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GROUPS OF LINEAR TRANSFORMATIONS

BY

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I. Linear groups in arbitrary fields.

The source for the theory of linear groups in finite fields (i.e., Galois fields) is, to this day, the book of DICKSON ([1]). Later on, DICKSON himself adapted many of his results to infinite fields. However, a complete overview of this domain that likewise clearly emphasizes the relationships with the theory of continuous groups and projective geometry does not exist. On those grounds, the subject of the exposition that follows will be that of treating the recent work once more from its foundations, in which, however, some of the details – in particular, the proofs of the simplicity of the groups examined – will be referred to the DICKSON book. The isomorphisms of the orthogonal groups in the singular cases $n = 3, 4, 5, 6$, which make up an attractive part of the DICKSON book, will be derived below from the ground up, while emphasizing their abundant geometric and algebraic relationships.

The last paragraphs will treat the encyclopedia article of A. WIMAN and R. FRICKE on the discrete groups of linear transformations with complex number coefficients, while expanding it with discussions of recent investigations.

§ 1. Linear transformations ([2]).

One understands an *n-dimensional vector space* $E_n(\mathbb{K})$ over a field $\mathbb{K}$ to mean an additive Abelian group (whose elements are called *vectors*) with $\mathbb{K}$ as an operator domain that (in addition to the axioms of an Abelian group) satisfies the following axioms (u, v, … are vectors, while $1, \alpha, \beta, \ldots$ are elements of $\mathbb{K}$):

1. $(u + v)\,\alpha = u\alpha + v\alpha$,
2. $u\,(\alpha + \beta) = u\alpha + u\beta$,
3. $u\,(\alpha\beta) = (u\alpha)\,\beta$,
4. $u\,1 = u$.
5. There are n "basis vectors" $u_1, \ldots, u_n$ such that any vector v can be written as a unique linear combination:

$$v = \sum_{\nu=1}^{n} u_\nu \xi_\nu \,.$$

Two vector spaces are operator-isomorphic over $\mathbb{K}$ if and only if they have the same dimension n (i.e., the same *linear rank*). One can then take an arbitrary n-dimensional

([1]) L. E. DICKSON, *Linear Groups, with an exposition of the Galois field theory*, Leipzig, 1901.

([2]) The basic concepts of linear algebra that will be needed in what follows will all be briefly summarized in this paragraph. For a thorough presentation, see, perhaps, B. L. VAN DER WAERDEN: *Moderne Algebra II*, Berlin, 1931, chap. 15, or L. E. DICKSON, *Modern algebraic theories*, Chicago, 1926.

vector space to be a model for all of them by defining a vector to be – say – a linear form $\sum_{\nu=1}^{n} u_\nu \xi_\nu$ in n indeterminates $u_1, \ldots, u_n$.

The admissible subgroups of a vector space $\mathfrak{R}$ (relative to $\mathbb{K}$ as an operator domain) are called *linear subspaces* or *subspaces* of $\mathfrak{R}$. Moreover, the proper subspaces are vector spaces of dimension $m < n$. It follows from this that any decreasing or increasing sequence of subspaces will truncate after a finite number of them.

The homomorphic maps of a vector space $\mathfrak{R}$ to a vector space $\mathfrak{S}$ will be called *linear transformations* of $\mathfrak{R}$ to $\mathfrak{S}$. A linear transformation is then a map A of $\mathfrak{R}$ to $\mathfrak{S}$ for which one has:

$$\mathsf{A}\,(u + v) = \mathsf{A}u + \mathsf{A}v,$$
$$\mathsf{A}\,(u\,\alpha) = (\mathsf{A}\,u)\,\alpha.$$

For an arbitrary choice of bases $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_m)$ for $\mathfrak{R}$ and $\mathfrak{S}$, resp., a linear transformation A that takes u_k to:

$$\mathsf{A}\,u_k = \sum_j v_j \alpha_{jk}$$

will be given completely by its *matrix* $A = (\alpha_{jk})$ (j is the row index, and k is the column index): Namely, it will then necessarily take the vector $\sum u_k \xi_k$ with components ξ_k to the vector $\sum (\mathsf{A}u_k)\,\xi_k = \sum v_k\,\xi'_k$ with the components:

$$\xi'_j = \sum_j \alpha_{jk} \xi_k\,. \tag{1}$$

The product AB of two linear transformations will then correspond to the product AB of the matrices (naturally, assuming that the product is meaningful; i.e., that A indeed operates on the image space of B).

On the basis for formula (1), one can also regard any linear transformation as a *linear substitution* of the variables $\xi_1, \ldots, \xi_n$ that takes $\xi_1, \ldots, \xi_n$ to $\xi'_1, \ldots, \xi'_m$. This way of looking at things will then be employed, in particular, when $m = n$, so A will become a square matrix of *degree n* (i.e., with n rows and columns).

If the transformation A takes the n basis vectors $u_1, \ldots, u_n$ to linearly-dependent basis vectors, so the vector space $\mathfrak{R}$ goes to a space of lower dimension, then the transformation will be called *singular*. A non-singular transformation A will map $\mathfrak{R}$ to an image space of the same dimension in a one-to-one manner, and will possess an inverse A^{-1} such that $\mathsf{A}^{-1}\,\mathsf{A} = \mathsf{A}\,\mathsf{A}^{-1} = I$ (i.e., the identity).

The linear transformations of a vector space $E_n(\mathbb{K})$ into itself (or their matrices) define a ring: viz., the *full matrix ring of degree n over* $\mathbb{K}$. This ring can be regarded as a hypercomplex system with n^2 basis elements, for which one can choose, e.g., the n^2 *matrix units* C_{ik} that have a one in the i^{th} row and k^{th} column, but zero everywhere else. These matrices C_{ik} satisfy the rules of calculation:

$$C_{ik}\,C_{kl} = C_{il}\,,$$
$$C_{ij}\,C_{kl} = 0 \qquad \text{for } j \neq k.$$

The unity element of the ring is $I = C_{11} + C_{22} + \ldots + C_{nn}$.

From now on, we will consider only linear transformations of a vector space $\mathfrak{R} = E_n(\mathbb{K})$ *into itself* and also assume that the field $\mathbb{K}$ is *commutative*.

One understands the *characteristic polynomial* $\chi(t)$ of a square matrix A to mean the determinant of $tI - A$. The individual coefficients of this characteristic polynomial – in particular, the *trace* $S(A) = \sum \alpha_{\nu\nu}$ and the *norm*, or determinant, $|A|$ – are invariant under the transformations TAT^{-1}. The zeroes of $\chi(t)$, in a suitable extension field of $\mathbb{K}$, are called the *characteristic roots* of the matrix A.

One achieves the classification of linear transformations with the help of their elementary parts most easily when one regards the vector space $\mathfrak{R}$ in which a given linear transformation A lives as an additive Abelian group with the polynomial domain $\mathbb{K}[\mathsf{A}]$ as its operator domain and then applies the main theorem of the decomposition of Abelian groups into cyclic ones. Here, I will briefly give only the main result, and refer to the textbooks ([2]) for the proof.

The *minimal polynomial* of A – i.e., the polynomial of smallest degree $\varphi(t)$ for which one has $\varphi(\mathsf{A}) = 0$ – is a divisor of the characteristic polynomial of the matrix A. If one decomposes $\varphi(t)$ into factors that are powers of prime polynomials:

$$\varphi(t) = \varphi_1(t) \ldots \varphi_s(t), \quad \varphi_k(t) = \pi_k(t)^{r_k}$$

then the space $\mathfrak{R}$ will decompose uniquely into subspaces:

$$\mathfrak{R} = \mathfrak{R}_1 + \mathfrak{R}_2 + \ldots + \mathfrak{R}_s$$

(i.e., a direct sum, in the sense of group theory), in which $\mathfrak{R}_k$ will be annihilated by $\varphi_k(\mathsf{A})$: $\varphi_k(\mathsf{A})\,\mathfrak{R}_k = 0$. Any space $\mathfrak{R}_k$ will decompose further into "cyclic" subspaces $\mathfrak{r}_\nu$, each of which will be spanned by a vector v_ν and its transforms $\mathsf{A}v_\nu$. Each will be associated with an annihilating polynomial of lowest degree $\psi_\nu(t) = \pi_k(t)^{e_\nu}$. The minimal polynomials ψ_ν will be the *elementary divisors* of the matrix $tI - A$. Their product will be the characteristic polynomial $\chi(t)$.

Any matrix A can be brought into a normal form that depends upon only elementary divisors by a transformation TAT^{-1}; in this expression, T, as well as A, will be a matrix with coefficients in $\mathbb{K}$. On the basis of that normal form (or on the basis of the reasoning that leads up to it), one can exhibit the linear transformations that commute with a given linear transformation A [3].

An important special case of the theory of elementary divisors then emerges when all elementary divisors become linear by the adjunction of the characteristic roots to the field $\mathbb{K}$. In that case, the normal form of A will be a diagonal matrix in whose diagonal one will find the characteristic roots. This case shows up, in particular, when $A^h = I$ and h is relatively prime to the characteristic of $\mathbb{K}$. If several matrices with linear elementary divisors commute with each other then one can bring them into diagonal form simultaneously.

Special case: *For the fields of complex or algebraic numbers, any periodic linear transformation, and similarly, any finite Abelian group of linear transformations, can be transformed into diagonal form* (with roots of unity in the diagonal).

A generalization of linear transformations that has been examined only slightly up to now, but which nevertheless plays a role in very many places in mathematics, is defined by the *semi-linear transformations,* which one obtains when one combines linear transformations with automorphisms of the ground field $\mathbb{K}$. If S_α is such an automorphism then the formula:

(2) $$\xi'_i = \sum \alpha_{ik} \xi_k^S$$

will define a semi-linear transformation. In particular, if $\mathbb{K}$ is the field of complex numbers and S is the transition to complex conjugates then one will speak of an *anti-linear transformation*. If A and B are semi-linear transformations that belong to the automorphisms S and T, resp., and are given by the matrices A and B, resp., moreover, then the product AB will belong to the automorphism ST and the matrix AB^S, where B^S arises from B by subjecting all elements of the matrix B to the automorphism S. In particular, the product of two anti-linear transformations will be a linear transformation with the matrix AB.

A classification of the semi-linear transformations – or even just the anti-linear ones, in particular – by the theory of elementary divisors still does not seem to exist. It is only for those anti-linear transformations whose square is a transformation λI that one knows normal forms into which they can all be transformed [4].

The *dual* space to a vector space E_n consists of all linear functions of a vector (or its components) whose values belong to the same field $\mathbb{K}$. If $v = \sum u_\nu \xi_\nu$ is an arbitrary vector then:

$$l = \sum_{\nu=1}^{n} \lambda^\nu \xi_\nu$$

---

[3] A long series of papers by various authors treated this theme, starting with G. FROBENIUS: J. reine angew. Math. **84** (1878), 1-63. For the literature, see C. C. MacDUFFEE: *Theory of Matrices*, Ergebn. d. Math. **2**, H. 5 (1933), 93. For extensions of that, let us mention the papers of O. SCHREIER and B. L. VAN DER WAERDEN: Abh. Math. Inst. Hamburg **6** (1928), 308-310 and K. SHODA: Math. Z. **29** (1929), 696-712.

[4] E. JACOBSTHAL: S.-B. Berl. math. Ges. **33** (1934), 15-34.

will be an arbitrary linear function of v. A vector in the dual space will then given by n components $\lambda^1, \ldots, \lambda^n$. A non-singular linear transformation of the given vector space into itself will necessarily induce a linear transformation of the dual space whose matrix is the transposed inverse of the matrix of the given transformation.

A non-singular linear transformation of the vector space E_n into its dual space is called a *duality*. It will be given by the formula:

$$'\lambda^i = \sum \delta^{ik} \xi_k \,. \tag{3}$$

Such a transformation of the space E_n into its dual space is necessarily coupled with a transformation of the dual space into the original E_n whose matrix is again the transposed inverse of the matrix (δ^{ik}) of the given duality. We will refer to these two associated transformations together as *one* duality. One can now multiply dualities and linear transformations, which must likewise be taken together with the linear transformations that they induce on the dual space, with each other arbitrarily. The composition of two dualities – e.g. – will yield a linear transformation of E_n into itself.

If one composes the dualities with the automorphisms S of the ground field $\mathbb{K}$ then one will obtain dualities in the extended sense:

$$'\lambda^i = \sum \delta^{ik} \xi_k^S \,. \tag{4}$$

The non-singular, semi-linear transformations and the dualities in the extended sense collectively define a group.

§ 2. The general and special linear group.

The non-singular, linear transformations of the vector space $E_n(\mathbb{K})$ into itself define a group: viz., the *general linear group* $GL(n, \mathbb{K})$ ([5]). As always in what follows, if we assume that the field $\mathbb{K}$ is commutative then the transformations with determinant one will define a subgroup: viz., the *special linear group* $SL(n, \mathbb{K})$. For $n > 1$, $SL(n, \mathbb{K})$ will be the commutator group of $GL(n, \mathbb{K})$, although in the one case of $n = 2$, one assumes that $\mathbb{K} = GF(2)$ ([6]). The group $SL(n, \mathbb{K})$ will be generated by the transformations:

([5]) The notation is borrowed from the American school (cf., L. E. DICKSON: *Linear Groups*, Leipzig, 1901); nonetheless, some notations will be simplified and others converted systematically. Thus, we shall write *GL*, instead of *GLH* (= general linear homogeneous) and *SL*, instead of *SLH*.

([6]) Here and in what follows, $GF(q)$ will always denote the Galois field with q elements. Cf., B. L. VAN DER WAERDEN, *Moderne Algebra I*, § 31.

$$B_{r,\,s,\,\lambda}:\begin{cases}\xi'_r = \xi_\nu + \lambda\xi_s \\ \xi'_\nu = \xi_\nu \qquad \text{for } \nu \neq r.\end{cases}$$

In order to generate $GL(n, \mathbb{K})$, one must add the transformations:

$$\begin{cases}\xi'_1 = \lambda\xi_1 \qquad (\lambda \neq 0), \\ \xi'_\nu = \xi_\nu \quad \text{for } \nu \neq 1.\end{cases}$$

L. E. DICKSON ([7]) has presented the defining relations of the groups $SL(n, \mathbb{K})$.

The center of $GL(m, \mathbb{K})$ consists of the transformations λI, where I is the identity. The center of $SL(n, \mathbb{K})$ consists of the transformations λI, where λ is an n^{th} root of unity. We will thoroughly discuss the factor group of $SL(n, \mathbb{K})$ by its center in § 3.

The classification of linear transformations by their elementary divisors that was discussed in § 1 simultaneously provides the partitioning of the elements of the group $GL(n, \mathbb{K})$ into conjugacy classes.

If $\mathbb{K}$ is a finite field $GF(q)$, $q = p^m$ then $GL(n, \mathbb{K})$ and $SL(n, \mathbb{K})$ will be finite groups of order:

$$(q^n - 1)\,(q^n - q)\,\ldots\,(q^n - q^{n-1}) \qquad (q = p^m),$$

or

$$(q^n - 1)\,(q^n - q)\,\ldots\,(q^n - q^{n-2})\,q^{n-1},$$

resp.

These groups will also be denoted by $GL(n, p^m)$ [$SL(n, p^m)$, resp.]. The group $SL(2, p)$ is the "group of binary congruences" with prime number modulus p.

L. E. DICKSON ([8]) has determined the subgroups of $SL(n, p^m)$. C. JORDAN ([9]) and G. BUCHT ([10]) treated the maximal solvable subgroups of the group $GL(n, p)$. We will learn about some other important subgroups in §§ 4-6. For the case of the field of complex numbers, see also §§ 7 and 8.

One obtains extensions of the group $GL(n, \mathbb{K})$ by adding the semi-linear transformations (dualities, resp.) (cf., § 1).

([7]) L. E. DICKSON: Bull. Amer. math. Soc. (2) **13** (1907), 386-389 – Quart; J. Math. **38** (1907), 141-145.

([8]) L. E. DICKSON: Amer. J. Math. **33** (1911), 175-192.

([9]) C. JORDAN: J. de Math. (7) **8** (1917), 263-374.

([10]) G. BUCHT: Ark. Mat. Astron. Fys. **11** (1917), no. 26.

§ 3. The projective group.

As is known, the totality of all rays or one-dimensional subspaces that go through the origin of the vector space $E_n(\mathbb{K})$ is called the *projective space* $P_{n-1}(\mathbb{K})$. The non-singular linear transformations of E_n into itself induce *projective transformations* of $P_{n-1}(\mathbb{K})$. Thus, the linear transformations A and $\lambda\mathsf{A}$, where λ is a number, will always yield the same projective transformation.

The totality of all projective transformations of $P_{n-1}(\mathbb{K})$ is called the *n-ary projective group* $PGL(n, \mathbb{K})$ ([11]). It is isomorphic to the factor group of $GL(n, \mathbb{K})$ by the subgroup of λI (i.e., the center). Likewise, the factor group of $SL(n, \mathbb{K})$ by its center is called the *special projective group* $PSL(n, \mathbb{K})$ ([12]).

In the case of a finite field $GF(q)$ with $q = p^m$ elements, $PSL(n, \mathbb{K}) = PSL(n, q)$ is a finite group of order:

$$\frac{(q^n - 1)(q^n - 1)\cdots(q^n - q^{n-1})}{d(q-1)},$$

where d means the number of n^{th} roots of unity in $\mathbb{K}$: viz., $d = (n, q - 1)$. Since the projective space contains $\frac{q^n - 1}{q - 1}$ points in this case, $PGL(n, q)$ and $PSL(n, q)$ are permutation groups of degree $\frac{q^n - 1}{q - 1}$. In the case $n = 2$, one deals with permutation groups of degree $q + 1$ and order $\frac{(q^2 - 1)q}{d}$, in particular. For $q = 2$, $PSL(2, q)$ is the symmetric group $\mathfrak{S}_3$, for $q = 3, 4$, it is the alternating group $\mathfrak{A}_4$ ($\mathfrak{A}_5$, resp.). $PSL(2, q)$ is not a simple group in either case $q = 2, 3$. However, one now has the theorem:

If $\mathbb{K}$ *is a field of characteristic* $\neq 2$ *or a complete field* ([12a]) *and* $n > 1$ *then the group* $PSL(n, \mathbb{K})$ *will be a simple group, except for the lowest cases* $PSL(2, 2)$ *and* $PSL(2, 3)$ *that were just mentioned.*

([11]) PGL = projective general linear.

([12]) PSL = projective special linear. The American school writes LF = linear fractional. We have preferred to make the transition from a linear group to the factor group by the substitutions λI that it contains systematically recognizable everywhere by prefixing a P.

([12a]) For this concept, see E. STEINITZ: J. reine angew. Math. **137** (1910), 181 and 218 or VAN DER WAERDEN ([6]), § 25 and § 33.

For the proof, see L. E. DICKSON: *Linear Groups* (Leipzig, 1901), § 104-105 ([13]).

For $\mathbb{K} = GF(p^r)$, one will arrive at an important infinite system of finite, simple groups on the grounds of this theorem. The smallest of them are the well-known simple groups $PSL(2, 4) \cong PSL(2, 5)$ and $PSL(2, 7) \cong PSL(3, 2)$ of orders 60 and 168, resp.

E. H. MOORE and A. WIMAN have enumerated the subgroups of the group $PSL(2, q)$ ([14]). H. H. MITCHELL ([15]) determined the finite subgroups of $PSL(2, \mathbb{K})$ for an arbitrary ground field $\mathbb{K}$ by a surprisingly simple method. He thus naturally once more obtained the results of MOORE and WIMAN as special cases, as well as a known theorem of KLEIN (cf., § 8) for $\mathbb{K}$ = field of complex numbers. In the same paper, MITCHELL determined the finite subgroups of the group $PSL(3, \mathbb{K})$ for all fields of characteristic $\neq 2$. R. W. HARTLEY ([16]) determined subgroups of $PSL(3, 2^n)$. For the subgroups of $PSL(4, \mathbb{K})$, see H. H. MITCHELL ([17]), as well as the literature that is given in that paper. For the case of the field of complex numbers, see also § 8.

One can infer the validity of the assertion that the group $PSL(2, q)$ contains no subgroup of index smaller than $q + 1$ from the list of subgroups of $PSL(2, q)$, which was first stated without proof by GALOIS, so it can also not be represented as a permutation group of less than $q + 1$ objects, except in the cases $q = 2, 3, 5, 7, 9, 11$, for which there will be subgroups of index 2, 3, 5, 7, 6, 11, resp. The associated representations as permutation groups of only q elements are 1-isomorphic, except in the cases $q = 2, 3$. For $q = 5$ and $q = 9$, one will be dealing with the representations of the group $PSL(2, q)$ as an alternating group of 5 (6, resp.) objects:

$$PSL(2, 5) \cong PSL(2, 4) = \mathfrak{A}_5 , \tag{1}$$

$$PSL(2, 9) \cong \mathfrak{A}_6 , \tag{2}$$

and for $q = 7$, the representation of the known simple group of order 168 as the permutation group of 7 points in a projective plane:

$$PSL(2, 7) \cong PSL(3, 2) . \tag{3}$$

([13]) In order to make the proof valid for the case of infinite fields, as well, one must replace $\tau_1^2 + \tau_2^2$ with $\tau_1^2 - \tau_2^2$ on pp. 97 of the DICKSON book that was cited above. [Cf., L. E. DICKSON: Trans. Amer. Math. Soc. **2** (1901), 368.]

([14]) E. H. MOORE: Chicago decennial publ. **9** (1904), 141-190. – A. WIMAN: Handl. Svenska Vet.-Akad. **25** (1899), 1-47. The special case $q = p$ (prime number), which is important for the theory of module substitutions, was already resolved before by GIERSTER: Math. Ann. **18** (1881), 319-325.

([15]) H. H. MITCHELL, Trans. Amer. Math. Soc. **12** (1911), 208-211.

([16]) R. W. HARTLEY: Ann. of Math. **27** (1925), 140-158.

([17]) H. H. MITCHELL: Trans. Amer. Math. Soc. **14** (1913), 123-142.

L. E. DICKSON ([18]) and W. H. BUSSEY ([19]) have presented systems of defining relations for the groups $PSL(n, q)$. In the special case of the modular group $PSL(2, p)$, where p is an odd prime number, the BUSSEY relations read simply ([20]):

$$S^p = T^2 = (ST)^3 = 1,$$
$$(S^\tau T S^\sigma T)^2 = 1 \qquad \text{for} \qquad \sigma\tau \equiv 2 \pmod{p}.$$

The projective transformations are not, as one often intends, the only transformations of the projective space into itself that transform points to points, lines to lines, planes to planes, etc. They are probably the only ones that will leave the double ratio of four points invariant, in addition. However, along with them, there are transformations that subject the double ratio to an automorphism of the field $\mathbb{K}$ ([20a]). One gets them when one applies a semi-linear transformation (§ 1, formula 2) to the coordinates ξ. We would like to call the transformations of projective space thus obtained *collineations*. The *correlations* stand beside them, which take points to hyperplanes, and which are induced by dualities, in the extended sense (§ 1, formula 4). Naturally, in fields like the real numbers, in which no automorphism exists besides the identity, any collineation is a projective transformation.

The collineations and correlations collectively define a group. According to SCHREIER and V. D. WAERDEN ([21]), that group will likewise be the group of automorphisms of the special projective group $PSL(n, \mathbb{K})$. That is: Any automorphism of the group $PSL(n, \mathbb{K})$ will have the form:

$$X \to CXC^{-1},$$

where C is a collineation or a correlation. The same investigation yielded that the various $PSL(n, \mathbb{K})$ exhibited no other isomorphisms between themselves than the ones that were written down in (1) and (3), and furthermore that only the following ones of the groups PSL were isomorphic to alternating groups $\mathfrak{A}_n$:

$$PSL(2, 3) \cong \mathfrak{A}_4, \qquad PSL(2, 9) \cong \mathfrak{A}_6,$$
$$PSL(2, 5) \cong PSL(2, 4) \cong \mathfrak{A}_5, \qquad PSL(4, 2) \cong \mathfrak{A}_8 .$$

In particular, the two simple groups $PSL(2, 4)$ and $PSL(3, 4)$ of order $\frac{1}{2}\cdot 8!$ are not isomorphic to each other ([22]).

([18]) L. E. DICKSON: *Linear Groups*, Leipzig, 1901. § 278 – Proc. London math. Soc. **35** (1903), 292-305, 306-319, and 443-454.

([19]) W. H. BUSSEY: Proc. London Math. Soc. (2) **3** (1915), 296-315.

([20]) On this, cf. also H. FRASCH: Math. Ann. **108** (1933), 249-252. Other relations were given by J. A. TODD: J. London Math. Soc. **7** (1932), 195-200.

([20a]) Cf., F. LEVI: *Geometrische Konfigurationen*, 1929, § 7.

([21]) Abh. Math. Sem. Hamburg **6** (1928), 303-322.

([22]) Cf., also J. M. SCHOTTENFELS: Bull. Amer. Math. Soc. (2) **8** (1902), 25-26.

§ 4. The complex group.

The group of all linear substitutions of the variables $x_1, \ldots, x_n$ with coefficients in $\mathbb{K}$ that are performed on two cogredient (i.e., they transform the same) sequences of variables ξ, η and leave an alternating form:

$$\varphi = \sum_{i=1}^{m} (\xi_{2i-1}\eta_{2i} - \xi_{2i}\eta_{2i-1}) \tag{1}$$

invariant is called the *complex group* $C(2m, \mathbb{K})$ [23]. It is also called the ABELian *linear group*, after ABEL, who was the first to examine it, but we would like to avoid that terminology, since the group is in no way Abelian. The restriction to the special form (1) is not an essential restriction, since any alternating bilinear form $\varphi = \sum \varepsilon_{ik}\, \xi_i\, \eta_k$ with a determinant $|\,\varepsilon_{ik}\,| \neq 0$ can be put into the form (1).

Obviously, for $m = 1$, one will have:

$$C(2, \mathbb{K}) = SL(2, \mathbb{K}).$$

For $m \neq 1$, $C(2m, \mathbb{K})$ will be generated by the transformations:

$$\mathsf{M}_i: \quad \xi'_{2i+1} = \xi_{2i}, \qquad \xi'_{2i} = -\xi_{2i-1}, \quad \text{and the remaining } \xi'_k = \xi_k,$$

$$\Lambda_{i,\lambda}: \quad \xi'_{2i-1} = \xi_{2i-1} + \lambda\xi_{2i}, \qquad \text{and the remaining } \xi'_k = \xi_k,$$

$$\mathsf{N}_{ij,\lambda}: \quad \begin{cases} \xi'_{2i-1} = \xi_{2i-1} + \lambda\xi_{2j}, \\ \xi'_{2j-1} = \xi_{2j+1} + \lambda\xi_{2i}, \end{cases} \qquad \text{and the remaining } \xi'_k = \xi_k .$$

All transformations of $C(2m, \mathbb{K})$ then have determinant 1. Moreover, that will follow from the fact that the form (1) possesses a relative invariant that takes on the factor Δ under linear transformations with determinant Δ [24], but remains absolutely invariant when the form itself is absolutely invariant.

If $\mathbb{K}$ is a finite field $GF(q)$ then the order of $C(2m, \mathbb{K}) = C(2m, q)$ will be equal to:

$$(q^{2m} - 1)\, q^{2m-1}\, (^{2m-2} - 1)\, q^{2m-3} \ldots (q^2 - 1)\, q.$$

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[23] In the American literature, the groups C and PC are denoted by SA (special Abelian) and A (Abelian).

[24] If one sets $\varphi = \sum \varepsilon_{ik}\ \xi_i\ \eta_k$ then $I = \sum \varepsilon_{i_1 i_2}\varepsilon_{i_3 i_4} \cdots \varepsilon_{i_{n-1} i_n} \operatorname{sign}(i_1 i_2 \cdots i_n)$ will be the invariant that was claimed.

In the case $q = p$, the group $C(2m, p)$ takes the form of the GALOIS group of the equation of the p-splitting of the periodic, hyperelliptic functions ([25]).

The center of $C(2m, \mathbb{K})$ consists of the transformations I and $-I$. If $\mathbb{K}$ has characteristic 2 then $I = -I$; i.e., the center will consist of only I. The factor group by the center will be denoted by $PC(2m, \mathbb{K})$ ([23]).

If $\mathbb{K}$ *is a field of characteristic* $\neq 2$ *or a complete field then* $PC(2m, \mathbb{K})$ *will be a simple group, except in the two cases* $PC(2, 2)$ *and* $PC(2, 3)$ *that were mentioned already in* § 3, *along with a new exceptional case of* $PC(4, 2) \cong \mathfrak{A}_6$.

For the proof, see L. E. DICKSON, *Linear Groups*, § 110 – 116 or J.-A. DE SÉGUIER, J. Math. pures appl. (7) **2** (1916), 281-366. The smallest new simple group that is contained in the infinite system of groups $PC(2m, q)$ is the group $PC(4, 3)$ of order 25930, which appears in the problem of the 27 lines on the cubic surface as the GALOIS group, and is thus the subject of an extensive volume of literature ([26]).

L. E. DICKSON ([27]) has exhibited the classes of conjugate elements in the groups $C(4, q)$ and $C(6, q)$. For the subgroups of the groups C and PC, see L. AUTONNE ([28]), H. H. MITCHELL ([29]), and C. JORDAN ([30]), as well as the literature cited therein. J.-A. DE SÉGUIER ([31]) has examined the elements of order 2 that we discussed in this paragraph, as well as the finite groups in the following ones.

The theory of invariants and representations of the complex group has been investigated by H. WEYL ([32]), above all.

§ 5. The unitary group.

Let $\mathbb{K}$ be a field of degree 2 over a sub-field $\mathbb{P}$ [e.g., $\mathbb{K} = GF(p^{2s})$, $\mathbb{P} = GF(p^s)$, or also $\mathbb{K}$ is the field of complex numbers and $\mathbb{P}$ is that of the real numbers.]. $\bar{\alpha}$ will always mean the quantity that is conjugate to α relative to $\mathbb{P}$. [In the case $GF(p^s)$, one can set $\bar{\alpha} = \alpha^{p^s}$.] The group of all linear transformations of the space $E_n(\mathbb{K})$ that leave the form:

([25]) See C. JORDAN: *Traité des Substitutions*, Paris, 1870, pp. 171-168 [sic] and 354-369.
([26]) See MILLER, BLICHFELDT, and DICKSON: *Finite Groups*, New York, 1916, ch. XIX, as well as the literature that is cited therein.
([27]) L. E. DICKSON, Trans. Amer. Math. Soc. **2** (1901), 103-138 – Amer. J. Math. **26** (1904), 243-318.
([28]) L. AUTONNE: J. Math. pures appl. (5) **7** (1901), 351-394.
([29]) H. H. MITCHELL: Trans. Amer. Math. Soc. **15** (1914), 379-396.
([30]) C. JORDAN: J. de Math. (7) **3** (1917), 263-374.
([31]) J.-A. DE SÉGUIER: Ann. École norm. (3) **50** (1933), 217-243; (3) **51** (1934), 79-147.
([32]) Math. Z. **23** (1925), 271-309 and **24** (1925), 328-395; Nachr. Ges. Wiss. Göttingen (1926), 235-243; Acta math. **48** (1926), 255-278; Math. Z. **35** (1932), 300-320.

$$\Phi = \bar{\xi}_1 \xi_1 + \bar{\xi}_2 \xi_2 + \cdots + \bar{\xi}_n \xi_n$$

invariant is called the *unitary* or *hyper-orthogonal* group $U(n, \mathbb{P}, \mathbb{K})$. The subgroup of transformations of determinant one in $U(n, \mathbb{P}, \mathbb{K})$ is called the *special unitary group* $SU(n, \mathbb{P}, \mathbb{K})$. The factor group of SU by the subgroup of substitutions λI ($\lambda^m = 1$, $\lambda\bar{\lambda} = 1$) will again be denoted by $PSU(m, \mathbb{P}, \mathbb{K})$ ([33]).

In the case where the sub-field $\mathbb{P}$ is determined uniquely by $\mathbb{K}$ [as in the case of a Galois field $\mathbb{K} = GF(p^{2s})$, $\mathbb{P} = GF(p^s)$], one can omit the symbol $\mathbb{P}$ in the parentheses and write:

$$U(n, \mathbb{K}), \qquad SU(n, \mathbb{K}), \qquad PSU(n, \mathbb{K}),$$

or in the case $\mathbb{K} = GF(p^{2s})$:

$$U(n, p^{2s}), \qquad SU(n, p^{2s}), \qquad PSU(n, p^{2s}).$$

In the latter case, one can also write:

$$\Phi = \xi_1^{p^s+1} + \xi_2^{p^s+1} + \cdots + \xi_n^{p^s+1}$$

for the form Φ.

