"Zur allgemeinen projektiven Differentialgeometrie. I. Einordnung in die Affingeometrie." Proc. Akad. Amst. **35** (1932), 525-542.

On general projective differential geometry. I. Relationship to affine geometry.

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§ 1. *Introduction.*

In a work that appeared recently [1]), I have shown that it is possible to construct a theory of projectively connected manifolds that subsumes and generalizes all of the older theories [2]). The essential step was the replacement of the group $\mathfrak{G}_n$ of all [3]) transformations in n variables with the group $\mathfrak{H}_{n+1}$ of all [4]) *homogeneous transformation of first degree* [5]) in n+1 variables. Furthermore, the notion of a general projective connection was introduced, for which neither geodetic lines nor a displacement (covariant differential, resp.) needs to exist, such that it can in no way be obtained from an (n-dimensional) affine connection. Thus, general projective geometry [6]) becomes an autonomous part of differential geometry [7]), and it also suggests the question of how to extend the Kleinian program to these curved manifolds.

The problem that arises is: to relate the theory of affine connections to the theory of projective connections. As is well known, (ordinary) projective geometry arises from (ordinary) affine geometry by singling out an ("infinitely distant" or "imaginary") hyperplane. In general differential geometry, the relationship is completely analogous: affine geometry [8]) can be obtained from projective differential geometry by singling out a hyperplane in each *local* manifold that does not include the contact point (§ 3). Moreover, in this case, an affine connection [8]) can be uniquely derived from each projective connection. In conclusion, the most important thing that is determined by a

† Translated by D.H. Delphenich.

[1]) D. VAN DANTZIG. Theorie des projektiven Zusammenhangs n-dimensionaler Räume. Math. Ann. **106** (1932), 400-454; denoted by TPZ. The notation of the *Supplement* to TPZ will be used, by omitting the * that was used there. For various deviations from this, cf. footnote ?), ?), as well as the beginning of § 2.

[2]) For the most important literature, cf. TPZ, as well as ENEA BORTOLOTTI, Connessioni proiettive, Boll. Un. Mat. Ital. **9** (1930) 258-294; **10** (1931) 28-34; 38-90.

[3]) All of the functions that are used are (unless otherwise noted) assumed to be analytic and regular in the region considered; all of the transformations are assumed to have a unique inverse there.

[4]) But not necessarily *linear!*

[5]) We use this expression for the theory of projectively connected manifolds, as opposed to the ordinary projective differential geometry of flat ("Euclidian") manifolds.

[6]) In the older theory, the differentiation indices still had a vector character, except for VEBLEN, in which, however, a formal difference still exists.

[7]) Briefly, for manifolds with a general group $\mathfrak{G}_n$.

[8]) Briefly, for the general linear (not necessarily symmetric) displacement L_n (III Aα) in the classification of J. A. Schouten, Der Ricci-Kalkül, Berlin, Springer (1924), pp. 75.

projective connection, the system of curves (“geodetic lines”), will be examined (§ 4). In the introductory section (§ 2), a brief overview of the most important basic notions will be given; for a thorough presentation, cf., TPZ, as well as a work by Prof. J.A. Schouten that will appear soon.

As for the relationship between projective differential geometry with affine geometry in one higher dimension, as well as the relationship of projective to conformal geometry, I hope to go into this on a later occasion.

§ 2. *Generalities.*

We use the following notations:

$\mathfrak{G}_n$ the group of all [3]) transformations $\xi^k \to \xi^{k'} = \xi^{k'}$ $(\xi^1, \ldots, \xi^n)$ in n variables ($h, \ldots, l = \mathbf{1}, \ldots, \mathbf{n}$; $h', \cdots, l' = \mathbf{1}', \cdots, \mathbf{n}'$);

$\mathfrak{H}_{n+1}$ the group of all [3]) *homogeneous* transformations of first degree [4]) $x^\nu \to x^{\nu'} = x^{\nu'}(x^0, x^1, \ldots, x^n)$ in n+1 variables ($\iota, \kappa, \ldots, \omega = 0, 1, \ldots, n; \iota', \kappa', \ldots, \omega' = 0', 1' \ldots, n'$);

$\mathfrak{F}$ the group of all point transformations $x^\nu \to \bar{x}^\nu = \rho\, x^\nu$ (ρ homogeneous of null degree in the x^ν);

X_n an n-dimensional manifold with ur-variables ξ^k and $\mathfrak{G}_n$ as its group of coordinate transformations;

H_n an n-dimensional manifold with n+1 (excess, “homogeneous”) coordinates x^ν and $\mathfrak{H}_{n+1}$ ($\mathfrak{F}$, resp.) as its group of coordinate (point, resp.) transformations (in TPZ, it was denoted by ${}^{n+1}H$);

L_n and X_n with a general linear displacement;

P_n an H_n with a general projective connection (in TPZ, it was denoted by ${}^{n+1}P$; in the older works of J.A. Schouten, it was denoted by $\mathfrak{P}_n$);

E_n a (“Euclidian”) plane in L_n;

E_n^* a (“projective Euclidian”) plane in P_n (i.e., a manifold with ordinary projective geometry) (in TPZ, it was denoted by ${}^{n+1}E$, and for J.A. Schouten, it was previously denoted by P_n).

Overview of the transformation laws:

	for $\mathfrak{H}_{n+1}$	for $\mathfrak{F}$
Scalar of $\mathfrak{r}^{\text{th}}$ degree	$\overset{(\nu')}{p} = \overset{(\nu)}{p}$	$\bar{p} = \rho^{\mathfrak{r}}\, p$
Contravariant point [9]) of $\mathfrak{r}^{\text{th}}$ degree	$v^{\nu'} = \mathfrak{J}^{\nu'}_{\nu} v^{\nu}$	$\bar{v}^{\nu} = \rho^{\mathfrak{r}}\, v^{\nu}$
Covariant point [9]) of $\mathfrak{r}^{\text{th}}$ degree	$w_{\nu'} = \mathfrak{J}^{\nu}_{\nu'} w_{\nu}$	$\bar{w}_{\nu} = \rho^{\mathfrak{r}}\, w_{\nu}$
Projector of $\mathfrak{r}^{\text{th}}$ degree (e.g.)	$X_{\lambda'\mu'}^{\cdot\,\cdot\,\nu'} = \mathfrak{J}^{\lambda\mu\nu'}_{\lambda'\mu'\nu}\, X_{\lambda\mu}^{\cdot\,\cdot\,\nu}$	$\bar{X}_{\lambda\mu}^{\cdot\,\cdot\,\nu} = \rho^{\mathfrak{r}}\, X_{\lambda\mu}^{\cdot\,\cdot\,\nu}$
(Projective) density of $\mathfrak{r}^{\text{th}}$ degree and weight $\mathfrak{r}$	$\overset{(\nu')}{\mathfrak{p}} = \Delta^{-\mathfrak{k}}\, \overset{(\nu)}{\mathfrak{p}}$	$\bar{\mathfrak{p}} = \rho^{\mathfrak{r}}\, \mathfrak{p}$
Projector density of $\mathfrak{r}^{\text{th}}$ degree and weight $\mathfrak{k}$ (e.g.)	$\mathfrak{J}^{\lambda'\cdot\nu'}_{\cdot\mu'} = \Delta^{-\mathfrak{k}}\, \mathfrak{J}^{\lambda'\mu\nu'}_{\lambda\mu'\nu}\, \mathfrak{X}^{\lambda\cdot\nu}_{\cdot\mu}$	$\bar{\mathfrak{X}}^{\lambda\cdot\nu}_{\cdot\mu} = \rho^{\mathfrak{r}}\, \mathfrak{X}^{\lambda\cdot\nu}_{\cdot\mu}$
(Projective) geometric object	(anything)	(anything)

in which:

$$\left.\begin{aligned} \mathfrak{J}^{\nu'}_{\nu} &\doteq \partial_{\nu} x^{\nu'}, \quad \partial_{\nu} = \frac{\partial}{\partial x^{\nu}} \\ \mathfrak{J}^{\nu}_{\nu'} &\doteq \partial_{\nu'} x^{\nu}, \quad \partial_{\nu'} = \frac{\partial}{\partial x^{\nu'}} \end{aligned}\right\} \quad {}^{10})\ {}^{11}) \quad \ldots . \quad (1)$$

are the mixed values (functional matrix) of the *unit projector* $\mathfrak{J}^{\nu'}_{\nu}$, and:

$$\Delta = \operatorname{Det}(\mathfrak{J}^{\nu'}_{\nu}) \quad \ldots . \quad (2)$$

is the functional determinant.

