frac{\partial S_0}{\partial q''_\alpha} + \sum_{i=1}^{p} \frac{\partial S_0}{\partial q''_{k+i}} \frac{\partial q''_{k+i}}{\partial q''_\alpha} = \frac{\partial S_0}{\partial q''_\alpha} + \frac{\partial S_1}{\partial q''_\alpha},$$

which is equal to Q_α, from (10). Hence, one will have the Appell equations:

$$\frac{\partial S}{\partial q''_\alpha} = Q_\alpha \qquad (\alpha = 1, 2, \ldots, k).$$

Conversely, since:

$$\frac{\partial S}{\partial q''_\alpha} = \frac{\partial S_0}{\partial q''_\alpha} + \frac{\partial S_1}{\partial q''_\alpha} \qquad \text{and} \qquad \frac{\partial S}{\partial q''_\alpha} = \frac{d}{dt}\frac{\partial T_0}{\partial q'_\alpha} - \frac{\partial T_0}{\partial q_\alpha},$$

we will have:

$$\frac{d}{dt}\frac{\partial T_0}{\partial q'_\alpha} - \frac{\partial T_0}{\partial q_\alpha} + \frac{\partial S_1}{\partial q''_\alpha} = Q_\alpha \qquad (\alpha = 1, 2, \ldots, k).$$

One easily infers equations (2) upon taking into account the fact that:

$$\frac{\partial T}{\partial q'_\alpha} = \frac{\partial T_0}{\partial q'_\alpha} + \frac{\partial T_1}{\partial q'_\alpha} \qquad \text{and} \qquad \frac{\partial T}{\partial q_\alpha} = \frac{\partial T_0}{\partial q_\alpha} + \frac{\partial T_1}{\partial q_\alpha}.$$

5. – We shall *obtain the equations of motion* (2) *or* (2′) *in another manner* by starting from the general equation of dynamics, which is written in the form:

$$\sum_{\alpha=1}^{k}\left(\frac{d}{dt}\frac{\partial T_0}{\partial q'_\alpha} - \frac{\partial T_0}{\partial q_\alpha}\right)\delta q_\alpha + \sum_{i=1}^{p}\left(\frac{d}{dt}\frac{\partial T_0}{\partial q'_{k+i}} - \frac{\partial T_0}{\partial q_{k+i}}\right)\delta q_{k+i} = \sum_{\alpha=1}^{k} Q_\alpha^0\,\delta q_\alpha + \sum_{i=1}^{p} Q_{k+i}^0\,\delta q_{k+i}.$$

If one now takes into account the differential constraints (1):

$$q'_{k+i} = \sum_{\alpha=1}^{k} a_{i\alpha}\, q'_\alpha + a_i \qquad (i = 1, 2, \ldots, p)$$

then

$$\delta q_{k+i} = \sum_{\alpha=1}^{k} a_{i\alpha}\,\delta q_\alpha,$$

and the general equation of dynamics will take the form:

$$\sum_{\alpha=1}^{k}\left[\frac{d}{dt}\frac{\partial T_0}{\partial q'_\alpha}-\frac{\partial T_0}{\partial q_\alpha}+\sum_{i=1}^{p}\left(\frac{d}{dt}\frac{\partial T_0}{\partial q'_{k+i}}-\frac{\partial T_0}{\partial q_{k+i}}\right)a_{i\alpha}\right]\delta q_\alpha=\sum_{\alpha=1}^{k}Q_\alpha\,\delta q_\alpha\,.$$

When one takes the relations (7) for $s = k + 1, \ldots, k + p$ into account, and the fact that:

$$a_{i\alpha}=\frac{\partial q''_{k+i}}{\partial q''_\alpha}\,,$$

so

$$\sum_{i=1}^{p}\left(\frac{d}{dt}\frac{\partial T_0}{\partial q'_{k+i}}-\frac{\partial T_0}{\partial q_{k+i}}\right)a_{i\alpha}=\sum_{i=1}^{p}\frac{\partial S_0}{\partial q'_{k+i}}\frac{\partial q''_{k+i}}{\partial q_\alpha}=\frac{\partial S_1}{\partial q''_\alpha}\,,$$

and we will have:

$$\sum_{i=1}^{p}\left(\frac{d}{dt}\frac{\partial T_0}{\partial q'_\alpha}-\frac{\partial T_0}{\partial q_\alpha}+\frac{\partial S_1}{\partial q''_\alpha}\right)\delta q_\alpha=\sum_{\alpha=1}^{k}Q_\alpha\,\delta q_\alpha\,,$$

since the δq_α are arbitrary, we will get equations (2′).