The condition for a linear transformation A to belong to the matrix A in the group U is:

(1) $$AA^\dagger = I, \qquad \text{or} \qquad A^\dagger = A^{-1}, \qquad \text{or} \qquad A^\dagger A = I,$$

where $A^\dagger$ is the transposed conjugate to A. When written out, that will be:

$$\sum_i \alpha_{ji}\bar{\alpha}_{ki} = \delta_{jk} \qquad \text{or} \qquad \sum_i \bar{\alpha}_{ji}\alpha_{ki} = \delta_{jk} .$$

If the determinant of $I + A$ is non-zero and the characteristic of the field is $\neq 2$ then one can define a matrix using A:

(2) $$C = (I - A)\,(I + A)^{-1},$$

and conversely express A in terms of C:

(3) $$A = (I + C)^{-1}(I - C).$$

From (1) and (2), it will then easily follow that:

([33]) In the American literature, the group PSU is denoted by HO (hyper-orthogonal).

$$(4) \qquad C = - C^{\dagger};$$

conversely, (1) will follow from (3) and (4). We thus have a one-to-one relationship between the unitary matrices A and the "skew-Hermitian" matrices C, in which only those A for which $| I + A | = 0$, as well as those C for which $| I + C | = 0$, have been left out ([34]). According to LOEWY ([35]), the exceptional case can be avoided when one writes:

$$A = \zeta (I + C)^{-1} (I - C), \qquad \zeta\bar{\zeta} = 1,$$

instead of (3).

In the case of the field of complex numbers, any unitary matrix A will be unitarily-equivalent to a diagonal matrix $D = UAU^{-1}$. The diagonal elements will be the characteristic roots of A and will have absolute value one. In fact, the equivalence will be valid inside of the special unitary group ([36]).

The order of the group $PSU(n, p^{2s})$ is ([37]):

$$\frac{1}{d} (q^n - (-1)^n)\, q^{n-1} (q^{n-1} - (-1)^{n-1})\, q^{n-2} \dots (q^2 - 1)\, q \qquad [q = p^s;\ d = (n, q + 1)].$$

If the field $\mathbb{K}$ contains a number ρ with the property that $\rho\bar{\rho} = -1$, as well as a number s with the property that $\sigma\bar{\sigma} = 1$, $\sigma \neq \bar{\sigma}$ [both assumptions are fulfilled automatically in the case $\mathbb{K} = GF(p^{2s})$] then one can transform the sum $\xi_1 \bar{\xi}_1 + \xi_2 \bar{\xi}_2$ by the substitution:

$$\xi_1 = \sigma\, \eta_1 + \eta_2\,,$$
$$\xi_2 = \rho\, (\bar{\sigma}\eta_1 + \eta_2)$$

into

$$\xi_1 \bar{\xi}_1 + \xi_2 \bar{\xi}_2 = (\sigma - \bar{\sigma})(\eta_1 \bar{\eta}_2 - \eta_2 \bar{\eta}_1)\,.$$

Under these assumptions, the unitary group $U(2m, \mathbb{P}, \mathbb{K})$ will then be isomorphic to the *hyper-Abelian group* $H(2m, \mathbb{P}, \mathbb{K})$, which leaves the form:

$$\Psi = (\xi_1 \bar{\xi}_2 - \xi_2 \bar{\xi}_1) + \dots + (\xi_{2m-1} \bar{\xi}_{2m} - \xi_{2m} \bar{\xi}_{2m-1})$$

([34]) A. LOEWY: C. R. Acad. Sci., Paris **123** (1896), 171.

([35]) A. LOEWY: Nova Acta. Abh. Kaiserl. Leop.-Carol. Acad. **71** (1898), 379-446. Math. Ann. **50** (1898), 557-576.

([36]) O. TOEPLITZ: Math. Z., **2** (1918), 187-197.

([37]) L. E. DICKSON: *Linear Groups*, §§ 146, 148.

invariant. Under this isomorphism, the subgroup $SU(2m, \mathbb{P}, \mathbb{K})$ will correspond to the special hyper-Abelian group [38] $SH(2m, \mathbb{P}, \mathbb{K})$ and the factor group PSU will correspond to the projective, hyper-Abelian group $PSH(2m, \mathbb{P}, \mathbb{K})$.

The simplicity of the groups $PSH(2m, \mathbb{P}, \mathbb{K})$ for $n > 1$ was proved by L. E. DICKSON [39] for arbitrary fields of characteristic $\neq 2$, as well as for finite fields of arbitrary characteristic; the latter proof is achieved by reverting back to $PSU(2m, p^{2s})$. *The group* $PSU(2m, p^{2s})$ *is, in fact, always simple for* $n > 2$ [40], *except for the case of* $PSU(3, 2^2)$, *where one is dealing with a solvable group of order* 72. However, the latter $n = 2$ case that was left unconsidered is trivial if one goes over to the isomorphic group $PSH(2, \mathbb{P}, \mathbb{K})$, since the invariance of:

$$\Psi = (\xi_1 \bar{\xi}_2 - \xi_2 \bar{\xi}_1)$$

under a substitution with determinant one means that ξ_1, ξ_2 will be transformed precisely as $\bar{\xi}_1$, $\bar{\xi}_2$ are, or that the transformation (α_{ik}) will be identical with the conjugate one $(\bar{\alpha}_{ik})$, so it will belong to the field $\mathbb{P}$. Therefore, $SH(2, \mathbb{P}, \mathbb{K}) = SL(2, \mathbb{P})$ and $PSH(2, \mathbb{P}, \mathbb{K}) = PSL(2, \mathbb{P})$.

At this point, one might refer to the isomorphism of the group $PSU(4, 2^2)$ with the simple group $PC(4, 3)$ of order 25920 that was mentioned already in § 4 that was found by DICKSON [41]

§ 6. The orthogonal groups.

Now, let $\mathbb{K}$ be a field with characteristic $\neq 2$. The group of linear transformations of $E_n(\mathbb{K})$ that leave a non-singular quadratic form:

$$Q = \sum \sum \alpha_{jk} \xi_j \xi_k$$

invariant might be called the *extended orthogonal group*. Its transformations are known to have determinants of ± 1. The ones with determinant $+ 1$ define the *restricted orthogonal group* $O(n, \mathbb{K}, Q)$.

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(38) Denoted *HA* by DICKSON. The group is called hyper-Abelian, because it contains the complex group – or ABELian linear group – as a subgroup.

(39) L. E. DICKSON: Proc. London Math. Soc. **34** (1901), 185-205.

(40) L. E. DICKSON: *Linear Groups*, § 145-151. – J.-A DE SÉGUIER: J. Math. pures appl. (7) **2** (1916), 281-366.

(41) L. E. DICKSON: *Linear Groups*, § 270-277.

If Q is the unit form $\sum \xi_i^2$, in particular, then we will have the *first orthogonal group* $O_1(n, \mathbb{K})$. The transformations of $O_1(n, \mathbb{K})$ define a rational, $\binom{n}{2}$–dimensional, algebraic manifold with the parametric representation:

$$A = (I + C)^{-1} (I - C); \qquad C = - C^{\mathrm{T}}$$

[cf., equation (3), § 5], which will, however, actually represent only elements of the manifold for which $| I + A | \neq 0$. R. LIPSCHITZ ([42]) gave a parametric representation with no exceptions. When he defined the expressions:

$$\begin{aligned} X &= \xi_1\, 1 + \xi_2\; i_{12} + \ldots + \xi_n\, i_{1n}\,, \\ Y &= \xi_1'\, 1 + \xi_2'\; i_{12} + \ldots + \xi_n'\; i_{1n}\,, \\ \Lambda &= \lambda_0 + \sum \lambda_{ab}\, i_{ab} + \sum \lambda_{abcd}\; i_{abcd} + \ldots, \\ \Lambda_1 &= \lambda_0 - \sum \lambda_{ab}\, i_{ab} + \sum \lambda_{abcd}\; i_{abcd} - \ldots, \end{aligned}$$

with the help of the 2^{n-1} basis elements $1, i_{ab}, i_{abcd}, \ldots$ ($a, b, c, \ldots = 1, 2, \ldots, n$; $a < b < c < \ldots$) of a well-defined hypercomplex system, and in which the 2^{n-1} coefficients λ_0, λ_{ab}, λ_{abcd}, … depended upon $\binom{n}{2} + 1$ of them rationally, he arrived at the representation of any orthogonal transformation $X \to Y$ by the formula:

$$\Lambda X = Y \Lambda_1\,.$$

The Λ then defined a group.

According to KRONECKER ([43]), in the cases of the fields of real or complex numbers, the group $O_1(n, \mathbb{K})$ is generated by the substitutions:

$$\left\{ \begin{aligned} \xi_1' &= c\xi_1 - s\xi_j, \\ \xi_j' &= s\xi_1 + s\xi_j, \\ \xi_k' &= \xi_k \quad \text{for the remaining ones,} \end{aligned} \right. \qquad\qquad (c^2 + s^2 = 1).$$

From the diagonal transformation of the unitary matrices that was mentioned in the previous paragraph, it follows easily that a real orthogonal matrix A can be brought into a normal BAB^{-1} by transforming with just such a matrix that consists of a sequence of two-rowed boxes $\begin{pmatrix} c & -s \\ s & c \end{pmatrix}$ with $c^2 + s^2 = 1$, and possibly the numbers ± 1, along the diagonal.

([42]) R. LIPSCHITZ: *Untersuchungen über die Summen von Quadraten*, Bonn, 1884.
([43]) L. KRONECKER: S.-B. preuß. Akad. Wiss. (1890), 1063-1080.

We now return to arbitrary fields $\mathbb{K}$ with characteristic $\neq 2$ and arbitrary quadratic forms Q. Any such form Q can be transformed into the form:

$$\sum \alpha_\nu \xi_\nu^2 \tag{1}$$

in $\mathbb{K}$. If $\mathbb{K}$ is algebraically closed then all α_i can be chosen to be equal to one. If $\mathbb{K}$ is the field of real numbers then one will choose all $\alpha_i = \pm 1$. The number of negatives among them is called the *signature* – or *index of inertia* – of the form Q, and is invariant under linear transformations. If $\mathbb{K}$ is a Galois field $GF(q)$ then one can choose all $\alpha_i = 1$, except for the last one, which will then be equal to the discriminant D of the form Q. In that case, the group $O(n, \mathbb{K}, Q)$ can also be denoted by $O_D(n, \mathbb{K})$ or $O_D(n, q)$. One will be dealing with the *first* or the *second* orthogonal group $O_1(n, \mathbb{K})$ or $O_\nu(n, q)$, resp. according to whether D is or is not a square, resp. An arbitrary non-square in the field $GF(q)$ can be employed as the index ν. For odd n, there is no difference between $O_1(n, \mathbb{K})$ and $O_\nu(n, q)$, since a form with a discriminant ν can then be converted into one with a quadratic discriminant by multiplying by ν.

For odd n, the orders of the groups $O_D(n, q)$ are ([44]):

$$(q^{n-1} - 1)\, q^{n-2}\, (q^{n-3} - 1)\, q^{n-4} \dots (q^2 - 1)\, q,$$

and for even n, they are:

$$(q^{n-1} - \eta\, \varepsilon^{n/2})\, (q^{n-2} - 1)\, q^{n-3} \dots (q^2 - 1)\, q,$$

$$\varepsilon = (-1)^{\frac{q-1}{2}}, \quad \eta = 1 \quad \text{for } O_1(n, q), \quad \eta = -1 \quad \text{for } O_\nu(n, q).$$

The generators of the group $O_D(n, q)$ are given by DICKSON (*Linear Groups*, § 173). The groups $O(2, \mathbb{K}, Q)$ are Abelian, so they are not interesting. From now on, we then assume that $n > 2$. C. JORDAN ([45]) studied the maximal solvable subgroups of the groups $O(2, p, Q)$. H. B. HEYWOOD ([46]) has exhibited the Abelian subgroups of the complex orthogonal groups.

If one forms the factor group of the group $O(n, \mathbb{K}, Q)$ by the transformation λI ($\lambda = \pm 1$, while for odd n, one has only $\lambda = 1$) then one will obtain a projective group $PO(n, \mathbb{K}, Q)$ that leaves a projective hypersurface $Q = 0$ invariant. For odd n, $PO(n, \mathbb{K}, Q)$ is

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([44]) L. E. DICKSON: *Linear Groups*, § 172.
([45]) C. JORDAN: J. de Math. (7) **3** (1917), 263-374.
([46]) H. B. HEYWOOD: Messenger of Math. (2) **43** (1913), 14-21.

isomorphic to $O(n, \mathbb{K}, Q)$ in one step. For even $n = 2m$, the transformations of $PO(2m, \mathbb{K}, Q)$ are distinguished from the transformations that are orthogonal in the extended sense by the fact that they will not permute the two families of linear spaces P_{n-1} that are found in the hypersurface $Q = 0$ in the space P_{2m-1} (possibly by extending the ground field). If $\mathbb{K}$ is the field of real numbers, and if Q has the index of inertia 0 or 1 then one will call $PO(n, \mathbb{K}, Q)$ a *non-Euclidian* (elliptic or hyperbolic) group of motions.

In the case of a finite field $\mathbb{K}$, as DE SÉGUIER and JORDAN ([47]) first showed, the group $O_D(n, \mathbb{K})$ possesses a subgroup $O'_D(n, \mathbb{K})$ of index two whose generators DICKSON ([48]) has previously obtained. The transformations of these subgroups are characterized by the fact that they transform the points of the hypersurface $Q = 1$ in the space $E_n(\mathbb{K})$ to each other by an even permutation.

The case of $n = 4$ plays a special role in the structure of the groups $PO(n, \mathbb{K}, Q)$, because in that case the group will be essentially a direct product of two simple non-Abelian groups (see § 7). By contrast, for $n > 4$, the groups $PO'_D(n, q)$ that have index 1 or 2 in $PO_D(n, q)$ will all be simple ([49]). The same thing is also true for $n = 3$, with the exception of $PO'(3, 3) \cong PSL(2, 3) \cong \mathfrak{A}_4$ (cf., § 7). The situation has still not been clarified completely for arbitrary ground fields $\mathbb{K}$. If one assumes that the form Q has one of the following three forms:

$$Q = \xi_1^2 + \xi_2\,\xi_3 + \ldots + \xi_{n-1}\,\xi_n \qquad (n \text{ odd}),$$
$$Q = \xi_1\,\xi_2 + \xi_2\,\xi_3 + \ldots + \xi_{n-1}\,\xi_n \qquad (n \text{ even}),$$
$$Q = \varphi(\xi_1, \xi_2) + \xi_2\,\xi_3 + \ldots + \xi_{n-1}\,\xi_n \qquad (n \text{ odd})$$

then there will once more be a subgroup $PO'(n, \mathbb{K}, Q)$ with an Abelian factor group whose index cannot be given in general, and which will be simple for $n \neq 4$, according to DICKSON ([50]). For the case in which $\mathbb{K}$ is the field of real numbers, it will follow from the theory of continuous groups that the part of $PO(n, \mathbb{K}, Q)$ that is continuously connected to the identity will be simple for $n \neq 4$ ([51]).

([47]) J.-A. DE SÉGUIER: C. R. Acad. Sci., Paris **157** (1913), 430-432. – C. JORDAN: J. Math. pures appl. (7) **2** (1916), 233-280.
([48]) L. E. DICKSON: *Linear Groups*, § 181.
([49]) L. E. DICKSON: *Linear Groups*, § 191-192. Cf., also J.-A. DE SÉGUIER: J. Math. pures appl. (7) **2** (1916), 281-365.
([50]) L. E. DICKSON: Trans. Amer. Math. Soc. **2** (1901), 363-394. – Proc. London Math. Soc. **34** (1902), 185-205.
([51]) E. CARTAN: Ann. École norm. **31** (1914), 263-355. – B. L. VAN DER WAERDEN: Math. Z. **36** (1933), 780-786.

The behavior for fields of characteristic 2 is somewhat different from the cases that were already considered, for which the characteristic was ≠ 2. Thus, let $\mathbb{K}$ be a complete field of characteristic 2. Any quadratic form in n variables with coefficients in $\mathbb{K}$ that cannot be written as a form in less than n variables can then be brought into one of the two normal forms ([52]):

$$Q = \xi_1\,\xi_2 + \xi_2\,\xi_3 + \dots + \xi_{n-2}\,\xi_{n-1} + \xi_n^2 \qquad (n \text{ odd}),$$
$$Q = \xi_1\,\xi_2 + \dots + \xi_{n-3}\,\xi_{n-2} + \varphi(\xi_{n-1}, \xi_n) \qquad (n \text{ even}),$$

where φ is a quadratic form in ξ_{n-1} and ξ_n . If φ is decomposable in the field $\mathbb{K}$ (viz., the *first case*) then one can assume that $\varphi = \xi_{n-1}\,\xi_n$, while in the other (viz., *second*) case, one can arrive at:

$$\varphi = \xi_{n-1}\,\xi_n + \lambda(\xi_{n-1}^2 + \xi_n^2) \qquad \text{with } \lambda \neq 0$$

by a simple transformation. The first case is subordinate to the second one for $\lambda = 0$. For even n, one can then set:

$$Q = \xi_1\,\xi_2 + \dots + \xi_{n-3}\,\xi_{n-2} + \xi_{n-1}\,\xi_n + \lambda(\xi_{n-1}^2 + \xi_n^2)$$

in any case.

Those transformations that leave the form Q invariant again define the *orthogonal group* $O(n, \mathbb{K}, Q)$. For odd n and Q in the normal form above, one can simply write $O(n, \mathbb{K})$. For even n, one writes $O_\lambda(n, \mathbb{K})$. Any transformation of $O(n, \mathbb{K}, Q)$ will also leave the polar form of Q:

$$P = (\xi_1\,\eta_2 - \xi_2\,\eta_1) + \dots + (\xi_{n-2}\,\eta_{n-1} - \xi_{n-1}\,\eta_{n-2}) \qquad (n \text{ odd}),$$
$$P = (\xi_1\,\eta_2 - \xi_2\,\eta_1) + \dots + (\xi_{n-1}\,\eta_n - \xi_n\,\eta_{n-1}) \qquad (n \text{ even})$$

invariant ([53]). If ξ belongs to the "ray" $\xi_1 = \xi_2 = \dots = \xi_{n-1} = 0$ then when n is odd the polar form P will be identically zero in η. Therefore, our transformations must also leave that ray invariant – i.e., they must transform $\xi_1, \dots, \xi_{n-1}$ only amongst themselves, and indeed by a transformation of the complex group $C(n-1, \mathbb{K})$. Thus, for odd n, the group $O(n, \mathbb{K}, Q)$ can be mapped homomorphically onto the complex group $C(n-1, \mathbb{K})$. If one investigates which transformations of $O(n, \mathbb{K}, Q)$ are mapped to the identity under this map then one will find that one is dealing with a 1-isomorphism:

$$O(2m + 1, \mathbb{K}) \cong C(2m, \mathbb{K}).$$

([52]) See, perhaps, L. E. DICKSON: *Linear Groups*, § 199.

([53]) $\xi_1\,\eta_2 - \xi_2\,\eta_1$ is the same as $\xi_1\,\eta_2 + \xi_2\,\eta_1$, since the characteristic is 2.

For even $n = 2m$, the orthogonal group $O(n, \mathbb{K}, Q) = O_1(n, \mathbb{K})$ will be a proper subgroup of the complex group $C(2m, \mathbb{K})$; it will then also be referred to as a *hypo-Abelian group* (*first* or *second* hypo-Abelian group, according to whether $\lambda = 0$ or $\lambda \neq 0$, resp.).

The hypersurface $Q = 0$ in the projective space P_{2m-1} contains two ("real" or conjugate) families of linear spaces P_{m-1} that will be transformed into themselves or each other under the group $O(2m, \mathbb{K}, Q)$. The subgroup that transforms them individually into themselves is called the *restricted orthogonal* – or JORDAN *hypo-Abelian – group* $J_\lambda(2m, \mathbb{K})$ ([54]). For $\mathbb{K} = GF(q)$, $q = 2r$, we again write $J_\lambda(2m, q)$, instead of $J_\lambda(2m, \mathbb{K})$. The orders of these groups are:

$$(q^m - \varepsilon)\,(q^{2(m-1)} - 1)\,q^{2(m-1)}\,(q^{2(m-1)} - 1)\,q^{2(m-1)} \dots (q^2 - 1)\,q^2,$$

where $\varepsilon = 1$ for $\lambda = 0$ and $\varepsilon = -1$ for $\lambda \neq 0$.

According to L. E. DICKSON ([55]), for $2m > 4$ and all complete fields $\mathbb{K}$, the groups $J_\lambda(2m, q)$ are *all simple*. In the exceptional case of $2m = 4$, $J_\lambda(2m, q)$ is a direct product (cf., § 7). For $2m = 2$, as is easy to see, the group is isomorphic to the multiplicative group of the field $\mathbb{K}$, and thus Abelian.

In the case of field of complex numbers, the groups $PSL(n, \mathbb{K})$, $PC(n, \mathbb{K})$, and $PO_1(n, \mathbb{K})$ define three infinite sequences of simple, continuous groups. According to CARTAN ([56]), in addition to these, there are only five types of simple, analytic, continuous groups, the simplest of which is a linear group of degree 7 whose elements depend analytically upon 14 complex parameters. L. E. DICKSON ([57]) found an analogue for this group for arbitrary ground fields $\mathbb{K}$ and provided a general proof of simplicity.

§ 7. The isomorphisms of the orthogonal groups in dimensions 3, 4, 5, and 6.

In the cases $n = 3, 4, 5, 6$, the orthogonal groups $PO(n, \mathbb{K}, Q)$ are isomorphic to certain linear groups of lower degrees. Here, one is dealing with entirely singular, non-generic phenomena that have no analogues for arbitrary dimension numbers.

([54]) DICKSON writes $FH(2m, q)$ and $SH(2m, q)$ for $\mathbb{K} = GF(q)$ and $\lambda = 0$ ($\lambda = \mu$, resp.) (viz., the first and second hypo-Abelian groups, resp.).

([55]) L. E. DICKSON: *Linear Groups*, § 209. Indeed, DICKSON considered only finite fields $\mathbb{K}$; however, his proof is still valid for all complete fields of characteristic 2 with no changes.

([56]) E. CARTAN: Thése. Paris, 1894 (2nd ed., Paris, 1933). Cf., also B. L. VAN DER WAERDEN: Math. Z. **37** (1933), 446-462.

([57]) L. E. DICKSON, Trans. Amer. Math. Soc. **2** (1901), 383-391.

For the case of the field of complex numbers, as well as for some real cases, these isomorphisms were probably first given by F. KLEIN ([58]). The real, three-dimensional case – viz., the isomorphism of the group of ordinary sphere rotations with a group of fractional linear transformations of one complex variable – is known, in general. One real, four-dimensional case was already known to GOURSAT ([59]), while another one was likewise known at the time of KLEIN ([60]) that played a great role in relativistic quantum mechanics ([61]). The real, five and six-dimensional cases were treated by CARTAN ([62]), STUDY ([63]), and SCHOUTEN ([64]). L. E. DICKSON ([65]) gave an exhaustive discussion of the isomorphisms for the case of Galois fields. We will derive them here with a unified method for arbitrary fields.

I. The cases $n = 4$ and $n = 6$.

$n = 4$. We first assume that the form Q can be brought into the form:

$$Q_1 = \xi_1 \xi_2 - \xi_3 \xi_4 \tag{1}$$

by a transformation in $\mathbb{K}$. The quadratic surface $Q = 0$ in projective space P_3 then possesses the parametric representation:

$$\xi_1 = \lambda_1 \mu_1, \qquad \xi_2 = \lambda_2 \mu_2, \qquad \xi_3 = \lambda_1 \mu_2, \qquad \xi_4 = \lambda_2 \mu_1 . \tag{2}$$

The geometric meaning of the parameters λ_i and μ_k is immediately obvious: λ_i = const. and μ_k = const. are the two families of lines on the surface, and the ratios $\lambda_1 : \lambda_2$ and $\mu_1 : \mu_2$ are projective parameters along the point-sequences that the lines of one family cut out from a line of the other family. It follows immediately from this that: For a projective transformation of the surface $Q_1 = 0$ into itself that does not permute the two families, the two parameter ratios $\lambda_1 : \lambda_2$ and $\mu_1 : \mu_2$ (which are independent of each other) will be transformed projectively:

$$\lambda_i' = \sum a_{ij} \lambda_j , \qquad \mu_i' = \sum b_{kl} \lambda_l . \tag{3}$$

([58]) F. KLEIN: Math. Ann. **5** (1872), 256-277; **23** (1884), 539-578; **43** (1893), 63-100 (Erlanger Programm of 1871).

([59]) E. GOURSAT: Ann. École norm. (3) **6** (1889), 9-102. Cf., also F. KLEIN: Math. Ann. **37** (1890), 546-554, as well as E. STUDY: Amer. J. Math. **19** (1906), 116.

([60]) See, perhaps, R. FRICKE and F. KLEIN: *Vorlesungen über automorphe Funktionen I*, Braunschweig, 1897.

([61]) See, perhaps, B. L. VAN DER WAERDEN: *Die gruppentheoretische Methode in der Quantenmechanik*, Berlin, 1932, § 20.

([62]) E. CARTAN: Ann. École norm. **31** (1914), 353-355.

([63]) E. STUDY: Math. Z. **18** (1923), 55-86 and 201-229; **21** (1924), 45-71 und 174-194. – J. reine angew. Math. **157** (1927), 33-59.

([64]) J. A. SCHOUTEN and J. HAANTJES: "Konforme Feldtheorie II," appeared in 1935 in the Ann. Scuola norm. super., Pisa.

([65]) L. E. DICKSON: *Linear Groups*, Leipzig, 1902, § 178-208.

If, conversely, two projective transformations of the parameter ratios $\lambda_1 : \lambda_2$ and $\mu_1 : \mu_2$ are given that can be represented by formulas (3) then, on the basis of the transformation (3), the products $\lambda_i\, \mu_k$, and therefore the coordinates of the associated points of the surface, will be transformed linearly:

$$\lambda_i' \mu_k' = \sum\sum a_{ij}\, b_{kl}\, \lambda_j\, \mu_l \,. \tag{4}$$

One can extend this transformation of the surface to the entire space by linearly transforming the coordinates ξ_i of arbitrary points precisely like the coordinates of the points of the surface according to (4). One writes the formulas for that transformation most conveniently when one denotes the ξ_i with double indices:

$$\xi_1 = \omega_{11}, \qquad \xi_2 = \omega_{22}, \qquad \xi_3 = \omega_{12}, \qquad \xi_4 = \omega_{21}.$$

One then has to transform the coordinates ω_{ik} in precisely the same way as the products $\lambda_i\, \mu_k$ using (4) ([66]):

$$\omega_{ik}' = \sum\sum a_{ij}\, b_{kl}\, \omega_{jl} \,. \tag{5}$$

This transformation will transform the surface into itself, and indeed, in such a way that the parameter values λ_i , μ_k of its points will be transformed as in (3). With that, it is proved:

The group of projective transformations of the space P_3 *that take the two families of lines in the surface* Q_1 *into themselves individually is isomorphic to the direct product* $PGL(2, \mathbb{K}) \times PGL(2, \mathbb{K})$ *of the projective groups of the parameter ratios* $\lambda_1 : \lambda_2$ and $\mu_1 : \mu_2$.

The transformations (5) will indeed transform the surface $Q_1 = 0$ into itself, but they do not need to leave the form:

$$Q_1 = \omega_{11}\, \omega_{22} - \omega_{12}\, \omega_{21},$$

absolutely invariant. A simple calculation teaches us that this form will be multiplied by the product $\alpha\beta$ of the determinants of the matrices A and B under the transformation (5). Now, in order for the transformation (5) to belong to the group $O(n, \mathbb{K}, Q_1)$, so the associated projective transformation will belong to $PO(n, \mathbb{K}, Q_1)$, it must leave Q_1 absolutely invariant; one must have $\alpha\beta = 1$. Thus:

The group $PO(4, \mathbb{K}, Q_1)$ *is isomorphic to the group of pairs of binary, projective transformations whose determinants yield the product one.*

([66]) Since one is dealing with a projective transformation, one must actually prefix an arbitrary factor λ to the right-hand side, but one can absorb it into the matrix *B*.

One will obtain a subgroup (with an Abelian factor group) when one restricts oneself to pairs of binary transformations with determinants equal to one. The subgroup of $PO(4, \mathbb{K}, Q)$ thus defined will be denoted by $PO'(4, \mathbb{K}, Q)$, while the corresponding linear group will be denoted by $O'(4, \mathbb{K}, Q)$. Moreover, one has the isomorphism:

$$PO'(4, \mathbb{K}, Q) \cong PGL(2, \mathbb{K}) \times PGL(2, \mathbb{K}). \tag{6}$$

Remark. If one takes two semi-linear transformations:

$$\lambda'_i = \sum b_{ij} \lambda_j^S , \qquad \mu'_i = \sum c_{kl} \lambda_l^S$$

with the same S, instead of (3), then one will also obtain a semi-linear transformation of the surface $Q_1 = 0$ into itself, instead of (5). If one takes a linear or semi-linear transformation that takes the λ to the μ' and the μ to the λ', in place of (3), then one will obtain a linear or semi-linear transformation of the ω that switches the two families of lines.

$n = 6$. We again assume that the form Q can be brought into the form:

$$Q_1 = \xi_1 \xi_2 + \xi_3 \xi_4 + \xi_5 \xi_6 . \tag{7}$$

We now introduce the new relations:

$$\pi_{12} = \xi_1, \quad \pi_{34} = \xi_2, \quad \pi_{13} = \xi_4, \quad \pi_{14} = \xi_5, \quad \pi_{23} = \xi_6 ; \qquad \pi_{ik} = -\pi_{ki} , \tag{8}$$

with which Q_1 goes to:

$$Q_1 = \pi_{12} \pi_{34} + \pi_{13} \pi_{24} + \pi_{14} \pi_{23} . \tag{9}$$

Now, the condition $Q_1 = 0$ is necessary and sufficient for π_{ik} to be the PLÜCKERian coordinates of a line in the space P_3 . That is, the parameter representation:

$$\pi_{ik} = x_i y_k - x_k y_i \tag{10}$$

will represent the entire hypersurface $Q_1 = 0$. If one holds the x in (10) constant, and thus considers all lines through a fixed point x in P_3, then the point ξ with coordinates π_{ik} will run through a plane that lies within the hypersurface $Q_1 = 0$ completely. In this way, any point of the space P_3 will correspond to a plane in the hypersurface. If a point and a plane P_2 are incident then the planes in the hypersurface that correspond to them will intersect in a line, and conversely.

A collineation of the space P_5 that transforms the hypersurface $Q_1 = 0$ into itself and transforms the two families of planes into themselves individually will therefore also induce a transformation of the points and planes in the space P_3 that preserves incidence, and thus, a collineation ([67]):

(11) $$x_i' = b_i^{\,k} x_k^{\,S}\,.$$

Likewise, a collineation of the space P_5 that transforms the hypersurface into itself and switches the families of planes will induce a transformation in P_3 that takes points to planes, and conversely, and preserves incidence; i.e., a correlation:

(12) $$'u^i = d^{ik}\, x_k^{\,S}.$$

Conversely, if a collineation (11) or a correlation (12) is given then it will induce a semi-linear transformation of the line coordinates:

(13) $$\pi_{ik}' = b_i^{\,j}\, b_k^{\,l}\, \pi_{jl}^{\,S}\,,$$

or

(14) $$'\pi^{ik} = d^{ij} d^{kl} \pi_{jl}^{\,S}\,,$$

resp., where $'\pi^{ik}$ are the contragredient line coordinates, which are coupled to the cogredient ones π_{ik}' by the formulas:

(15) $$'\pi^{12} = \pi_{34}'\,, \quad '\pi^{13} = \pi_{42}'\,, \quad '\pi^{14} = \pi_{23}'\,, \quad '\pi^{34} = \pi_{12}'\,, \quad '\pi^{42} = \pi_{13}'\,, \quad '\pi^{23} = \pi_{14}'\,.$$

With that, we have proved:

The group of collineations of the space P_5 that leave the hypersurface $Q_1 = 0$ invariant is isomorphic to the group of collineations and correlations of the space P_3. Therefore, the collineations of P_3 will correspond to those collineations of P_5 that do not permute the two families of planes in the hypersurface, and in particular, the projective transformations ($S = I$) will correspond to projective transformations. The group of the automorphic collineations of the hypersurface $Q_1 = 0$ that do not permute the two families of planes will then be isomorphic to the projective group $PGL(4, \mathbb{K})$.