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[9]) In TPZ, this was denoted by "contravariant positions of $\mathfrak{r}^{\text{th}}$ degree." It is, however, better to reserve the expression "position" for sets of coincident points $\lambda\, v^{\nu}$ with a *completely arbitrary* factor λ, which will also be done here. With no loss of generality, one can, however, assume that λ is homogeneous of null degree.

[10]) In TPZ, the notation for $\mathfrak{J}^{\nu'}_{\nu}$ was $A^{\nu'}_{\nu}$. Here, however, we would like to reserves the ordinary symbol A for the unit *affinor*.

[11]) We employ the sign $\doteq$ when the left-hand or right-hand side of an equation is a projector, and the equation is *not* invariant under *all holonomic or anholonomic transformations* in the system of reference (and indeed for each individual index in it).

If a geometric object acquires the factor ρ^{ε} when one first performs the transformation $x^{\nu} \to \rho x^{\nu}$ from $\mathfrak{F}$ and then the transformation $x^{\nu} \to x^{\nu'} = \rho^{-1} x^{\nu}$ from $\mathfrak{H}_{n+1}$ [12]) *for a constant* ε, then ε is called the *excess* of the geometric object. For (a projector or) a projector density of $\mathfrak{r}^{\text{th}}$ degree $\mathfrak{X}_{\mu_1 \cdots \mu_s}^{\cdot \cdot \cdot \cdot \nu_1 \cdots \nu_t}$ of weight $\mathfrak{k}$ (t is called the *contravariant valence*, s, the *covariant valence*, $s + t$, the *valence*) one has:

$$\varepsilon = \mathfrak{r} - (t - s) - (n + 1)\mathfrak{k} \qquad . \ . \ . \ . \qquad (3)$$

The *weight* [13]) of a covariant point w_{μ} is the scalar:

$$w = w_{\mu} x^{\mu} . \qquad . \ . \ . \ . \qquad (4)$$

Each hyperplane in the local E_n^* that does not include that contact point x^{ν} can be *uniquely* associated with a covariant point of weight 1. For contravariant points, such an association is only possible in the projective-affine case (§ 3).

If ξ^k ($h, \ldots, l = \mathbf{1}, \ldots, \mathbf{n}$) are any n independent homogeneous functions of null degree in the x^{ν} [14]) (i.e., $x^{\mu} \partial_{\mu} x^k = 0$) that come from the group $\mathfrak{G}_n$ then they determine a X_n whose points correspond to the *positions* of H_n in a one-to-one way; hence, they can be identified with them [15]). Let the transformations of $\mathfrak{G}_n$ and $\mathfrak{H}_{n+1}$ be completely independent of each other, i.e., let the ξ^k (x^0, resp.) be scalars relative to $\mathfrak{H}_{n+1}$ ($\mathfrak{G}_n$, resp.). If one sets:

$$E_{\mu}^{k} \doteq \partial_{\mu} \xi^{k} \qquad . \ . \ . \ . \qquad (5)$$

then the E_{μ}^{k} transform like an affinor in the index k, and like a projector in the index μ:

$$E_{\mu''}^{k'} \doteq \mathfrak{I}_{\mu''}^{\mu} E_{\mu}^{k} A_{k}^{k'} . \qquad . \ . \ . \ . \qquad (6)$$

Thus, $A_{k}^{k'}$ denotes a unit affinor in X_n . E_{μ}^{k} defines the connecting term between affinors and projectors; it uniquely associates each *contravariant point* with a contravariant vector and each *covariant vector* with a covariant point of weight 0:

$$'v^k = E_{\mu}^{k} v^{\mu} , \quad 'w_{\mu} = E_{\mu}^{k} w_{\kappa} , \quad 'w_{\mu} x^{\mu} = 0 . \qquad . \ . \ . \ . \qquad (7)$$

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[12]) This process is a special case of the well-known "dragging" of a coordinate system in the study of deformations.

[13]) In the sense of Möbius. Not to be confused with the weight of a density. VEBLEN's "weight" or "index" does not correspond to our weight, but to our excess.

[14]) In TPZ, § 14, only the special functions $\xi^k \doteq \dfrac{x^k}{x^0}$ were considered.

[15]) Thus, the difference between X_n and H_n will no longer be tacit, since *geometric objects* in X_n can be completely different from ones in H_n.

One must note that $'v^k$ does not depend upon the position λv^ν, but on the point v^ν itself.

§ 3. *Projective-affine manifolds.*

In each local E_n^* in H_n, let there be given, once and for all, a hyperplane that does not pass through x^ν. From what was said above, we can represent it uniquely by a covariant point t_ν of weight 1 and excess 0:

$$t_\nu x^\nu = 1. \qquad \ldots . \qquad (8)$$

We call such a field of hyperplanes in H_n *projective-affine*. In a *projective-affine* H_n one can also associate each *contravariant* point v^ν with a weight:

$$v = v^\nu t_\nu , \qquad \ldots . \qquad (9)$$

and each contravariant *position* that does not lie in t_ν is uniquely associated with its contravariant point of weight 1. The affine projector E_μ^k may now be associated with a unique corresponding dual E_k^ν that is a solution of the equations:

$$E_j^\nu \; E_\nu^k = A_j^k , \quad t_\nu \; E_j^\nu = 0. \qquad \ldots \qquad (10)$$

In a projective-affine H_n there is a *one-to-one* map from the covariant and contravariant *points of null weight and excess* in E_n^* to the covariant and contravariant *vectors* in E_n :

$$\left.\begin{array}{lll} v^k = E_\mu^k v^\mu & \text{for } v^\mu t_\mu = 0; & v^\nu = E_k^\nu v^k , \\ w_j = E_j^\nu w_\nu & \text{for } w_\nu x^\nu = 0; & w_\mu = E_\mu^k w_k . \end{array}\right\} \qquad \ldots \qquad (11)$$

We can thus identify these objects with each other (so we express them by the same symbol) and consider affinors to be a special type of projector; in particular, vectors are a special type of point, namely, a point of null weight and excess. By means of this association, the affine projector E_μ^k is identical with the unit affinor (but not with the unit projector $\mathfrak{J}_\mu^\nu$!); correspondingly, we will replace the symbol E with A:

$$A_\mu^k = \partial_\mu \, \xi^k , \qquad x^\mu \; A_\mu^k = 0 \; ; \qquad A_{j\nu}^{\nu k} = A_j^k , \qquad t_\nu \; A_k^\nu = 0 . \quad \ldots \; (12)$$
$$A_\mu^\nu = \mathfrak{J}_\mu^\nu - x^\nu \, t_\nu .$$