6. – We can *deduce the vis viva theorem* from equations (2) or (2′). The theorem, which is deduced from d'Alembert's principle, is stated thus: If the constraints are independent of time then the differential of the semi-*vis viva* will be equal to the sum of the elementary works done by the given forces. Since the constraints are independent of time, the real displacements are found among the virtual displacements that are compatible with those constraints; that is why one can replace δq_s ($s = 1, 2, \ldots, k + p$) with dq_s. In that case, equations (1) will be:

$$q'_{k+i}=\sum_{\alpha=1}^{k}a_{i\alpha}\,q'_\alpha=\sum_{\alpha=1}^{k}\frac{\partial q'_{k+i}}{\partial q'_\alpha}q'_\alpha \qquad (i = 1, 2, \ldots, p), \tag{11}$$

since the coefficients $a_{i\alpha}$ are functions of $q_1, \ldots, q_{k+p}$.

The *vis viva* theorem is written:

$$\frac{dT}{dt}=\sum_{\alpha=1}^{k}Q_\alpha\,q'_\alpha\,.$$

We now come down to deducing that equation from equations (2):

$$\frac{d}{dt}\frac{\partial T}{\partial q'_\alpha}-\frac{\partial T}{\partial q_\alpha}+\frac{\partial T_1}{\partial q_\alpha}-\frac{d}{dt}\frac{\partial T_1}{\partial q'_\alpha}+\frac{\partial S_1}{\partial q''_\alpha}=Q_\alpha \qquad (\alpha = 1, 2, \ldots, k).$$

We multiply them by q'_α and add them. Upon supposing that equations (2) are replaced with (9), we will then get:

(12)
$$\sum_{\alpha=1}^{k}\frac{d}{dt}\frac{\partial T}{\partial q'_\alpha}q'_\alpha-\frac{\partial T}{\partial q_\alpha}q'_\alpha+\sum_{\alpha=1}^{k}\frac{\partial T_1}{\partial q_\alpha}q'_\alpha+\sum_{i=1}^{p}\frac{\partial T_0}{\partial q_{k+i}}\left(\frac{\partial q'_{k+i}}{\partial q_\alpha}-\frac{d}{dt}\frac{\partial q_{k+i}}{\partial q_\alpha}\right)q'_\alpha$$
$$-\sum_{i=1}^{p}\frac{\partial T_0}{\partial q_{k+i}}\sum_{\alpha=1}^{k}\frac{\partial q'_{k+i}}{\partial q'_\alpha}q'_\alpha=\sum_{\alpha=1}^{k}Q_\alpha\,q'_\alpha\,.$$

Since the function T is a quadratic form in the $q'_1, \dots, q'_k$, one will have:

$$\sum_{\alpha=1}^{k}\frac{\partial T}{\partial q'_\alpha}q'_\alpha=2T,$$

and consequently:

$$\sum_{\alpha=1}^{k}\frac{d}{dt}\frac{\partial T}{\partial q'_\alpha}q'_\alpha=\frac{d}{dt}\sum_{\alpha=1}^{k}\frac{\partial T}{\partial q'_\alpha}q'_\alpha-\sum_{\alpha=1}^{k}\frac{\partial T}{\partial q'_\alpha}q''_\alpha=\frac{d_2T}{dt}-\sum_{\alpha=1}^{k}\frac{\partial T}{\partial q'_\alpha}q'_\alpha$$

$$=\frac{d_2T}{dt}-\sum_{\alpha=1}^{k}\frac{\partial T}{\partial q'_\alpha}q'_\alpha-\sum_{\alpha=1}^{k}q''_\alpha\sum_{i=1}^{p}\frac{\partial T_0}{\partial q'_{k+i}}\frac{\partial q'_{k+i}}{\partial q'_\alpha}$$

$$=\frac{d_2T}{dt}-\sum_{\alpha=1}^{k}\frac{\partial T_0}{\partial q'_\alpha}q''_\alpha-\sum_{i=1}^{p}\frac{\partial T_0}{\partial q'_{k+i}}\sum_{\alpha=1}^{k}\frac{\partial q'_{k+i}}{\partial q'_\alpha}q''_\alpha\,.$$