We now restrict ourselves to the projective transformations, so we assume that $S = I$, and prefix an arbitrary factor ρ to (13) on the right:

(16) $$\pi_{ik}' = \rho\, b_i^{\,j}\, b_k^{\,l}\, \pi_{jl}^{\,S}\,.$$

Under the linear transformation (16), the form (9) will be multiplied by the factor $\rho^2\beta$, where β is the determinant of the matrix B. In order for this linear transformation to

([67]) We now introduce upper and lower indices, and to abbreviate, specify that indices that appear both above and below will be summed over.

belong to the group $O(6, \mathbb{K}, Q_1)$, and thus, for the corresponding projective transformation to belong to $PO(6, \mathbb{K}, Q_1)$, one must have $\rho^2\beta = 1$, so:

$$\beta = \rho^{-2} \tag{17}$$

must be a square. Therefore:

The group $PO(6, \mathbb{K}, Q_1)$ *is isomorphic to the group of all quaternary projective transformations whose determinants are squares.*

One subgroup of the latter group (namely, its commutator subgroup) is the special projective group $PSL(4, \mathbb{K})$. Under the isomorphism, it will correspond to a subgroup $PO'(6, \mathbb{K}, Q_1)$, namely, the commutator group of $PO(6, \mathbb{K}, Q_1)$. One will then have:

$$PO'(6, \mathbb{K}, Q_1) \cong PSL(4, \mathbb{K}). \tag{18}$$

II. Before we go on to the remaining cases of $n = 3$ and $n = 5$, we shall discuss the extension of the results up to now to those forms Q that cannot be brought into the forms (1) [(7), resp.]. We thus begin with the most instructive case of $n = 6$.

From now on, the ground field might be denoted by $\mathbb{P}$. In the event that $\mathbb{P}$ does not have characteristic 2, Q can, in any case, be brought into the form:

$$Q = \alpha_1\,\xi_1^2 + \alpha_2\,\xi_2^2 + \cdots + \alpha_6\,\xi_6^2\ ; \tag{19}$$

however, if $\mathbb{P}$ does have characteristic 2 then, from § 6, we will assume that:

$$Q = \xi_1\,\xi_2 + \xi_3\,\xi_4 + \xi_5\,\xi_6 + \lambda(\xi_5^2 + \xi_6^2) \tag{20}$$

is the normal form. In the case (19), after adjoining the three square roots:

$$w_1 = \sqrt{-\frac{\alpha_2}{\alpha_1}}\,,\quad w_2 = \sqrt{-\frac{\alpha_4}{\alpha_3}}\,,\quad w_3 = \sqrt{-\frac{\alpha_6}{\alpha_5}}\,,$$

one can write:

$$\begin{aligned} Q = \alpha_1\,(\xi_1 + w_1\,\xi_2)\,(\xi_1 - w_1\,\xi_2) + \alpha_3\,(\xi_3 + w_2\,\xi_4)\,(\xi_5 - w_2\,\xi_4) \\ + \alpha_4\,(\xi_5 + w_3\,\xi_6)\,(\xi_5 - w_3\,\xi_6). \end{aligned}$$

In the case (20), one likewise adjoins the roots θ_1 and θ_2 of the equation:

$$\theta + \lambda\,(1 + \theta^2) = 0,$$

and obtains:

$$Q = \xi_1\,\xi_2 + \xi_3\,\xi_4 + \lambda\,(\xi_5 - \theta_1\,\xi_6)\,(\xi_5 - \theta_2\,\xi_6).$$

One thus always obtain a separable extension field $\mathbb{K}$, in which the form Q can be brought into the normal forms (7) or (9), for which the variables π_{ik} that enter into this normal form will be linear functions of the original ξ_i with coefficients in $\mathbb{K}$:

(21) $$\pi_{ik} = e_{ik}^{\ \nu}\,\xi_\nu\,.$$

It is therefore remarkable that one will achieve a *quadratic* extension of $\mathbb{K}$ to the field $\mathbb{P}$ in the case of Galois fields, as well as in the case of the field of real numbers.

All of the isomorphisms that were proved above will be true in the extension field $\mathbb{K}$; in particular, $PO(6, \mathbb{K}, Q) \cong PO(6, \mathbb{K}, Q_1)$ is isomorphic to a subgroup of $PGL(4, \mathbb{K})$. If we now once more go from $PO(6, \mathbb{K}, Q)$ to the subgroup $PO(6, \mathbb{P}, Q)$ then we will have to examine which subgroup of $PGL(4, \mathbb{K})$ will correspond to it under the isomorphism.

A projective transformation T with coefficients in $\mathbb{K}$ belongs to the ground field $\mathbb{P}$ if and only if it commutes with all automorphisms S of the GALOIS group $\mathfrak{G}$ of $\mathbb{K}/\mathbb{P}$, or more precisely, with all collineations ([68]):

(22) $$\xi_i' = \xi_i^S \qquad (S \text{ in } \mathfrak{G}).$$

This commutability is preserved under the isomorphic transition from the group of correlations and collineations of P_3 . The correlations (22), which always leave the hypersurface $Q = 0$ invariant, might correspond to a collineation or a correlation C_S of the space P_3 . It then follows that:

Under the isomorphism, the group $PO(6, \mathbb{P}, Q)$ *will correspond to the subgroup of those transformations in* $PGL(4, \mathbb{K})$ *whose determinants are squares and which commute with all correlations (correlations* C_S*, resp.) that belong to the substitutions* S *of the* GALOIS *group of* $\mathbb{K}/\mathbb{P}$.

([68]) In order to prove this, one remarks that one must always be able to choose a matrix element of the projective transformation T that equals one. If T then commutes with the collineation (22) then all matrix elements must admit the substitutions S, and therefore belong to $\mathbb{P}$, since $\mathbb{K}$ is separable over $\mathbb{P}$.

These conditions will mean different things depending upon the nature of the C_S . If C_S is a correlation (12) then one can also characterize the collineations that commute with it as the ones that leave the form:

(23) $$d^{ik} x_i \, x_k^S$$

invariant, up to a factor.

Let $\mathbb{P}$ be – e.g. – the field of real numbers and let:

(24) $$Q = \xi_1^2 + \xi_2^2 + \xi_3^2 + \xi_4^2 + \xi_5^2 + \xi_6^2 .$$

The substitution (21), which brings Q into the form (9), then reads like:

$$\begin{array}{lll} \pi_{12} = \xi_1 + i\,\xi_2, & \pi_{13} = \xi_3 + i\,\xi_4, & \pi_{14} = \xi_5 + i\,\xi_6, \\ \pi_{34} = \xi_1 - i\,\xi_2, & \pi_{42} = \xi_3 - i\,\xi_4, & \pi_{23} = \xi_5 - i\,\xi_6. \end{array}$$

Now, the collineation (22) takes π_{12} to π_{34}^S, π_{13} to π_{42}^S, π_{14} to π_{23}^S, and conversely, where S is the transition to the complex conjugate. Under the isomorphism, it will correspond to the correlation:

$$'u^i = x_i^S,$$

from which, one can define the HERMITIAN form:

(25) $$\sum x_i x_j^S = \sum x_i \, \bar{x}_i .$$

If a real projective transformation leaves this form invariant, up to a factor, then that factor must be positive, since the form (25) is positive-definite. Therefore, one can also choose the factor to be equal to one. We then have the isomorphism:

(26) $$PO_1(6, \mathbb{P}) \cong PU(4, \mathbb{K}).$$

The same argument will always be true with small modifications when the form Q has one of the following two forms:

(27) $$Q_2 = \varphi(\xi_1, \xi_2) + \varphi(\xi_3, \xi_4) + \varphi(\xi_5, \xi_6),$$
(28) $$Q_3 = \varphi(\xi_1, \xi_2) + \xi_3\,\xi_4 + \xi_5\,\xi_6 ,$$

where φ is a quadratic form that is indecomposable in $\mathbb{P}$. In the first case (27), the associated form (23) has the form (25), while in the second case (28), it has the form:

(29) $$x_1\bar{x}_2 - x_2\bar{x}_1 + x_4\bar{x}_3 - x_3\bar{x}_4 .$$

One will thus always obtain a *unitary* group for Q_2, and a *hyper-Abelian* group for Q_3 . If one goes to those subgroups PSU (PSH, resp.) whose elements leave the form (25) [(29), resp.] absolutely invariant and have the determinant one then under the isomorphism it will also correspond to a subgroup PO' with Abelian factor groups:

(30) $$PO'(6, \mathbb{P}, Q_2) \cong PSU(4, \mathbb{K}),$$

(31) $$PO'(6, \mathbb{P}, Q_3) \cong PSH(4, \mathbb{K}).$$

If $\mathbb{K}$ is a GALOIS field then, from § 5, the groups PSU and PSH on the right-hand side will be isomorphic to each other. The forms Q_1 and Q_2 (or also Q_1 and Q_3) with discriminants – 1 and – v will also already exhaust all types of quadratic forms over a Galois field $GF(q)$. The same will also be true for complete fields of characteristic 2, where Q_1 and Q_2 belong to the groups J_0 and J_λ , resp. If $\mathbb{P}$ is the field of real numbers then the forms Q_2 and Q_3 will have the indices of inertia 1 and 0, resp., while the form Q_1 will have the index 3. A form of index of inertia 1 can be treated by the same method: One obtains the group of projective transformations that commute with an anti-collineation of the form:

$$x_1' = \bar{x}_4, \qquad x_2' = -\bar{x}_3, \qquad x_3' = +\bar{x}_2, \qquad x_4' = -\bar{x}_1.$$

From STUDY ([63]), one can represent these transformations very elegantly by quaternion matrices.

The case $n = 4$ is completely analogous. Here, as well, by the introduction of new variables:

$$\omega_{ik} = d_{ik}^{\ \nu}\xi_\nu ,$$

one brings the form Q into the form:

$$Q = \omega_{11}\,\omega_{22} - \omega_{12}\,\omega_{21} ,$$

then looks for a semi-linear transformation C_S of the λ and μ that corresponds to the collineation:

$$\xi_\nu' = \xi_\nu^S,$$

and finally defines the group of pairs of projective transformations of the parameter ratios $\lambda_1 : \lambda_2$ and $\mu_1 : \mu_2$ that commute will C_S .

In addition to (1), two forms of the form Q come into consideration:

(32) $$Q_2 = \xi_1\,\xi_2 + \varphi(\xi_3, \xi_4),$$
(33) $$Q_3 = \varphi(\xi_1, \xi_2) + \varphi(\xi_3, \xi_4).$$

For the field of real numbers, Q_1, Q_2, Q_3 will be typical forms with indices of inertia 2, 1, and 0, resp. For the Galois field $GF(q)$, Q_1 and Q_2 will already exhaust the possible cases (discriminants square or not, resp.).

In the case of the form Q_2, C_S will be the transformation:

$$\begin{cases} \lambda_1' = \mu_1^S, \\ \lambda_2' = \mu_2^S, \end{cases} \qquad \begin{cases} \mu_1' = \lambda_1^S, \\ \mu_2' = \lambda_2^S. \end{cases}$$

In order for a pair of binary, projective transformations A, B to commute with C_S, one must have:

$$B = \rho A^S$$

for their matrices, while A remains arbitrary. Instead of the direct product $PGL(2, \mathbb{K}) \times PGL(2, \mathbb{K})$ in the isomorphism that was mentioned at the beginning of this paragraph, one will obtain only the one group $PGL(2, \mathbb{K})$. The subgroup $PO(4, \mathbb{P}, Q_2)$ will be isomorphic to the group of those transformations in $PGL(2, \mathbb{K})$ whose determinants α have the property that:

$$\alpha\,\alpha^S = \rho^{-2} = \text{square in } \mathbb{K}.$$

This condition is fulfilled automatically in the case of the Galois field or the field of real numbers. Thus, one will have the isomorphism:

$$PO(4, \mathbb{P}, Q_2) \cong PGL(2, \mathbb{K}) \tag{34}$$

in these two cases.

In the real case, the group on the left-hand side is essentially the Lorentz group of the special theory of relativity.

In the case Q_3, C_S will be the transformation:

$$\begin{cases} \lambda_1' = \lambda_2^S, \\ \lambda_2' = -\lambda_1^S, \end{cases} \qquad \begin{cases} \mu_1' = \mu_2^S, \\ \mu_2' = -\mu_2^S. \end{cases}$$

The condition that the pair (A, B) should commute with this transformation will now yield two separate conditions for A and B. The condition for A is that $(\lambda_2^S, -\lambda_1^S)$ should transform like λ_1, λ_2, up to a factor, or that the form:

$$\lambda_1\,\lambda_2^S + \lambda_2\,\lambda_1^S \tag{35}$$

should remain invariant, up to a factor; the condition for B reads correspondingly. We are thus dealing with two *extended unitary groups*. Under the transition to the restricted unitary group, one will obtain a subgroup with an Abelian factor group:

$$PO'(4, \mathbb{P}, Q_2) \cong PSU(2, \mathbb{P}, \mathbb{K}) \times PSU(2, \mathbb{P}, \mathbb{K}). \tag{36}$$

One can also write the unitary transformation in this case in the form:

$$A = \begin{pmatrix} \alpha+i\beta & \gamma+i\delta \\ -\gamma+i\delta & \alpha-i\beta \end{pmatrix} = \alpha\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \beta\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} + \gamma\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \delta\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

$$= \alpha I + \beta J + \gamma K + \sigma L,$$

and then effortlessly obtain the well-known two-fold representation of the real four-dimensional rotations in the form:

$$X' = AXB^\dagger,$$

where $X = \xi_1 I + \xi_2 J + \xi_3 K + \xi_4 L$ is a variable quaternion and A and $B^\dagger$ are quaternions of norm one ([69]).

III. We now come to the cases $n = 3$ and $n = 5$. They will be treated simply due to the fact that the groups $O(3, \mathbb{K}, Q)$ and $O(5, \mathbb{K}, Q)$ can be considered to be subgroups of $O(4, \mathbb{K}, Q)$ [$O(6, \mathbb{K}, Q)$, resp.]. Since, from § 6, the case of characteristic 2 is not interesting for odd n, we can assume that the form Q has the form:

$$Q = \alpha_1 \xi_1^2 + \alpha_2 \xi_2^2 + \xi_3^2, \qquad Q = \alpha_1 \xi_1^2 + \alpha_2 \xi_2^2 + \alpha_3 \xi_3^2 + \alpha_4 \xi_4^2 + \xi_5^2, \text{ resp.,}$$

in the cases $n = 1$ and $n = 5$. One now extends Q to a quaternary (senary, resp.) form Q^* by adding a term $-\xi_4^2$ ($-\xi_6^2$, resp.). When one necessarily extends the ground field $\mathbb{P}$ to a field $\mathbb{K}$ by the adjunction of $\sqrt{-\frac{\alpha_2}{\alpha_1}}$ and $\sqrt{-\frac{\alpha_4}{\alpha_3}}$, one can bring the form Q^* into the form:

$$Q^* = -\,\omega_{11}\,\omega_{22} + \omega_{12}\,\omega_{21}, \qquad [Q^* = \pi_{12}\,\pi_{34} + \pi_{13}\,\pi_{42} + \pi_{14}\,\pi_{23}, \text{ resp.}]$$

in $\mathbb{K}$, where one can choose:

$$\omega_{12} = \xi_3 + \xi_4, \qquad \omega_{21} = \xi_3 - \xi_4,$$

or

$$\omega_{14} = \xi_5 + \xi_6, \qquad \omega_{23} = \xi_5 - \xi_6,$$

resp. Now, $O(3, \mathbb{K}, Q)$ is the subgroup of $O(4, \mathbb{K}, Q)$ that leaves ξ_4 invariant, and therefore also $2\xi_4 = \omega_{12} - \omega_{21}$. Likewise, $O(5, \mathbb{K}, Q)$ is the subgroup of $O(6, \mathbb{K}, Q)$ that leaves $2\xi_6 = \pi_{14} - \pi_{23}$ invariant. If one seeks the corresponding projective groups that these subgroups are isomorphic to, the subgroups of $PO(4, \mathbb{K}, Q)$ [$PO(6, \mathbb{K}, Q)$, resp.] and the corresponding subgroups of $PGL(2, \mathbb{K}) \times PGL(2, \mathbb{K})$ [$PGL(4, \mathbb{K})$, resp.] by

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([69]) A. CAYLEY: J. reine angew. Math. **50** (1885), 312-313. Cf., also F. KLEIN: Math. Ann. **37** (1890), 546-554, as well as J. BOUMAN: Niew Arch. Wiskde (2) **17** (1932), 240-266.

means of the aforementioned isomorphism, then one will ultimately find, in the case n = 3, the group of those pairs of projective transformations of $\lambda_1 : \lambda_2$ and $\mu_1 : \mu_2$ that leave the form:

$$\lambda_1 \mu_2 - \lambda_2 \mu_1 \tag{38}$$

invariant, up to a factor, and in the case n = 5, likewise that quaternary projective group that leave the form:

$$(x_1 y_4 - x_4 y_1) + (x_3 y_2 - x_2 y_3), \tag{39}$$

up to a factor.

The invariance of the form (38) says that $\mu_1 : \mu_2$ will be transformed in precisely the same way as $\lambda_1 : \lambda_2$: one will then have the isomorphism:

$$O(3, \mathbb{K}, Q) \cong PGL(2, \mathbb{K}). \tag{40}$$

One can also infer this directly from the parametric representation of the conic section Q = 0. The projective transformations of the conic section into itself will, in fact, induce fractional linear parameter transformations, and conversely.

The condition for the invariance of (39), up to a factor, defines an extension of the complex group. If one restricts oneself to those transformations that leave the form (39) absolutely invariant then one will obtain a normal subgroup with an Abelian factor group:

$$O'(5, \mathbb{K}, Q) \cong PC(4, \mathbb{K}). \tag{41}$$

The reversion of the super-field $\mathbb{K}$ to a sub-field $\mathbb{P}$ can, when necessary, be performed by the process that was explained in II for n = 4 and n = 6. In the case of the Galois field, the transition is not necessary, just as in the case of the real "group of hyperbolic motions":

$$Q = -\xi_1^2 + \xi_2^2 + \xi_3^2 .$$

In the case of the field of real numbers and the form:

$$Q = \xi_1^2 + \xi_2^2 + \xi_3^2 ,$$

one will obtain the well-known isomorphism (cf., the beginning of this paragraph):

$$O_1(3, \mathbb{K}) \cong PSU(2, \mathbb{P}, \mathbb{K}). \tag{42}$$

§ 8. Linear groups in complex number fields. Reducible and irreducible, primitive, imprimitive, and monomial groups.

The groups of linear substitutions with complex number coefficients, which will be called *linear groups* in what follows, are somewhat easier to survey when one restricts oneself to *closed* groups – i.e., to groups that include all of their accumulation elements – as long as they are not singular. From a theorem of J. V. NEUMANN ([70]), any closed linear group $\mathfrak{G}$ behaves as follows: $\mathfrak{G}$ contains an r-parameter continuous (LIE) subgroup $\mathfrak{H}$, which is a normal subgroup of $\mathfrak{G}$, and the factor classes of $\mathfrak{H}$ are isolated from each other – i.e., none of them contain accumulation elements of the union of the other ones. This theorem is a special case of a theorem on closed subgroups of LIE groups that E. CARTAN ([71]) has proved very simply. The two extreme cases for closed, linear groups are thus: The *continuous* groups, where $\mathfrak{H} = \mathfrak{G}$ and the *discrete* (or *discontinuous*) ones, where $\mathfrak{H}$ consists of only the unity element I. A group is then called *discrete* when the unity element (and thus, also any other group element) is not an accumulation element of other group elements ([72]).

The structure of continuous groups will be examined in the LIE theory, to which another booklet in this series will be directed. We thus content ourselves here with the proof that an r-parameter continuous linear group will be determined by r linearly-independent matrices $A_1, \ldots, A_r$ – viz., the matrices of the "infinitesimal generators" – in such a way that the r one-parameter subgroups that are defined by the matrices e^{tA_i} ($i = 1, 2, \ldots, r$), where t runs through all real numbers, collectively generate the r-parameter group. Any group element in the neighborhood of one can be represented "canonically" by:

$$e^{t_1 A_1 + \cdots + t_r A_r} .$$

Therefore, e^A will be defined by the exponential series. The matrices $A_1, \ldots, A_r$ must fulfill the relations:

$$A_i\, A_k - A_k\, A_i = \sum c_{ik}^l A_l ,$$

where the real constants c_{ik}^l depend upon only the "structure" group – i.e., they will be the same for two groups that are "continuously isomorphic in the small ([71])." Especially important are the *semi-simple* continuous groups; i.e., the ones that contain no solvable, continuous, normal subgroup. E. CARTAN ([73]) has enumerated these groups completely, and their representations by linear transformations are also all known, in principle, ([74]).

([70]) J. V. NEUMANN, Math. Z. **30** (1929), 3-42.

([71]) E. CARTAN: Mém. Sci. math. **42** (1930), in particular, § 27.

([72]) It only makes sense to speak of a discrete group when a topology is defined on the group (which is indeed the case for linear groups). There is no meaning to calling an abstract group discrete or discontinuous.

([73]) E. CARTAN: Thése. Paris 1894 (2nd ed., Paris, 1933) – Ann. École norm. **31** (1914), 263-355. Cf., also VAN DER WAERDEN: Math. Z. **37** (1933), 446-462 and W. LANDHERR, Bull. Soc. Math. Semin. Hamburg. Univ. **11** (1934), 41-64.

([74]) E. CARTAN: Bull. Soc. Math. France **41** (1913), 53-96. – J. Math. pures appl. (6) **10** (1914), 149-186. – H. WEYL: Math. Z. **23** (1925), 271-309; **24** (1925), 328-395.

A discrete linear group with restricted matrix elements is obviously finite; in particular, a discrete group of unitary transformations is therefore always finite. The following converse of this theorem is true:

Any finite linear group can leave a positive-definite HERMITIAN *form* $\sum \sum \alpha_{ik} \xi_i \xi_k$ *invariant.* In order to prove this, like R. L. MOORE ([75]), one needs only to subject the form $\sum \xi_i \xi_i$ to all transformations of the group and define the sum. The same method of proof is valid when one, like HURWITZ ([76]), replaces the summation with a suitable integration, and also for compact, continuous groups – indeed for arbitrary compact groups, according to HAAR ([76a]). More generally, one has: *If the matrix elements of a linear group* $\mathfrak{G}$ *are uniformly restricted then* $\mathfrak{G}$ *will possess an invariant positive* HERMITIAN *form.* H. AUERBACH ([77]) gave a direct proof of this. One can, however, also derive the theorem from the theorem above on the structure of the closed groups when one defines the (compact) closed hull of $\mathfrak{G}$, which possesses a continuous normal subgroup $\mathfrak{H}$ with mutually isolated cosets, hence, only finitely many of them, due to compactness. MOORE's proof above can be duplicated by integrating over the subgroup $\mathfrak{H}$ and summing over the cosets.

Since one can easily transform any positive-definite HERMITIAN form into the form:

$$\sum_{i=1}^{n} \xi_i \overline{\xi_i}$$

by introducing new coordinates (the proof is completely analogous to the one that is known for quadratic forms), from the discussion of finite (or more generally restricted) linear groups, one can always restrict oneself to groups of unitary transformations. For finite groups, one can replace the HERMITE form with the corresponding quadratic form, and thus assume that the transformations of the group are orthogonal.

A linear group is called *reducible* (or in a terminology that has justifiably fallen out of use: intransitive) when it leaves a proper subspace E_m of the vector space E_n $(0 < m < n)$ invariant. MASCHKE's *Theorem* follows immediately from the existence of an invariant HERMITIAN form ([78]): *If a finite (or, more generally, a restricted) linear group is reducible then the vector space* E_n *can be decomposed into two invariant subspace:* $E_n = E_m + E_{n-m}$ ([79]). E_{n-m} is, in fact, the space that is "totally perpendicular" to E_m for the metric that is determined by the HERMITIAN form.

We shall return to the general properties of reducible and irreducible linear groups (in arbitrary fields) in § 11 – 15. According to E. CARTAN ([80]), an irreducible *continuous* linear group is either semi-simple or the product of a semi-simple normal subgroup with an Abelian continuous group that consists of multiples λI of the identity. Since one

([75]) R. L. MOORE: Math. Ann. **50** (1898), 213-214. There is further literature in this paper.
([76]) A. HURWITZ: Nachr. Ges. Wiss. Göttingen **1897**, 71-90, Cf., also WEYL ([74]).
([76a]) A. HAAR: Ann. of Math. II, pp. 34 (1933), 147-169.
([77]) H. AUERBACH: C. R. Acad. Sci., Paris **195** (1932), 1367.
([78]) H. MASCHKE, Math. Ann. **52** (1899), 363-368.
([79]) + means the direct sum (in the sense of additive groups).
([80]) E. CARTAN: Ann. École norm. **26** (1909), 147-148 – Bull. Soc. Math. France **41** (1913), 53-96.

knows the semi-simple linear groups, in principle (see above), the irreducible continuous groups will also be known, in principle.

A linear group is called *imprimitive* when there is a decomposition of the space E_n into subspaces $E_h + E_k + \ldots + E_l$ that are only permuted with each other by the group; otherwise, it will be called *primitive*. If the group $\mathfrak{G}$ is irreducible, but imprimitive, then $h = k = \ldots = l$. In particular, if $h = k = \ldots = l = 1$ then the group will be called *monomial*.

An imprimitive group $\mathfrak{G}$ possesses a reducible normal subgroup $\mathfrak{H}$ that leaves the spaces E_h , E_k , …, E_l individually invariant. $\mathfrak{G} / \mathfrak{H}$ is isomorphic to a permutation group. If $\mathfrak{G}$ is monomial then $\mathfrak{H}$ will be Abelian. Conversely, if a linear group $\mathfrak{G}$ possesses any Abelian normal subgroup that does not consist of just λI then $\mathfrak{G}$ will be imprimitive. The proof is implied easily from the fact that one can bring all of the transformations of $\mathfrak{H}$ into diagonal form simultaneously. It then follows: *A linear group is imprimitive when it includes a finite, Abelian, normal subgroup that does not contain the center.* H. F. BLICHFELDT ([81]) and K. SHODA ([82]) have presented further theorems on imprimitive groups and their normal subgroup.

It follows easily from the criterion for imprimitivity that was formulated that a linear group of prime-power order is always monomial ([83]); likewise, any two-level (i.e., meta-Abelian) linear group ([84]), and in particular, any linear group of quadratic order, is monomial ([85]). The method of proof is always the same: The trivial Abelian case is omitted. In the non-Abelian case, there exists an Abelian normal subgroup that does not contain the center. Imprimitivity follows from that, and thus, the existence of an invariant decomposition $E_n = E_h + E_k + \ldots$ If one then restricts oneself to the subgroup that leaves E_h invariant then it will again be imprimitive in E_h, on the same basis; one can then further decompose E_h, and correspondingly, E_k, …, until one has obtained an invariant decomposition into one-dimensional subspaces.

According to C. JORDAN ([86]), any finite linear group $\mathfrak{G}$ possesses an Abelian normal subgroup $\mathfrak{H}$ whose index i does not exceed a limit that depends upon only n. The order of $\mathfrak{G}$ is then equal to the order h of $\mathfrak{H}$, multiplied by the restricted number i. In particular, if $\mathfrak{G}$ is primitive then, from the theorem above, $\mathfrak{H}$ will consist of only multiples of the identity, so the order of the projective group that corresponds to $\mathfrak{G}$ will be restricted (namely, it will be equal to i).

L. BIEBERBACH ([87]) has given a simple proof of the aforementioned theorem of JORDAN with explicit assumed limits that was simplified by G. FROBENIUS ([88]) and sharpened by A. SPEISER ([89]). The cited proofs all rest upon the fact that two

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([81]) H. F. BLICHFELDT: Trans. Amer. Math. Soc. **4** (1903), 387-397; **5** (1904), 310-325.
([82]) K. SHODA: J. Fac. Sci. Univ. Tokyo **2** (1931), 180-209.
([83]) See footnote ([81]). Cf., also MILLER-BLICHFELDT-DICKSON: *Theory and application of finite groups*, New York, 1916.
([84]) K. TAKETA: Proc. Imp. Acad. Tokyo **6** (1930), 31-33.
([85]) W. BURNSIDE: Messenger Math. (2) **35** (1906), 46-50.
([86]) C. JORDAN: J. reine angew. Math. **84** (1878), 89-213.
([87]) L. BIEBERBACH: S.-B. preuss. Akad. Wiss. (1911), 231-240.
([88]) G. FROBENIUS: S.-B. preuss. Akad. Wiss (1911), 241-248.
([89]) A. SPEISER: *Theorie der Gruppen von endlicher Ordnung*, 2nd ed., Berlin, 1927, § 68.

substitutions in a finite linear group that are sufficiently close to the identity will necessarily commute. FROBENIUS ([90]) gave the sharpest definition of "sufficiently close": It suffices that the characteristic roots of the one substitution do not occupy all of one-sixth of the unit circle, and those of the other substitution do no occupy all of one-half of it.

H. F. BLICHFELDT ([91]) has presented sharper limits for the order of a primitive, unimodular, linear group and the prime-powers that come about for them. His method rests upon the arithmetic discussion of an algebraic equation that couples the traces of the transformations Σ, $\Sigma\,T$, $\Sigma\,T^2$, …, $\Sigma\,T^{n-1}$, and $\Sigma\,T^r$ ($r > n - 1$) with each other and with the characteristic roots of T, if Σ and T are arbitrary elements of the group. This method also produces another proof of JORDAN's theorem above.

The theorems of MASCHKE and JORDAN were adapted to infinite groups of periodic linear substitutions (i.e., linear substitutions of finite order) by I. SCHUR ([92]). On these groups, cf., furthermore W. BURNSIDE: Proc. London Mat. Soc. (2) **3** (1905), 435-440. For another generalization of the finite linear groups, see A. LOEWY: Math. Ann. **64** (1907), 264-272.

I. SCHUR ([93]) has proved, by arithmetic methods, that the order of a finite group of given degree is restricted, as long as one is given which part of the circle that the traces of the group elements belong to.

§ 9. Finite, linear groups of given degree.

For the presentation of the finite, linear groups of given degree (i.e., given dimension number) over the field of complex numbers, one restricts oneself to linear transformations with determinant 1 (or possibly ± 1), for the sake of convenience. We will tacitly make this restriction in what follows. Moreover, following § 8, the transformations will all be assumed to be unitary (orthogonal, resp., for real groups).

A projective group $\mathfrak{G}'$ belongs to any linear group $\mathfrak{G}$: namely, the factor group of $\mathfrak{G}$ with respect to the subgroup of transformations λI in $\mathfrak{G}$. The presentation of linear groups of a given degree mostly precedes the presentation of the associated projective groups. For each such projective group $\mathfrak{G}'$, there is a greatest associated linear group $\mathfrak{G}$ that consists of all linear transformations with determinant 1, whose associated projective transformations lie in $\mathfrak{G}'$. This group is mapped to $\mathfrak{G}'$ in an n-to-one homomorphic, since each projective transformation corresponds to n linear ones with determinant 1. All linear groups that correspond to the same projective group $\mathfrak{G}'$ are included in this one group $\mathfrak{G}$.

([90]) S.-B. preuss. Akad. Wiss. (1911), 373-378.
([91]) H. F. BLICHFELDT: Trans. Amer. Math. Soc. **4** (1903), 387-397; **5** (1904), 310-325; **12** (1911), 39-42.
([92]) I. SCHUR, S.-B. preuss. Akad. Wiss. (1911), 619-627.
([93]) I. SCHUR: S.-B. preuss. Akad. Wiss (1905), 77-91.

F. Klein ([94]) has determined the finite *binary* projective groups by converting the binary projective groups into ternary real rotation groups by means of the isomorphism $O(3; \mathbb{P}) \cong PU(2, \mathbb{K})$ that was discussed in § 7. He then found the well-known types: cyclic groups, dihedral groups, tetrahedral groups, octahedral groups, icosahedral groups. H. H. MITCHELL ([95]) gave a very simple direct derivation of these types. The previously-remarked binary, linear groups of double order belong to these projective groups. Only the cyclic and dihedral groups also belong to linear groups of the same order. There are no other finite, binary, linear groups with determinant 1.

The finite, ternary, projective groups were presented incompletely by C. JORDAN ([96]) and H. VALENTINER ([97]), and completely by H. F. BLICHFELDT ([98]) [cf., also H. H. MITCHELL ([95])]. Since the imprimitive groups are either reducible or monomial, and thus relatively easy to find, it will suffice to give the primitive groups. They are:

1. The projective, ternary icosahedral group G_{60}, which corresponds to the real, orthogonal, icosahedral group.
2. The group that JORDAN called the "HESSIAN group" G_{216}, which takes the inflection point configuration of a plane curve of order three to itself ([99]).
3. A normal divisor G_{72} of G_{216} ([99]).
4. A normal divisor G_{36} of G_{216} ([99]).
5. A group G_{168} that is isomorphic to $PSL(2, 7)$ and was discovered by KLEIN ([100]).
6. A group that was discovered by VALENTINER ([97]) and then WIMAN ([101]) that is isomorphic to the alternating subgroup $\mathfrak{A}_6$ of G_{360} .

These projective groups correspond naturally to linear groups of three-fold order that are the 3-homomorphic images of them. Only the groups G_{168} and G_{60} have 1-isomorphic linear groups.