The difference of two covariant (contravariant, resp.) points is a vector when and only when the points have equal weight. It then follows that one can associate each covariant (contravariant, resp.) *position* μw_ν ($\lambda \; v^\nu$, resp.) that does not pass through x^ν (does not lie in t_ν, resp.) uniquely with a vector w'_μ (v'^ν , resp.) by means of the relation:

$$\left.\begin{aligned} v'^{\nu} &= \frac{1}{v}v^{\nu} - x^{\nu} = \frac{1}{v}A^{\nu}_{\mu}v^{\mu}, \qquad v = v^{\nu}t_{\nu} \neq 0, \\ w'_{\mu} &= \frac{1}{w}w_{\mu} - t_{\mu} = \frac{1}{w}A^{\nu}_{\mu}w_{\nu}, \quad w = w_{\mu}x^{\mu} \neq 0, \end{aligned}\right\} \quad . \; . \; . \quad (13)$$

or with affine values:

$$v'^{k} = \frac{1}{v}A^{k}_{\nu}v^{\nu}, \quad w'_{\mu} = \frac{1}{w}A^{j}_{\mu}w_{j}. \quad . \; . \; . \quad (14)$$

v'^k and w'_μ are invariant under the replacement of v^ν (w_μ, resp.) by the points that are coincident with them. Geometrically, $\lambda v'^\nu$ is the intersection of t_ν with the line that connects v^ν and x^ν; the factor λ changes when and only when the position of v^ν moves along this line, and will be null when and only when v^ν coincides with x^ν. Correspondingly, $\mu w'_\nu$ is the E^*_{n-1} that connects x^ν with the intersection E^*_{n-2} of the two E^*_{n-1}'s, w_ν and t_ν; the factor m changes when and only when these E^*_{n-1}'s rotate around this E^*_{n-2} in a clump of E^*_{n-1}'s, and are null when and only when they coincide with t_ν. From (13), the incidence condition $v^\nu w_\nu = 0$ for covariant (contravariant, resp.) points is equivalent to the incidence condition $v'^\nu w'_\nu = 1$ for the associated vectors. Each point v^ν (w_ν, resp.) is uniquely determined by its vector v'^ν (w'_ν, resp.) and its weight v (w, resp.):

$$v^{\nu} = \begin{cases} v(v'^{\nu} + x^{\nu}) & \text{for } v \neq 0 \\ v'^{\nu} & \text{for } v = 0 \end{cases} \qquad w_{\nu} = \begin{cases} w(w'_{\nu} + t_{\nu}) & \text{for } w \neq 0 \\ w'_{\nu} & \text{for } w = 0. \end{cases} \qquad (15)$$

If one defines the *difference of two positions* μu^ν, λv_ν by:

$$\frac{v^{\nu}}{v} - \frac{u^{\nu}}{u}, \quad u = u^{\nu}t_{\nu} \neq 0, \quad v = v^{\nu}t_{\nu} \neq 0 \; . \; . \; . \; . \; . \; . \; . \qquad (16)$$

then it is always a vector, and one easily verifies that it is identical with the point difference of the MÖBIUS-GRASSMANN point calculus. If one interprets t_ν as an "infinitely distant" hyperplane then one can construct infinitesimal parallelograms that are accurate up to quantities of second order, in not only the E^*_n's, but also in H_n itself [16]), that obey the addition properties of line elements. Namely, if y^ν, z^ν, u^ν, v^ν are four positions in H_n that lie in an infinitesimal neighborhood of the position x^ν in H_n then one easily verifies [17]) that the condition for the parallelogram:

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[16]) The existence of a displacement does not naturally follow from this, since that demands a parallelism that is accurate up to quantities of *third* order.

[17]) For this, one needs to regard the transformation formula for $y^\nu = x^\nu + dx^\nu$, etc., as points in H_n (not in E^*_n):

$$y^{\nu'} = x^{\nu'}(y) = x^{\nu'}(x) + dx^{\lambda}\,\mathfrak{I}^{\nu'}_{\lambda} + \frac{1}{2}dx^{\kappa}dx^{\lambda}\,\partial_{\kappa\lambda}\,x^{\nu'} + \ldots = y^{\mu}\,\mathfrak{I}^{\nu'}_{\lambda} + \frac{1}{2}dx^{\kappa}dx^{\lambda}\,\partial_{\kappa\lambda}\,x^{\nu'} + \ldots$$

$$\frac{y^\nu}{y^\mu t_\mu} - \frac{z^\nu}{z^\mu t_\mu} = \frac{u^\nu}{u^\mu t_\mu} - \frac{v^\nu}{v^\mu t_\mu}, \qquad (t_\mu = t_\mu(x))$$

is exactly invariant under $\mathfrak{F}$ and up to quantities of second order under $\mathfrak{H}_{n+1}$, and independent of the position of x_ν.

If y^ν in H_n is an infinitesimally neighboring point to x^ν then the differential $dx^\nu = y^\nu - x^\nu$ is well-known to not be invariant under $\mathfrak{F}$:

$$d\bar{x}^\nu = \rho\, dx^\nu + x^\nu\, d\rho\,. \qquad \ldots\ldots \quad (17)$$

In a *projective-affine* H_n one can associate it with a unique vector differential $d'x^\nu$ that is totally (relatively) invariant under $\mathfrak{F}$, depends only upon the *position* of y^ν, agrees with the ordinary line element:

$$d'x^\nu = A^\nu_\mu\, dx^\mu = dx^\nu - (t_\mu\, dx^\mu)\, x^\nu = \frac{y^\nu}{y^\mu t_\mu} - x^\nu = A^\nu_\kappa\, d\xi^k\,. \qquad (18)$$

The affinization of projective differential geometry by means of a covariant point t_μ of weight 1 also clarifies the meaning of the projective densities in the older theories [18]):

If $\mathfrak{G}$ is an arbitrary projective density of degree $\mathfrak{r}$ and weight $\neq 0$ then:

$$t_\mu \doteq \frac{1}{\mathfrak{G}}\,\partial_\mu\,\mathfrak{G}$$

is a piecewise undetermined hyperplane that does not go through the contact point. In fact, in the older theories (in a certain formulation) $\mathfrak{G}$ ultimately appears in the link (21). From the fact that:

$$x^\mu\, \partial_\mu\, \mathfrak{G} = \mathfrak{r}\ \mathfrak{G} = \varepsilon\, \mathfrak{G} + \mathfrak{k}\,(n+1)\,\mathfrak{G}\,,$$

one has that t_μ is normalized for densities of null excess when and only when $\mathfrak{k} = \dfrac{1}{n+1}$.

§ 4. *Projective and affine connections.*

Now, let an arbitrary projective connection be given by a system of homogeneous functions in H_n, $\Pi^\nu_{\lambda\mu}$, of degree –1, and with the transformation law:

$$\Pi^{\nu'}_{\lambda'\mu'} = \mathfrak{I}^{\lambda\mu\nu'}_{\lambda'\mu'\nu}\Pi^\nu_{\lambda\mu} + \mathfrak{I}^{\nu'}_\rho\,\partial\mathfrak{I}^\rho_{\lambda'} \qquad \ldots\ldots \quad (19)$$

For the sake of later applications, we mention the formulas:

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[18]) Cf., TPZ, § 14.

$$x^{\mu}\,\nabla_{\mu}\,v^{\nu} = +P^{\nu}_{\cdot\,\mu}\,v^{\mu}\,; \qquad x^{\mu}\,\nabla_{\mu}\,w_{\lambda} = -P^{\nu}_{\cdot\,\lambda}\,w_{\nu} \quad . \; . \; . \; . \tag{20}$$

$$P^{\nu}_{\cdot\,\mu} \doteq \Pi^{\nu}_{\lambda\mu}x^{\mu} + \mathfrak{J}^{\nu}_{\lambda}\ {}^{19)} \qquad Q^{\nu}_{\cdot\,\mu} \doteq \Pi^{\nu}_{\lambda\mu}x^{\lambda} + \mathfrak{J}^{\nu}_{\mu}\,; \quad b^{\nu} = P^{\nu}_{\cdot\,\mu}x^{\lambda} = Q^{\nu}_{\cdot\,\mu}x^{\mu}\,. \tag{21}$$