The second term in (12) is written:

$$\sum_{\alpha=1}^{k}\frac{\partial T}{\partial q'_\alpha}q'_\alpha=\sum_{\alpha=1}^{k}\frac{\partial T_0}{\partial q_\alpha}q'_\alpha+\sum_{i=1}^{p}\frac{\partial T_0}{\partial q_{k+i}}\sum_{\alpha=1}^{k}\frac{\partial q'_{k+i}}{\partial q'_\alpha}q'_\alpha\,.$$

Equation (12) will then take the form:

$$\frac{d_2T}{dt}-\sum_{\alpha=1}^{k}\frac{\partial T_0}{\partial q'_\alpha}q''_\alpha-\sum_{\alpha=1}^{k}\frac{\partial T_0}{\partial q_\alpha}q'_\alpha-\sum_{i=1}^{p}\frac{\partial T_0}{\partial q'_{k+i}}\sum_{\alpha=1}^{k}\left(\frac{\partial q'_{k+i}}{\partial q'_\alpha}q''_\alpha+\frac{d}{dt}\frac{\partial q'_{k+i}}{\partial q'_\alpha}q'_\alpha\right)$$

$$-\sum_{i=1}^{p}\frac{\partial T_0}{\partial q_{k+i}}\sum_{\alpha=1}^{k}\frac{\partial q'_{k+i}}{\partial q'_\alpha}q'_\alpha=\sum_{\alpha=1}^{k}Q_\alpha\,q'_\alpha\,,$$

or rather, when one takes equations (11) into account:

$$\frac{d_2T}{dt}-\sum_{s=1}^{k+p}\frac{\partial T_0}{\partial q'_s}q''_s-\sum_{s=1}^{k+p}\frac{\partial T_0}{\partial q_s}q'_s=\sum_{\alpha=1}^{k}Q_\alpha\,q'_\alpha\,;$$

finally:

$$\frac{d_2T}{dt}-\frac{dT_0}{dt}=\frac{dT}{dt}=\sum_{\alpha=1}^{k}Q_\alpha\,q'_\alpha\,.$$ Q. E. D.

7. – We now seek the form that must be given to the differential constraints that are imposed upon the constraints in order for *the Lagrange equation to apply to one of the parameters.*

We shall consider the special case in which all of the constraints are independent of time and the quantities q_{k+i} , …, q_{k+p} do not enter into the equation of motion. Equations (1) will then have the form:

(13) $$dq_{k+i} = \sum_{\alpha=1}^{k} a_{i\alpha}\, dq_\alpha \qquad (i = 1, 2, \ldots, p),$$

in which the $a_{i\alpha}$ depend upon only the q_1 , q_2 , …, q_k .

In that case, equations (9) take the form:

$$\frac{d}{dt}\frac{\partial T}{\partial q'_\alpha} - \frac{\partial T}{\partial q_\alpha} - \sum_{i=1}^{p} \frac{\partial T}{\partial q'_{k+i}} \left(\frac{\partial q'_{k+i}}{\partial q_\alpha} - \frac{d}{dt}\frac{\partial q'_{k+i}}{\partial q'_\alpha} \right) = Q_a \qquad (a = 1, 2, \ldots, k),$$

$$q'_{k+i} = \sum_{\alpha=1}^{k} a_{i\alpha}\, q'_\alpha \qquad (i = 1, 2, \ldots, p).$$

However:

$$\frac{\partial q'_{k+i}}{\partial q_\alpha} - \frac{d}{dt}\frac{\partial q'_{k+i}}{\partial q'_\alpha} = \frac{\partial q'_{k+i}}{\partial q_\alpha} - \frac{d}{dt} a_{i\alpha} = \sum_{j=1}^{k} \frac{\partial a_{ij}}{\partial q_\alpha} q'_j - \sum_{j=1}^{k} \frac{\partial a_{i\alpha}}{\partial q_j} q'_j$$

$$= \sum_{j=1}^{k} \left(\frac{\partial a_{ij}}{\partial q_\alpha} - \frac{\partial a_{i\alpha}}{\partial q_j} \right) q'_j ,$$

and the equations of motion:

(14) $$\begin{cases} \dfrac{d}{dt}\dfrac{\partial T}{\partial q'_\alpha} - \dfrac{\partial T}{\partial q_\alpha} + \displaystyle\sum_{i=1}^{p} \frac{\partial T_0}{\partial q'_{k+i}} \sum_{j=1}^{k} \left(\frac{\partial a_{ij}}{\partial q_\alpha} - \frac{\partial a_{i\alpha}}{\partial q_j} \right) q'_j = Q_\alpha & (\alpha = 1, 2, \ldots, k), \\ q'_{k+i} = \displaystyle\sum_{\alpha=1}^{k} a_{i\alpha}\, q'_\alpha & (i = 1, 2, \ldots, p). \end{cases}$$

If one has:

(15) $$\frac{\partial a_{ij}}{\partial q_\alpha} = \frac{\partial a_{i\alpha}}{\partial q_j} \qquad \begin{pmatrix} i = 1, 2, \ldots, p \\ j = 1, 2, \ldots, k \end{pmatrix}$$

then the Lagrange equation with respect to the parameter q_α will give a differential equation of motion for the non-holonomic system.

We shall define p functions F_1, F_2, …, F_p of q_1, q_2, …, q_k by the conditions:

$$F_s = \int_{q_\alpha^0}^{q_\alpha} a_{s\alpha}\, dq_\alpha \qquad (s = 1, 2, \ldots, p) ;$$

one then deduces that:

$$\frac{\partial F_s}{\partial q_\alpha} = a_{s\alpha} .$$

Similarly:

$$\frac{\partial F_s}{\partial q_\gamma} = \int_{q_\alpha^0}^{q_\alpha} \frac{\partial a_{s\alpha}}{\partial q_\gamma} dq_\alpha = \int_{q_\alpha^0}^{q_\alpha} \frac{\partial a_{s\gamma}}{\partial q_\alpha} dq_\alpha = a_{s\gamma} - a_{s\gamma}^0 ,$$

from (15) and upon letting denote the value of $a_{s\gamma}$ for $q_\alpha = q_\alpha^0$.

Hence:

$$\frac{dF_s}{dt} = a_{s1} q_1' + \cdots + a_{s\alpha} q_\alpha' + \cdots + a_{sk} q_k' - a_{s1}^0 q_1' - \cdots - 0 - \cdots - a_{sk}^0 q_k' .$$

Equation (13) gives us:

$$dq_{k+s}' = \frac{dF_s}{dt} + a_{s1}^0 q_1' + \cdots + a_{\overline{s\alpha-1}}^0 q_{\alpha-1}' + a_{\overline{s\alpha-1}}^0 q_{\alpha+1}' + \cdots + a_{sk}^0 q_k' .$$

Therefore, the differential constraints take the form:

$$dq_{k+i} = dF_i + \sum_{r=1}^{\alpha-1} a_{ir}^0 \, dq_r + \sum_{r=\alpha+1}^{k} a_{ir}^0 \, dq_r \qquad (i = 1, 2, \ldots, p).$$

Consequently, *the Lagrange equation is applicable to the parameter* q_α *if the constraints imposed* (13) *can be put into the form of an exact total differential followed by a differential expression that does not contain* q_α .

8. – One knows that the Lagrange equations can be put into a different form that Hamilton gave, and which one calls their *canonical form.* We propose to do that with equations (9):

$$(16) \quad \begin{cases} \dfrac{d}{dt}\dfrac{\partial T}{\partial q_\alpha'} - \dfrac{\partial T}{\partial q_\alpha} + \displaystyle\sum_{i=1}^{p} \left[\dfrac{\partial T_0}{\partial q_{k+i}'} \left(\dfrac{\partial q_{k+i}'}{\partial q_\alpha} - \dfrac{\partial q_{k+i}''}{\partial q_\alpha'} \right) - \dfrac{\partial T_0}{\partial q_{k+i}} \dfrac{\partial q_{k+i}'}{\partial q_\alpha'} \right] = Q_\alpha, & (\alpha = 1, 2, \ldots, k) \\ \dfrac{dq_\alpha}{dt} = q_\alpha' & (\alpha = 1, 2, \ldots, k) \\ \dfrac{dq_{k+i}}{dt} = \displaystyle\sum_{\alpha=1}^{k} a_{i\alpha} q_\alpha' + a_1 & (i = 1, 2, \ldots, p). \end{cases}$$

Those equations, which are $2k + p$ in number, determine $q_1 , \ldots, q_k , \ldots, q_{k+p} , , \ldots, q_1'$, $\ldots, q_k'$ as functions of t.