E. Goursat ([102]) has determined the finite, real (orthogonal), quaternary, projective group. On the basis of the isomorphism:

$$PO(4, \mathbb{P}) \cong PU(2, \mathbb{K}) \times PU(2, \mathbb{K})$$

that was discussed in § 7, the determination of the finite, real, quaternary groups comes down to the determination of all groups of pairs of binary, unitary substitutions. GOURSAT has also given all extensions of the groups found to orthogonal substitutions

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([94]) F. KLEIN: Math. Ann. **9** (1876), 183-208.
([95]) H. H. MITCHELL: Trans. Amer. Math. Soc. **12** (1911), 208-211.
([96]) C. JORDAN: J. reine angew. Math. **84** (1878), 89-215.
([97]) H. VALENTINER: Skr. Widensk.-Selsk. Kopenhagen (6) **5** (1889), 64-235.
([98]) H. F. BLICHFELDT: Trans. Amer. Math. Soc. **5** (1904), 321-325 – Math. Ann. **63** (1907), 552-572.
([99]) For a more precise discussion of these groups, we refer to the encyclopedia article of A. WIMAN: "Endliche Gruppen linearer Substitutionen," Enc. math. Wiss. IB, 3f. Cf., also K. RÖSSLER: Čas. pést. Mat. a Fys. **60** (1931), 166-172.
([100]) F. KLEIN: Math. Ann. **14** (1879), 438.
([101]) A. WIMAN: Math. Ann. **47** (1896), 531-556.
([102]) E. GOURSAT: Ann. École norm. (3) **6** (1889), 9-102. Cf., also G. BAGNERA: Rend. Circ. mat. Palermo **15** (1901), 161-309.

of determinant – 1. W. THRELFALL and H. SEIFERT ([103]) have determined the orthogonal groups that belong to the orthogonal projective groups with determinant 1 that were found by GOURSAT. Among these groups, one naturally also finds groups of deck motions of the regular polytopes in R_4 that were exhibited by various authors ([104]) since GOURSAT up to recently.

H. F. BLICHFELDT ([105]) has presented the complex, quaternary, primitive groups by his arithmetic methods that were already mentioned in § 8. G. BAGNERA and H. H. MITCHELL ([106]) repeated the determination by geometric methods. Some of the more remarkable of these groups are three simple groups of orders 168, 2520, and 25,920 ([107]), a group of order 16,720 that was discovered by KLEIN ([99]) that has normal subgroups of order 16 · 360 and 16, as well as two groups that are isomorphic to $\mathfrak{S}_6$ ($\mathfrak{A}_7$, resp.) and their subgroups ([108]) that are isomorphic to $\mathfrak{A}_6$, $\mathfrak{S}_5$, and $\mathfrak{A}_5$. W. BURNSIDE ([109]) and H. H. MITCHELL ([110]) have given a series of remarkable linear groups in more than four variables. For the groups of the regular polytopes (simplexes and hyperoctahedra) in $n >$ 4 dimensions, see ([104]) and ([111]).

One can determine the solvable linear groups of prime degree following BURNSIDE ([111a]). One finds further theorems on the structure of finite groups of prime degree in K. SHODA ([82]).

§ 10. Infinite, discrete groups of fractional linear transformations; in particular, discrete groups of motions.

For the older literature on this topic, we refer, once and for all, to the encyclopedia article of FRICKE on automorphic functions ([112]).

A group $\mathfrak{G}$ of one-to-one, continuous transformations in a spatial region D is called *properly discontinuous* when each point P of a domain D possesses a neighborhood U that has only finitely many points in common with the image neighborhoods SU (S comes

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([103]) W. THRELFALL and H. SEIFERT: Math. Ann. **104** (1931), 1-70.

([104]) Of the recent ones, let us mention only: D. E. LITTLEWOOD: Proc. London Math. Soc. **32** (1930), 10-20. – J. A. TODD: Proc. Cambridge Philos. Soc. **27** (1931), 212-231.

([105]) H. F. BLICHFELDT: Trans. Amer. Math. Soc. **6** (1905), 230-236. – Math. Ann. **60** (1905), 204-231.

([106]) G. BAGNERA: Rend. Circ. mat. Palermo **19** (19105), 1-56. – H. H. MITCHELL: Trans. Amer. Math. Soc. **14** (1913), 123-142.

([107]) Cf., A. WITTING: Diss. Göttingen, 1887. See also the encyclopedia article IB, 3f, of A. WIMAN, no. 23.

([108]) See H. MASCHKE: Math. Ann. **51** (1899), 253-298, as well A. WIMAN: Math. Ann. **52** (1899), 243-270.

([109]) W. BURNSIDE: Proc. London Math. Soc. (2) **10** (1911), 284-308.

([110]) H. H. MITCHELL: Trans. Amer. Math. Soc. **16** (1914), 1-12.

([111]) G. DE B. ROBERTSON: Proc. Cambridge Philos. Soc. **26** (1930), 94-98. – D. M. Y. SOMMERVILLE: Proc. London Math. Soc. (2) **35** (1933), 101-115.

([111a]) W. BURNSIDE: Acta math. **27** (1903), 217-224.

([112]) R. FRICKE: Enc. math. Wiss. IIB, 4 (1913).

from $\mathfrak{G}$) ([113]). The group is obviously discrete then, so no discrete group is properly discontinuous (example given below).

One understands a *fundamental domain* or *discontinuity domain* of a properly discontinuous group in D to mean an open subset that is disjoint from the image subsets that contains a point SP in its interior or on its boundary that is equivalent to each point P of D. From BAER and LEVI ([114]), a fundamental domain always exists when one demands, besides the proper discontinuity of the group, that for any two inequivalent points P and Q there must be neighborhoods $U(P)$ and $U(Q)$ such that the image of $U(P)$ does not enter into $U(Q)$. For a properly discontinuous group of hyperbolic, Euclidian, or elliptic motions, the totality of all points that have a smaller distance to a fixed point P_0 in D than the distance from all image points to P_0 is a *normal, fundamental domain* that is bounded by hyperplanes. A properly discontinuous group of fractional linear transformations of one complex variable possesses a fundamental domain that is bounded by circles ([115]).

P. J. MYRBERG ([116]) introduced a sharpening of the concept of proper discontinuity with consideration given to the theory of automorphic functions of several variables. According to MYRBERG, a discrete transformation group is called *normally discontinuous* in D when there is a subsequence (S_ν) in any infinite sequence of transformations of the group that converges uniformly in any closed sub-domain of D. The limit transformation to which the sequence converges does not belong to the group, and is not one-to-one, since otherwise $S_\nu^{-1}S_{\nu-1}$ would converge to I, which is impossible in a discrete group. In the case of a projective group, the limit transformations are singular, linear transformations:

$$\xi_i' = \sum \alpha_{ik}\,\xi_k \qquad \text{with} \qquad |\alpha_{ik}| = 0,$$

which take all points of the space P_{n-1}, with the exception of the points of a linear subspace $\bar{M}_{n-\rho}$ (whose equations read $\sum \alpha_{ik}\,\xi_k = 0$), to the points of a linear space $M_{\rho-1}$, where ρ is the rank of the matrix (α_{ik}). The spaces $M_{\rho-1}$ and $\bar{M}_{n-\rho}$ are called the *first* and

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([113]) This definition, which I discovered in the book ([117]) of FUBINI, is somewhat sharper than the one that was given originally by POINCARÉ ([115]), which only demands that the point P should not be the accumulation point P of its image points SP. Example: The projective transformations:

$$A = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix} \qquad \text{and} \qquad B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

generate a group that is properly discontinuous in the neighborhood of the point (1, 1, 0) in the sense of POINCARÉ, but not in the sense of FUBINI.

([114]) R. BAER and F. LEVI: Math. Z. **34** (1931), 110-130.

([115]) H. POINCARÉ: Acta math. **3** (1183), 49-92. R. L. FORD gave a very simple proof in his book *Automorphic Functions*, New York. 1929.

([116]) P. J. MYRBERG: Acta math. **46** (1925), 215-336. Cf., also Math. Ann. **93** (1924), 61-97 and Math. Z. **21** (1924), 224-253.

second limit element of the sequence (S_ν), respectively. If the inverse sequence (S_ν^{-1}) again converges then its first limit element will be contained in $\bar{M}_{n-\rho}$, while its second limit element will be subsumed by $M_{\rho-1}$.

A discrete group of projective transformations is normally discontinuous in D when the second limit element $\bar{M}_{n-\rho}$ of the convergent sequence (S_ν) contains no point of D outside the group. The *domain of normal discontinuity* of a discrete, projective group is then the totality of all points that belong to no $\bar{M}_{n-\rho}$ (and, as a result, also to no $M_{\rho-1}$).

The concept of normal discontinuity will coincide with that of proper discontinuity for $n = 2$, and thus for the case of fractional linear substitutions of one real or complex variable. In the general case, proper discontinuity will follow from the normal kind, but not the converse: *The domain D of normal discontinuity is a subset of the domain of proper discontinuity.* Proof: A point P of D has a neighborhood U, whose closed hull still belongs to D. If infinitely many images SU still had points in common with U then one could select a convergent sequence (S_ν) from the S. Since U lies separate from the second limit element $\bar{M}_{n-\rho}$ of this sequence, the image sets $S_\nu U$ will gravitate towards $M_{\rho-1}$; thus, not all $S_\nu U$ can have points in common with U.

One proves in a completely analogous way that the BAER-LEVI condition that was cited above for the existence of a fundamental domain under normal discontinuous groups is always fulfilled.

According to MYRBERG ([116]), a discrete projective group, among others, is normally discontinuous in a domain D when D remains invariant under the group and has no points in common with a system of n hypersurfaces that do not go through a point. The latter condition is then the case, in particular, when the invariant domain D lies entirely are finite points.

The real, discrete, projective groups that leave invariant an indefinite, quadratic form whose index of inertia is 1 or 2 possess a domain of normal discontinuity in real P_{n-1} . In particular, *a discrete group of hyperbolic motions is normally discontinuous in the entire interior of any quadratic fundamental surface* ([117]). Likewise, the complex, discrete, projective groups that leave a HERMITIAN form:

$$H = \bar{\xi}_1\xi_1 + \bar{\xi}_2\xi_2 + \cdots + \bar{\xi}_{n-1}\xi_{n-1} - \bar{\xi}_n\xi_n$$

invariant in the domain $H < 0$ (and also in the domain $H > 0$ for $n = 2$) will be normally discontinuous. One calls these groups *Fuchsian* for $n = 2$ and for *hyper-Fuchsian* for $n > 2$. One finds further examples of normally discontinuous groups – in particular, the ones that leave no domain D invariant – in MYRBERG ([116]). I add that *the discrete groups of real, Euclidian motions:*

$$\begin{cases} \xi_i' = \sum \alpha_{ik}\xi_k + \gamma_i\xi_0, \\ \xi_0' = \xi_0, \end{cases} \qquad (\alpha_{ik}) \text{ orthogonal}$$

([117]) G. FUBINI: *Introduzione alla teoria dei gruppi discontinui e delle funzioni automorfe,* Pisa, 1908.

are all normally discontinuous in the domain of finite points ($\xi_0 \neq 0$). In fact, a convergent sequence of such motions will always converge for a non-singular motion, unless some γ_i increases to infinity, in which case, after multiplying by a γ_i^{-1} that converges to zero strongly, a limit transformation:

$$\begin{cases} \xi_i' = \beta_i \xi_0 \\ \xi_0' = 0 \end{cases}$$

will come about whose limit elements $M_{\rho-1}$ and $\bar{M}_{n-\rho}$ both lie at infinity.

A real, linear transformation of $\xi_1, \ldots, \xi_n$ induces a linear transformation of the coefficients $z_1, \ldots, z_N$ of a quadratic form in the x. Likewise, a complex, linear transformation of $\xi_1, \ldots, \xi_n$ induces a real, linear transformation of the real and imaginary parts of the coefficients of a HERMITIAN form, which we again denote by $z_1, \ldots, z_N$. Therefore, those pairs of linear transformations that differ by only a factor λ will again induce similar pairs of transformations of $z_1, \ldots, z_N$; one can then say that the real (complex, resp.) projective transformations of a space P_{n-1} into a space P_{N-1} induce real projective transformations in both cases. There is a sub-domain D in P_{N-1} whose points belong to the definite forms. This sub-domain is always connected, and will be transformed into itself by all of the transformations considered. One now has: *Under the association above, a discrete group of real (complex, resp.) projective transformations corresponds to a discrete group of real, projective transformations of P_{N-1}, which is normally discontinuous in the domain D* ([117]).

In particular, for $n = 2$, the fractional linear transformations of one complex variable $\xi = \xi_1 : \xi_2$:

(1)
$$\begin{cases} \xi_1' = \alpha\xi_1 + \beta\xi_2, \\ \xi_2' = \gamma\xi_1 + \delta\xi_2, \end{cases} \qquad \text{or} \qquad \zeta' = \frac{\alpha\zeta + \beta}{\gamma\zeta + \delta}$$

correspond to real transformations of the space P_3 whose coordinates are the real and imaginary parts of z_1, z_2, z_3, z_4 of the coefficients of the HERMITIAN form:

$$z_1 \bar{\xi}_1 \xi_1 + (z_2 + iz_3) \bar{\xi}_1 \xi_2 + (z_2 - iz_3) \bar{\xi}_2 \xi_1 + z_4 \bar{\xi}_2 \xi_2 .$$

The definite forms are characterized by:

$$Q = z_2^2 + z_3^2 - z_1 z_4 < 0;$$

our transformations then leave the interior of the surface $Q = 0$ invariant, and are then hyperbolic motions. *Each discrete group of ζ-substitutions* (1) *corresponds to a normally discontinuous group of three-dimensional, hyperbolic motions.*

The connection between the hyperbolic motions and the ζ-substitutions (1) becomes most intuitive when one thinks of the surface Q as being taken to a sphere – viz., the ζ-sphere – which one can then project stereographically onto the ζ-plane. The hyperbolic motions generate conformal mappings of the ζ-sphere into itself, which will yield fractional linear ζ-substitutions under stereographic projection.

According to whether the discrete group of hyperbolic motions in question leaves invariant a point inside the ζ-sphere, a point outside of it, or no such point, one can distinguish:

1. *Platonic groups*, which are the finite groups of rotation or binary, projective, unitary groups that were enumerated already in § 9.

2. *Doubly-periodic groups*, which leave a point of the ζ-sphere invariant, for which one chooses the point $\zeta = \infty$ for the sake of convenience, with which, a *planar, Euclidian groups of motions* in the ζ-plane will arise.

3. *Great circle groups*, which leave a circle on the ζ-sphere invariant, which one takes to the real ζ-axis by stereographic projection for the sake of convenience, with which, a *group of real ζ-substitutions* (1) *will arise*.

4. *Non-rotation groups*, which leave no point of space invariant.

One finds all of the discrete groups of planar, Euclidian motions (along with a bibliography) in A. SPEISER ([118]).

Those great circle groups that transform the upper ζ-half plane into itself are called *Fuchsian groups* (cf., *supra*). Since these groups leave a plane in ζ-space invariant, one can also consider them to be groups of planar, hyperbolic motions. As such, they have a normal, polygonal, fundamental domain that corresponds to a fundamental domain in the upper ζ-half plane that is bounded by a circle. One can read off its generators and defining relations from the boundary relationships of the fundamental domain. We refer to the book of KLEIN-FRICKE ([119]) for the further discussion and classification of these groups.

An important example of a great circle group is defined by the *modulus group*, which consist of all ζ-substitutions $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with entire rational coefficients and determinant one. It will be generated by the two substitutions:

$$S = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad T = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Its defining relations read:

$$T^2 = (TS)^3 = 1.$$

Among its subgroups, the *congruence subgroups* of level m are noteworthy, whose matrices are constrained by congruences modulo m. The *principal congruence group of* level m consists of the substitutions $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $a \equiv d \equiv 1$, $b \equiv c \equiv 0 \pmod{m}$. H. RADEMACHER ([120]) and H. FRASCH ([121]) have given systems of generators for the

([118]) A. SPEISER: *Theorie der Gruppen von endlicher Ordnung*, 2nd ed., Berlin, 1927, § 28 and § 29.

([119]) R. FRICKE and F. KLEIN: *Vorlesungen über die Theorie der automorphen Funktionen I*, Braunschweig, 1897.

([120]) H. RADEMACHER: Abh. math. Semin. Hamburg. Univ. **7** (1929), 134-148.

principal congruence group and other congruence groups of prime level. The factor group of the modulus group with respect to the principal congruence group of level p is the modular group $PSL(2, p)$, whose structure and subgroups were discussed already in § 3.

G. PICK and R. FRICKE ([122]) gave examples of subgroups of the modulus group that are not congruence subgroups. G. BOL ([123]) has determined all groups of fractional linear ζ-substitutions that isomorphic to the modulus group.

The groups that were classified by 1, 2, 3 above are all already properly discontinuous on the ζ-sphere, but not the ones in 4. Those non-rotation groups that first become properly discontinuous inside the ζ-sphere are called (from the form of their fundamental domains) *polyhedral groups*. An example of this is defined by the PICARD *group* of those substitutions (1) with determinant one, for which α, β, γ, δ are whole numbers of the form $a + bi$. Following BIANCHI ([112]), one can generalize the Ansatz by taking α, β, γ, δ to be whole numbers in an imaginary-quadratic number field $k(\sqrt{-r})$. On the basis of the isomorphism (34), § 7, one can also obtain the same group as the groups of quaternary, whole-number, projective transformations with determinant one that leave a quadratic form $Q_2 = \xi_1\,\xi_2 + \xi_3^3 + r\xi_4^2$ invariant. One obtains great circle groups when one restricts oneself to those substitutions (1) that leave an indefinite, binary, HERMITIAN form invariant. One obtains even more general arithmetically-defined groups by considering the ternary, whole-number, projective transformations with coefficients in a given field that leave a ternary, quadratic form invariant ([119]).

Discrete groups of Cremona transformations fall outside the scope of this discussion. We thus mention the *hyper-Abelian groups* only quite briefly, which are discrete groups of real fractional linear substitutions of n complex variables:

$$\text{(2)} \qquad \zeta'_\nu = \frac{\alpha_\nu \zeta_\nu + \beta_\nu}{\gamma_\nu \zeta_\nu + \delta_\nu}, \qquad D_\nu = \alpha_\nu\,\delta_\nu - \beta_\nu\,\gamma_\nu > 0, \qquad \nu = 1, 2, \ldots, n.$$

From MYERBERG ([116]), these groups are all normally discontinuous in the domain:

$$I(\zeta_1)\, I(\zeta_2) \,\ldots\, I(\zeta_n) \neq 0.$$

Examples of this are defined by the *higher modulus groups* that were discussed by VON BLUMENTHAL ([112]), for which, the coefficients of the substitutions that are conjugate to (2) run through entire, algebraic numbers in n conjugate real number fields of degree n, while the determinants D_ν define a system of conjugate units.

From POLYA and NIGGLI ([124]), there are 17 affine-distinct discrete groups of planar, Euclidian motions and transfers that leave no point and no line invariant. NIGGLI ([124a]) then arrived at five groups that leave a line invariant.

([121]) H. FRASCH: Math. Ann. **108** (1933), 229-252.
([122]) G. PICK: Math. Ann. **28** (1886), 119-124. – R. FRICKE: *ibidem*, 99-118.
([123]) G. POL: Nieuw Arch. Wiskde. **17** (1932), 55-61.
([124]) G. POLYA: Z. Kristallogr. **60** (1924), 278-282. – P. NIGGLI: *ibidem*, 282 to 298.
([124a]) P. NIGGLI: Z. Kristallogr. **63** (1926), 255-272. See also ([113]).

A. SCHOENFLIES ([125]), as well as VON FEDOROW ([126]) have presented the three-dimensional discrete groups of Euclidian motions and transfers that leave no point and no line or plane invariant. As both authors found in agreement with each other, there are 230 *space groups* that divide into 32 *crystal classes* ([127]). That means the following: Two space groups $\mathfrak{G}$ of the kind considered contain three linearly-independent transformations. The subgroup of all translations of $\mathfrak{G}$ then generates a three-dimensional lattice Γ when it is applied to a fixed starting point *O*. If one subdivides each motion or transfer in $\mathfrak{G}$ into a translation and a rotation or reversal with the fixed point *O* then the rotational components define a group $\mathfrak{H}$ in itself (which is homomorphic to $\mathfrak{G}$), which shall be called the "point group," and which leaves the lattice Γ invariant. If one chooses the lattice vectors to be coordinate vectors then $\mathfrak{H}$ will become a *finite group of unimodular, integer, linear, vector transformations that leave a definite quadratic form invariant.* Two such point groups will be counted in the same *class* when they can be transformed into each other by a linear transformation *U*. In that sense, there are 32 classes ([128]). However, one can also grasp the concept of class more precisely when one demands that *U* also be unimodular and integer ([129]).

C. HERMANN, L. WEBER, as well as E. ALEXANDER and K. HERMMAN ([130]) have determined the discrete groups of three-dimensional motions and transfers that leave a plane invariant, and likewise C. HERMANN and E. ALEXANDER ([131]) have determined the ones that leave a line fixed. On the four-dimensional groups that leave an R_3 invariant, see H. HEESCH ([131a]), as well as J. J. BURCKHARDT ([135])

L. BIEBERBACH ([132]) has examined the discrete, Euclidian groups of motions in *n* dimensions. The main result is:

1. A discrete group of motions is either *decomposable* – i.e., it leaves a proper linear subspace R_m of R_n invariant – or it contains *n* linearly-independent translations. (In the first case, the fundamental domain obviously extends to infinity, but in the second case, it is obviously finite). In the indecomposable case, the rotational components of the motions of the group define finite rotation groups of restricted order.

2. There are (up to affine transformations) only finitely many different discrete groups of motions with *n* linearly-independent translations.

([125]) A. SCHOENFLIES: *Kristallsysteme und Kristallstruktur*, Leipzig, 1891.

([126]) E. VON FEDOROW: Z. Kristallogr. **20** (1892), 25-75.

([127]) Cf., on this, also, P. NIGGLI: *Geometrische Kristallographie des Diskontinuums*, Leipzig, 1919. – C. HERMANN: Z. Kristallogr. **69** (1928), 266-249. – H. HEESCH: Z. Kristallogr. **72** (1929), 177-201. – E. SCHIBOLD: *Neue Herleitung und Nomenklatur der 230 kristallographischen Raumgruppen*, Leipzig, 1929. – R. W. G. WYCKOFF: *The analytic expression of the results of the theory of space groups*, Washington, 1930.

([128]) See also G. FROBENIUS: S.-B. preuss. Akad. Wiss. (1911), 681-691.

([129]) J. J. BURCKHARDT: Comment. math. helv. **6** (1933), 159-184.

([130]) C. HERMANN: Z. Kristallogr. **69** (1928), 250-270. – L. WEBER: Z. Kristallogr. **70** (1929), 309-327. – E. ALEXANDER and K. HERMMANN: *ibidem*, 328-345 and 460.

([131]) C. HERMANN: Z. Kristallogr. **69** (1928), 250-270. – E. ALEXANDER: Z. Kristallogr. **70** (1929), 367-382.

([131a]) H. HEESCH: Z. Kristallogr. **73** (1930), 325-346.

([132]) L. BIEBERBACH: Nachr. Ges. Wiss. Göttingen (1910), 75-84. – Math. Ann. **70** (1910), 297-336; **72** (1912), 400-412.

G. FROBENIUS ([133]) has proved 1 quite simply and adapted it to groups of complex, affine transformations whose homogeneous components leave a definite HERMITIAN form invariant. One also finds a simpler proof of 2 in A. SPEISER ([134]). J. J. BURCKHARDT ([135]) has shown how one can develop the BEIBERBACH-FROBENIUS method far enough that it makes the complete determination of the groups of motions possible. As an application, he determined all hexagonal and rhombohedral four-dimensional groups that leave a one-dimensional space invariant.

COXETER ([136]) has determined the discrete groups of motions of R_n whose fundamental domains are simplexes. In connection with that, he also enumerated and examined the discrete groups of motions that are generated by reflections ([137]).

([133]) G. FROBENIUS: S.-B. preuss. Akad. Wiss. (1911), 654-665.

([134]) A. SPEISER: *Theorie der Gruppen von endlicher Ordnung*, 2nd ed., Berlin, 1927, § 70.

([135]) J. J. BURCKHARDT: Comment. math. helv. **6** (1934), 159-184. See also F. SEITZ: Z. Kristallogr. **88** (1934), 433-459.

([136]) H. S. M. COXETER: J. London Math. Soc. **6** (1931), 132-136. – Proc. London Math. Soc. II, s. **34** (1932), 126-189.

([137]) H. S. M. COXETER: Ann. of Math. II.s. **35** (1934), 588-621.

II. Representations of rings and groups.

Whereas in Part I all linear groups were considered to have a *given degree*, the problem of representation theory reads: Discover all linear groups *of given structure*, hence, all of the linear groups that are isomorphic to a given group or, more generally, homomorphic to it. This theory was created by G. FROBENIUS ([138]), and then more recently founded and developed further by W. BURNSIDE ([139]) and I. SCHUR ([140]). Here, we give the essence of the construction of the theory by E. NOETHER ([141]), which was based upon its organic connection with the theory of representations of hypercomplex systems.

§ 11. Representations and representation modules.

One understands a *representation* $\mathfrak{D}$ of a group $\mathfrak{g}$ (by linear transformations) to mean a homomorphic map of the group into a system $\mathfrak{S}$ of linear transformations of a vector space $\mathfrak{M}$:

$$a \to \mathsf{A}, \qquad b \to \mathsf{B}, \qquad ab \to \mathsf{AB}.$$

The same thing is true when $\mathfrak{g}$ is only a *semi-group* – i.e., when all products $a \cdot b$ are defined in $\mathfrak{g}$ and the associativity law is satisfied, but the existence of the inverses is not required.

If the semi-group $\mathfrak{g}$ is given as a *ring*, in particular, then the additive isomorphism:

([138]) G. FROBENIUS: "Über Gruppencharaktere," S.-B. preuss. Akad. Wiss. (1896), 985-1021. – "Über die Primfaktoren der Gruppendeterminante," *ibidem* (1896), 1343-1382; (1903), 401-409. – "Über die Darstellung der endlichen Gruppen durch lineare Substitutionen," *ibidem* (1897), 904-1015; (1899), 482-500. – "Über die Komposition der Charaktere einer Gruppe," *ibidem* (1899), 330-339. – G. FROBENIUS and I. SCHUR: "Über die Äquivalenz der Gruppen linearer Substitutionen," *ibidem* (1906), 209-217. – On the genesis of representation theory, cf., also the exchange of letters between DEDEKIND and FROBENIUS in DEDEKIND's Werken **2**.

([139]) W. BURNSIDE: "On the continuous group that is defined by any group of finite order," Proc. London Soc. **29** (1898), 207-224 and 546-565. – "On the composition of group characteristics," *ibidem* **34** (1901), 41-48. – "On the representation of a group of finite order as an irreducible group of linear substitutions and the direct establishment of the relations between group-characteristics," *ibidem* (2) **1** (1903), 117-123. – *Theory of Groups*, 2nd edition, Cambridge, 1911.

([140]) I. SCHUR: "Neue Begründung der Theorie der Gruppencharakteren," S.-B. preuss. Akad. Wiss. (1905), 406-432. – "Arithmetische Untersuchungen über endliche Gruppen linearer Substitutionen," *ibidem* (1906), 164-184. – "Über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen," J. reine angew. Math. **127** (1904), 20-50; **132** (1907), 85-137.

([141]) E. NOETHER: "Hyperkomplexe Grössen und Darstellungstheorie," Math. Z. **30** (1929), 641-692. – Cf., also TH. MOLIEN: Math. Ann. **41** (1892), 83-156. – M. HERZBERGER: "Über Systeme hyperkomplexer Grössen," Diss. Berlin, 1923, as well as the first papers of FROBENIUS (footnote 1).

$$a + b \to \mathsf{A} + \mathsf{B}$$

will also be demanded of a representation. If $\mathfrak{g}$ is a *hypercomplex system* over a field $\mathbb{P}$, more especially, then we will demand, in addition, that $\mathbb{P}$ must be included in the center of the representation field $\mathbb{K}$, and that one must have:

$$a\lambda \to \mathsf{A}\lambda \qquad \text{for all } \lambda \text{ in } \mathbb{P}.$$

The *degree* of a representation is the dimension of the space $\mathfrak{M}$. A representation is called *faithful* when it is 1-isomorphic.

It is now preferable to define a product au for every a in $\mathfrak{g}$ and every u in $\mathfrak{M}$ by way of:

(1) $$au = \mathsf{A}u,$$

where A is the representative transformation of a.

One then has the rules:

$$a(u + v) = au + av,$$
$$a(u\lambda) = (au)\lambda,$$
$$(ab)\,u = a(bu) \qquad \text{for groups and semi-groups,}$$

and:

$$(a + b)\,u = au + bu \qquad \text{for rings } \mathfrak{g},$$
$$(a\lambda)\,u = a(u\lambda) = (au)\lambda \qquad \text{for hypercomplex systems } \mathfrak{g}.$$

The symbol A of the representative transformation will be made superfluous with this notation, (which is, in fact, an advantage when several representations are considered simultaneously), and the entire problem of representation theory comes down to the examination of a module (i.e., an additive group) $\mathfrak{M}$ that is endowed with two kinds of operators: The elements of $\mathbb{K}$, which will be written to the right, and those of $\mathfrak{g}$, which will be written to the left. This double module – viz., the *representation module* – will determine the representation uniquely by means of (1).

One can also make the vector space $\mathfrak{M}$ into a double module by giving an arbitrary system $\mathfrak{S}$ of linear transformations of $\mathfrak{M}$ into itself, for which, one assumes that $\mathfrak{S}$ is an operator domain of $\mathfrak{M}$, or – what amounts to the same thing – when one considers $\mathfrak{S}$ to be own representation. Indeed, the product $\mathsf{A}u$ is meaningful for an arbitrary A in $\mathfrak{S}$ and u in $\mathfrak{M}$, and fulfills all of the rules of calculation above.

If we apply the basic concepts of the theory of groups ([142]) to the double module $\mathfrak{M}$ then that will yield the following concepts.

([142]) See, perhaps, B. L. VAN DER WAERDEN: *Moderne Algebra I*, chap. 2 and 6, or the booklet by VAN DER WAERDEN and LEVI that will appear soon in this series.

1. *Allowable subgroups* are those linear subspaces of $\mathfrak{M}$ that admit the transformations of $\mathfrak{g}$ – i.e., they will be transformed into themselves by these transformations. One calls them *invariant subspaces* of $\mathfrak{M}$ in this case (under $\mathfrak{g}$). If $\mathfrak{N}$ is such a subspace, $(v_1, \ldots, v_m)$ is a basis of $\mathfrak{N}$, and $(u_1, \ldots, u_l, v_1, \ldots, v_m)$ is a basis for $\mathfrak{M}$ then the matrix of A will take the following form:

$$A = \begin{pmatrix} P & 0 \\ Q & R \end{pmatrix} \tag{2}$$

when it is referred to these bases.

The sub-matrix R gives how the subspace $\mathfrak{N}$ is transformed by A; likewise, P determines the transformation of the factor module $\mathfrak{M} / \mathfrak{N}$.

If the module $\mathfrak{M}$ is simple – i.e., no sub-module exists that admits all operators – then one will call the system $\mathfrak{S}$, or the representation $\mathfrak{D}$, or also the space $\mathfrak{M}$, *irreducible*; by contrast, if an invariant subspace exists, so the transformations A of $\mathfrak{S}$ can all be represented simultaneously by matrices then $\mathfrak{S}$, $\mathfrak{D}$, and $\mathfrak{M}$ will be called *reducible*.

2. If $\mathfrak{M}$ is a *direct sum* of two allowable subgroups $\mathfrak{N}_1 = (v_1, \ldots, v_m)$ and $\mathfrak{N}_2 = (w_1, \ldots, w_h)$ then one will say that the module $\mathfrak{M}$ *decomposes* into $\mathfrak{N}_1$ and $\mathfrak{N}_2$. One will then have $Q = 0$ in the matrices (2), and one will say that the system of these matrices or the representation $\mathfrak{D}$ *decomposes* into the systems of matrices P and R, and analogously for a direct sum of more than two summands.

3. If one defines a *composition series* of invariant subspaces for $\mathfrak{M}$:

$$\mathfrak{M} = \mathfrak{M}_0 \supset \mathfrak{M}_1 \supset \ldots \mathfrak{M}_r = (0),$$

in such a way that $\mathfrak{M}_{\nu+1}$ is a maximal invariant subspace in $\mathfrak{M}_\nu$, and therefore $\mathfrak{M}_\nu / \mathfrak{M}_{\nu+1}$ is simple (i.e., irreducible), then one can put the representative matrix A for the system $\mathfrak{S}$ into the form:

$$\begin{pmatrix} A_{11} & 0 & \cdots & 0 \\ A_{21} & A_{22} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ A_{r1} & A_{r2} & \cdots & A_{rr} \end{pmatrix} \tag{3}$$

for a suitable choice of basis, where the "diagonal boxes" $A_{\nu\nu}$ represent the transformations that are induced in the factor modules $\mathfrak{M}_{\nu-1} / \mathfrak{M}_\nu$. Since these factor modules are simple, the matrix system ($A_{\nu\nu}$) will be irreducible. One calls them the *irreducible diagonal components* of the matrix system $\mathfrak{S}$, and one says that the system $\mathfrak{S}$ has been *reduced* to the form (3).