The projective connection uniquely determines a *correspondence* that associates each point v^{ν} with the point $x^{\mu}\,\nabla_{\mu}\,v^{\nu}$. Since $x^{\mu}\,\nabla_{\mu}\lambda$ vanishes for every scalar of null excess, one has $x^{\mu}\,\nabla_{\mu}\,\lambda v^{\nu} = \lambda\,x^{\mu}\,\nabla_{\mu}\,v^{\nu}$, such that the correspondence uniquely determines a projective map (for the *positions*) from E^{*}_{n} to itself. For its own part, this map determines the correspondence only up to an arbitrary factor that can be established up to an arbitrary non-vanishing invariant of $P^{\nu}_{\cdot\,\mu}$, e.g. $P^{\rho}_{\cdot\,\rho}$ or $P^{\nu}_{\cdot\,\mu}\,P^{\mu}_{\cdot\,\nu}$ [20]).

From (16), $x^{\mu}\,\nabla_{\mu}\,v^{\nu}$ does not, in general, determine any well-defined point; the expression is determined only up to an arbitrary multiplicity of corresponding points $x^{\mu}\,\nabla_{\mu}\,v^{\nu}$. There thus exists, in general, *no* covariant differential. In order to obtain such a thing there are two possibilities: first, the additional terms that are responsible for the indeterminacy can vanish because $P^{\nu}_{\cdot\,\lambda} = 0$ [21]); second, however, one can succeed in choosing one of the infinitely many possible points uniquely. This case comes about in a projective-affine P_n, since one can normalize the differential by means of (18) in such a space. We denote the thus-defined covariant differential operator by δ:

$$\delta = d'\,x^{\lambda}\,\nabla_{\lambda}\,\nabla\,dx^{\mu}\;A^{\rho}_{\mu}\,\nabla_{\rho} = dx^{\lambda}\,\nabla_{\lambda} - (t_{\mu}\;dx^{\mu}\,)\,x^{\lambda}\,\nabla_{\lambda}\ {}^{22)}. \qquad . \; . \; . \tag{22}$$

In general, this covariant differential defines no affine displacement, since the covariant differential of an affinor is not generally an affinor, but a projector. However, an affine connection is completely determined by way of the projective connection, along with the differential operator $\overset{A}{\nabla}_{\lambda}$ that one obtains when one takes the affinor part of the covariant derivative of an *affinor*:

$$\left.\begin{array}{ll} \overset{A}{\nabla}_{\mu}\,v^{\nu} = A^{\rho\nu}_{\mu\sigma}\nabla_{\rho}v^{\sigma} \quad \text{for } v^{\nu}t_{\nu} = 0, & \\ & \overset{A}{\nabla}_{\mu}\,x^{\nu} = 0. \\ \overset{A}{\nabla}_{\mu}\,w_{\nu} = A^{\rho\sigma}_{\mu\nu}\nabla_{\rho}w_{\sigma} \quad \text{for } w_{\nu}x^{\nu} = 0; & \end{array}\right\} \qquad . \; . \; . \tag{23}$$

If $\overset{A}{\Pi}{}^{\nu}_{\lambda\mu} = A^{ij\nu}_{\lambda\mu k}\Gamma^{k}_{ij} + A^{\nu}_{k}\partial A^{k}_{\lambda}$ is the associated displacement parameter and one has:

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[19]) In the supplement to TPZ this is denoted by $\overset{*}{P}{}^{\cdot\,\nu}_{\lambda}$.

[20]) For the most important physical application, $P^{\nu}_{\cdot\,\mu}\,P^{\mu}_{\cdot\,\nu}$ is, up to a constant factor, the energy density of the electromagnetic field.

[21]) Cf., TPZ, § 7, 8.

[22]) One can also arrange this so that the differential operator $=_{\mu}$ is replaced with $\nabla^{*}_{\mu} = A^{\rho}_{\mu}\,\nabla_{\rho}$. One then has $\Pi^{*\nu}_{\lambda\mu} = \Pi^{\nu}_{\lambda\mu} - P^{\nu}_{\cdot\,\lambda}t_{\mu}$, such that $P^{*\nu}_{\cdot\,\lambda} = 0$. For ∇^{*}_{μ}, the analysis of TPZ is still valid, and the covariant differential that is defined by (22) is equal to the ordinary differential (TPZ, § 7) relative to ∇^{*}_{μ}.

$$X_{\lambda\mu}^{\cdot\cdot\nu} = \Pi_{\lambda\mu}^{\nu} - \overset{A}{\Pi}{}_{\lambda\mu}^{\nu}, \qquad (24)$$

then one has:

$$X_{\lambda\mu}^{\cdot\cdot\nu} = -x^{\nu}\,\nabla'_{\mu} t_{\lambda} + P'^{\nu}_{\cdot\,\mu} t_{\mu} + t_{\lambda} Q'^{\nu}_{\cdot\,\mu} + A^{\rho}_{\lambda} P^{\sigma}_{\cdot\,\rho}\, t_{\sigma} t_{\mu} x^{\nu} + t_{\lambda} t_{\mu} b'^{\nu} + x^{\nu} t_{\lambda} Q^{\rho}_{\cdot\,\mu} t_{\rho}, \qquad (25)$$

in which:

$$\nabla'_{\mu} t_{\lambda} = A^{\sigma\rho}_{\mu\lambda} \nabla_{\sigma} t_{\rho}, \qquad P'^{\nu}_{\cdot\,\mu} = A^{\sigma\rho}_{\mu\lambda} P^{\sigma}_{\cdot\,\rho}, \qquad Q'^{\nu}_{\cdot\,\mu} = A^{\sigma\rho}_{\mu\lambda} Q^{\sigma}_{\cdot\,\rho}, \qquad b'^{\nu} = A^{\nu}_{\rho} b^{\rho} \qquad (26)$$

denote the affinor parts of the unprimed quantities. In order for the differential operator ∇_{μ} itself to be identical with $\overset{A}{\nabla}_{\mu}$ *when applied to affinors*, it is necessary and sufficient that $\nabla_{\mu} x^{\nu}$, $\nabla_{\mu} t_{\nu}$, and $P^{\nu}_{\cdot\,\lambda}$ have the following form:

$$\nabla_{\mu} x^{\nu} = q_{\mu} x^{\nu}, \qquad \nabla_{\mu} t_{\nu} = -q_{\mu} t^{\nu}, \qquad (27)$$

$$P^{\nu}_{\cdot\,\lambda} = P\, x^{\nu} t_{\lambda}, \qquad P = q_{\rho} x^{\rho}. \qquad (28)$$

In this case, one can thus take the projective connection to be an *affine connection*, as well. Conversely, an affine connection with the parameters Π_{ij}^{k} uniquely determines a projective connection (if one further demands that $q_{\mu} = 0$, hence, that one also has $P = 0$), *as long as the hyperplane in E_n^* that is identified with the "imaginary" one is given.* (27) states that the contact point and the imaginary hyperplane (affine: the contravariant and covariant null vectors) is covariantly constant under displacement. On the other hand, (28) states that the correspondence degenerates completely, namely, each covariant (contravariant, resp.) position goes to infinity (the contact point, resp.) Only for $P = 0$ is the displacement identical with the one that was obtained by the first method.