Take the quantities:

(17) $$p_1 = \frac{\partial T}{\partial q_1'}, \qquad p_2 = \frac{\partial T}{\partial q_2'}, \qquad \ldots, \qquad p_k = \frac{\partial T}{\partial q_k'}$$

to be the new variables, in place of $q_1', \ldots, q_k'$. Inversely, one infers $q_1', \ldots, q_k'$ as functions of $p_1, p_2, \ldots, p_k$, and equations (16) determine $q_1, \ldots, q_k, \ldots, q_{k+p}, p_1, \ldots, p_k$ as functions of t. We seek the form that equations (16) take when one performs a change of variables.

The function T depends upon $q_1, \ldots, q_{k+p}, q_1', \ldots, q_k'$. Fix t and give infinitely-small increments $\delta q_1, \ldots, \delta q_k, \delta p_1, \ldots, \delta p_k$ to the variables $q_1, \ldots, q_k, p_1, \ldots, p_k$ that are independent and arbitrary. One infers the increments $\delta q_{k+1}, \ldots, \delta q_{k+p}$ from the equations:

(18) $$\delta q_{k+i} = \sum_{\alpha=1}^{k} a_{i\alpha}\,\delta q_\alpha \qquad (i = 1, 2, \ldots, p),$$

and the increments $\delta q_1', \ldots, \delta q_k'$ from equations (17), which are assumed to have been solved for the $q_1', \ldots, q_k'$.

The variation of T will be:

$$\delta T = \sum_{\alpha=1}^{k} \frac{\partial T}{\partial q_\alpha}\delta q_\alpha + \sum_{\alpha=1}^{k} \frac{\partial T}{\partial q_\alpha'}\delta q_\alpha' + \sum_{i=1}^{p} \frac{\partial T}{\partial q_{k+i}}\delta q_{k+i},$$

or rather, upon taking into account equations (17) and (18):

$$\delta T = \sum_{\alpha=1}^{k} p_\alpha\,\delta q_\alpha' = \sum_{\alpha=1}^{k} p_\alpha\,\delta q_\alpha' + \sum_{\alpha=1}^{k}\left(\frac{\partial T}{\partial q_\alpha} + \sum_{i=1}^{p}\frac{\partial T}{\partial q_{k+i}}a_{i\alpha}\right)\delta q_\alpha .$$

If one sets:

(19) $$K = \sum_{\alpha=1}^{k} p_\alpha\, q_\alpha' - T$$

then one will have:

$$\delta K = \sum_{\alpha=1}^{k} p_\alpha'\,\delta q_\alpha - \sum_{\alpha=1}^{k}\left(\frac{\partial T}{\partial q_\alpha} + \sum_{i=1}^{p}\frac{\partial T}{\partial q_{k+i}}\frac{\partial q_{k+i}}{\partial q_\alpha'}\right)\delta q_\alpha .$$

We have then obtained an expression for the total differential δK that replaces δT. We will have a new expression for δK when we suppose that K is expressed as a function of $t, q_1, \ldots, q_k, \ldots, q_{k+p}, p_1, \ldots, p_k$, and that $q_1, \ldots, q_k, p_1, \ldots, p_k$ are subjected to arbitrary variations $\delta q_1, \ldots, \delta q_k, \delta p_1, \ldots, \delta p_k$. Hence:

$$\delta K = \sum_{\alpha=1}^{k} \frac{\partial K}{\partial p_\alpha}\delta q_\alpha + \sum_{\alpha=1}^{k}\left(\frac{\partial K}{\partial q_\alpha} + \sum_{i=1}^{p}\frac{\partial K}{\partial q_{k+i}}\frac{\partial q_{k+i}'}{\partial q_\alpha'}\right)\delta q_\alpha .$$