4. If the module $\mathfrak{M}$ is *completely reducible* – i.e., the direct sum of simple (or irreducible) invariant subspaces – then one will find zeroes everywhere in the matrix (3) outside of the main diagonal, and one will also call the system $\mathfrak{S}$ (or the representation $\mathfrak{D}$) *completely reducible*. The system $\mathfrak{S}$ *decomposes into its irreducible components.*

5. Just like the concept of composition series, one can also adapt the LOEWY *composition series* ([143]). The last group in that series will be the sum of the minimal allowable sub-modules (viz., the REMAK base). One obtains the remaining groups in succession by applying the same process to the factor groups. The base, and likewise the other composition factors, are completely reducible. One thus obtains a matrix form that is similar to the one that was given in 3, but for which the diagonal component $A_{\nu\nu}$ is not reducible: They are the *successive greatest complete reducible components of the representation* ([144]). We will make no further use of these concepts.

6. An operator homomorphism that maps a module $\mathfrak{M}_1$ to another module $\mathfrak{M}_2$ (with the same operator domains $\mathfrak{g}$ and $\mathbb{K}$) is obviously nothing but a linear transformation T of $\mathfrak{M}_1$ into $\mathfrak{M}_2$ with the property that $\mathsf{T}av = a\mathsf{T}v$ (for any a and any v in M), or – what amounts to the same thing:

$$\text{(4)} \qquad \mathsf{TA}_1 = \mathsf{A}_2\mathsf{T} \qquad \text{for all } a \text{ in } \mathfrak{g},$$

where A_1 and A_2 are the transformations in $\mathfrak{M}_1$ and $\mathfrak{M}_2$, resp., that are induced by a.

In particular, if $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are 1-isomorphic and T is a 1-isomorphism then one can also write:

$$\mathsf{A}_2 = \mathsf{T}\,\mathsf{A}_1\mathsf{T}^{-1},$$

instead of (4). The representations $a \to \mathsf{A}_1$ and $a \to \mathsf{A}_2$ are called *equivalent* in this case.

In particular, if $\mathfrak{M}_1 = \mathfrak{M}_2$, $\mathsf{A}_1 = \mathsf{A}_2 = \mathsf{A}$ then (2) will become $\mathsf{TA} = \mathsf{AT}$, so: *The operator automorphisms of the representation module $\mathfrak{M}$ are the linear transformations that commute with all transformations of the representation.*

Once the basic concepts of group theory have been adapted to representation modules, we can also adapt the important theorems that relate to groups and their homomorphisms:

([143]) W. KRULL: S.-B. Heidelberg. Akad. Wiss. (1926), ser. 1.
([144]) A. LOEWY: Trans. Amer. Math. Soc. **4** (1903), 171-177.

1. The JORDAN-HÖLDER theorem says, in our case, that the diagonal components $A_{\nu\nu}$ (and especially the irreducible components of a completely reducible representation) that enter into a composition series of $\mathfrak{M}$ are independent of the arbitrariness in the complete reduction, up to its sequence, and are determined uniquely up to equivalence. Likewise, the completely reducible components that appear in a LOEWY composition series are determined uniquely up to equivalence.

2. The REMAK-SCHMIDT, or KRULL-SCHMIDT, Theorem ([145]) on the uniqueness of the directly-indecomposable summands of a group with operators asserts, in our case, the uniqueness (up to equivalence and sequence) of the indecomposable components in a decomposition of a system of linear transformations ([146]).

3. A module $\mathfrak{M}$ is completely reducible if and only if it admits a decomposition $\mathfrak{M} = \mathfrak{N} + \mathfrak{N}'$ into allowable sub-modules $\mathfrak{N}$. $\mathfrak{N}$ and $\mathfrak{N}'$ are then also themselves completely reducible, and the same will be true for the factor module $\mathfrak{M} / \mathfrak{N}$, since $\mathfrak{M} / \mathfrak{N} \equiv \mathfrak{N}'$, and thus for any module that is homomorphic to $\mathfrak{M}$.

4. If $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are two irreducible modules, and $\mathfrak{M}_1$ is mapped homomorphically to $\mathfrak{M}_2$ then the image set will either be the zero module or the entire module $\mathfrak{M}_2$. A homomorphism of a simple module $\mathfrak{M}_1$ is, however, always a 1-isomorphism when it is not the zero homomorphism. If one translates this into the language of representation theory then that will say that when $a \to \mathsf{A}_1$ and $a \to \mathsf{A}_2$ are irreducible representations that are mediated by $\mathfrak{M}_1$ and $\mathfrak{M}_2$ then: *Any linear transformation* T *of* $\mathfrak{M}_1$ *into* $\mathfrak{M}_2$ *that has the property:*

(4) $$\mathsf{T}\,\mathsf{A}_1 = \mathsf{A}_2\,\mathsf{T} \quad \textit{for all } a \textit{ in } \mathfrak{g}$$

will either be the zero map or it will be non-singular; in the latter case, the two given irreducible representations $a \to \mathsf{A}_1$ and $a \to \mathsf{A}_2$ *are equivalent* [the SCHUR lemma ([147])].

In the same way, one proves the more general assertion, which likewise goes back to I. SCHUR: If there is a transformation T that is not zero and has the property (4) then the representations $a \to \mathsf{A}_1$ and $a \to \mathsf{A}_2$ will have some common irreducible diagonal components whose total degree will be equal to the rank of the matrix T.

5. The linear transformations that commute with a representation $\mathfrak{D}$ – or in fact, a system of linear transformations – define a ring: viz., the *automorphism ring* of the representation module. H. FITTING ([148]) has developed the theory of automorphism rings of arbitrary Abelian groups with operators. For the case of a *completely reducible* module $\mathfrak{M}$, the first main result of this theory reads ([149]): If one combines the equivalent,

([145]) R. REMAK: J. reine angew. Math. **139** (1911), 293. – W. KRULL: Math. Z. **23** (1925), 161-186. – O. SCHMIDT: *ibidem* **29** (1929), 34-44.

([146]) Cf., also, R. BRAUER and I. SCHUR: S.-B. preuss. Akad. Wiss. (1930), 209-226.

([147]) I. SCHUR: S.-B. preuss. Akad. Wiss. (1905), 406-432.

([148]) H. FITTING: Math. Ann. **107** (1932), 514-542.

([149]) This also presented in B. L. VAN DER WAERDEN: *Moderne Algebra II*, § 117.

irreducible components in the decomposition $\mathfrak{M} = \mathfrak{m}_1 + \mathfrak{m}_2 + \dots$ into a sum $\mathfrak{M}_i$ such that $\mathfrak{M} = \mathfrak{M}_1 + \mathfrak{M}_2 + \dots$ then the automorphism ring will become a direct sum of rings $\mathfrak{R}_1$, $\mathfrak{R}_2$, … that one can regard as automorphism rings of $\mathfrak{M}_1$, $\mathfrak{M}_2$, … (Any element of $\mathfrak{R}_1$ transforms $\mathfrak{M}_1$ into itself and annuls $\mathfrak{M}_2$, $\mathfrak{M}_3$, …) If one decomposes $\mathfrak{M}_1$ into r equivalent components $\mathfrak{m}_1 + \mathfrak{m}_2 + \dots + \mathfrak{m}_r$ then $\mathfrak{R}_1$ will be isomorphic to a full matrix ring of degree r over a skew field Λ_1, namely, the automorphism field of $\mathfrak{m}_1$, and analogously for $\mathfrak{M}_i$. If one chooses a basis for the representation module $\mathfrak{M}_1$ that is adapted to the decomposition $\mathfrak{M}_1 = \mathfrak{m}_1 + \dots + \mathfrak{m}_r$, for which the bases of the individual (equivalent) $\mathfrak{m}_i$ are chosen in such a way that they are transformed the same by all transformations of the representation then the matrix of such a transformation of $\mathfrak{M}_1$ will look like:

$$\begin{pmatrix} A_1 & 0 & \cdots & \cdot \\ 0 & A_1 & & \vdots \\ \vdots & & \ddots & \vdots \\ \cdot & \cdots & \cdots & A_1 \end{pmatrix}$$

and the matrix of transformation in the automorphism ring $\mathfrak{R}_1$ that commutes with this will look like:

$$\begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1r} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ T_{r1} & T_{r2} & \cdots & T_{rr} \end{pmatrix}, \tag{5}$$

where the T_{jk} are matrices that commute with all matrices of the irreducible representation $\mathfrak{D}_1$ that is mediated by $\mathfrak{m}_1$, and range through the automorphism field Λ_1 independently of each other. By writing the matrices (5) that refer to the individual $\mathfrak{M}_i$ one after each other, one will obtain the matrix of the most general transformation of $\mathfrak{R}$.

6. In particular, the automorphism ring of an irreducible module is a skew field. Thus: *The linear transformations that commute with all transformations of an irreducible system define a skew field* Λ_1 . That also follows immediately from the SCHUR lemma.

If – as we would like to assume from now – the ground field is commutative then the skew field Λ_1 will contain the transformations λI in its center, in particular. As a matrix ring, Λ_1 can contain only finitely many elements that are linearly independent over $\mathbb{K}$, so it will be a skew field of finite rank over $\mathbb{K}I$. Any element of Λ_1 satisfies an irreducible algebraic equation with coefficients in $\mathbb{K}I$. In particular, if $\mathbb{K}$ is algebraically closed then one must have $\Lambda_1 = \mathbb{K}I$, i.e.: *the operator automorphisms of an irreducible representation module with an algebraically closed coefficient field* $\mathbb{K}$ *are scalar multiples* λI *of the identity* I ([147]). The same thing is true for arbitrary ground fields when the representation is absolutely irreducible – i.e., it remains irreducible under an arbitrary algebraic extension of the ground field $\mathbb{K}$.

7. H. FITTING ([148]) stated a further theorem: If the representation module $\mathfrak{M}$ is completely reducible then there will be a one-to-one correspondence between the invariant subspaces $\mathfrak{N}$ of $\mathfrak{M}$ and the right-ideals $\mathfrak{r}$ of the automorphism ring $\mathfrak{A}$, in which, in particular, every decomposition of $\mathfrak{M}$ into irreducible subspaces will correspond to a decomposition of $\mathfrak{A}$ into minimal right-ideals (and conversely). For a given $\mathfrak{r}$, one will have $\mathfrak{N} = \mathfrak{r}\ \mathfrak{M}$, and for a given $\mathfrak{N}$, $\mathfrak{r}$ will consist of those homomorphisms that map $\mathfrak{M}$ into $\mathfrak{N}$.

§ 12. Representations of hypercomplex systems. Semi-groups of linear transformations.

A *hypercomplex system* – or *an algebra* – of rank h over $\mathbb{K}$ is a ring that is also an h-dimensional vector space relative to the commutative field $\mathbb{K}$. Thus, a hypercomplex system is given by a basis $(u_1, \ldots, u_n)$, and a multiplication table:

$$u_j\, u_k = \sum u_l\, \gamma_{jk}^l .$$

Any hypercomplex system $\mathfrak{S}$ possesses an immediately-associated representation, namely, the *regular representation*, which one obtains when one regards the system $\mathfrak{S}$ itself as the representation module (with $\mathfrak{S}$ as the left operator domain and $\mathbb{K}$ as the right one). The representative matrix of a quantity $\sum u_j \xi_j$ in the regular representation is obviously $\sum \gamma_{jk}^l \xi_j$ (l is the row index, k is the column index). The invariant subspaces of the representation module are the *left ideals*, which contain all multiples $r \cdot a$ (r in $\mathfrak{S}$) and $a\lambda$ (λ in $\mathbb{K}$), along with every element a. The irreducible subspaces are the *minimal left ideals*. We will also call two operator-isomorphic left ideals that mediate equivalent representations *equivalent.*

If the system $\mathfrak{S}$ has a unity – which we will always assume in what follows – then the regular representation will always be faithful.

The types of hypercomplex systems that will be most important for us are:

1. The *division algebras* (i.e., fields), in which any non-zero element possesses an inverse, and therefore unrestricted division will be possible.

2. The *simple systems* or *full matrix rings* of degree n over a division algebra Λ, which consist of all matrices with elements in Λ.

3. The *semi-simple systems* (or *systems without radical*), which decompose completely into minimal left ideals. Any semi-simple system is the direct sum of full matrix rings $\mathfrak{S}_1 + \mathfrak{S}_2 + \ldots + \mathfrak{S}_m$, which are mutually annihilating ([150]):

(1) $$\mathfrak{S} = \sum_{\nu} \mathfrak{S}_{\nu} = \sum_{\nu}\sum_{i,k} c_{ik}^{(\nu)} \Lambda_{\nu}\;; \quad \Lambda_{\nu} \text{ is a division algebra.}$$

If n_ν is the degree of the matrix ring $\mathfrak{S}_\nu$ then $\mathfrak{S}_\nu$ will decompose into n_ν equivalent minimal left ideals, while the left ideals of different $\mathfrak{S}$'s will be inequivalent. If r_ν is the rank of Λ_ν then, from (1), the rank of $\mathfrak{S}$ will be equal to:

$$h = \sum n_\nu^2 \, r_\nu \,.$$

A decomposition of a semi-simple system $\mathfrak{S}$ into left ideals:

$$\mathfrak{R} = \mathfrak{l}_1 + \mathfrak{l}_2 + \ldots + \mathfrak{l}_s$$

is also associated with a decomposition of the unity into idempotents:

$$1 = e_1 + e_2 + \ldots + e_s\,,$$
$$e_i^2 = e_i\,; \qquad e_i\, e_k = 0 \qquad \text{for } i \neq k.$$

An arbitrary hypercomplex system $\mathfrak{S}$ possesses a *radical* – i.e., a maximal, nilpotent, left ideal $\mathfrak{c}$:

$$\mathfrak{c}^{\rho} = 0.$$

$\mathfrak{c}$ is a two-sided ideal in $\mathfrak{S}$, and the residue class ring $\mathfrak{S} / \mathfrak{c}$ is semi-simple ([150]).

The theory of representations of hypercomplex systems will now be governed by the following theorems:

Lemma. *Any representation of a semi-group with unity decomposes into two components (one of which can be missing): In one of them, the unity will be represented*

([150]) The theorems presented go back to J. H. MACLAGAN-WEDDERBURN: Proc. London Math. Soc. **6** (1907), 77-118. For simple proofs, see B. L. VAN DER WAERDEN: *Moderne Algebra II*, chap. 16, or H. FITTING: Math. Ann. **107** (1932), 514-542. Cf., also the booklet on algebra by M. DUERING in this series (**4** Heft 1).

by the identity matrix, while the other one will consist of nothing but zeroes (i.e., the zero representation).

Proof. From the lemma, one can restrict oneself to the case in which the unity of $\mathfrak{S}$ induces the identity transformation in the representation module $\mathfrak{M}$. Now, let:

$$\mathfrak{S} = \mathfrak{l}_1 + \mathfrak{l}_2 + \ldots + \mathfrak{l}_s ,$$
$$\mathfrak{M} = (u_1, \ldots, u_m) = \sum \mathfrak{S} u_k = \sum_{i,k} \mathfrak{l}_i u_k ,$$

where the sum that is suggested by $\mathfrak{S}$ does not need to be direct. Each of the modules $\mathfrak{l}_i$ u_k is an operator-homomorphic image of $\mathfrak{l}_i$ under the association $x \to x\, u_k$, so it is either the zero module or it is operator-homomorphic to $\mathfrak{l}_i$, and therefore minimal. Therefore, each of them either has only zero in common with the sum of the foregoing, or it is contained entirely within it. If one now drops those summands $\mathfrak{l}_i\, u_k$ in the sum that are already contained in the sum of the foregoing then the sum will be direct.

If one actually presents the left ideal $\mathfrak{l}_\nu = c_{11}^{(\nu)}\Lambda_\nu + c_{21}^{(\nu)}\Lambda_\nu + \ldots + c_{n1}^{(\nu)}\Lambda_\nu$ in terms of the irreducible representation $\mathfrak{D}_\nu$ then that will imply the following *additional theorems and corollaries* ([151]):

The representation $\mathfrak{D}_\nu$ of the element $a = \sum_\nu \sum_{j,k} c_{jk}^{(\nu)} \alpha_{jk}^{(\nu)}$ of $\mathfrak{S}$ [cf., (1)] will be obtained when one defines the matrix $A_\nu = (\alpha_{ik}^{(\nu)})$ and replaces every element $\alpha_{ik}^{(\nu)}$ of the division algebra Λ_ν with its representative matrix in the regular representation of Λ_ν. In the case $\Lambda_\nu = \mathbb{K}$, the A_ν already define the representation $\mathfrak{D}_\nu$, in their own right. The irreducible representation $\mathfrak{D}_\nu$ then represents the sub-ring $\mathfrak{S}_\nu$ [cf., (1)] faithfully, and represents the rings $\mathfrak{S}_\mu$ $(\mu \neq \nu)$ by zero. It is of degree $n_\nu\, r_\nu$, so it appears in the regular representation n_ν times, and because it represents $\mathfrak{S}_\nu$ faithfully, it will contain $n_\nu^2\, r_\nu$ linearly-independent matrices. The field Λ_ν is inversely isomorphic to the field of the matrices that commute with the representation $\mathfrak{D}_\nu$.

Second representation theorem. *The radical $\mathfrak{c}$ will be represented by zero for an irreducible – and therefore also for a completely-reducible – representation of an arbitrary hypercomplex system $\mathfrak{S}$; i.e., the representation can be regarded as the representation of a semi-simple system.*

Proof. Let $\mathfrak{M}$ be an irreducible representation module. Now, if one had $\mathfrak{c}\mathfrak{M} \neq 0$ then one would have $\mathfrak{c}\mathfrak{M} = \mathfrak{M}$, so:

([151]) One finds that the arguments are detailed completely in the already-cited *Moderne Algebra II*, § 121 and § 118.

$$\mathfrak{M} = \mathfrak{c}\mathfrak{M} = \mathfrak{c}^2\,\mathfrak{M} = \ldots = \mathfrak{c}^\rho\,\mathfrak{M} = (0),$$

which is not true.

Corollaries. A completely reducible representation $\mathfrak{D}$ contains $\sum n_\nu^2\, r_\nu$ linearly-independent matrices, where the summation is extended over the irreducible representations $\mathfrak{D}_\nu$ of $\mathfrak{S}$ / $\mathfrak{c}$ that enter into $\mathfrak{D}$ as components at least once. The representation $\mathfrak{D}$ then represents $\mathfrak{S}$ / $\mathfrak{c}$ faithfully if and only if all $\mathfrak{D}_\nu$ enter into it at least once.

It follows from both representation theorems that: A faithful representation of a hypercomplex system $\mathfrak{S}$ is completely reducible if and only if the system $\mathfrak{S}$ is semi-simple. It follows further from this that: *A semi-group* $\mathfrak{g}$ *of linear transformations is completely reducible if and only if the hypercomplex system that consists of all linear combinations of transformations* $\sum \mathsf{A}_\mu \lambda_\mu$ *of* $\mathfrak{g}$ *(viz., the "linear hull" of* $\mathfrak{g}$*) is semi-simple.*

One obtains a homomorphic image $\mathfrak{g}'$ from an arbitrary reducible semi-group $\mathfrak{g}$ of linear transformations when one replaces all matrix elements in the matrices of $\mathfrak{g}$ outside of the irreducible diagonal boxes with zeroes. When one then necessarily goes to the linear hulls, one can then assume that $\mathfrak{g}$ is a hypercomplex system. If $\mathfrak{g}$ is completely reducible then $\mathfrak{g}$ will obviously be mapped to $\mathfrak{g}'$ 1-isomorphically; by contrast, if $\mathfrak{g}$ is not completely reducible then $\mathfrak{g}$ will have a radical, which goes to zero under the map to $\mathfrak{g}'$, from the second representation theorem. It follows from this that: *The semi-group* $\mathfrak{g}$ *is not completely reducible if and only if the non-zero linear combinations of the matrices of* $\mathfrak{g}$ *consist of only ones that have nothing but zeroes in all of their irreducible diagonal boxes. These linear combinations will define the radical of the linear hull of* $\mathfrak{g}$.

The number of linearly-independent matrices in the semi-group $\mathfrak{g}$ is therefore equal to the sum of the numbers of linearly-independent matrices of its essentially different irreducible components in the completely reducible case, but larger than it in the other case ([152]).

A representation $\mathfrak{D}_\nu$ is called *absolutely irreducible* when it remains irreducible under an extension of the ground field $\mathbb{P}$ to an algebraically closed field Ω. From the first representation theorem (when applied to the ground field Ω), the number of linearly-independent matrices in this case is equal to the square of the degree of the representation (BURNSIDE's theorem). It then follows from this that:

$$(r_\nu\, n_\nu)^2 = r_\nu\, n_\nu^2 \qquad \text{or} \qquad r_\nu = 1.$$

([152]) G. FROBENIUS and I. SCHUR: S.-B. preuss. Akad. Wiss. (1906), 209-217.

The same thing also follows from the fact that, from § 11, 6, the matrices that commute with the representation are scalar multiples of the identity matrix. The argument can be easily inverted, and one finds that:

A representation $\mathfrak{D}_\nu$ is absolutely irreducible if and only if either the number of its linearly-independent matrices is equal to the square of its degree, or if $\Lambda_\nu = \mathbb{P}$, or if all of the matrices that commute with the representation are scalar multiples λI of the identity matrix I.

One understands a *general element* of a hypercomplex system to mean a linear combination of the basis elements with undetermined coefficients. The arbitrariness in the choice of basis is expressed by the fact that an arbitrary linear substitution of the indeterminates is permissible. The *system matrix* of a representation is the representative matrix of the general element. For the calculation of the system matrix, we assume that the coefficient domain is algebraically closed, and that the system is semi-simple (the other cases can be brought back to this case quite easily), and we employ the basis ($c_{jk}^{(\nu)}$) that is given by (1). The general element is then $\sum c_{jk}^{(\nu)} \xi_{jk}^{(\nu)}$, where $\xi_{jk}^{(\nu)}$ are undetermined. If Δ_ν is the determinant $| \xi_{jk}^{(\nu)} |$ then the system determinant of an arbitrary representation that contains the irreducible represent $\mathfrak{D}_\nu$ – say – s_ν times will be equal to:

(2) $$\Delta = \prod_\nu \Delta_\nu^{s_\nu} .$$

In particular, in the case of a *regular system determinant* (viz., the regular representation), one will have $s_\nu = n_\nu$. The Δ_ν are obviously different, irreducible forms in the indeterminates $\xi_{jk}^{(\nu)}$, and that will still be true after a linear substitution of the indeterminates.

For FROBENIUS ([138]), the factor decomposition (2) of the system determinant defined the starting point for the theory of representations.

The theorem of RABINOWITSCH ([153]) follows from the theorems of this and the previous paragraphs:

If $\mathfrak{S}$ is the semi-simple system with unity (or the linear hull of a completely reducible semi-group with unity) of linear transformations of a vector space $\mathfrak{M}$ into itself, and $\mathfrak{T}$ is

([153]) This theorem was made known to me some years ago by verbal communication. Cf., also the somewhat more specialized theorems on commuting sub-rings of simple system of R. BRAUER: J. reine angew. Math. **166** (1932), 245; K. SHODA: Math. Ann. **107** (1932), 252-258, and E. NOETHER: Math. Z. **37** (1933), 514-541. BRAUER and SHODA assumed that the center of $\mathbb{K}$ was complete, while BRAUER and NOETHER assumed that $\mathfrak{S}$ was simple. These assumptions are unnecessary, since one can reduce the semi-simple case to the simple case (which was treated by NOETHER) by a decomposition of $\mathfrak{S}$ into simple systems: $\mathfrak{S} = \sum \mathfrak{S}_\nu$, which involve the decompositions $\mathfrak{M} = \sum \mathfrak{M}_\nu$ and $\mathfrak{T} = \sum \mathfrak{T}_\nu$ ($\mathfrak{M}_\nu = \mathfrak{S}_\nu \mathfrak{M}$; $\mathfrak{T}_\nu$ = automorphism ring of $\mathfrak{M}_\nu$), in which the $\mathfrak{T}_\nu$ are again simple systems, from § 1.

the system of the linear transformations that commute with all transformations of $\mathfrak{S}$ *then, conversely,* $\mathfrak{S}$ *will be the system of linear transformations that commute with all transformations of* $\mathfrak{T}$.

§ 13. Representations of finite groups.

The general theorems that were proved in the foregoing on semi-groups of linear substitutions are naturally true for the representation of groups, in particular. For finite groups, one has the following theorem of MASCHKE:

Any representation of a finite group $\mathfrak{g}$ *in a field* $\mathbb{P}$ *whose characteristic does not divide the order h of the group is completely reducible.*

We already mentioned in § 8 how one can carry out the proof in the case of the field of complex numbers by constructing an invariant, positive, HERMITIAN form. For arbitrary fields, one employs a proof of I. SCHUR that reads (when briefly summarized): On the basis of the lemmas of § 12, one can first assume that the group identity is represented by the identity matrix, and therefore, the inverse group elements s and s^{-1} are also represented by inverse matrices. Now, if:

$$A(s) = \begin{pmatrix} P(s) & 0 \\ Q(s) & R(s) \end{pmatrix}$$

are the matrices of a reducible representation then one will define the matrix:

$$S = \frac{1}{h} \sum_{s \text{ in } \mathfrak{g}} R(s)^{-1} Q(s) .$$

One will then have:

$$\begin{pmatrix} I & 0 \\ S & I \end{pmatrix} \begin{pmatrix} P(t) & 0 \\ Q(t) & R(t) \end{pmatrix} = \begin{pmatrix} P(t) & 0 \\ 0 & R(t) \end{pmatrix} \begin{pmatrix} I & 0 \\ S & I \end{pmatrix},$$

so the matrix system $A(s)$ will be equivalent to a decomposable system.

The problem of representing finite groups can be immediately converted, moreover, into the problem of representing hypercomplex system that was resolved already when one defines the *group ring* $\mathfrak{R}$ (or $\mathfrak{R}_0$) of the group $\mathfrak{g}$; i.e., the hypercomplex system whose basis elements are the elements $s_1, \ldots, s_h$ of $\mathfrak{g}$. Any representation:

$$s \rightarrow A(s)$$

of $\mathfrak{g}$ can obviously be extended to a representation:

$$\sum \lambda_i\, s_i \to \sum \lambda_i\, A(s_i)$$

of $\mathfrak{R}$. Conversely, any representation of $\mathfrak{R}$ is also a representation of $\mathfrak{g}$, since $\mathfrak{g}$ is naturally contained in $\mathfrak{R}$. One naturally chooses the ground field of the hypercomplex system to be the field in which $\mathfrak{g}$ should be represented.

In particular, the regular representation of $\mathfrak{R}$ – under which, $\mathfrak{R}$ is its own representation module – produces a representation of degree h of $\mathfrak{g}$ that one likewise call the *regular representation.* The matrix elements of $A(s)$ are:

$$\alpha_{ik}(s) = \begin{cases} 1 \text{ for } ss_k = s_i, \\ 0 \text{ otherwise,} \end{cases}$$

in this case.

Since, from MASCHKE's theorem, any representation of $\mathfrak{g}$ is completely reducible, the regular representation will also be completely reducible; i.e., $\mathfrak{R}$ will decompose completely into irreducible left ideals: $\mathfrak{R}$ *is semi-simple*, where one always assumes that the characteristic of $\mathbb{K}$ does not divide h. It then follows from the first representation theorem (§ 12) that:

All irreducible representation of $\mathfrak{g}$ *are already contained in the regular one and will be generated by the left ideals of* $\mathfrak{R}$. *If the irreducible representation* $\mathfrak{D}_\nu$ *is contained in the regular one – perhaps n times – then its degree will be* $n_\nu r_\nu$. *The rank of* $\mathfrak{R}$ *is:*

$$h = \sum_{\nu=1}^{s} n_\nu^2\, r_\nu\,,$$

or

$$h = \sum_{\nu=1}^{s} n_\nu^2\,,$$

resp., in the case of absolutely irreducible representations ($r_\nu = 1$).

One obtains another similar relation in the absolutely irreducible case from enumerating the rank of the center of $\mathfrak{R}$ (cf., below, § 15). This rank is, on the one hand, equal to the number s of inequivalent representations, and on the other, equal to the number of classes of conjugate group elements. *Therefore, the number of inequivalent absolutely irreducible representations is equal to the number of classes of conjugate group elements.*

If one extends the ground field $\mathbb{K}$ in such a way that all representations decompose into absolutely irreducible ones then the group ring $\mathfrak{R}$ will become a direct sum of full matrix rings $\mathfrak{S}$ over $\mathbb{K}$ with matrix units $c_{ik}^{(\nu)}$, and one will have an expression:

$$s = \sum_{\nu} \sum_{i,j} \alpha_{ij}^{(\nu)}(s)\, c_{ij}^{(\nu)} \tag{1}$$

for each group element s. From § 12, the $\alpha_{ik}^{(\nu)}(s)$ will be precisely the matrix elements of the representative matrix $A_\nu(s)$ of s in the representation $\mathfrak{T}_\nu$.

For the *Abelian groups*, the absolutely irreducible representations are of degree 2; i.e., the matrices have only one element, which, when regarded as a function of the group element a, is called a *character* $\chi(a)$. The characters of an Abelian group are then functions $\chi(a)$ of the group element a that have the property:

$$\chi(ab) = \chi(a) \cdot \chi(b).$$

Since a finite Abelian group is a direct product of cyclic groups $\mathfrak{C}_1\ \mathfrak{C}_2 \ldots \mathfrak{C}_r$ with the generators $c_1, \ldots, c_r$, and the orders $l_1, \ldots, l_r$, its character can be exhibited effortlessly: One associates each c_r with an arbitrary l^{th} root of unity ζ_ν and sets:

$$\chi(c_1^{\rho_1} c_2^{\rho_2} \cdots c_r^{\rho_r}) = \zeta_1^{\rho_1} \zeta_2^{\rho_2} \cdots \zeta_r^{\rho_r}.$$

The product of two characters is again a character. The characters of a finite, Abelian group define an Abelian group $\mathfrak{C}$ that is isomorphic to the given group. Any subgroup $\mathfrak{h}$ of the given group $\mathfrak{g}$ will be in one-to-one correspondence with a subgroup $\mathfrak{U}$ of the character group, which is characterized by:

$$\chi(a) = 1 \qquad \text{for } a \text{ in } \mathfrak{h},\ \chi \text{ in } \mathfrak{U}.$$

Therefore, $\mathfrak{C} / \mathfrak{U} \cong \mathfrak{h}$ and $\mathfrak{g} / \mathfrak{h} \cong \mathfrak{U}$, because $\mathfrak{C} / \mathfrak{U}$ is the character group of $\mathfrak{h}$, and $\mathfrak{U}$ is that of $\mathfrak{g} / \mathfrak{h}$.

In a precisely corresponding way, one can also characterize every normal subgroup $\mathfrak{h}$ of a finite group $\mathfrak{g}$ by the fact that the elements of $\mathfrak{h}$ will correspond to the identity matrix for some completely-determined representation of $\mathfrak{g}$, or – what amounts to the same thing in the case of field of characteristic zero – the fact that the traces of the representative matrices of the elements of $\mathfrak{h}$ are equal to the degree of the representation (or the trace of the identity matrix). Certain applications of the theory of representations rest upon this method, for which one is concerned with inferring the existence of normal subgroups from the properties of the group character ([154]).