A *geodetic position field* will be given by an equation of the form:

$$v^{\mu}\,\nabla_{\mu} v^{\nu} = \varphi\, v^{\nu}. \qquad (29)$$

This equation preserves its form under the replacement of v^{ν} with λv^{ν}. For a certain choice of λ one can arrange that $\varphi = 0$; if one links each position λv^{ν} in E_n^* with the associated x^{ν} by a line then a direction field is determined in H_n; the associated curves, which we call *pseudo-geodetic lines*, depend only upon the position λv^{ν} (not on the weight of v^{ν}). In general, the pseudo-geodetic lines change, however, when one replaces the initial position λv^{ν} with one that is collinear with it and x^{ν} (hence, under preservation of the initial direction). In order for the pseudo-geodetic lines to remain invariant under all such changes (in which case, we call it *geodetic*), it is necessary and sufficient that $P^{\nu}_{\cdot\lambda} + Q^{\nu}_{\cdot\lambda}$ has the form [23]):

$$P^{\nu}_{\cdot\lambda} + Q^{\nu}_{\cdot\lambda} = x^{\nu} z_{\lambda} + R\,\mathfrak{I}^{\nu}_{\lambda}. \qquad (30)$$

This is the *first possibility* for uniquely determining a system of curves in a P_n; pseudo-geodetic lines exist in *each* P_n. The *second* possibility is, in turn, to choose one position λv^{ν} uniquely out of all possible ones. In the projective-affine case, this possible through the condition that $v^{\nu}\,\nabla v'^{\nu}$ be a *vector*. The defining equation of the line that results in that manner:

$$v'^{\mu}\nabla_{\mu} v'^{\nu} = \varphi\, v'^{\nu} \qquad (31)$$

is integrable when and only when $\nabla'_{(\mu} t_{\nu)} = 0$ (cf., 26), hence, when $\nabla_{(\mu}\, t_{\lambda)}$ has the form:

$$\nabla_{(\mu}\, t_{\lambda)} = u_{(\mu}\, t_{\lambda)}\,. \qquad (32)$$

One then has $u_{\mu} = -\,t_{\rho}\, P^{\rho}_{\cdot\mu} - t_{\rho}\, Q^{\rho}_{\cdot\mu} + t_{\rho}\, b^{\rho}\, t_{\mu}$. In the event that solutions of (31) exist, they are identical with the *affine geodetic lines*, i.e., the (ordinary) *geodetic lines* of the affine connection, which are defined by:

$$v'^{\mu}\, \overset{A}{\nabla}_{\mu}\, v'^{\nu} = \varphi\, v'^{\nu}. \qquad (33)$$

In the event that the projective connection itself is affine, then, from (27), this is always the case. Except for the pseudo-geodetic, the pseudo-affine geodetic, and affine geodetic lines, only the *semi-affine geodetic lines* have any particular meaning for the unification problem in physics [24]). There definition reads like:

$$v'^{\mu}\nabla_{\mu} v'^{\nu} = \varphi\, v'^{\nu}. \qquad (34)$$

They arise from performing a *displacement* of a position (setting the covariant differential (22) to zero) in the direction of its vector (the line connecting it with x^{ν}), and are independent of the weight of v^{ν}, but dependent on where the position is along the connecting line. They agree with the affine (the pseudo-affine, resp.) geodetic lines when and only when $Q'^{\nu}_{\cdot\mu} = 0$, i.e., when $Q'^{\nu}_{\cdot\mu}$ has the form:

$$Q^{\nu}_{\cdot\mu} = b^{\nu}\, t_{\mu} + x^{\nu}\, q_{\mu} - q\,\mathfrak{I}^{\nu}_{\mu} \qquad (q = q_{\mu}\, x^{\mu}), \qquad (35)$$

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[23]) In TPZ, individual condition equations of the form (30) were given for $P^{\nu}_{\cdot\lambda}$ and $Q^{\nu}_{\cdot\lambda}$. The geodetic lines that are defined here are *not* identical with the ones that were defined there; instead of the defining equation $H_{ab}^{\cdot\cdot\nu} = 0$ found there (TPZ, § 10), they satisfy only the weaker condition $H_{(ab)}^{\cdot\cdot\nu} = 0$.

[24]) They essentially agree with the ones that were introduced by EINSTEIN and MAYER; cf., also, J.A. Schouten and D. VAN DANTZIG, Über eine vierdimensionale Deutung der neuesten Feldtheorie, these Proceedings, **34** (1931) 1398-1407.

(when (35) and (32) are valid, resp.). In the latter case, the semi-affine geodetic, pseudo-affine geodetic, and the affine geodetic (but not the pseudo-geodetic) lines are identical. In the physical interpretation, the deviation of the semi-affine geodetic lines from the affine geodetic ones implies the presence of an electromagnetic field.

One can introduce a curve parameter s (a scalar of null degree) on any geodetic line by the defining equation:

$$v'^{\mu}\nabla_{\mu}s = 1, \qquad \ldots\ldots\ldots \qquad (36)$$

that depends *only* upon the current *direction* of the curve [25]). If t is an arbitrary parameter (a scalar of null degree) along the curve then $\frac{d'x^{\nu}}{dt} = \beta\, v^{\nu}$, and the integral in (36) reads like $s = \int \beta\, dt$. From (36), it follows that:

$$d'x^{\nu} = A^{\nu}_{k} d\xi^{k} = A^{\nu}_{k}\frac{d\xi^{k}}{ds}ds = A^{\nu}_{k}(v'^{\mu}\partial_{\mu}\xi^{k})ds = v'^{\nu}\,ds, \qquad \ldots\ldots \qquad (37)$$

such that one finds the differential operator $v'^{\mu}\nabla_{\mu}$ to be:

$$v'^{\mu}\nabla_{\mu} = \frac{\delta}{ds}. \qquad \ldots\ldots\ldots \qquad (38)$$

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[25]) $e^{\int t_{\mu}dx^{\mu}}$ (N.B.: dx^{ν} itself, not $d'x^{\nu}$, exists on any curve as a homogeneous function of the first degree (excess = 1), up to an arbitrary factor of null degree.)

On general projective differential geometry.
II. X_{n+1} with a one-parameter group.

By D. VAN DANTZIG [†]

(Communicated at the meeting of April 30, 1932.)

I. In the previous part [1]), I showed that affine geometry [2]) can be related to general projective geometry in a manner that is analogous to the way that ordinary affine geometry relates to ordinary projective geometry in flat spaces, namely, by singling out a field of hyperplanes in the local E_n^*, with the single condition that they not pass through the contact point. In the present part, I will treat the converse problem: the relationship between projective differential geometry to affine geometry in one higher dimension. [3])

It is well-known that ordinary projective geometry exists within affine geometry in one higher dimension, when one regards the lines through a fixed point as the elements of new space. In the present note, I will show that a completely analogous fact is true in the general differential geometry of curved spaces: general projective geometry may be obtained from the affine geometry of one higher dimension, in which *one regards the curves of an arbitrary one-parameter group* (in the flat case: the homothety group of a fixed point) *as the elements of a new space.* For the basic notions and notation, confer TPZ [4]) and APD I.

2. The n+1 "homogeneous" coordinates x^ν in a H_n can be regarded as the ur-variables of an X_{n+1}. The group $\mathfrak{H}_{n+1}$ thus goes to that subgroup of $\mathfrak{G}_{n+1}$ that fixed the coordinates of the null point and preserves the form of the equation $x^\nu = t\, a^\nu$ (a^ν = const.) of the lines through the null point; the group F corresponds to the group of those point transformations of X_{n+1} that leave each line through the null point invariant and induce a homothety on each such line. These lines themselves correspond to the *positions* in H_n. Now, if Ξ^N ($I, K, \ldots, \Omega = \overline{0}, \overline{1}, \cdots, \overline{n}$) are any ("curvilinear") coordinates in the X_{n+1} then one can *transform the projectors as affinors in* X_{n+1}, i.e., introduce affinors whose values relative to the special coordinate system x^ν in which the projectors in question are equal to, e.g.:

[†]) Translated by D.H. Delphenich.