Upon identifying those two expressions for δK, one will get:

$$(20)\qquad \begin{cases} -\dfrac{\partial T}{\partial q_\alpha}-\sum_{i=1}^{p}\dfrac{\partial T}{\partial q_{k+i}}\dfrac{\partial q'_{k+i}}{\partial q_\alpha}=\dfrac{\partial K}{\partial q_\alpha}+\sum_{i=1}^{p}\dfrac{\partial K}{\partial q_{k+i}}\dfrac{\partial q'_{k+i}}{\partial q_\alpha}, \\ q'_\alpha=\dfrac{\partial K}{\partial p_\alpha} \end{cases}\qquad (\alpha=1,2,\ldots,k).$$

In those equations, the partial derivatives of T are taken by considering T to be a function of $t, q_1, \ldots, q_k, q'_1, \ldots, q'_k$, while K is assumed to be a function of $t, q_1, \ldots, q_k, p_1, \ldots, p_k$.

Therefore, from (20), when one notes that:

$$\frac{\partial T}{\partial q_{k+i}}=\frac{\partial T_0}{\partial q_{k+i}}+\sum_{j=1}^{p}\frac{\partial T_0}{\partial q'_{k+j}}\frac{\partial q'_{k+j}}{\partial q_{k+i}}$$

and

$$\sum_{i=1}^{p}\frac{\partial T_0}{\partial q'_{k+i}}\frac{\partial q'_{k+i}}{\partial q_\alpha}=\sum_{i=1}^{p}\frac{\partial T_0}{\partial q_{k+j}}\frac{\partial q'_{k+j}}{\partial q'_\alpha}+\sum_{i=1}^{p}\frac{\partial T_0}{\partial q'_{k+j}}\sum_{j=1}^{p}\frac{\partial q'_{k+i}}{\partial q_{k+j}}\frac{\partial q'_{k+j}}{\partial q'_\alpha},$$

equations (16) will take the following form:

$$(21)\qquad \begin{cases} \dfrac{dp_\alpha}{dt}+\dfrac{\partial K}{\partial q_\alpha}+\sum_{i=1}^{p}\dfrac{\partial K}{\partial q_{k+i}}\dfrac{\partial q'_{k+i}}{\partial q_\alpha} \\ \qquad +\sum_{i=1}^{p}\left[\dfrac{\partial T_0}{\partial q'_{k+i}}\left(\dfrac{\partial q'_{k+i}}{\partial q_\alpha}-\dfrac{d}{dt}\dfrac{\partial q'_{k+i}}{\partial q'_\alpha}+\sum_{j=1}^{p}\dfrac{\partial q'_{k+i}}{\partial q_{k+j}}\dfrac{\partial q'_{k+i}}{\partial q'_\alpha}\right)\right]=Q_\alpha \quad (\alpha=1,2,\ldots,k), \\ \dfrac{dq_\alpha}{dt}=\dfrac{\partial K}{\partial p_\alpha} \quad (\alpha=1,2,\ldots,k), \\ \dfrac{dq_{k+i}}{dt}=\sum_{i=1}^{p}a_{i\alpha}\dfrac{\partial K}{\partial p_\alpha}+a_i \quad (i=1,2,\ldots,p). \end{cases}$$

The fourth term in the first equation is a function of $t, q_1, q_2, \ldots, q_k, \ldots, q_{k+p}, p_1, \ldots, p_k$, by virtue of the second and third equations.

Equations (21), which are $2k + p$ in number, are therefore of order one in the $2k + p$ variables $q_1, \ldots, q_{k+p}, \ldots, p_1, \ldots, p_k$. One can solve them and obtain the variables above as functions of t and $2k + p$ arbitrary constants.

Now suppose that the given forces derive from a function U that depends upon $t, q_1, q_2, \ldots, q_{k+p}$, but not upon $p_1, p_2, \ldots, p_k$; hence, $\partial U / \partial p_\alpha = 0$. One has:

$$Q^0_\alpha=\frac{\partial U}{\partial q_\alpha}\qquad (\alpha=1,2,\ldots,k),$$

$$Q^0_{k+i} = \frac{\partial U}{\partial q_{k+i}} \qquad (i = 1, 2, \dots, p),$$

$$Q_\alpha = Q^0_\alpha + \sum Q^0_{k+i} \frac{\partial q'_{k+i}}{\partial q'_\alpha} \qquad (\alpha = 1, 2, \dots, k).$$