If $s \to A(s)$ and $s \to B(s)$ are two absolutely irreducible representation of a finite group $\mathfrak{g}$ and C is an entirely arbitrary matrix then:

$$P = \sum_{t \text{ in } \mathfrak{g}} A(t)\, C\, B(t^{-1})$$

will be a matrix with the property:

$$A(s)\, P = P\, B(s).$$

It follows from the SCHUR lemma that $P = 0$ when the representations $A(s)$ and $B(s)$ are inequivalent; however, if they are equal then $P = \lambda I$, from § 11.6. If one writes the matrix equations $P = 0$ ($P = \lambda I$, resp.) and observes that the matrix element of C is completely arbitrary then it will follow that:

$$\sum_t \alpha_{ij}(t)\beta_{kl}(t^{-1}) = \begin{cases} 0 & \text{when } A(s) \text{ is not equiv. to } B(s), \\ \omega_{jk}\delta_{il} & \text{for } A(s) = B(s). \end{cases}$$

Since the left-hand side admits the permutation $(ik)(jl)$ for $\alpha_{ij} = \beta_{ij}$, ω_{ij} can only be equal to $\omega\delta_{ij}$. Therefore, we can also write our relation as:

$$\text{(2)} \qquad \sum_t \alpha_{ij}^{(\nu)}(t)\alpha_{kl}^{(\mu)}(t^{-1}) = \begin{cases} \omega & \text{for } \nu = \mu, i = l, j = k, \\ 0 & \text{otherwise.} \end{cases}$$

If one sets $j = k$ and sums over k then if h is the order of the group and n_ν is the degree of the representation then that will yield:

$$h \cdot 1 = n_\nu\, \omega.$$

If h is not divisible by the characteristic of the field then it can also not be n_ν, and we will obtain:

$$\omega = \frac{h}{n_\nu} 1.$$

When one multiplies (2) by $\alpha_{ik}^{(\nu)}(s)$ and sums over i, what will follow is the general relation:

$$\text{(3)} \qquad \sum_t \alpha_{hj}^{(\nu)}(st)\, \alpha_{kl}^{(\mu)}(t^{-1}) = \begin{cases} \omega\alpha_{hl}^{(\nu)} & \text{for } \mu = \nu,\ j = k, \\ 0 & \text{otherwise.} \end{cases}$$

([154]) See, perhaps, A. SPEISER: *Theorie der Gruppen von endlicher Ordnung*, 2nd ed., Berlin, 1927, chap. 13.

The relation (2) can then be employed to solve (1) for $c_{ik}^{(\nu)}$. If one multiplies (1) $\frac{1}{h} n_\nu$ $\alpha_{kl}^{(\nu)}(s^{-1})$ and sums over s then one will get:

$$c_{ik}^{(\nu)} = \frac{n_\nu}{h} \sum_s \alpha_{kl}^{(\nu)}(s^{-1})\, s \,. \tag{4}$$

§ 14. Restricted representations of arbitrary groups.

According to J. VON NEUMANN ([155]), the theory of representations of finite groups that was developed in § 13 can be adapted to restricted representations of arbitrary groups $\mathfrak{G}$ in the field of complex numbers; i.e., representations for which matrix elements $d_{ik}(x)$ of the representative matrix $D(x)$ of the group element x are uniformly restricted complex-valued functions of x.

In place of the group ring that was employed in § 13, one generally finds the *ring of almost-periodic functions on* $\mathfrak{G}$. A complex-valued function $f(x)$ that is defined for all x in $\mathfrak{G}$ is called *almost-periodic* (a. p.) on $\mathfrak{G}$ when one can select a uniformly-convergent subsequence from any sequence of functions $f(a_\nu\, x\, b_\nu)$ ([156]).

A *mean value* of an a. p. function $f(x)$ will be defined as a constant A that (when regarded as a constant function on $\mathfrak{G}$) can be uniformly approximated by functions of the form:

$$c_1 f(a_1\, x\, b_1) + c_2 f(a_2\, x, b_2) + \ldots + c_n f(a_n\, x\, b_n),$$

with $c_1 + \ldots + c_n = 1$. One proves that there is one and only one mean value that is a function of $f(x)$ ([155]). We denote it by Mf, or $M_x f(x)$, when the variable x should be interpreted in relation to the one whose mean value is being defined.

The matrix elements $d_{ik}(x)$ of a restricted representation are a. p. functions of x, so the $d_{ik}(a_\nu\, x b_\nu)$ are linear combinations of the finitely-many functions $d_{jl}(x)$ with restricted coefficients:

$$d_{ik}(a_\nu\, x b_\nu) = \sum_j \sum_l d_{ij}(a_\nu)\, d_{jl}(x)\, d_{lk}(b_\nu)\,.$$

With the help of the definition of mean value that was described above, one proves precisely, as in § 8, that every restricted representation of $\mathfrak{G}$ leaves a positive HERMITIAN form invariant, and is thus equivalent to a unitary one. From this, or the direct proof of SCHUR (§ 13), it then follows further that every reducible, restricted representation is completely reducible.

Ultimately, as is § 13, one proves the relations:

([155]) J. VON NEUMANN: Trans. Amer. Math. Soc. **36** (1934), 445-492.

([156]) According to BOCHNER: Math. Ann. **96** (1927), 119-147, this definition is equivalent to BOHR's original definition in the case where $\mathfrak{G}$ is the additive group of real numbers and $f(x)$ is a continuous function. On that, see BOHR: Ergeb. Math. I, 5 (1932).

$$(1)\qquad \begin{cases} M_y d_{ij}(xy^{-1}) d_{kl}(y) = \dfrac{1}{n}\delta_{jk} d_{il}(x), \\ M_y d_{ij}(xy^{-1}) d'_{kl}(y) = 0, \qquad \text{for } \mathfrak{D} \text{ inequivalent to } \mathfrak{D}', \end{cases}$$

in which $d_{ij}(x)$ means the matrix element of an irreducible representation $\mathfrak{D}$ of degree n, and d'_{kl} means that of another irreducible representation $\mathfrak{D}'$.

The *product* $f \times g$ of two a. p. functions will be defined by:

$$f \times g(x) = M_y(f(x\,y^{-1})\,g(y)) = M_y(f(y)\,g(y^{-1}\,x)).$$

It defines an analogy with the product of two elements $\frac{1}{h}\cdot\sum f(y)y$ and $\frac{1}{h}\cdot\sum g(z)z$ of the group rings $\mathfrak{R}_{\mathfrak{g}}$ in § 13, which will, in fact, be defined by:

$$\left(\frac{1}{h}\cdot\sum f(y)y\right)\left(\frac{1}{h}\cdot\sum g(z)z\right) = \frac{1}{h}\sum_x\left\{\frac{1}{h}\sum_{yz=x} f(y)g(z)\right\}x$$
$$= \frac{1}{h}\sum_x\left\{\frac{1}{h}\sum_y f(y)g(y^{-1}x)\right\}x.$$

The product is associative and distributive over ordinary addition $f(x) + g(x)$, so the a. p. functions define a ring under this multiplication and addition that we will denote by $\mathfrak{R}_{\mathfrak{G}}$.

With the help of the product sign, one can also write (1) as:

$$(2)\qquad \begin{cases} d_{ij} \times d_{kl}(x) = \dfrac{1}{n}\delta_{jk} d_{il}(x), \\ d_{ij} \times d'_{kl}(x) = 0. \end{cases}$$

These relations state that the functions:

$$(3)\qquad c_{ij}(x) = n\, d_{ij}(x)$$

fulfill the equations exactly that are characteristic of the matrix units of a full matrix ring over the field $\mathbb{K}$ (cf., § 12). Therefore, any irreducible representation $\mathfrak{D}_\nu$ belongs to a full matrix ring $\mathfrak{S}_\nu$ in $\mathfrak{R}_{\mathfrak{G}}$, whereby the same matrix ring belongs to equivalent $\mathfrak{D}$, and therefore, from (2), two different $\mathfrak{S}_\nu$ will mutually annihilate each other.

The representations $\mathfrak{D}$ can also be generated by representation modules that can be chosen to be right ideals in $\mathfrak{R}_{\mathfrak{G}}$. To that end, we define the multiplication of a function $f(x)$ times a group element a by way of ([157]):

$$a\,f(x) = f(x\,a). \tag{4}$$

One then has the rules:

$$\begin{aligned} a(f+g) &= af + ag, \\ a(f \times g) &= f \times ag, \\ a(bf) &= (ab)\,f. \end{aligned}$$

The ring $\mathfrak{R}_{\mathfrak{G}}$ is thus a $\mathfrak{G}$-module. If a sub-module $\mathfrak{m} = (g_1, \ldots, g_n)$ of finite rank admits multiplication by the group elements then it will mediate a representation $y \to d_{ik}(y)$ by means of:

$$y \cdot g_k(y) = g_k(xy) = \sum_i g_i(x)\, d_{ik}(y). \tag{5}$$

Such a module $\mathfrak{m}$ is simultaneously also a right ideal in $\mathfrak{R}_{\mathfrak{G}}$, due to the fact that:

$$g_k \times f(x) = M_y\, g_k(xy^{-1})\, f(y) = M_y \sum g_i(x)\, d_{ik}(y^{-1})\, f(y) = \sum g_i(x) \cdot \beta_{ik},$$

with:

$$\beta_{ik} = M_y\, (d_{ik}(y^{-1})\, f(y)).$$

If one decomposes $\mathfrak{m}$ into irreducible representation modules $\mathfrak{m}_\nu$, for which $\mathfrak{m}_\nu$ gets the representation $\mathfrak{D}_\nu$, say, then $\mathfrak{m}_\nu$ will be contained in the ring $\mathfrak{S}_\nu$, and it will follow from (5) for $x = 1$, on account of (3), that:

$$g_k(y) = \sum g_i(1)\, d_{ik}(y) = \sum \frac{g_1(1)}{n}\, c_{ik}(y).$$

Conversely, a minimal right ideal of the ring $\mathfrak{S}_\nu$ – e.g., the ideal $\mathfrak{r}_\nu = (c_{11}, c_{12}, \ldots, c_{1n})$ – will mediate the representation $\mathfrak{D}_\nu$ precisely, as one easily confirms.

The irreducible representations of $\mathfrak{R}_{\mathfrak{G}}$ will then be mediated by the minimal right ideals of the ring $\mathfrak{S}_\nu$, corresponding to § 14, precisely.

The ring $\mathfrak{S}_\nu$ itself is also a $\mathfrak{G}$-module, and thus, a right, but likewise also a left, ideal in $\mathfrak{R}_{\mathfrak{G}}$.

([157]) In order to maintain the analogy with formulas (4) of § 13, we actually must write $c_{ij}(x) = n\, d_{ij}(x^{-1})$, instead of (3) and $a\,f(x) = f(a^{-1}\,x)$, instead of (4). The formulas will then become simpler when one does things as above.

The capstone of the theory is defined by the proof of the completeness of the systems of functions d_{ik} or c_{ik} . Consequently, from here on, that will be understood.

If one defines the *scalar product* of two functions f, g in $\mathfrak{R}_{\mathfrak{G}}$ by:

$$(f, g) = M_y\, f(y)\,\overline{g(y)} = g^\dagger \times f(1) = f \times g^\dagger(1), \qquad \text{where } g^\dagger(x) = \overline{g(x^{-1})},$$

and the *norm*, or *length*, by:

$$N(f) = \sqrt{(f, f)} = \sqrt{M_y\,|\,f(y)|^2}$$

then $\mathfrak{R}_{\mathfrak{G}}$ will become a generalized HILBERT space ([158]), into which one can introduce a topology on the basis of the definition of the distance by $N(f - g)$. A system of functions $f_1, f_2, \ldots$ is now called *complete* when the linear combinations $\gamma_1 f_1 + \gamma_2 f_2 + \ldots + \gamma_r f_r$ are everywhere dense in $\mathfrak{R}_{\mathfrak{G}}$ – i.e., any a. p. function f comes arbitrarily close:

$$N(\gamma_1 f_1 + \gamma_2 f_2 + \ldots + \gamma_r f_r - f) < \varepsilon$$

for any $\varepsilon > 0$ with suitable γ_i .

In order to prove the completeness of the $c_{ik}(x)$, J. v. NEUMANN, following the example of PETER and WEYL ([159]), considered the "integral equation":

$$f \times f^\dagger \times \psi = \gamma\,\psi.$$

Here, we shall give an altered proof that follows G. KÖTHE ([160]) by employing the theory of integral equations as little as possible, while reaching a more algebraic conclusion.

The functions $c_{ik}^{(\nu)}$ generate the full matrix ring $\mathfrak{S}_\nu$, whose unity element is:

$$e_\nu = \sum c_{ik}^{(\nu)} .$$

From the unitarity of the representation $\mathfrak{D}_\nu$ it easily follows that $e_\nu^\dagger = e_\nu$.

Any a. p. function f can now be decomposed into a component in $\mathfrak{S}_\nu$ and one that it orthogonal to it:

$$f = f \times e_\nu + (f - f \times e_\nu).$$

One easily convinces oneself that the scalar product of $f - f \times e_\nu$ with an element $g \times e_\nu$ of $\mathfrak{S}_\nu$ is, in fact, zero ([161]):

$$(f - f \times e_\nu ,\, g \times e_\nu) = (f - f \times e_\nu) \times e_\nu^\dagger \times g^\dagger(1)$$

([158]) F. RELLICH: Math. Ann. **110** (1934), 342-356.
([159]) F. PETER and H. WEYL: Math. Ann. **97** (1927), 737-755.
([160]) G. KÖTHE: Math. Ann. **103** (1930), 545-572.
([161]) The rule of computation $(g \times h)^\dagger = h^\dagger \times g^\dagger$ is employed in this.

$$= f \times e_\nu \times g^\dagger(1) - f \times e_\nu \times e_\nu \times g^\dagger(1) = 0.$$

The component of f in $\mathfrak{S}_\nu$ can also be written in the form $e_\nu \times f$. Namely, since $e_\nu \times f$ and $f \times e_\nu$ both belong to $\mathfrak{S}_\nu$, and since e_ν is the unity element of $\mathfrak{S}_\nu$, one will have:

$$f \times e_\nu = e_\nu \times f \times e_\nu = e_\nu \times f .$$

The components of f in different $\mathfrak{S}_\nu$ are mutually orthogonal. We define the sum:

$$f_\mu = f \times e_1 + f \times e_2 + \ldots + f \times e_\mu .$$

$f - f_\mu$ is then orthogonal to f_μ, so:

$$(6) \qquad \left\{ \begin{aligned} N(f) &= N(f_\mu) + N(f - f_\mu) \\ &\geq N(f_\mu) = N(f \times e_1) + N(f \times e_2) + \cdots + N(f \times e_\mu). \end{aligned} \right.$$

It follows from (6) that at most a restricted number of norms can satisfy $N(f \times e_\nu) \geq \frac{Nf}{n}$. When one sets $n = 1, 2, 3, \ldots$, in succession it will follow that the e_ν with $f \times e_\nu \neq 0$ – i.e., the ones with $N(f \times e_\nu) > 0$ – can be put into a denumerable sequence. We call it $e_1, e_2, e_3, \ldots$ It now follows from (6) that the series:

$$(7) \qquad N(f \times e_1) + N(f \times e_2) + \ldots$$

converges with a sum $\leq N f$ (BESSEL's inequality). The completeness of the system of functions $c_{ik}^{(\nu)}$ will be proved when we can show that $\lim_{\nu \to \infty} N(f_\nu - f) = 0$.

It likewise follows from the convergence of the series $N(f \times e_1) + N(f \times e_2) + \ldots$ that the sequence of f_ν fulfills the CAUCHY *convergence condition:*

$$N(f_\nu - f_\mu) = N(f \times e_{\mu+1} + \ldots + f \times e_\nu) < e \qquad \text{for} \qquad \nu > \mu > n(\varepsilon).$$

We now prove a *lemma:*

If the sequences of a. p. functions f_ν and g_ν both fulfill the CAUCHY *convergence condition then $h_\nu(x) = f_\nu \times g_\nu(x)$ will converge uniformly to an a. p. function $h(x)$.*

Proof. Let the upper bound of $N(f_\nu)$ and $N(g_\nu)$ be M. One will then have:

$$\begin{aligned} | f_\nu \times g_\nu(x) - f_\mu \times g_\mu(x) | &\leq | (f_\nu - f_\mu) \times g_\nu(x) | + | f_\mu \times (g_\nu - g_\mu)(x) | \\ &\leq N(f_\nu - f_\mu) \cdot N g_\nu + M f_\mu \cdot N(g_\nu - g_\mu) < 2 M \varepsilon \end{aligned}$$

for $\nu > \mu > n(\varepsilon)$. The sequence of $h_\nu(x)$ thus converges uniformly to a limit function $h(x)$. However, a uniform limit of a. p. functions is again an a. p. function, which follows immediately from the definition of a. p. functions.

We apply the lemma to the sequence of functions $f_\nu - f$ and the "adjoint" sequence $f_\nu^\dagger - f^\dagger$, so we set:

$$h_\nu = (f_\nu - f) \times (f_\nu^\dagger - f^\dagger).$$

One then has $h_\nu^\dagger = h_\nu$ ([161]), so one likewise has $h^\dagger = h$ for the limit function $h(x)$. Furthermore, h is also orthogonal to all $\mathfrak{S}_\mu$ with $\mu \leq \nu$:

$$e_\mu \times h_\nu = e_\mu \times (f_\nu - f) \times (f_\nu^\dagger - f^\dagger) = 0,$$

and the same thing is true for all e_μ that do not appear in the sequence $e_1, e_2, \ldots$ Therefore, h is also orthogonal to all $\mathfrak{S}_\mu$:

$$e_\mu \times h = 0. \tag{8}$$

Finally, $h_\nu(1) = N(f_\nu - f)$, so $h(1) = \lim N(f_\nu - f)$. If $N(f_\nu - f)$ did not tend to zero then $h(x)$ would also be an a. p. function that is non-zero and orthogonal to all $\mathfrak{S}_\nu$. We will show that this is impossible.

To that end, we consider the eigenvalue problem:

$$h \times \psi = \lambda\, \psi. \tag{9}$$

With the methods of E. SCHMIDT's theory of integral equations, one can prove, as WEYL and PETER ([159]), as well as v. NEUMANN ([155]), did more rigorously, that there is at least one non-zero eigenvalue and an associated eigenfunction. The same thing also follows from the general theory of completely continuous linear operators in a general HILBERT space ([158]). As is shown in the same theory, the eigenfunctions ψ that are associated with the eigenvalue λ define a module $\mathfrak{m}$ of finite rank that admits multiplication by the elements y of $\mathfrak{G}$; it then follows from $h \times \psi = \lambda\, \psi$ that:

$$h \times y\psi = y(h \times y) = y\lambda\, \psi = \lambda \cdot y\psi.$$

From our theorems, the module $\mathfrak{m}$ contains an irreducible sub-module $\mathfrak{m}_\nu$ that is contained in a ring $\mathfrak{S}_\nu$. That is, an element ψ_ν appears among the eigenfunctions ψ such that:

$$h \times \psi_\nu = \lambda\psi_\nu\,; \quad \lambda \neq 0, \quad \psi_\nu \text{ in } \mathfrak{S}_\nu. \tag{10}$$

The unity element e_ν of $\mathfrak{S}_\nu$ annuls the left-hand side of (10), due to (8), but it does not annul the right-hand side. That is impossible. Therefore, $N(f_\nu - f)$ tends to zero.

Using completeness, v. NEUMANN ([155]) has proved that a. p. functions can be uniformly approximated by linear combinations of the $d_{ik}(x)$ with a method of N. WIENER.

All of the theorems and proofs above remain true verbatim when one restricts oneself to a *topological group* ([162]) with *continuous a. p. functions* and a continuous representation.

There are topological groups for which all restricted representations, and thus also all a. p. functions, are continuous ([163]), namely, the semi-simple, continuous groups, whose representation theory has been developed by CARTAN and WEYL ([164]). There are even groups that possess no restricted representations besides the identity representation and on which the only a. p. are therefore the constants. The real projective group $PSL(n, \mathbb{P})$ belongs to them. However, there are also groups on which all continuous functions are a. p. Clearly, that is the case for compact, topological groups. The completeness theorem is even true for all continuous function for these groups. We refer to v. NEUMANN ([155]) for a thorough investigation of these different possibilities.

The absolutely irreducible restricted representations of *Abelian groups* $\mathfrak{G}$ are given by one-rowed matrices, and thus by complex numbers of modulus one; these are again called *characters* $\chi(a)$. Following PALEY and WIENER ([165]) or ALEXANDER ([165a]), one obtains them when one totally orders the generators of $\mathfrak{G}$ and determines the value of a character χ for each generator a in such a way that it follows from:

$$a^h = \prod_\nu a_\nu^{h_\nu} ,$$

where the a_ν run through the a in the total ordering, that:

$$\chi(a)^h = \prod_\nu \chi(a_\nu)^{h_\nu} .$$

Another method, which was given by A. HAAR ([166]) for denumerable Abelian groups and was extended to separable, compact-in-the-small, Abelian, topological groups by v. NEUMANN ([155]), generally does not yield all characters, but only a family of characters $\varphi(a, \lambda)$ that are BAIRE functions of a real parameter λ that are also continuous functions of a in the topological case that have the property that $\lim \varphi(a_\nu, \lambda) = 1$ implies that $\lim a_\nu = 1$ for all λ. If $\mathfrak{G}$ is denumerable then the $\chi(a, \lambda)$, as functions of λ, define a complete orthogonal system relative to a monotone regulating function (Ger: *Belegunsfunktion*) ([167]).

([162]) For this concept, see F. LEJA: Fundam. Math. **9** (1927), 37-44. – R. BAER: J. reine angew. Math. **160** (1929), 208-226. – D. VAN DANTZIG: “Studien over topologische algebra,” Diss. Groningen 1931.

([163]) B. L. VAN DER WAERDEN: Math. Z. **36** (1933), 780-786.

([164]) See footnotes 73 and 74 in I, § 8. See also footnote 159.

([165]) N. WIENER and R. E. A. C. PALEY: Proc. Nat. Acad. Sci. U. S. A **19** (1933), 253-257.

([165a]) J. F. ALEXANDER: Ann. of Math., II. s. **35** (1934), 389-395.

([166]) A. HAAR: Math. Z. **33** (1931), 129-159.

([167]) L. PONTRJAGIN: Ann. of Math., II. s. **35** (1935), 361-388.

The *continuous characters* of an *Abelian, topological group* again define an Abelian, topological group Γ when products and limits in Γ are defined by:

$$\begin{cases} \psi \cdot \chi(a) = \psi(a) \cdot \chi(a), \\ \lim \chi_\nu = \chi, \text{ when } \lim \chi_\nu(a) \text{ for all } a. \end{cases}$$

If $\mathfrak{G}$ is discrete and denumerable then Γ will clearly be compact; on the other hand, if $\mathfrak{G}$ is compact then Γ will discrete and denumerable ([167]). If $\mathfrak{G}$ is compact in the small and separable then Γ will also be so ([167a]). In all of these cases, the groups $\mathfrak{G}$ and Γ define a *group pair*, in the sense of L. PONTRJAGIN ([167]); i.e., a *product* $\chi \cdot a = \chi(a)$ is defined for every a in $\mathfrak{G}$ and χ in Γ that is a real number with absolute value one that depends continuously upon χ and a individually and possesses the distributive properties:

$$\chi a \cdot \chi b = \chi \cdot ab; \qquad \psi\, a \cdot \chi a = \psi\chi \cdot a.$$

In addition, the group pair is *orthogonal;* i.e., if $\chi a = 1$ for some χ and all a then it will follow that $\chi = 1$, and if $\chi a =$ for some a and all χ then it will follow that $a = 1$.

According to PONTRJAGIN ([167]), in the case where $\mathfrak{G}$ is discrete and denumerable – and thus compact – any subgroup $\mathfrak{H}$ of $\mathfrak{G}$ will be in one-to-one correspondence with a closed subgroup Φ of Γ, such that Φ will consist of the χ with $\chi a = 1$ for all a in $\mathfrak{H}$, and conversely $\mathfrak{H}$ will consist of the a with $\chi a = 1$ for all a in Φ. One has, moreover: *If* $\mathfrak{G}$ *and* Γ *define an orthogonal group pair and if* $\mathfrak{G}$ *is denumerable and* Γ *is compact then* Γ *will be the character group of* $\mathfrak{G}$ *and* $\mathfrak{G}$ *will be the group of continuous characters of* Γ. It follows from this that: *If* Γ *is the character group of* $\mathfrak{G}$ *then* $\mathfrak{G}$ *will be the group of continuous characters of* Γ, *and conversely.*

E. R. VAN KAMPEN ([167a]) has adapted these theorems to pairs of compact-in-the-small, separable, Abelian groups.

§ 15. Traces and characters.

1. Definition and general properties.

If a representation $\mathfrak{D}$ of a semigroup $\mathfrak{g}$ is given then we will consider the trace of the representative matrix A of an element a to be a function of a and denote it by $S_{\mathfrak{D}}(a)$ or by $S(a)$. In particular, if $\mathfrak{g}$ is a group then $S(b^{-1}a\, b) = S(a)$. The trace then depends upon only the class of the group element a.

([167a]) E. R. VAN KAMPEN: Proc. Nat. Acad. Sci. U. S. A. **20** (1934), 434-436. A further paper by the same author in which the theory of characters will be developed systematically will appear in Ann. of Math. **36** (1935).

The trace of a matrix of a reducible system is the sum of the traces of the irreducible components. Thus, if the irreducible representations $\mathfrak{D}_\nu$ enter into a representation of a semigroup k_ν times as diagonal components then:

(1) $$S_{\mathfrak{D}}(a) = \sum k_\nu \, S_{\mathfrak{D}_\nu}(a) \, .$$

If $\mathfrak{g}$ is a hypercomplex system then the traces of the elements of the radical will always be zero, since they will be represented by zeroes in all irreducible representations. The trace in the regular representation is called the *regular trace.*

The trace of an element s of a finite group in the regular representation is zero for $s \neq 1$ and equal to the order h of the group for $s = 1$, as one infers immediately from the formula for the matrix elements of the regular representation (§ 13).

In fields with characteristic zero, one has the theorem:

Two completely reducible representations $\mathfrak{D}$, $\mathfrak{D}'$ of a semigroup $\mathfrak{g}$ are equivalent only if their traces coincide ([152]).

Without the assumption of characteristic zero, the cited theorem is not true, in general, but it is true for two irreducible representations. The traces of the absolutely irreducible representations of a semigroup are called *characters*, and will be denoted by $\chi(a)$ or $\chi_\nu(a)$. Since one can absolutely reduce any representation by going to an algebraically closed field, *any trace will be a sum of characters* ([168]). The trace of an individual group element is likewise the sum of characters of a cyclic group, so in the case of a finite group, it will be a sum of roots of unity.

2. The KRONECKER product representation.

If two representations of a semigroup $\mathfrak{g}$ by linear transformations of the vector spaces $(u_1, \ldots, u_m)$ and $(v_1, \ldots, v_n)$ are given then one can regard the basis vectors u_i, v_j as indeterminates and define the $m \cdot n$ products $u_i \, v_j$; these will likewise be linearly transformed by the group $\mathfrak{g}$. If $A = (a_{ik})$ and $B = (b_{jl})$ are the representative matrices of a group element s in the two given representations then $\gamma_{ij,kl} = \alpha_{ik} \, \beta_{jl}$ (i, j are row and k, l are column indices) is the matrix by which the $u_i \, v_j$ will be transformed; one calls it the KRONECKER *product matrix* $A \times B$. The product transformations again define a representation of $\mathfrak{g}$: viz., the *product representation*. The trace of the product matrix is equal to the product of the traces of the matrices A and B.

If one denotes the irreducible representations of a semigroup by $\mathfrak{D}_1$, $\mathfrak{D}_2$, …, and one assumes that the product representation $\mathfrak{D}_\lambda \times \mathfrak{D}_\mu$ includes the irreducible component $\mathfrak{D}_\nu$ – say – $c^{\nu}_{\lambda\mu}$ times then one can write:

([168]) One also refers to an arbitrary integer linear combination of characters – and in particular, the trace of an arbitrary representation – as a *composite character.*

$$\mathfrak{D}_\lambda \times \mathfrak{D}_\mu = \sum_\nu c^\nu_{\lambda\mu} \mathfrak{D}_\nu .$$

For the characters χ_ν of the representations $\mathfrak{D}_\nu$, it follows from this that:

(2) $$\chi_\lambda(s)\, \chi_\mu(s) = \sum_\nu c^\nu_{\lambda\mu}\, \chi_\nu(s) .$$

3. The system discriminant and complementary bases.

A sufficient condition for the semi-simplicity of a hypercomplex system is the non-vanishing of the regular trace determinant – or *regular discriminant:*

(3) $$D = |\, S(u_\mu\, v_\nu)\, |$$

that is defined by any two bases $(u_1, \ldots, u_n)$ and $(v_1, \ldots, v_n)$ of the system, or – what amounts to the same thing – the existence of a *complementary basis* $(w_1, \ldots, w_n)$ to any basis $(u_1, \ldots, u_n)$, which has the property:

$$S(u_\mu\, w_\nu) = \delta_{\mu\nu} \qquad (= 0 \text{ or } 1).$$

The fact that one always has $D = 0$ for a system with a radical will become clear when one chooses u_1 in the radical, since all $S(u_1\, w_\nu) = 0$ then.

In the case of the group ring of a group whose order h is not divisible by the characteristic, one will always have $D \neq 0$, so the elements $\frac{1}{h} s^{-1}$ will define a complementary basis to the basis of the group element s:

(4) $$S(st) = \begin{cases} h \cdot 1 & \text{for } t = s^{-1}, \\ 0 & \text{for } t \neq s^{-1}. \end{cases}$$

The semi-simplicity of the group ring will follow from this once more.

If one expresses the trace in (4) in terms of the characters then one will obtain:

$$\sum n_\nu\, \chi_\nu(st) = \begin{cases} h \cdot 1 & \text{for } t = s^{-1}, \\ 0 & \text{for } t \neq s^{-1}, \end{cases}$$

or, when one introduces the matrices $(\alpha_{jk}^{(\nu)})$ of the absolutely irreducible representations:

(5) $$\sum_\nu \sum_{j,k} n_\nu\, \alpha_{jk}^{(\nu)}(s)\, \alpha_{kj}^{(\nu)}(t) = \begin{cases} h \cdot 1 & \text{for } (t = s^{-1}), \\ 0 & \text{for } (t \neq s^{-1}). \end{cases}$$

The basis that is complementary to the basis $(c_{jk}^{(\nu)})$ is $(n_\nu^{-1}\, c_{jk}^{(\nu)})$, as one easily verifies.

4. The relations between the characters.

When one sets $h = j$, $k = l$ and sums over j and l, it will follow from equation (3), § 13 that:

(6) $$\sum_t \chi_\nu(st)\,\chi_\mu(t^{-1}) = \begin{cases} \omega\chi_\nu(s) & \text{for } \nu = \mu, \\ 0 & \text{for } \nu \neq \mu. \end{cases}$$

In particular, for $s = 1$, one will obtain the *orthogonality relations of the character:*

(7) $$\sum_t \chi_\nu(t)\,\chi_\mu(t^{-1}) = \begin{cases} h\cdot 1 & (\nu = \mu), \\ 0 & (\nu \neq \mu). \end{cases}$$

The orthogonality relations can thus be employed to make the decomposition of a given representation $\mathfrak{D}$ into absolutely irreducible ones possible by mere trace calculations. Namely, if:

(8) $$\mathfrak{D} = \sum_\nu c_\nu \mathfrak{D}_\nu\,, \qquad \text{so} \qquad S_{\mathfrak{D}}(s) = \sum_\nu c_\nu \chi_\nu(s)\,,$$

then it will follow from (7) that:

$$\sum_s \chi_\nu(s^{-1})\,S_{\mathfrak{D}}(s) = h\,c_\nu \cdot 1.$$

One determines the numbers c_ν from this (in the case of characteristic zero).

$$\sum_s S_{\mathfrak{D}}(s)\,S_{\mathfrak{D}}(s^{-1}) = h \cdot \sum c_\nu{}^2 \cdot 1,$$

so:

A representation $\mathfrak{D}$ in a field of characteristic zero is absolutely irreducible if and only if one has:

$$\sum_s S_{\mathfrak{D}}(s)\,S_{\mathfrak{D}}(s^{-1}) = h \cdot 1$$

for its trace.

The trace relation (8) often finds applications in the theory of invariants when one must determine the number of linearly-independent vectors that remain invariant under a representation $\mathfrak{D}$ of a group $\mathfrak{g}$. Namely, this number is obviously equal to the coefficient c_1 of the identity representation $\mathfrak{D}_1$ in the decomposition (8).

We now assume that the ground field is a number field. The characters χ, as sums of roots of unity, are algebraic numbers then, and indeed $\chi(s^{-1})$ is complex conjugate to $\chi(s)$. It now follows from (6) (with $\mu = \nu$) in a known way that ω, as the root of the secular equation:

$$|\chi(st^{-1}) - \delta_{st}\,\omega| = 0,$$

is likewise a whole algebraic number; it then follows that: *The degree of an absolutely reducible representation is a divisor of the order of the group*.

8. The center of the group ring.

The center of a semi-simple hypercomplex system $\mathfrak{S}$ over an algebraically closed field consists of the elements z that commute with all elements of $\mathfrak{S}$, so they will be represented by a multiple of the identity matrix in any absolutely irreducible representation. The formula:

$$r = \sum_{\nu}\sum_{j,k}\alpha_{jk}{}^{(\nu)}(r)\,c_{jk}{}^{(\nu)},$$

which is true for any element r of $\mathfrak{S}$, will then reduce to:

$$z = \sum_{\nu}\alpha_{\nu}(z)\sum_{k}c_{kk}{}^{(\nu)} = \sum_{\nu}\alpha_{\nu}(z)\,I_{\nu}\,, \tag{9}$$

for $r = z$.