[1]) D. VAN DANTZIG, Zur allgemeinen projectiven Differentialgeometrie, I. Einordnung der Affinegeometrie, these Proceedings **35** (1932), denoted by APD I.

[2]) Briefly, for manifolds with general linear (not necessarily symmetric) displacement (III A*a*, in the classification of J.A. SCHOUTEN, Der Ricci-Kalkül, Springer (1924), pp. 75).

[3]) Cf., also D. VAN DANTZIG, Theorie des projektiven Zusammenhangs n-dimensionaler Räume, Math. Ann. **106** (1932), 400-454, denoted by TPZ, pp. 408. In that work, I briefly touched on the questions without completely following through, since in that work the general coordinates X^N for n+1 were not introduced. For a special class of projective displacements (cf., TPZ, footnote 44a), J.H.C. WHITEHEAD, The representation of projective spaces, Ann. of Math. **32** (1931), 327-360, treated a closely related problem, without which the aforementioned questions would be regarded as completely resolved.

[4]) The notation of the *supplement* will be used, with the omission of the * that appears there; some deviations were given in APD I.

$$P^{N}_{\cdot M} = \mathfrak{I}^{N\mu}_{\nu M}\, P^{\nu}_{\cdot\mu}\,; \qquad \mathfrak{I}^{N}_{\nu} \doteq \partial_{\nu}\Xi^{N}, \qquad \mathfrak{I}^{\nu}_{N} \doteq \partial_{N} x^{\nu}\,; \qquad \partial_{\nu} \doteq \frac{\partial}{\partial x^{\nu}}, \qquad \partial_{N} \doteq \frac{\partial}{\partial \Xi^{N}}. \tag{1}$$

In particular, the field of "contact points" x^ν, when regarded as a points in the local E_n^* (not the H_n itself!), goes to a contravariant vector field x^N:

$$x^N = \mathfrak{I}^N_\nu x^\nu, \qquad \ldots\ldots\ldots\ldots \tag{2}$$

that has the character of a field of *radius vectors* in X_{n+1}, relative to the special coordinates x^ν, but seems to be a completely arbitrary field relative to more general ones. Naturally, one thus has, in general:

$$x^N \neq \Xi^N. \qquad \ldots\ldots\ldots\ldots \tag{3}$$

3. The vector field x^N determines an infinitesimal transformation $\mathfrak{T}$ with the LIE symbol:

$$Xf = x^N \partial_N f. \qquad \ldots\ldots\ldots\ldots \tag{4}$$

The expression (4) has an invariant meaning only when f is a scalar. However, it may be extended to an invariant operator on arbitrary affinors that we will call *the Lie derivative*, and denote by the symbol $\underset{L}{D}$. [1]) For its definition, one needs to be given only a contravariant vector field x^N, but no displacement in X_{n+1}.

Namely, if one considers the Ξ^N to be new coordinates of those points whose old coordinates were $\Xi^N + x^N\,dt$ then we say that the coordinate system has been *dragged along* by the infinitesimal transformation. The LIE derivative of any geometrical object at a point of X_{n+1} will now be defined as the difference of its values relative to the dragged and the original coordinate systems (both of them at the point in question). The operator satisfies the following requirements:

I. If $X\,\mathbf{:}$ and $Y\,\mathbf{:}$ are any sort of affinors [2]) then:

$$\underset{L}{D}(X\,\mathbf{:} + \mathrm{Y}\,\mathbf{:}) = \underset{L}{D}X\,\mathbf{:} + \underset{L}{D}Y\,\mathbf{:}\,.$$

II. For products and contractions, one has the LEIBNIZ rule for differentiation:

$$\underset{L}{D}X\,\mathbf{:}\,Y\,\mathbf{:} = (\underset{L}{D}X\,\mathbf{:})\,Y\,\mathbf{:} + X\,\mathbf{:}\,\underset{L}{D}Y\,\mathbf{:}\,.$$

III. For a scalar f one has:

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[1]) The operator was first introduced by W. ŚLEBODZINSKI. Sur les équations canoniques de Hamilton, Bull. Acad. Roy. Belg. (5) **17** (1931), 864-870. For the definition that is given here, I would like to thank a still quite obscure work of J.A. SCHOUTEN and E.R. VAN KAMPEN on deformations of a X_n.

[2]) By the word "affinors" here we mean affinors in X_{n+1}, not affinors in X_n as in APD I. The points remain for arbitrary sets of indices.

$$\underset{L}{D} f = X f .$$

IV. The LIE derivative of a contravariant vector is equal to the LIE bracket:

$$\underset{L}{D} v^N = [x, v]N = x^\Lambda \partial_\Lambda v^N - v^\Lambda \partial_\Lambda x^N . \qquad (5)$$

For a covariant vector, one finds that:

$$\underset{L}{D} w_M = x^\Lambda \partial_\Lambda w_M + w_\Lambda \partial_M x^\Lambda = \partial_M (w_\Lambda x^\Lambda) + 2\, x^\Lambda \partial_{[\Lambda\ wM]} , \qquad (6)$$

for a general affinor:

$$\left.\begin{aligned} \underset{L}{D} X_{M_1\cdots M_s}^{\cdots\cdots\cdots N_1\cdots N_t} &= x^\Lambda \partial_\Lambda X_{M_1\cdots M_s}^{\cdots\cdots\cdots N_1\cdots N_t} - \\ &-\sum_{j=1}^{t} X_{M_1\cdots M_s}^{\cdots\cdots\cdots N_1\cdots N_{j-1}\Lambda N_{j+1}\cdots N_t} \partial_\Lambda x^{N_j} + \\ &+\sum_{i=1}^{s} X_{M_1\cdots M_{i-1}\Lambda M_{i+1}\cdots M_t}^{\cdots\cdots\cdots\cdots\cdots N_1\cdots N_t} \partial_{M_i} x^\Lambda , \end{aligned}\right\} \qquad (7)\ [1]$$

and for the parameters $\Pi_{M\Lambda}^N$ of an affine transformation:

$$\underset{L}{D} \Pi_{M\Lambda}^N = x^K \partial_K \Pi_{M\Lambda}^N - \Pi_{M\Lambda}^N \partial_\Pi x^N + \Pi_{M\Lambda}^N \partial_\Pi x^N + \Pi_{M\Pi}^N \partial_\Lambda x^\Pi + \Pi_{\Pi\Lambda}^N \partial_M x^\Pi . \qquad (8)$$

It is noteworthy that $\underset{L}{D} \Pi_{M\Lambda}^N$ is an affinor [2]) whether or not this is the case with the $\Pi_{M\Lambda}^N$ themselves, namely:

$$\underset{L}{D} \Pi_{M\Lambda}^N = x^K N_{K\Lambda M}^{\cdot\ \cdot\ \cdot\ N} - \nabla_\Lambda P_{\cdot\ M}^N , \qquad (9)$$

in which:

$$P_{\cdot\ \Lambda}^N = \partial_\Lambda x^N + \Pi_{\Lambda M}^N x^M = \nabla_\Lambda x^N + 2\, S_{\Lambda M}^{\cdot\ \cdot\ N} x^M \qquad (10)$$

and:

$$N_{K\Lambda M}^{\cdot\ \cdot\ \cdot\ N} = -\, 2\partial_{[K} \Pi_{|M|\Lambda]}^N - 2\, \Pi_{\Sigma[K}^N \Pi_{|M|\Lambda]}^\Sigma \qquad (11)$$

is the curvature quantity associated with $\Pi_{M\Lambda}^N$.