Hence, upon setting:

$$H = K - U,$$

we will have:

$$\frac{\partial H}{\partial p_\alpha} = \frac{\partial K}{\partial p_\alpha}, \qquad \frac{\partial H}{\partial q_\alpha} = \frac{\partial K}{\partial q_\alpha} - \frac{\partial U}{\partial q_\alpha}, \qquad (\alpha = 1, 2, \dots, k),$$

$$\frac{\partial H}{\partial q_{k+i}} = \frac{\partial K}{\partial q_{k+i}} - \frac{\partial U}{\partial q_{k+i}}, \qquad (i = 1, 2, \dots, p),$$

and equations (21) will take the form:

(22)
$$\begin{cases} \dfrac{dp_\alpha}{dt} + \displaystyle\sum_{i=1}^{p} \frac{\partial H}{\partial q_{k+i}} \frac{\partial q'_{k+i}}{\partial q'_\alpha} & (\alpha = 1, 2, \dots, k), \\ \qquad + \displaystyle\sum_{i=1}^{p} \left[\frac{\partial T_0}{\partial q'_{k+i}} \left(\frac{\partial q'_{k+i}}{\partial q_\alpha} - \frac{\partial q'_{k+i}}{\partial q'_\alpha} + \sum_{j=1}^{p} \frac{\partial q''_{k+i}}{\partial q_{k+j}} \frac{\partial q'_{k+j}}{\partial q'_\alpha} \right) \right] = \frac{\partial H}{\partial q_\alpha} & \\ \dfrac{dq_\alpha}{dt} = \dfrac{\partial H}{\partial p_\alpha} & (\alpha = 1, 2, \dots, k), \\ \dfrac{dq_{k+i}}{dt} = \displaystyle\sum_{\alpha=1}^{k} a_{i\alpha} \frac{\partial H}{\partial p_\alpha} + a_i & (i = 1, 2, \dots, p). \end{cases}$$

Those equations represent the canonical form for the equations of motion of non-holonomic systems with a complementary term.

Particular case. – Suppose that the constraints are independent of time; t will not enter into H and T_0 then. The a_i will be zero. T will then be a quadratic form with respect to $q'_1, \dots, q'_k$, and consequently:

$$\sum_{\alpha=1}^{k} q'_\alpha \frac{\partial T}{\partial q'_\alpha} = \sum q'_\alpha \, p_\alpha = 2T,$$

$$K = \sum p_\alpha \, q'_\alpha - T = 2T - T = T,$$

and

(23)
$$H = K - U = T - U.$$

The *vis viva* theorem, which is given by the equation:

$$\frac{dT}{dt} = \sum Q_\alpha\, q'_\alpha\,,$$

will become:

$$\frac{dT}{dt} = \sum_{\alpha=1}^{k}\left(Q_\alpha^0 + \sum_{i=1}^{p} Q_{k+i}^0\, a_{i\alpha}\right) q'_\alpha = \sum_{\alpha=1}^{k}\frac{\partial U}{\partial q_\alpha} q'_\alpha + \sum_{i=1}^{p}\frac{\partial U}{\partial q_{k+i}} q'_{k+i} = \frac{dU}{dt}$$

here, or rather, upon integrating:

$$T = U + h, \qquad \text{so} \qquad H = h.$$

One can deduce that result from equations (22). Indeed, the function H depends upon $q_1, \ldots, q_{k+p}$, $p_1, \ldots, p_k$. During the motion, those quantities will be functions of time, and consequently:

$$\frac{dH}{dt} = \sum_{\alpha=1}^{k}\left(\frac{\partial H}{\partial q_\alpha}\frac{dq_\alpha}{dt} + \frac{\partial H}{\partial p_\alpha}\frac{dp_\alpha}{dt}\right) + \sum_{i=1}^{p}\frac{\partial H}{\partial q_{k+i}}\frac{dq_{k+i}}{dt}\,,$$

so when one takes (22) into account, along with the fact that $a_i = 0$, that will become:

$$\frac{dH}{dt} = -\sum_{\alpha=1}^{k} q'_\alpha \sum_{i=1}^{p}\frac{\partial T_0}{\partial q'_{k+i}}\left(\frac{\partial q'_{k+i}}{\partial q_\alpha} - \frac{d}{dt}\frac{\partial q'_{k+i}}{\partial q'_\alpha} + \sum_{j=1}^{p}\frac{\partial q'_{k+i}}{\partial q_{k+j}}\frac{\partial q'_{k+j}}{\partial q'_\alpha}\right)$$

$$= -\sum_{i=1}^{p}\frac{\partial T_0}{\partial q'_{k+i}}\left(\sum_{j=1}^{p}\frac{\partial q'_{k+i}}{\partial q_\alpha} q'_\alpha - \sum_{\alpha=1}^{k} q'_\alpha\frac{d}{dt}\frac{\partial q'_{k+i}}{\partial q'_\alpha} + \sum_{j=1}^{p}\frac{\partial q'_{k+i}}{\partial q_{k+j}} q'_{k+j}\right).$$

However, one has:

$$q''_{k+i} = \sum_{\alpha=1}^{p}\frac{\partial q'_{k+i}}{\partial q_\alpha} q'_\alpha + \sum_{j=1}^{p}\frac{\partial q'_{k+i}}{\partial q'_\alpha} q'_{k+j} + \sum_{\alpha=1}^{k}\frac{\partial q'_{k+i}}{\partial q'_\alpha} q''_\alpha\,,$$

and one the other hand:

$$q''_{k+i} = \sum_{\alpha=1}^{k}\frac{d}{dt}\frac{\partial q'_{k+i}}{\partial q'_\alpha} q'_\alpha + \sum_{\alpha=1}^{k}\frac{\partial q'_{k+i}}{\partial q'_\alpha} q''_\alpha\,.$$

Upon comparing these, one infers:

$$\sum_{\alpha=1}^{k}\frac{\partial q'_{k+i}}{\partial q_\alpha} q'_\alpha + \sum_{j=1}^{p}\frac{\partial q'_{k+i}}{\partial q'_{k+j}} q'_{k+j} - \sum_{\alpha=1}^{k}\frac{d}{dt}\frac{\partial q'_{k+i}}{\partial q'_\alpha} q'_\alpha = 0 \qquad (i = 1, 2, \ldots, p),$$

so:

$$\frac{dH}{dt} = 0 \qquad \text{or} \qquad H = h.$$

In the more special case for which T_0 and T do not depend upon the parameters q_{k+1}, …, q_{k+p}, equations (22) take the form:

$$\frac{dp_\alpha}{dt} + \sum_{i=1}^{p} \frac{\partial T_0}{\partial q'_{k+i}} \sum_{j=1}^{k} \left(\frac{\partial a_{ij}}{\partial q_\alpha} - \frac{\partial a_{i\alpha}}{\partial q_j} \right) \frac{\partial H}{\partial p_j} = - \frac{\partial H}{\partial q_\alpha} \qquad (\alpha = 1, 2, \ldots, k),$$

$$\frac{dq_\alpha}{dt} = \frac{\partial H}{\partial p_\alpha}, \qquad \frac{dq_{k+i}}{dt} = \sum_{\alpha=1}^{k} a_{i\alpha} \frac{\partial H}{\partial p_\alpha} \qquad (i = 1, 2, \ldots, p).$$

9. – One knows that in the case of holonomic systems, one can obtain the Lagrange equations by starting from the integral:

$$\int_{t_0}^{t_1} \left(\delta T + \sum_{\alpha=1}^{k} Q_\alpha \, \delta q_\alpha \right) dt \, .$$

It is easy to show that in the case of non-holonomic systems, one must add the functions – T_1 and T_1^0 to the semi-*vis viva* T, where T_1^0 denotes the function T_0 when it is considered to be a function of only q'_{k+1}, …, q'_{k+p}, and consequently T_1 is the function T_1^0, when one takes into account the differential constraints (1). Hence, the equations of motion of a non-holonomic system are deduced from the equation:

$$\delta I = \int_{t_0}^{t_1} [\delta(T - T_1 + T_1^0) + \sum Q_\alpha \delta q_\alpha] \, dt = 0$$

on the condition that:

$$\delta q_{k+i} = \sum_{\alpha=1}^{k} a_{i\alpha} \, \delta q_\alpha \qquad (i = 1, 2, \ldots, p),$$

where the δq_α are arbitrary.