The $I_\nu = \sum_k c_{kk}{}^{(\nu)}$ are idempotent elements of the center, namely, the unity elements of the full matrix rings $\mathfrak{S}_\nu$ into which $\mathfrak{S}$ decomposes. The $\alpha_\nu(z)$ are the irreducible representations (of degree 1) of the center. Obviously, the relation:

$$\chi_\nu(z) = n_\nu\,\alpha_\nu(z) \tag{10}$$

exists between the characters $\chi_\nu(z) = \sum_k \alpha_{kk}{}^{(\nu)}$ and the $\alpha_\nu(z)$.

In the case of the group ring, $z = \sum \lambda_s s$ belongs to the center if and only if $tzt^{-1} = z$ for any group element t, and that comes down to saying that all of the elements tst^{-1} that are conjugate to an s have the same coefficients λ_s. If one then sets k_s equal to the sum of all *different* elements tst^{-1} of the class of s then the k_s will generate the center, and relation (9) will become:

$$k_s = \sum \alpha_\nu(k_s)\,I_\nu = \sum \frac{\chi_\nu(k_s)}{n_\nu}I_\nu = \sum_{\nu}\frac{h_s}{n_\nu}\chi_\nu(s)I_\nu\,, \tag{11}$$

in which h_s is the number of elements in the class of s.

The solution of this formula for I_ν is obtained from (4), § 13, when one sets $l = k$ in it and sums over k:

$$I_\nu = \frac{n_\nu}{h}\sum_{s}\chi_\nu(s^{-1})\,s = \frac{n_\nu}{h}\sum_{s}{}'\chi_\nu(s^{-1})\,k_s\,. \tag{12}$$

In the last summation $\sum'$, s runs through a system of representatives for all classes of the group. A comparison of (11) with (12) yields that the matrices $(\chi_\nu(s))$ and $\left(\frac{h_s}{h}\chi_\nu(s^{-1})\right)$, in which the row (column, resp.) index s runs through a system of representatives of all classes, are inverse to each other. The orthogonality relations (11) say the same thing. One can also express this by the formula:

$$\sum_\nu \chi_\nu(s)\chi_\nu(t) = \begin{cases} 0 & \text{for } t \neq s^{-1}, \\ h/h_s & \text{for } t = s^{-1}. \end{cases} \tag{13}$$

The product $k_s\, k_t$ of two generators of the center is again an element of the center, and thus, a (integer) linear combination of the generators k_r :

$$k_s\, k_t = \sum g_{st}^{\ r} k_r .$$

The fact that the functions $\alpha_\nu(z)$ define a representation of the center is expressed by the formula:

$$\alpha_\nu(k_s)\ \alpha_\nu(k_s) = \sum g_{st}^{\ r}\alpha_\nu(k_r),$$

which is converted into:

$$h_s\, h_t\chi_\nu(s)\ \chi_\nu(t) = n_\nu \sum g_{st}^{\ r}\, h_r\, \chi_\nu(r) \tag{14}$$

(i.e., summation over a system of representatives of the classes) after multiplying by n_ν^2, due to (10).

G. FROBENIUS ([169]) first defined the character $\chi(s)$ by formula (14). A. HAAR ([170]) gave another basis for the theory of characters that is independent of the theory of representations, and which is also valid for certain infinite groups.

Most of the formulas in this paragraph were derived only under the assumption that the order h of the group was not divisible by the characteristic of the field, and thus that the group ring was semi-simple. However, that is not true for formulas (2), (6), (7), (8), (10), (14), which have general validity.

§ 16. The decomposition of irreducible representations by extension of the ground field.

The question of how an irreducible semigroup of linear transformations can decompose under an extension of the ground field $\mathbb{P}$ to a commutative field $\mathbb{K}$ reverts immediately to the question of the behavior of a simple hypercomplex system $\mathfrak{S}$ under an extension of the ground field. Namely, if $\mathfrak{S}$ is the linear hull of the given semigroup $\mathfrak{G}$

([169]) G. FROBENIUS: S.-B. preuss. Akad. Wiss. (1896), 985-1021.
([170]) A. HAAR: Acta Litt. Sci. Szeged **5** (1932), 172-186.

then $\mathfrak{S}$ will be a simple hypercomplex system over $\mathbb{P}$, and the given representation of $\mathfrak{S}$ by linear transformations will be mediated by a left ideal $\mathfrak{l}$ of $\mathfrak{S}$. Under an extension of $\mathbb{P}$ to $\mathbb{K}$, $\mathfrak{S}$ will go to a hypercomplex system $\mathfrak{S}_{\mathbb{K}}$, and $\mathfrak{l}$ will go to a left ideal $\mathfrak{l}_{\mathbb{K}}$ of $\mathfrak{S}$; one then simply treats the problem of how this left ideal in $\mathfrak{S}_{\mathbb{K}}$ will decompose into irreducible left ideals. Since this question was discussed thoroughly in the book on algebra (volume IV, book 1) in these "Ergebnisse," here, it will suffice to summarize briefly the most important results without proof ([171]). Let $\mathfrak{S}$ be a full matrix ring of degree *n* over a division algebra Λ, let Z be the center of Λ, and let $\mathfrak{D}$ be the given representation of $\mathfrak{S}$ as an irreducible semigroup of linear substitutions.

1. *If Z or $\mathbb{K}$ or both of them are separable over $\mathbb{P}$ then the system $\mathfrak{S}_{\mathbb{K}}$ will be semi-simple, so any representation of $\mathfrak{S}$ in $\mathbb{K}$ will be completely reducible. In particular: Any irreducible or completely reducible representation will remain completely reducible under a separable extension of the ground field.*

In the sequel, we will assume that Z is separable over $\mathbb{P}$.

2. *The ideal $\mathfrak{l}_{\mathbb{K}}$ decomposes into just as many irreducible left ideals as the ring $\Lambda_{\mathbb{K}}$, only of n times greater rank. The ring $\Lambda_{\mathbb{K}}$ decomposes into just as many simple systems (two-sided ideals of the ring) as its center $\mathsf{Z}_{\mathbb{K}}$.*

Any of these simple systems can be further decomposed into only *equivalent* left ideals; however, the left ideals of different systems are inequivalent. For the representation $\mathfrak{D}$, this says that it decomposes into just as many inequivalent components as $\mathsf{Z}_{\mathbb{K}}$; each of these components can then be further decomposed into equivalent irreducible representations.

3. *If $\mathbb{K}$ is Galoisian, in particular, then the different simple subsystems of $\Lambda_{\mathbb{K}}$, and therefore also the inequivalent components of the representation $\mathfrak{D}$, will be conjugate relative to $\mathbb{P}$; i.e., they will go to each other under the automorphisms of $\mathbb{K}$.*

([171]) The theorems of these paragraphs (to the extent that they relate directly to the semigroup $\mathfrak{G}$) go back to I. SCHUR: S.-B. preuss. Akad. Wiss. (1906), 64-184; Trans. Amer. Math. Soc. **10** (1909), 159-175; their hypercomplex basis and refinement goes back to E. NOETHER; Math. Z. **37** (1933), 514-541. For the historical development, see H. TABER: C. R. Acad., Paris **142** (1906), 948-951; L. E. DICKSON: Trans. Amer. Math. Soc. **4** (1903), 434-436.

If one is interested in the absolutely irreducible representation $\mathfrak{D}'$ into which $\mathfrak{D}$ decomposes under a sufficient further (e.g., algebraically closed) extension of $\mathbb{P}$, then from 3, it will suffice to consider one of these representations $\mathfrak{D}'$; the remaining ones will certainly be conjugate to it and will appear in $\mathfrak{D}$ equally often.

A field $\mathbb{K}$ in which an absolutely irreducible representation $\mathfrak{D}'$ of $\mathfrak{D}$ splits is called a *splitting field.* A field in which $\mathfrak{D}$ decomposes completely into absolutely irreducible representations is called a *decomposition field.* The number m that gives how often the absolutely irreducible representation $\mathfrak{D}'$ appears in $\mathfrak{D}$ is called the *SCHUR index* of $\mathfrak{D}'$ or $\mathfrak{D}$ relative to the field $\mathbb{P}$.

4. *Any splitting field $\mathbb{K}$ envelops a field Z_1 that is equivalent to Z. A component $\mathfrak{D}_1$ of the given representation $\mathfrak{D}$ that is irreducible in Z_1 splits off from $\mathfrak{D}$ in Z_1, which further splits into m equivalent components $\mathfrak{D}'$ in $\mathbb{K}$.*

In the event that the index m is not divisible by the characteristic of the field, the field Z_1 will be generated by the characters of the absolutely irreducible representation $\mathfrak{D}'$. One can also obtain the representation $\mathfrak{D}$ when one identifies Z with Z_1 and regards $\mathfrak{S}$ as a hypercomplex system over Z and $\mathfrak{l}$ as a representation module relative to Z. Regarding $\mathsf{Z} = \mathsf{Z}_1$ to be the ground field, instead of $\mathbb{P}$, simplifies the investigation insofar as $\mathfrak{S}$ will then become a *normal*, simple system – i.e., one whose center is the ground field. The concepts of decomposition field and splitting field coincide relative to these ground fields:

5. *The division algebra Λ has rank m^2 relative to $\bar{\mathsf{Z}}$. The degree of any splitting field $\mathbb{K}$ over Z is divisible by m. The splitting fields of smallest degree have degree m and are isomorphic to the maximal, commutative sub-field of Λ. Any splitting field of degree mq is isomorphic to a maximal, commutative sub-field of the full matrix ring of degree q over Λ, and any such maximal, commutative sub-field of Λ_q is a splitting field.*

In conclusion, we mention a theorem that is easy to prove for completely reducible representations:

If two representations of a semigroup $\mathfrak{G}$ by linear transformations in a field $\mathbb{P}$ are equivalent in an extension field $\mathbb{K}$ then they will also be equivalent in the ground field $\mathbb{P}$.

Proof ([172]): First of all, assume that the ground field $\mathbb{P}$ has infinitely many elements, or at least, more than the degree of the representation would imply. The equivalence of two representations $a \to A_1$ and $a \to A_2$ would be equivalent to the solubility of a system of linear equations $TA_1 = A_2T$ and an inequality $|T| \neq 0$ for the elements of the matrix T. If such a system were soluble in the field $\mathbb{K}$ then it would also be already soluble in the ground field $\mathbb{P}$, assuming that $\mathbb{P}$ contains more elements than the degree of the inequality would imply.

Secondly, assume $\mathbb{P}$ has finitely many elements. If two representations in $\mathbb{K}$ are equivalent then they will also be equivalent in a finite extension field Σ of $\mathbb{P}$ with sufficiently many elements. If $\mathfrak{M} = u_1\,\mathbb{P} + \ldots + u_m\,\mathbb{P}$ and $\mathfrak{N} = v_1\,\mathbb{P} + \ldots + v_m\,\mathbb{P}$ are the representation modules of the semigroup $\mathfrak{G}$ then their extension modules $\mathfrak{M}_\Sigma = u_1\,\Sigma + \ldots + u_m\,\Sigma$ and $\mathfrak{M}_\Sigma = v_1\,\Sigma + \ldots + v_m\,\Sigma$ will be operator isomorphic as $(\mathfrak{G}, \Sigma)$-modules, and therefore all the more so as $(\mathfrak{G}, \mathbb{P})$-modules. Now, if $(\sigma_1, \ldots, \sigma_g)$ is a $\mathbb{P}$-basis of Σ then $\mathfrak{M}_\Sigma$ can also be written in the form:

$$\mathfrak{M}_\Sigma = \mathfrak{M}\sigma_1 + \ldots + \mathfrak{M}\sigma_g\,. \tag{1}$$

The individual summand $\mathfrak{M}\sigma_i$ is operator isomorphic to $\mathfrak{M}$ by means of the association $u \to u\sigma_i$. If one thinks of $\mathfrak{M}$ and $\mathfrak{M}_\Sigma$ as being written in the form of direct sums of directly indecomposable summands, according to the REMAK-SCHMIDT theorem (§ 11.4), then, on the basis of (1), $\mathfrak{M}_\Sigma$ will contain each summand precisely g times as often as $\mathfrak{M}$. Now, if $\mathfrak{M}_\Sigma$ and $\mathfrak{N}_\Sigma$ contain directly indecomposable summands just as often then the same thing will also be true for $\mathfrak{M}$ and $\mathfrak{N}$.

§ 17. Factor systems.

Let $\mathfrak{D}$ be an absolutely irreducible representation of a normal, simple hypercomplex system $\mathfrak{S}$ in a finite, separable extension field $\mathbb{K} = \mathbb{P}(\vartheta)$ of the ground field $\mathbb{P}$. $\mathfrak{D}$ might go to $\mathfrak{D}_\alpha$ by means of the field isomorphisms Γ_α, which take $\vartheta = \vartheta_1$ to its conjugate quantities ϑ_α. Since the representations $\mathfrak{D}_\alpha$ are all equivalent, there will be non-singular matrices $P_{\alpha\beta}$ in the field $\mathbb{P}(\vartheta_\alpha, \vartheta_\beta)$ that transform $\mathfrak{D}_\beta$ into $\mathfrak{D}_\alpha$:

$$D_\alpha = P_{\alpha\beta}\,D_\beta\,P_{\alpha\beta}^{-1}\,. \tag{1}$$

([172]) The proof goes back to E. NOETHER and was partially presented by M. DEURING: Math. Ann. **107** (1932), 144.

One can obviously choose the $P_{\alpha\beta}$ in such a way that any isomorphism of $\mathbb{P}(\vartheta_\alpha, \vartheta_\beta)$ that takes ϑ_α, ϑ_β to a conjugate pair ϑ_γ, ϑ_δ also takes $P_{\alpha\beta}$ to $P_{\gamma\delta}$. To that end, one needs only to choose a pair $(\vartheta_\alpha, \vartheta_\beta)$ arbitrarily from any class of conjugate pairs in order to determine a $P_{\alpha\beta}$ and derive the remaining $P_{\gamma\delta}$ from $P_{\alpha\beta}$ by the isomorphisms in question. One can thus choose $P_{11} = I$.

The matrix $P_{\alpha\beta} P_{\beta\gamma}$ transforms $\mathfrak{D}_\gamma$ into $\mathfrak{D}_\alpha$; one then has:

$$P_{\alpha\beta} P_{\beta\gamma} = c_{\alpha\beta\gamma} P_{\alpha\gamma},$$

where $c_{\alpha\beta\gamma}$ is a non-zero number in $\mathbb{P}(\vartheta_\alpha, \vartheta_\beta, \vartheta_\gamma)$. These numbers define the *factor system* of the representation $\mathfrak{D}$ of $\mathfrak{S}$ in the field $\mathbb{K}$ over $\mathbb{P}$. The following conditions are characteristic of such a factor system ([173]):

1. $c_{111} = 1$,
2. $c_{\alpha\beta\gamma} c_{\alpha\gamma\delta} = c_{\alpha\beta\delta} c_{\beta\gamma\delta}$,
3. $S\, c_{\alpha\beta\gamma} = c_{\alpha'\beta'\gamma'}$, when S is an isomorphism that takes ϑ_α, ϑ_β, ϑ_γ to $\vartheta_{\alpha'}$, $\vartheta_{\beta'}$, $\vartheta_{\gamma'}$.

If one replaces $P_{\alpha\beta}$ with $k_{\alpha\beta} P_{\alpha\beta}$, where the numbers $k_{\alpha\beta}$ must fulfill the same conjugacy conditions as the $P_{\alpha\beta}$, then the c will go to an "associated factor system":

$$c'_{\alpha\beta\gamma} = \frac{k_{\alpha\beta} k_{\beta\gamma}}{k_{\alpha\gamma}} c_{\alpha\beta\gamma}.$$

If one regards associated factor systems as not being different then the factor system $c_{\alpha\beta\gamma}$ will be determined uniquely by the hypercomplex system $\mathfrak{S}$ and the field $\mathbb{K}(\vartheta)$. For a given $c_{\alpha\beta\gamma}$ that fulfills the conditions 1, 2, 3, one obtains a hypercomplex system $\mathfrak{S}$ with just this factor system when one constructs all matrices of the form:

$$(c_{\kappa\lambda 1}^{-1}\, l_{\kappa\lambda}) \qquad (\kappa \text{ rows, } \lambda \text{ columns}),$$

where the $l_{\kappa\lambda}$ run through all numbers in $\mathbb{P}(\vartheta_\alpha, \vartheta_\beta)$ that fulfill the same conjugacy conditions as the $k_{\kappa\lambda}$ do above. The totality of these matrices is absolutely irreducible and linearly closed; it then faithfully represents a simple hypercomplex system $\mathfrak{S}$. This representation is equivalent to a representation that is rational in $\mathbb{P}(\vartheta)$ and has the factor system $c_{\alpha\beta\gamma}$ ([174]).

The fundamental theorem in the theory of factor systems reads:

([173]) See R. BRAUER: Math. Z. **28** (1928), 677-698.
([174]) R. BRAUER: Math. Z. **30** (1929), 90.

$\mathfrak{S}$ is a full matrix ring over the ground field $\mathbb{P}$ if and only if the factor system is associated with the identity system $c_{\alpha\beta\gamma} = 1$ ([175]).

If one brings into play the easily-proved facts that the product $\mathfrak{S} \times \mathfrak{T}$ of two normal, simple, hypercomplex systems belongs to the product of the factor systems $c_{\alpha\beta\gamma} \cdot d_{\alpha\beta\gamma}$, and the inverse isomorphism system S' belongs to the inverse factor system $c_{\alpha\beta\gamma}^{-1}$ then it will follow that $\mathfrak{S} \times \mathfrak{S}'$ has the factor system one, so it will be a full matrix ring over $\mathbb{P}$ ([176]). If one now divides the algebra $\mathfrak{S}$ into classes, when one counts all full matrix rings over the same division algebra Λ in one class, then these classes will define a group under the multiplication $\mathfrak{S} \times \mathfrak{T}$: viz., the BRAUER *algebra class group*, in which the class of $\mathbb{P}$ plays the role of the identity element and the class of $\mathfrak{S}'$ plays the role of the inverse element to the class of $\mathfrak{S}$. The algebra classes with fixed decomposition field $\mathbb{K}$ define a subgroup of the algebra group that is homomorphic to its group of factor systems, and in fact is 1-isomorphic to it on the basis of the main theorem above. It follows from this that *any algebra class with a given center $\mathbb{P}$ and decomposition field $\mathbb{K}$ will be determined uniquely by its factor system $c_{\alpha\beta\gamma}$*; in particular, a division algebra with a given $\mathbb{P}$ and $\mathbb{K}$ will then be determined uniquely by its factor system.

Naturally, an extension field $\mathbb{K}'$ of $\mathbb{K}$ is also a decomposition field of $\mathfrak{S}$ with $\mathbb{K}$; the associated factor system is determined from that of $\mathbb{K}$ in a closely-related way: one will have $c'_{\alpha\beta\gamma} = c_{\alpha\beta\gamma}$, when the isomorphisms $\Gamma_{\alpha'}$, $\Gamma_{\beta'}$, $\Gamma_{\gamma'}$ of $\mathbb{K}'$ yield the isomorphisms Γ_{α}, Γ_{β}, Γ_{γ} when they are applied to $\mathbb{K}$.

If one chooses $\mathbb{K}'$ to be a Galois field Ω over $\mathbb{P}$, in particular, and employs the elements $S, T, U, \ldots$ of the Galois group as indices, instead of the numbers α', β', γ', then one can also construct a hypercomplex system $\mathfrak{S}$ that belongs to the given factor system in the following way: $\mathfrak{S}$ envelops Ω, and each automorphism S of $\mathbb{K}$ belongs to a basis element u_S of $\mathfrak{S}$ relative to Ω, such that one will have:

$$\mathfrak{S} = \sum_S \Omega u_S = \sum_S u_S \Omega .$$

For each ω in Ω, one will have:

$$\omega u_S = u_S (S\,\omega),$$

([175]) A. SPEISER: Math. Z. **5** (1919), 1-6; cf., also I. SCHUR: Math. Z. **5** (1919), 7-10 and R. BRAUER: S.-B. preuss. Akad. Wiss. (1926), 410-416.

([176]) A direct proof of this theorem is given by E. NOETHER: Math. Z. **37** (1933), 532.

$$u_S\, u_T = u_{ST}\ c_{ST,S,1}^{-1}.$$

$\mathfrak{S}$ is called the *folded product of the field* Ω *with its* GALOIS *group.*

For more on the theory of factor systems and the folded product, we refer to the cited literature, in particular, to the book on algebra by M. DUERING in this collection (Band IV, Heft 1).

§ 18. Integrality properties. Modular representations.

W. BURNSIDE has proved ([177]):

A semigroup of linear substitutions whose matrix elements are rational numbers with reduced denominators is equivalent to an integer subgroup.

The proof of this, which is presented by SPEISER ([178]), yields the following generalization of this theorem:

If the matrix elements of the semigroup $\mathfrak{g}$ *are numbers in a finite algebraic number field with reduced denominators then* $\mathfrak{g}$ *will be equivalent to a semigroup with integer algebraic matrix elements in a suitable extension field.*

In particular, the theorems that were mentioned for a representation of a finite group (more generally, for a representation of an "ordering" of a hypercomplex system, as well) will be true in an algebraic number field.

Two integer representations $A(s)$, $B(s)$ of a semigroup are called *integer equivalent* when they can be taken to each other by transformations with a unimodular, integer matrix, or – what amounts to the same thing – when their integer representation modules are operator isomorphic. Rational equivalence is necessary, but not sufficient, for integer equivalence. According to C. JORDAN, however, the integer representations of a semigroup that are rationally equivalent to a given rationally irreducible representation decompose into only *finitely many* classes of mutually integer equivalent representations ([179]).

If one extends the given representation to a semi-simple, hypercomplex system $\mathfrak{S}$ then one can choose the representation module for it and all representations that are equivalent to it to be a minimal left ideal $\mathfrak{l}$ of $\mathfrak{S}$. The integer linear combinations of the matrices of the given representation define an "ordering" $\mathfrak{o}$ in $\mathfrak{S}$, and the integer representations will be mediated by an $\mathfrak{o}$-module that is contained in $\mathfrak{l}$. When one

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([177]) W. BURNSIDE: Proc. London Math. Soc. (2) **7** (1909), 8-13.
([178]) A. SPEISER: *Theorie der Gruppen von endlicher Ordnung*, 2nd ed., Berlin, 1927, § 65.
([179]) C. JORDAN: J. École polytechn. **48** (1880), 111-150. – Another proof is in L. BIEBERBACH: Nachr. Akad. Wiss. Göttingen (1912), 207-216. – L. BIEBERBACH and I. SCHUR: S.-B. preuss. Akad. Wiss. (1928), 523-527.

multiplies these $\mathfrak{o}$-modules with suitable natural numbers, one can convert them into ideals of the ring $\mathfrak{o}$. JORDAN's theorem then means the same thing as another one that says that there are only finitely many classes of mutually isomorphic ideals in $\mathfrak{o}$ that are contained in the given $\mathfrak{l}$ of $\mathfrak{S}$. This finitude of the ideal classes can also be proved with the classical ideal-theoretic methods ([180]).

One understands the term *modular representations* of a group to mean the representations in a field of characteristic p, or especially a Galois field $GF(p^f)$. We shall now examine the relations that exist between modular and non-modular representations.

Let $\mathfrak{D}_1, \ldots, \mathfrak{D}_r$ be a complete system of inequivalent absolutely irreducible representations of a finite group $\mathfrak{g}$ that may be assumed to be integer algebraic for a suitable algebraic number field $\mathbb{K}$. If one now reduces these representations modulo a prime ideal $\mathfrak{p}$ of $\mathbb{K}$ then one will obtain just as many modular representations of $\mathfrak{g}$ in $GF(p^f)$. *If the characteristic p of GF does not go into the order h of the group then the representations $\mathfrak{D}_\nu$ will also remain irreducible and inequivalent modulo p, and they will exhaust all absolutely irreducible representations of $\mathfrak{g}$ in fields of characteristic p.*

Proof. From § 13, formula (4), only integer algebraic numbers, divided by the order h of the group, will appear as coefficients in the expressions for the matrix units $c_{ik}^{(\nu)}$ of the group ring as linear combinations of the group elements s. Thus, the formulas will remain meaningful modulo $\mathfrak{p}$. The rules of calculation $c_{ik}^{(\nu)} c_{kl}^{(\nu)} = c_{il}^{(\nu)}$, etc., as well as the formula $s = \sum \alpha_{ik}^{(\nu)}(s) \cdot c_{ik}^{(\nu)}$, will likewise remain true. However, these formulas also define the decomposition of the group ring into full matrix rings relative to the ground field $GF(p^f)$; therefore, the various absolutely irreducible representations of $\mathfrak{g}$ by matrices mod $\mathfrak{p}$ will also be given in GF.

The representations of finite groups $\mathfrak{g}$ in fields whose characteristic p goes into the order h of the group were investigated by DICKSON ([181]). In the extreme case $h = p^g$, the identity representation is the only irreducible representation; any representation can then be brought into the form ([182]):

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ \alpha_{21} & 1 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ \alpha_{n1} & \cdots & \cdots & 1 \end{pmatrix}.$$

DICKSON deduced from this:

If $\mathfrak{g}$ contains a Sylow group $\mathfrak{h}$ of order p^g then any irreducible representation of $\mathfrak{g}$ in a field of characteristic p will be contained in that representation that is induced by the

([180]) C. G. LATIMER: Bull. Amer. Math. Soc. **40** (1934), 433-435.
([181]) L. E. DICKSON: Trans. Amer. Math. Soc. **8** (1907), 389-398.
([182]) Another proof of this was given by E. SPEISER: *Theorie der Gruppen von endlicher Ordnung*, 2nd ed., § 69.

identity representation of the Sylow group $\mathfrak{h}$ (cf., § 19). *The regular representation contains this induced representation precisely* p^g *times as a diagonal component, but it is not, however, completely reducible.*

L. E. DICKSON ([183]) has calculated the characters of the modular representations for some examples of finite groups with the help of formulas (6), § 15. The example of $\mathfrak{A}_5 \cong SL(4, 2)$ shows that there are modular representation that cannot arise from integer algebraic representations by residue classes modulo p, so $\mathfrak{A}_5$ has no integer representation of degree 2.

R. BRAUER communicated to me the fact that the number of inequivalent absolutely reducible, modular representations of characteristic p is equal to the number of those classes of conjugate elements in which the order of the element is relatively prime to p. I hope that he will publish this result soon.

Finally, let us mention a theorem of MINKOWSKI here ([184]):

If one reduces a faithful, rational representation of a finite group modulo an odd prime number then a faithful, modular representation will arise.

If the order of the group is odd then the same thing will also be true modulo 2.

§ 19. Relations between the representations of a group and those of its subgroups. Imprimitive representations.

Let $\mathfrak{g}$ be a finite group and let $\mathfrak{h}$ be a subgroup of $\mathfrak{g}$. Any representation of $\mathfrak{g}$ also yields a representation of $\mathfrak{h}$ then; in particular, any absolutely irreducible representation $\mathfrak{D}_\mu$ of $\mathfrak{g}$ yields a representation of $\mathfrak{h}$ – which will be denoted by $\mathfrak{D}_\mu(\mathfrak{h})$ – and a result, it can be decomposed into irreducible representations of $\mathfrak{h}$ (in an algebraically closed ground field $\mathbb{P}$):

$$\mathfrak{D}_\mu(\mathfrak{h}) = \sum c_{\mu\nu}\, \mathfrak{d}_\nu\,.$$

Any irreducible representation $\mathfrak{d}_\nu$ of $\mathfrak{h}$ will be mediated by a left ideal $\mathfrak{l}_\nu$ of the group ring $\mathfrak{R}_\mathfrak{h}$ of $\mathfrak{h}$. However, this can be regarded as a subring of the group ring $\mathfrak{R}_\mathfrak{h}$; $\mathfrak{l}_\nu$ then generates a left ideal $\mathfrak{L}_\nu = \mathfrak{R}_\mathfrak{h}\mathfrak{l}_\nu$ that mediates a representation $\mathfrak{D}(\mathfrak{d}_\nu)$ of $\mathfrak{g}$. This representation $\mathfrak{D}(\mathfrak{d}_\nu)$ is called the *(imprimitive) representation of* $\mathfrak{g}$ *that is induced by the representation* $\mathfrak{d}_\nu$ *of* $\mathfrak{h}$. One sees immediately the fact that one is, in fact, dealing with an imprimitive system of linear transformations, in the sense of § 8, when one decomposes $\mathfrak{g}$

([183]) L. E. DICKSON: Bull. Amer. Math. Soc. (2) **13** (1907), 477-488.
([184]) H. MINKOWSKI: J. reine angew. Math. **100** (1887), 449-458; **101** (1887), 196-202 – Ges. Abh. I, 203-211 and 212-218.

into cosets $s_\mu \mathfrak{h}$, and thus also decomposes the space $\mathfrak{L}_\nu$ into subspaces $s_\mu \mathfrak{l}_\nu$ that will be permuted amongst themselves by the elements s of $\mathfrak{g}$.

One also easily sees that *any imprimitive representation of a group* $\mathfrak{g}$ *can be composed of such representations that are induced by subgroups* $\mathfrak{h}$. Namely, if $\mathfrak{M} = \mathfrak{m}_1 + \mathfrak{m}_2 + \ldots + \mathfrak{m}_s$ is a given $\mathfrak{g}$-invariant decomposition of an imprimitive representation space then we will understand $\mathfrak{h}$ to mean the subgroup of $\mathfrak{g}$ that leaves $\mathfrak{m}_1$ invariant. If we assume that $\mathfrak{m}_1$ is irreducible under $\mathfrak{h}$ and will be transformed into $\mathfrak{m}_\mu$ (μ = 1, …, r) by the coset $s_\mu \mathfrak{h}$, and we choose a $u \neq 0$ from $\mathfrak{m}_1$ arbitrarily then we will get $\mathfrak{R}_\mathfrak{h}\, u = \mathfrak{m}_1$ and $\mathfrak{R}_\mathfrak{g}\, u = \mathfrak{m}_1 + \ldots + \mathfrak{m}_r = \mathfrak{M}_1$. If one now decomposes $\mathfrak{R}_\mathfrak{h}$ into left ideals: $\mathfrak{R}_\mathfrak{h} = \sum \mathfrak{l}_\nu$ then at least one $\mathfrak{l}_\nu\, u \neq 0$ and therefore $\mathfrak{l}_\nu\, u = \mathfrak{m}_1$ and $\mathfrak{R}_\mathfrak{g} \mathfrak{l}\, u = \mathfrak{R}_\mathfrak{g} \mathfrak{m} = \mathfrak{M}_1$. The association $x \to xu_1$ (for x in $\mathfrak{R}_\mathfrak{g}$) will then mediate an operator isomorphism of $\mathfrak{L}_\nu$ with $\mathfrak{M}_1$. However, if $\mathfrak{m}_1$ is irreducible for $\mathfrak{h}$ then one can decompose $\mathfrak{m}_1$ – and therefore, also $\mathfrak{M}_1$ – into irreducible components of the required kind.

Some special imprimitive representations are given by the *monomial* representations, whose matrices have only one non-zero element in each row and column. From the above, the monomial representations will be induced by representations of the first degree of subgroups $\mathfrak{h}$. (In particular, the representations of $\mathfrak{g}$ will be induced as transitive permutation groups of the identity representation of the respective subgroup $\mathfrak{h}$.) Now, a representation of the first degree of a group $\mathfrak{h}$ is always a faithful representation of a cyclic factor group $\mathfrak{h} / \mathfrak{n}$. If this factor group has order f, and if $Z\mathfrak{n}$ is a generating residue class of $\mathfrak{h} / \mathfrak{n}$ that is represented by an f^{th} root of unity ζ, and finally, if $\Sigma_\mathfrak{n}$ is the sum of the elements of $\mathfrak{n}$ in the group ring of $\mathfrak{g}$ then the left ideal $\mathfrak{l}_\nu$ that mediates the representation of first degree of $\mathfrak{h}$ will be generated by:

$$\vartheta = \Sigma_\mathfrak{n} + \zeta^{-1} Z\, \Sigma_\mathfrak{n} + \zeta^{-2} Z^2\, \Sigma_\mathfrak{n} + \ldots + \zeta^{-(f-1)} Z^{f-1}\, \Sigma_\mathfrak{n}\,.$$

One indeed sees, with no further analysis, that $Z\vartheta = \vartheta\,\zeta$ and $H\vartheta = \vartheta$ for any H in $\mathfrak{n}$. The left ideal $\mathfrak{L}_\nu$ that is generated by $\mathfrak{l}_\nu$ will have the basis:

$$s_1 \vartheta,\ s_2 \vartheta,\ \ldots,\ s_j \vartheta,$$

where $s_1, \ldots, s_j$ are the representatives of the residue classes of $\mathfrak{h}$ in $\mathfrak{g}$. The monomial representation be written down, with no further ado, using the basis ([185]).