We call a geometric object *invariant* under the transformation $\mathfrak{T}$ when the LIE derivative vanishes. In particular, the displacement is called *invariant* when:

$$\underset{L}{D} \Pi_{M\Lambda}^N = 0 . \qquad (12)$$

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[1]) This equation was used as the definition by ŚLEBODZINSKI.
[2]) I would like to thank Professor J.A. SCHOUTEN for this remark, as well as the relation (8).

Equivalent to this is the condition that $\underset{L}{D}$ and $=_M$ commute.

With the intended interpretation of the H_n in a X_{n+1} with a restricted group, we infer the following: projectors of null excess are invariant affinors under $\mathfrak{T}$, which follows immediately by setting the LIE derivative equal to zero in the special coordinates, with the help of the EULER homogeneity condition. In general, a projector X **:** of excess ε is an affinor with:

$$\underset{L}{D} X : = \varepsilon X : \qquad (13)$$

(ε = constant). In this case, we say that X **:** is *relatively invariant* under $\mathfrak{T}$. The condition that $\Pi^N_{M\Lambda}$ has degree −1 (excess 0) is equivalent to (12) (which follows immediately by writing out the LIE derivative in the special coordinates x^ν). Substitution of (12) in (9) yields a well-known formula of projective differential geometry:

$$x^\nu N_{\kappa\lambda\mu}^{\cdot\cdot\cdot\nu} = \nabla_\lambda P^\nu_{\cdot\mu} \qquad (14)$$ [1]

4. Conversely, if an *arbitrary* vector field x^N is given in an X_{n+1} in general coordinates then there are always n + 1 independent solutions of the scalar equation:

$$\underset{L}{D} \Xi = X\,\Xi = \Xi. \qquad (15)$$

If we call them Ξ^ν ($\iota, \kappa, \ldots, \omega = 0, 1, \ldots, n$) and choose them to be new coordinates then, from (15), we have:

$$x^\nu = \mathfrak{J}^\nu_N x^N = x^N \partial_N \Xi^\nu = X\,\Xi^\nu = \Xi^\nu, \qquad (16)$$

i.e., the *values* of the vectors x relative to the special coordinates Ξ^ν are equal to the *coordinates* of the associated "eigenpoint." From (16), it immediately follows that:

$$\partial_\mu x^\nu = \mathfrak{J}^\nu_\mu . \qquad (17)$$

If one now regards the ∞^n curves of the transformation $\mathfrak{T}$ as elements (called "positions") of a new n-dimensional manifold H_n then one can regard the $x^\nu = \Xi^\nu$ as surplus ("homogeneous") coordinates in this H_n. Namely, if $\Xi^{\nu'}$ are n + 1 other solutions of equation (15) then one has:

$$\Xi^\nu \partial_\nu \Xi^{\nu'} = x^\nu \partial_\nu \Xi^{\nu'} = X\,\Xi^{\nu'} = \Xi^{\nu'}, \qquad (18)$$

i.e., (due to the EULER homogeneity condition) that the $\Xi^{\nu'}$ are homogeneous of first degree in the Ξ^ν. The $\Xi^\nu = x^\nu$ thus lie in the group $\mathfrak{H}_{n+1}$. (One obtains the VEBLEN coordinates $\underset{-}{\xi}^\nu$ ($\iota, \kappa, \ldots, \omega = 0, 1, \ldots, n$) (cf., TPZ, §14) when one chooses $\underset{-}{\xi}^1, \underset{-}{\xi}^2, \ldots, \underset{-}{\xi}^n$ to be any n independent integrals of the transformation $\mathfrak{T}$: $X\underset{-}{\xi}^k = 0$, whereas $\underset{-}{\xi}^0$ is the curve

[1]) Cf., TPZ (76), pp. 422 (76*), pp. 452. Whitehead, loc. cit. (1.7), pp. 334.

parameter, which is defined by $X\,\xi^0_- = 1$.) The infinitesimal transformation $\mathfrak{T}$ takes a point $\Xi^\nu = x^\nu$ to $\Xi^\nu + x^\nu\,dt = (1 + dt)\,x^\nu$; a finite transformation of the group that is generated by $\mathfrak{T}$ thus takes x^ν to a point whose coordinates are proportional to those of the prototype x^ν; i.e., two points of X_{n+1} lie on the same curve of the group (which belongs to the same position then and only then) when and only when one goes to the other under a transformation of the group. For an affinor $X_{\mu_1\cdots\mu_s}^{\cdots\cdots\cdots\nu_1\cdots\nu_t}$ on X_{n+1} that satisfies (13), we have, from (7):

$$x^\lambda\,\partial_\lambda\;X_{\mu_1\cdots\mu_s}^{\cdots\cdots\cdots\nu_1\cdots\nu_t} = (\varepsilon + t - s)\,X_{\mu_1\cdots\mu_s}^{\cdots\cdots\cdots\nu_1\cdots\nu_t}\,, \qquad (19)$$

i.e., the affinor goes to a projector of degree $\mathfrak{r} = \varepsilon + t - 2$, hence, of excess ε. (The most general projective connection that was considered in TPZ, § 6 *does not* correspond to the case of a general affine displacement in X_{n+1}, but to a generalization of it in which a displacement is always defined for vectors with different ε that are relatively invariant under $\mathfrak{T}$.) We call the process so described of generating an H_n from a X_{n+1} a *collapsing of X_{n+1} along the given system of curves.* In the particular case in which the X_{n+1} is an E_{n+1} and the curves are parallel lines, it becomes the process that was introduced by H. WEYL of "collapsing a E_{n+1} in a given direction."

5. A field of hyperplanes t_ν (covariant positions) in the local E_n^* of H_n yields a field of n-directions, by their interpretation in X_{n+1}. The condition that the hyperplane does not go through the contact point states that the n-direction does not include the direction of the curve. The local n-directions define a system X_{n+1}^{n} [1]) that is invariant under T and generally anholonomic. If $x^M\,t_M \neq 0$ then one can choose x^N to be an attaching (Einspannung = clamp, hold) vector. The normalization condition $x^M\,t_M = x^\mu\,t_\mu = 1$ [2]) corresponds to the well-known "first normalization condition") for an embedded structure in an X_{n+1} and one can (in the case for which a linear displacement L_{n+1} is given in X_{n+1} that is invariant under $\mathfrak{T}$) carry the entire well-known theory of the curvature of an L_{n+1}^n in L_{n+1} over to H_n (which then becomes a P_n) completely, but we shall go into this no further.

6. If a geodetic line in L_{n+1} is dragged along by the group $\mathfrak{F}$ then there exists a family of ∞^1 geodetic lines that are invariant under $\mathfrak{T}$ (naturally, under the assumption that the displacement in L_{n+1} is invariant under $\mathfrak{T}$) that corresponds to a *pseudo-geodetic* line [3]) in H_n. If a contravariant vector in L_{n+1} is assumed to be pseudo-parallel displaced in the direction of its projection on t_M at x^N (instead of its own direction), and we drag the resulting curve along $\mathfrak{F}$ then we produce a family of ∞^1 curves that are invariant under $\mathfrak{T}$

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[1]) For the theory of anholonomic systems, cf., J.A. SCHOUTEN, On non-holonomic connexions, these Proceedings, **31** (1928), 291-299; J.A. SCHOUTEN and E.R. VAN KAMPEN, Zur Einbettungs- und Krümmungstheorie nichtholonomer Gebilde, Math. Ann. **103** (1930), 752-783.

[2]) Cf., APD I § 3, (18).

[3]) For the various types of geodetic lines in a projective-affine H_n, cf., APD I § 5.

and which correspond to a *semi-affine geodetic* line in H_n . This definition essentially agrees with the one that is given by EINSTEIN and MAYER [1]). Likewise, the *pseudo-affine geodetic* lines correspond to families of geodetic lines in L_{n+1} whose direction always points in the local *n*-direction; on the other hand, the *affine geodetic* lines are simply the geodetic lines of the anholonomic system L^n_{n+1} itself.

Finally, under the action of $\mathfrak{T}$, the geodetic lines in H_n that were defined in TPZ, §10 correspond to *totally geodesic surfaces* in L_{n+1}, when such things exist.

7. The meaning given above for an H_n in an X_{n+1} with a one-parameter group also allows us to clarify the relation between the older five-dimensional [2]) projective relativity theory and the newer four-dimensional one. Here, the X_{n+1} is a V_5, in which a one-parameter group is given. The only difference between the five-dimensional interpretation of the formulas and the four-dimensional one thus amounts to whether one regards the V_5 *before* or *after* the collapsing, i.e., whether one regards the *points* or the *curves* of V_5 as the elements of the space. (Differences concerning this appear among the various authors, independently of the particular location of the curves and any possible asymmetry in the displacement.) Here, we restrict ourselves to the case in which the displacement in the X_5 is RIEMANNIAN, hence, symmetric. [3]) The condition for the displacement to be invariant under $\mathfrak{T}$ is then, from (9), (10), equivalent to:

$$X^K \; N_{K\Lambda M}^{\;\cdot\;\cdot\;\cdot\; N} - =_{\Lambda M} x^N = 0 \;.\;.\;.\;.\;.\;.\;.\;.\;.\;. \qquad (20)$$

A sharper requirement is that the fundamental tensor $G_{\Lambda M}$ on V_5 is itself invariant under $\mathfrak{T}$. The condition for this, namely, $\underset{L}{D}\, G_{\Lambda M} = 0$, is, from (7), equivalent to:

$$\nabla_{[\Lambda}\, x_{M]} = 0 \;, \qquad .\;.\;.\;.\;.\;.\;.\; \qquad (21)$$

viz., the KILLING equation. A necessary and sufficient condition for the invariant of $G_{\Lambda M}$ is then that $\mathfrak{T}$ should be an infinitesimal *motion*, which can be also be seen by a simple calculation. We further assume that the "velocity vector" x_N has a constant *length*, and does not point in a null direction of the fundamental tensor.

In a V_{n+1}, the field x_N uniquely determines the field $t_N = t\, x_N$, t = const. (namely, the local E_{n+1} that is perpendicular to the curve direction; in projective geometry: the polar

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[1]) A. EINSTEIN and W. MAYER, Einheitliche Theorie von Gravitation and Elektrizität, Berlin, Sitzungsberichte **25** (1931), 541-557; (for the case in which L_{n+1} is a V_5). However, it is not an anholonomic system of 4-directions in V_5 that is given, but a V_4 with local R_5, which one can think of as a V_4 that exists in V_5 .

[2]) TH. KALUZA, Zum Unitätsproblem der Physik, Berlin, Sitzungsber. Pr. Ak. (1921) 966-972; O. KLEIN, Quantentheorie und fünfdimensionale Relativitätstheorie, Z. f. Phys. **37** (1926), 895-906; L. ROSENFELD, L'univers à cinq dimensions et la mécanique ondulatoire, Bull. Acad. Roy. Belg. (5) **13** (1927); J.A. SCHOUTEN, Dirac equations in general relativity; 2. Five-dimensional theory, J. for Math. and Phys. **10** (1931), 272-283, and others.

[3]) This case is *not* to be combined with the complete set of physical requirements. Since it is, however, inessential for the sake of recognizing the differences between the five-dimensional theories, we would like to ignore the asymmetry, for the sake of simplicity.

E^*_{n+1} to the contact point x^N relative to the quadric). The most definitive condition for the existence of an electromagnetic field is the anholonomity of the field t_N , which is expressed, upon normalizing t_N, by the non-vanishing of the bivector:

$$t_{\Lambda M} = \partial_{[\Lambda}\, t_{M]} = \nabla_{[\Lambda}\, t_{M]}\,, \qquad \ldots\ldots\ldots (22)$$

which can be identified with the electromagnetic bivector, up to a constant factor

$$F_{\Lambda M} = f\, t_{\Lambda M}\,. \qquad \ldots\ldots\ldots (23)$$

If the KILLING equation (21) is satisfied then one has:

$$t_{\Lambda M} = \nabla_{\Lambda}\, t_M\,, \qquad \ldots\ldots\ldots (24)$$

and conversely, (21) (when $S_{\Lambda M}^{\cdot\,\cdot\,N} = 0$!) follows from (24). Condition (24) is therefore important since it allows one to bring the equation of motion for the electromagnetic field:

$$\frac{\delta i_\nu}{d\tau} = -\frac{e}{mc}\, i^\mu F_{\mu\nu} \qquad \ldots\ldots\ldots (25)$$

in the simpler form:

$$\frac{\delta}{d\tau}\left(i_\nu + \frac{e}{mc} f\, t_\nu\right) = 0. \qquad \ldots\ldots\ldots (26)$$

In the H_4 that comes about after collapsing there is not only the projective connection (viz., the RIEMANNIAN displacement in V_5), but a RIEMANNIAN displacement that is induced and can be identified with the well-known displacement of general relativity theory (which implies the geodetic precession). In the V_5, this translates into nothing but the displacement that is induced in V_5^4 (which is invariant under T). It is clear that the latter does not need to be Euclidian when the V_5 itself is Euclidian. (One already obtains the simplest counter-example in R_3 when one takes T to be an infinitesimal twist.) Since one easily sees that the "second fundamental tensor" of V_5^4 is $=_\Lambda t_M$ in this case, hence, under the assumption of (24), it is proportional to $F_{\Lambda M}$, it follows from the GAUSS equation (extended to the case of a V_5^4) [1]) that the RIEMANNIAN displacement in a H_4 that comes about by collapsing a R_5 by a group of motions with constant (scalar) velocity, is itself Euclidian when and only when the field t_N is holonomic. [2])

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[1]) Cf., J.A. SCHOUTEN, Über nicht-holonome Überträgungen in einer L_n, Math. Z. **30** (1929), 149-172; J.A. SCHOUTEN and E. R. VAN KAMPEN, Zur Einbettungs- und Krümmungstheorie nichtholonomer Gebilde, Math. Ann. **103** (1930), 752-783, Formulas (128), pp. 778.

[2]) Editor's footnote: In a note that recently appeared that has many points of contact with ours (Sur les transformations isomorphiques d'une variété à connexione affine. Prac Mat.-Fiz. (Warszawa) **39** (1932), 55-62), W. ŚLEBODZINSKI, et al., presented the integrability conditions for equations (12). They state that $R_{K\Lambda M}^{\cdot\,\cdot\,\cdot\,N}$ and $S_{\Lambda M}^{\cdot\,\cdot\,N}$, as well as all of their covariant derivatives, are invariant under T.

In conclusion, we would like to further remark that the oft-investigated notion of a *stationary* universe in the arena of relativity theory leads back in a completely analogous way to the notion of a V_4 with a one-parameter group of motions. The curves then determine a distinguished time direction and are the world-lines of particles at rest. The H_3 that results from collapsing is nothing but the ordinary three-dimensional space, which is regarded as the totality of all objects that are independent of time, not as "momentum space." If, moreover, the field t_N is holonomic then the universe is *static*; in this case, and only in this case, there is "momentum space" at each point of time, i.e., a V_3 that is perpendicular to all curves.

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