K. SHODA ([186]) has investigated the condition under which a monomial representation is irreducible and the conditions under which two monomial

([185]) Cf., on this, say, A. SPEISER: *Theorie der Gruppen von endl. Ordnung*, 2nd ed., § 46. One will also find a series of applications of the monomial representations to finite groups there.

([186]) K. SHODA: Proc. Phys.-Math. Soc. Jap. (3) **15** (1933), 249-257.

representations will be equivalent. A monomial representation that is defined by $\mathfrak{h}$, $\mathfrak{n}$, Z, ζ is reducible (in a suitable extension field) if and only if there is an element G that is not contained in $\mathfrak{h}$ that has the following properties:

$$\mathfrak{h} \cap G\,\mathfrak{n}\,G^{-1} = \mathfrak{n} \cap G\,\mathfrak{n}\,G^{-1},$$
$$\beta \equiv \gamma \qquad (\mathrm{mod}\ f),$$

where β means the exponent of the first power of $G\,Z\,G^{-1}$ that lies in $[\mathfrak{h} \cap G\,\mathfrak{h}\,G^{-1}] \cdot G\,\mathfrak{n}\,G^{-1}$, and indeed in $Z^{\gamma}\,\mathfrak{n}\,G\,\mathfrak{n}\,G^{-1}$. Two irreducible polynomials that are defined by $\mathfrak{h}_1$, $\mathfrak{n}_1$, Z_1, ζ_1 and $\mathfrak{h}_2$, $\mathfrak{n}_2$, Z_2, ζ_2 are equivalent if and only if there is an element G in $\mathfrak{g}$ with the properties:

$$\mathfrak{h}_1 \cap G\,\mathfrak{n}_1\,G^{-1} = \mathfrak{n}_2 \cap G\,\mathfrak{n}_1\,G^{-1},$$
$$\zeta_1^{\beta} = \zeta_2^{\gamma},$$

where β means the exponent of the first power of $G\,Z_1\,G^{-1}$ that lies in $[\mathfrak{h}_2 \cap G\,\mathfrak{h}_1\,G^{-1}] \cdot G\,\mathfrak{n}_1\,G^{-1}$, and indeed in $Z_2^{\gamma}\,\mathfrak{n}_2\,G\,\mathfrak{n}_1\,G^{-1}$.

The monomial representation can often be employed in the proof of theorems on finite groups ([186a]).

The decomposition of an imprimitive representation $\mathfrak{D}(\mathfrak{d}_\nu)$ into absolutely irreducible components will be ruled by the following theorem of FROBENIUS ([187]):

The number $c_{\mu\nu}$ that gives how often an irreducible representation $\mathfrak{d}_\nu$ of $\mathfrak{h}$ is contained in the representation $\mathfrak{D}_\mu$ of $\mathfrak{g}$ will also simultaneously give how often the irreducible representation $\mathfrak{D}_\mu$ of $\mathfrak{g}$ is contained in the imprimitive representation $\mathfrak{D}(\mathfrak{d}_\nu)$.

J. LEVITZKI ([188]) has extended this theorem to the semi-simple subrings of semi-simple hypercomplex systems and has presented some other number relations for this case.

E. ARTIN ([189]) has proved that any rational representation trace of a finite group is a rational-number linear combination of traces of representations that are induced from the identity representations of the cyclic subgroups.

A KULAKOFF ([190]) proved: If $\mathfrak{h}$ is a normal subgroup of $\mathfrak{g}$ then the identity representation of $\mathfrak{h}$ will either not appear in the decompositions of $\mathfrak{D}(\mathfrak{d}_\nu)$ at all or $\mathfrak{D}(\mathfrak{d}_\nu)$ will be a multiple of the identity representation.

([186a]) W. BURNSIDE: *Theory of Groups of Finite Order*, 2nd ed., Cambridge, 1911, 327. – W. K. TURKIN: Math. Z. **38** (1934), 301-305.
([187]) G. FROBENIUS: S.-B. preuss. Akad. Wiss. (1898), 501-515.
([188]) J. LEVITZKI: Math. Z. **33** (1931), 663-665.
([189]) E. ARTIN: J. reine angew. Math. **164** (1931), 1-11.
([190]) A. KULAKOFF: Rec. math. Soc. math. Moscou **36** (1928), 129-134.

I. SCHUR ([191]) and R. BRAUER ([192]) have investigated the relations between the SCHUR index of the representations of a semigroup $\mathfrak{g}$ and the representations of its sub-semigroups $\mathfrak{h}$. The main result reads:

If $\mathfrak{D}$ is an absolutely irreducible representation of a semigroup $\mathfrak{g}$ in a field of finite degree over $\mathbb{P}$, and if the representation $\mathfrak{D}$, when applied to a sub-semigroup $\mathfrak{h}$, contains an absolutely irreducible representation $\mathfrak{d}$ and its conjugate representations r times in all then the index m of $\mathfrak{D}$ will be a divisor of $m'r$, where m' is the index of d. In particular, if $\mathfrak{d}$ is rational in $\mathbb{P}$ then m / r will be, as well.

For finite groups, this theorem follows immediately from the theorem of FROBENIUS above and the properties of the SCHUR index *m* that were defined. One obtains useful special cases when one lets $\mathfrak{d}$ be the identity representation of $\mathfrak{h}$ ([193]) or when one chooses $\mathfrak{h}$ to be a cyclic group. In the latter case, one obtains the theorem:

The SCHUR *index of a representation $\mathfrak{D}$ of $\mathfrak{g}$ relative to a field $\mathbb{P}$ that contains the l^{th} root of unity is a divisor of all numbers c_ν that give how often the different l^{th} roots of unity ζ_ν appear as characteristic roots of the representative matrix of a group element s of order l.*

In particular, if the greatest common divisor of all of these numbers $c_{\mu\nu}$ for different group elements is equal to one then one will have $m_\mu = 1$ for a suitable circle field as ground field; i.e., the representation is $\mathfrak{D}_\mu$ will be representable in this circle field ([194]). Naturally, it suffices to base the field on the h^{th} root of unity, where *h* is the order of the group.

One suspects that all absolutely irreducible representations of a group of order *h* are realizable in the field of h^{th} roots of unity ([194]). That conjecture was proved for solvable groups by I. SCHUR using the methods of this paragraph ([195]). H. HASSE has shown ([196]) that in any event the field of h^{th} roots of unity is attained (for a sufficiently large λ).

In connection with that, we mention yet another theorem of A. SPEISER ([197]), which says that:

Any absolutely irreducible representation of a finite group of odd order with a real character is already realizable in the field of characters.

([191]) I. SCHUR: S.-B. preuss. Akad. Wiss. (1906), 164-184.
([192]) R. BRAUER: Math. Z. **31** (1929), 733-747, § 3.
([193]) Cf., G. FROBENIUS: S.-B. preuss. Akad. Wiss. (1903), 328.
([194]) Cf., W. BURNSIDE: Proc. London Math. Soc. (2) **3** (1905), 239-252.
([195]) I. SCHUR: S.-B. preuss. Akad. Wiss. (1906), 164-184.
([196]) R. BRAUER, H. HASSE, and E. NOETHER: J. reine angew. Math. **167** (1931), 399-404.
([197]) A. SPEISER: Math. Z. **5** (1919), 1-6.

§ 20. Representations of special groups.

We already gave the representations of Abelian groups in § 13. One finds the representations of the simplest non-Abelian groups – e.g., the tetrahedral group, the quaternion group, the icosahedral group $\mathfrak{A}_5$ – in the book of SPEISER ([198]), and those of the octahedral group $\mathfrak{S}_4$ in VAN DER WAERDEN ([199]).

W. BURNSIDE ([200]) gave the general form of the faithful representations of groups with nothing but cyclic Sylow groups (thus, the groups of square-free order, especially). These representations are all monomial. More generally, from § 8, the representations of a two-level, meta-Abelian group are all monomial and indeed, according to K. SHODA ([201]), the faithful, irreducible representations of such a group will be induced by the linear representations of those maximal Abelian subgroups that envelop the commutator group.

The representations of the groups of order p^n are also all monomial (cf., § 8), and are therefore easy to find in any concrete case.

For the groups with complex structure, the calculation of the characters mostly precedes the actual presentation of the representations. In order to calculate the characters, one chiefly resorts to two methods: The method of increases and the method of composition. With the method of increases, one starts from known characters of any subgroup and calculates the traces of the induced representations of the super-group from them. Occasionally, one also conversely goes down from a super-group to a subgroup. With the method of composition, one calculates the trace of a product representation by multiplying two known characters. In order to decompose the composite characters that are obtained by these methods into simple ones, one appeals to the orthogonality relations of the characters (§ 15). G. FROBENIUS has calculated the characters of the *binary tetrahedral, octahedral, and icosahedral groups* ([202]), as well as those of the *modular groups* $PSL(2, p)$ ([203]), and likewise I. SCHUR ([204]), and simultaneously H. E. JORDAN ([205]), calculated the characters of the groups $SL(2, p^m)$ and $GL(2, p^m)$, and furthermore, I. SCHUR ([204]) calculated those of a 2-isomorphic covering group of $SL(2, p^m)$. H. ROHRBACH ([206]) has determined the characters of the binary congruence group mod p^2 (which consists of the two-rowed matrices mod p^2 with determinant 1). If one defines the factor group of this congruence group with the matrices λI then one will obtain the *modular group mod* p^2, whose characters were determined recently by H. W. PRAETORIUS ([207]).

If a representation of a group as a permutations group of degree n is given then one can always also regard it as a representation by linear transformations. Since the sum of all permuting quantities is an invariant, the identity representation splits once. The

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([198]) A. SPEISER: *Theorie der Gruppen von endlicher Ordnung*, 2nd ed., § 59.
([199]) B. L. VAN DER WAERDEN: *Moderne Algebra II*, § 125.
([200]) W. BURNSIDE: Messenger of Math. (2) **35** (1906), 46-50.
([201]) K. SHODA: Proc. Phys.-Math. Soc. Jap. (5) **15** (1933), 249-257.
([202]) G. FROBENIUS: S.-B. preuss. Akad. Wiss. (1899), 330-339.
([203]) G. FROBENIUS: S.-B. preuss. Akad. Wiss. (1896), 1013-1021.
([204]) I. SCHUR: J. reine angew. Math. **132** (1907), 85-137.
([205]) H. E. JORDAN: Amer. J. Math. **29** (1907), 387-405.
([206]) H. ROHRBACH: "Die Charaktere der binären Kongruenzgruppen mod p^2," Diss., Berlin, 1932.
([207]) H. W. PRAETORIUS: Abh. math. Semin. Hamburg. Univ. **9** (1933), 365-394.

remaining representation of degree $n - 1$ is irreducible if and only if the permutation group is doubly transitive. This theorem is the first in a series of similar theorems by FROBENIUS ([208]) on multiply transitive groups. FROBENIUS ([208]) has calculated the characters of the two 5-fold transitive permutations groups of degree 12 and 24 (with orders $12 \cdot 11 \cdot 10 \cdot 9 \cdot 8$ and $24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 48$, resp.) that were discovered by MATHIEU.

The irreducible representations of the modular group $PSL(2, p)$ are of great importance for the theory of module functions, and have therefore been examined, in part, many times. The representation of PSL as a permutation group of $p + 1$ points of the projective line first yields an irreducible representation of degree p (cf., the previous section). Two complex conjugate representations of degree $\frac{p+\varepsilon}{2}$, where $\varepsilon = (-1)^{\frac{p-1}{2}}$, have been known since F. KLEIN ([209]). Furthermore, there are $\frac{p-\varepsilon-4}{4}$ representations of degree $p + 1$ that E. HECKE ([210]) presented and $\frac{p+\varepsilon-2}{4}$ representations of degree $p - 1$ that B. SCHOENBERG ([211]) has presented. According to H. W. PRAETORIUS ([207]), there is a representation of degree $p(p + 1)$ of the modular group mod p^2 that is analogous to the representation of degree $p + 1$.

The representations of the *symmetric and alternating groups* have been investigated most thoroughly. In the case of the symmetric group, G. FROBENIUS ([212]) first calculated the characters with the method of increases when he started with the identity representation of certain subgroups $\mathfrak{H}_\alpha$, to which we will return later on. Building upon the investigations of A. YOUNG ([213]), G. FROBENIUS ([214]) could give the minimal left ideals of the group ring that generates irreducible representations directly. In what follows, we will give only the result and refer to B. L. VAN DER WAERDEN: *Moderne Algebra II* (1931), § 127 for the proof.

We might understand a *tableau* $T_\alpha = T_{\alpha_1,\alpha_2,\dots,\alpha_h}$ to mean an arrangement of the numbers $1, 2, \dots, n$ into h rows (h arbitrary $\leq n$), for which α_ν numbers are in the ν^{th} row and which fulfill the conditions:

$$(1) \qquad \begin{cases} \alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_h \geq 0, \\ \alpha_1 + \alpha_2 + \dots + \alpha_h = n. \end{cases}$$

If $\alpha_h = 0$ then α_h can be dropped from the index sequence $\alpha_1, \dots, \alpha_h$; we can therefore always assume that $\alpha_h > 0$. The *columns* of the tableau consist of the first, second, etc., numbers in all rows.

([208]) G. FROBENIUS: S.-B. preuss. Akad. Wiss (1904), 558-571.
([209]) F. KLEIN: Math. Ann. **15** (1879), 275-278. – W. BURNSIDE: Proc. Cambridge Philos. Soc. **22** (1929), 779-787.
([210]) E. HECKE: Abh. math. Semin. Hamburg. Univ. **6** (1928), 256-257.
([211]) B. SCHOENBERG: Abh. math. Semin. Hamburg. Univ. **9** (1932), 1-14.
([212]) G. FROBENIUS: S.-B. preuss. Akad. Wiss. (1900), 516-534.
([213]) A. YOUNG: Proc. London Math. Soc. **33** (1900), 97-146; **34** (1902), 361-397.
([214]) G. FROBENIUS: S.-B. preuss. Akad. Wiss. (1903), 328-358.

P_α denotes the sum that is defined in the group ring $\mathfrak{o}$ of all of the permutations which only switch the numbers inside of the rows of the tableau, and likewise N_α denotes the alternating sum of all of the permutations that only switch the numbers inside the columns, where odd permutations will be given the minus sign. We arbitrarily select a tableau T_α from any solution $(\alpha_1, \ldots, \alpha_h)$ of (1), which we denote briefly by α; those solutions will be lexicographically ordered. The left ideal $\mathfrak{o}N_\alpha$ that is generated by N_α will then contain a minimal left ideal $\mathfrak{l}_\alpha$ which certainly appears in no $\mathfrak{o}N_\beta$ with $\beta < \alpha$. This left ideal $\mathfrak{l}_\alpha$ will be generated by $P_\alpha N_\alpha$. $P_\alpha N_\alpha$ is idempotent, up to a numerical factor ([215]):

$$(P_\alpha N_\alpha)^2 = \lambda_\alpha (P_\alpha N_\alpha).$$

Thus, an irreducible representation $\mathfrak{D}_\alpha$ that is mediated by $\mathfrak{l}_\alpha$ will belong to any integer solution α of the system of equations (1). One also obtains all irreducible representations in this way, since the number of solutions of (1) obviously agrees with the number of classes of conjugate group elements. The sum of all $\mathfrak{l}_\alpha$ and their transforms $s\,\mathfrak{l}_\alpha s^{-1}$ will be the entire group ring $\mathfrak{o}$.

One can also switch the roles of N_α and P_α in the foregoing: The left ideal $\mathfrak{o}P_\alpha$ contains a minimal left ideal $\mathfrak{l}'_\alpha$ that does not appear in the $\mathfrak{o}P_\beta$ with $\beta > \alpha$, and which will be generated by $M_\alpha P_\alpha$. $\mathfrak{l}'_\alpha$ is equivalent to $\mathfrak{l}_\alpha$ (i.e., operator isomorphic).

The representations are all rational. If the tableau consists of only one row (column, resp.) then the representation $\mathfrak{D}_\alpha$ will be the identity representation (the representation of degree one for which the even permutations are represented by 1 and the odd ones by -1, resp.). The tableau will belong to a *reflected* tableau (switching the rows with the columns); one will obtain the "associated" representation that goes with it by multiplying the representative matrices of the odd permutations by -1.

A. YOUNG ([216]) has carried out the calculations even further, when he actually gave the idempotents e that belong to the decomposition of $\mathfrak{o}$ into minimal left ideals, as well as the "matrix units" $c_{ik}^{(\nu)}$, which are expressed in terms of the group element s; from § 13, equation (4), these formulas also yield the irreducible representation in an explicit form that coincides with a matrix representation that was given by I. SCHUR ([217]).

In order to calculate the characters of the representation $\mathfrak{D}_\alpha$, FROBENIUS ([218]) proceeded as follows: One first calculates the trace of the representation $\mathfrak{P}_\alpha$ that is mediated by the ideal $\mathfrak{o}P_\alpha$, which contains $\mathfrak{D}_\alpha$ once as a component. Since P_α is the sum of the elements of those groups $\mathfrak{H}_\alpha$ that leave the rows of the tableau T_α invariant, P_α will be the imprimitive representation that is induced by the identity representation of $\mathfrak{H}_\alpha$, and

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([215]) The numerical factor λ_α is easily seen to be n_α^{-1}, where n_α is the degree of the representation $\mathfrak{D}_\alpha$.

([216]) A. YOUNG: J. London Math. Soc. **3** (1928), 14-19. – Proc. London Math. Soc. (2) **28** (1928), 255-292; **31** (1930), 253-272; **34** (1932), 196-230.

([217]) I. SCHUR: S.-B. preuss. Akad. Wiss. (1908), 664-678.

([218]) G. FROBENIUS: S.-B. preuss. Akad. Wiss. (1900), 516-534.

thus, a representation by permutations, whose traces are easy to calculate. The result is that the trace of a permutation s that decomposes into cycles of lengths γ_1, γ_2, … is equal to the coefficients of $x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_h^{\alpha_h}$ in the product $(x_1^{\gamma_1} + x_2^{\gamma_1} + \cdots x_h^{\gamma_1})\,(x_1^{\gamma_2} + x_2^{\gamma_2} + \cdots x_h^{\gamma_2})$. The characters can now be obtained from these traces by linear combinations, and indeed, as FROBENIUS proved by a clever calculation on the basis of the orthogonality relations, $\chi(s)$ is equal to the coefficient of $x_1^{\beta_1} x_2^{\beta_2} \cdots x_h^{\beta_h}$ in the polynomial:

$$\Lambda \cdot (x_1^{\gamma_1} + x_2^{\gamma_1} + \cdots x_h^{\gamma_1})\,(x_1^{\gamma_2} + x_2^{\gamma_2} + \cdots x_h^{\gamma_2}) \ldots \tag{2}$$

with

$$\Lambda = \prod_{\mu<\nu} (x_\mu - x_\nu)\,; \qquad \beta_\nu = \alpha_\nu + (h - \nu).$$

In particular, one finds the following formulas for the degree $n_\alpha = \chi_\alpha(1)$:

$$n_\alpha = \sum_{i=1}^{h} n_{\alpha_1\alpha_2\cdots(\alpha_i-1)\alpha_{i+1}\cdots\alpha_h}\,,$$

$$n_\beta = \frac{n!}{\beta_1!\,\beta_2!\cdots\beta_h!} \prod_{\mu<\nu} (\beta_\mu - \beta_\nu)\,.$$

I. SCHUR ([219]), H. WEYL ([220]), and A. YOUNG ([221]) gave other derivations of the FROBENIUS generating function (2). The derivations of SCHUR and WEYL employ the connection with the representations of the linear groups (cf., § 22), while A. YOUNG derived the following remarkable relation in the group ring:

$$\frac{n!}{n_\alpha} I_\alpha = \prod_{r<s} (1 - \Omega_{rs}) S_\alpha\,. \tag{3}$$

In it, I_α means the idempotent central elements (cf., § 15.56) that belong to the representation $\mathfrak{D}_\alpha$, S_α means the sum of all *different* P_α that arise by permutation of the numbers in a schema Σ_α, and furthermore, Ω_{rs} means an operation on the indices $\alpha_1, \ldots, \alpha_h$ that consists of increasing the index α_r by 1 and reducing α_s by 1. If the conditions (1) are violated after performing a product of operations Ω_{rs} then the term in question in (1) must be set to zero. On the basis of formula (16), § 15, formula (3):

$$\frac{n!}{n_\alpha} I_\alpha = \sum_s \chi_\alpha(s^{-1})\, s$$

permits the calculation of the characters $\chi_\alpha(s)$.

([219]) S.-B. preuss. Akad. Wiss. (1908), 664-678; (1927), 58-75 – Diss., Berlin, 1901, pp. 31.

([220]) H. WEYL: Math. Z. **23** (1925), 271-309 – *Gruppentheorie und Quantenmechanik*, 2nd ed., Leipzig, 1931, chap. 5.

([221]) A. YOUNG: Proc. London Math. Soc. **34** (1932), 195-230.

LITTLEWOOD and RICHARDSON ([222]) gave other ways of calculating the characters. In their paper, one also finds a table of characters for the $\mathfrak{S}_n$ for all $n \leq 9$.

Representations of the $\mathfrak{S}_n$ of lowest degree. In addition to the two trivial linear representations, there is only one unfaithful representation (of degree 2) for $n = 4$. All of the remaining representations have degree at least $n - 1$. There are only two representations of degree precisely $n - 1$ for $n \neq 6$: One of them is deduced immediately from the representation of the $\mathfrak{S}_n$ as a permutation group of degree n, and the other one is thus associated with it. For $n = 6$, the two representations of degree 5 that emerge from the two aforementioned ones by means of the known automorphisms combine.

The representations of the *alternating group* can be derived, for the most part, from those of the symmetric group ([223]). Namely, it follows easily from the orthogonality relations for the characters that the irreducible representations of the symmetric group $\mathfrak{S}_n$ also represent the alternating group $\mathfrak{A}_n$ irreducibly, except for the ones whose tableau goes to itself under reflection: Those ones decompose into two inequivalent, irreducible representations that differ from each other by the value of an irrational square root. G. FROBENIUS ([223]) has calculated their characters. The lowest degree of a faithful representation is also $n - 1$ now, except for the case of $n = 5$, for which a representation of degree 3 exists.

A. YOUNG ([224]) has also investigated the group ring of the hyper-octahedral group – i.e., the group of linear transformations of the n-dimensional generalization of the octahedron ([225]) in the same way as for the symmetric group. He again found explicit formulas for the matrix elements and characters of the irreducible representations. As in the case of the symmetric group, they are rational numbers. The same thing is true for a subgroup of index 2 that A. YOUNG has likewise examined.

The hyper-octahedral group is a special case of a genre of groups of orders $n!\, g^n$ whose representation were examined by W. SPECHT ([226]). In § 21, we shall return to a series of groups of orders $2n!$ ($n!$, resp.) that I. SCHUR considered, which likewise possess the $\mathfrak{S}_n$ ($\mathfrak{A}_n$, resp.) as factor groups.

§ 21. Representations of groups by projective transformations.

A homomorphic *representation of a group $\mathfrak{H}$ by projective transformations* – or briefly, a *projective representation* of $\mathfrak{H}$ – will be obtained when the elements $a, b, \ldots$ of $\mathfrak{H}$ are associated with non-singular matrices $A, B, \ldots$ (or linear transformations $\mathsf{A}, \mathsf{B}, \ldots$) in such a way that the product ab corresponds to the matrix $\varepsilon_{a,b}\, AB$. The non-zero

([222]) D. E. LITTLEWOOD and A. R. RICHARDSON: Philos. Trans. Roy. Soc. London (A) **233** (1934), 99-141.

([223]) G. FROBENIUS: S.-B. preuss. Akad. Wiss. (1901), 303-315.

([224]) A. YOUNG: Proc. London Math. Soc. (2) **31** (1930), 273-288.

([225]) One deals with the group of monomial substitutions whose matrices contain only the elements ± 1 and 0.

([226]) W. SPECHT: "Eine Verallgemeinerung der symmetrischen Gruppe," Diss., Berlin, 1932.

numbers $\varepsilon_{a,b}$ define the *factor system* of the representation. Two representations $a \to A$ and $a \to A'$ are called *associated* when one always has $A = \delta_a A'$ ($\delta_a \neq 0$); that therefore means that one is indeed dealing with different matrices, but the same projective transformation. One likewise calls the associated factor systems $\varepsilon_{a,b}$ and $\varepsilon'_{a,b}$ associated; the condition for that obviously reads:

$$\varepsilon'_{a,b} = \frac{\delta_a \delta_b}{\delta_{ab}} \varepsilon_{a,b} .$$

The factor system must satisfy the relations:

$$\varepsilon_{a,bc}\, \varepsilon_{b,c} = \varepsilon_{a,b}\, \varepsilon_{a,bc} . \tag{1}$$

If all $\varepsilon_{a,b} = 1$ then one will be dealing with a representation in the ordinary sense, or as we will now say, a *full representation*.

In his ground-breaking paper ([227]), I. SCHUR showed how one can get back to the problem of finding all projective representations of *finite* groups from the previously-solved problem of finding all full representations when one constructs a covering group $\mathfrak{G}$ for $\mathfrak{H}$ whose full representations mediate all projective representations of $\mathfrak{H}$ precisely. One then has $\mathfrak{H} \cong \mathfrak{G} / \mathfrak{A}$, and the normal subgroup $\mathfrak{A}$ is contained in the center of $\mathfrak{G}$. One arrives at this result in the following way, where we refer to the aforementioned paper by SCHUR for the precise details of the proof.

Any system of numbers $\varepsilon_{a,b}$ that fulfills the relations (1) is the factor system of a representation, and one indeed obtains such a representation when one chooses the group ring for the vector space and defines the transformation A that is associated with the group element a by ([228]):

$$\mathsf{A}b = \varepsilon_{a,b}\, ab .$$

Any irreducible representation that belongs to the same factor system is equivalent to a component of this representation.

Let h denote the order of the given group $\mathfrak{H}$. There is then an associated factor system to any factor system whose factors are h^{th} roots of unity. There are then only finitely many essentially different factor systems.

The product of two factor systems is again a factor system. The classes of associated factor systems thus define an Abelian group $\mathfrak{M}$ of finite order m that one calls the *multiplier* of $\mathfrak{H}$.

If $\mathfrak{G}$ is a group whose center contains a subgroup $\mathfrak{A}$ such that $\mathfrak{G} / \mathfrak{A} \cong \mathfrak{H}$ then any absolutely irreducible full representation of $\mathfrak{G}$ will mediate an irreducible projective representation of $\mathfrak{H}$. Namely, since the central elements will be necessarily represented

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([227]) I. SCHUR: "Über die Darstellung der endlichen Gruppen durch gebrochene lineare Substitutionen," J. f. M. **127** (1904), 20-50.

([228]) Cf., M. TAZAWA: Sci. Rep. Tôhuku Univ., ser. I, **23** (1934), 76-88.

under the representation by multiples λI of the identity matrix, the representative matrices of the elements of a residue class of $\mathfrak{A}$ will always differ by only a numerical factor λ.

Such a group $\mathfrak{G}$ is called a *covering of* $\mathfrak{H}$ *extended by the Abelian group* $\mathfrak{A}$. One easily proves: *All projective representations of* $\mathfrak{H}$ *can be obtained in the manner above from full representations of the covering of* $\mathfrak{H}$ *extended by Abelian groups.*

A *sufficiently extended* covering of $\mathfrak{H}$ is a group $\mathfrak{G}$ with the behavior above such that any irreducible projective representation of $\mathfrak{H}$ is mediated by a full representation of $\mathfrak{G}$. It is necessary and sufficient for this that the intersection of $\mathfrak{A}$ with the commutator group of $\mathfrak{G}$ have the same order as $\mathfrak{M}$.

A sufficiently extended covering group of smallest order is called a *representation group* of $\mathfrak{H}$. $\mathfrak{G}$ is a representation group when $\mathfrak{A}$ is contained in the commutator group of $\mathfrak{G}$ and has the same order as $\mathfrak{M}$. One will then have $\mathfrak{M} \cong \mathfrak{A}$. With that, a criterion for a representation group is found that is also practical to apply as long as one knows the order m of $\mathfrak{M}$.

We will now construct a representation group $\mathfrak{G}$ of order mh as follows ([229]): If $s_1, \ldots, s_n$ are the generators of $\mathfrak{H}$, and $f_\lambda(s_\mu) = 1$ ($\lambda = 1, \ldots, q$) are the defining relations then we will first define an infinite group $\mathfrak{G}'$ with the generating elements $Q_1, \ldots, Q_n$ by the relations that state that the expressions:

$$f_\lambda(Q_\mu) = J_\lambda$$

should commute with all Q_λ. These J_λ then generate an Abelian group $\mathfrak{B}'$ in the center of $\mathfrak{G}'$ that can be represented as a direct product of a group that is isomorphic to $\mathfrak{M}$ and an infinite group precisely n generators $Z_1, \ldots, Z_n$. If one adds the relations $Z_1 = 1, \ldots, Z_n = 1$ to the relations that were defined above then $\mathfrak{G}'$ will go to a representation group $\mathfrak{G}$, and $\mathfrak{B}'$ will go to the group $\mathfrak{A}$, which is isomorphic to $\mathfrak{M}$.

There can be several non-isomorphic representation groups, but their groups $\mathfrak{A}$, as well as their commutator groups $\mathfrak{R}$, will all be mutually isomorphic. In a paper ([229]) that was cited already, I. SCHUR gave restrictions on the number of essentially different representation groups.

A group $\mathfrak{H}$ is called *closed* when it is its own representation group; in that case, any projective representation will be associated with a full representation. If follows from the construction that was given above that a group is closed when one can reduce the group $\mathfrak{B}'$ that was constructed there to the identity group by the addition of precisely n further relations $\Pi J_\lambda = 1$. More generally, one has: *If one can reduce the group* $\mathfrak{B}'$ *to a group of order* μ *by the addition of precisely n relations then* $m \leq \mu$.

([229]) I. SCHUR: J. reine angew. Math. **132** (1907), 85-137.

With the help of this theorem, one proves, with no further analysis, that all cyclic groups, as well as a series of prime power groups – and among them, one finds the quaternion group – are closed. One has further: If all Sylow groups of $\mathfrak{H}$ are closed then the same thing will be true for $\mathfrak{H}$. In particular, all groups of square-free order are then closed.

If $\mathfrak{L}$ is covering of $\mathfrak{H}$ extended by $\mathfrak{A}$ whose commutator group contains $\mathfrak{A}$ then the order m of the multiplier of $\mathfrak{H}$ will be a divisor of the product of the orders of $\mathfrak{A}$ and the multiplier of $\mathfrak{L}$. In particular, if $\mathfrak{L}$ is closed then $\mathfrak{L}$ will be a representation group of $\mathfrak{H}$.

In the last-mentioned paper ([229]), I. SCHUR gave the representation groups, as well as the characters, for a series of special groups that included the groups $SL(2, p^m)$ and $PGL(2, p^m)$. The multiplier groups of the finite, Abelian groups are inferred (*ibid.*, pp. 113) from a general theorem that allows one to express the multiplier of a direct product in terms of the multipliers of the direct factors. R. FRUCHT ([230]) determined the representation groups and the projective representations of the finite, Abelian groups completely.

I SCHUR ([231]) determined the representation groups for the symmetric groups $\mathfrak{S}_n$ and the alternating groups $\mathfrak{A}_n$, with the following result:

The groups $\mathfrak{A}_3$ and $\mathfrak{S}_3$ are closed. For $n > 3$, the groups $\mathfrak{S}_n$ have two representation groups $\mathfrak{T}_n$ and $\mathfrak{T}'_n$ of order $2 \cdot n!$ that are two-to-one homomorphic to $\mathfrak{S}_n$. For $n = 4$, $n = 5$, and $n > 7$, $\mathfrak{A}_n$ has a representation group $\mathfrak{B}_n$ of order $n!$ that is two-to-one homomorphic to $\mathfrak{A}_n$, namely, the commutator group of $\mathfrak{T}_n$. The representation groups $\mathfrak{C}_6$ and $\mathfrak{C}_7$ of $\mathfrak{A}_6$ and $\mathfrak{A}_7$, by contrast, have orders $3 \cdot 6!$ ($3 \cdot 7!$, resp.) and are six-to-one homomorphic to $\mathfrak{A}_6$ ($\mathfrak{A}_7$, resp.).

In order to find the projective representation of $\mathfrak{S}_n$ and $\mathfrak{A}_n$, one must find the full representations of the groups $\mathfrak{T}_n$, $\mathfrak{B}_n$, $\mathfrak{C}_6$, $\mathfrak{C}_7$. I. SCHUR solved this problem by calculating the characters, in principle. The result is the following: If one ignores the full representations of $\mathfrak{S}_n$ ($\mathfrak{A}_n$, resp.) then the representation of $\mathfrak{T}_n$ of lowest order is a representation of degree 2^e, where one sets $e = \left[\frac{n-1}{2}\right]$. This was also given explicitly [*loc. cit.* ([231]), section VI]. The remaining irreducible representation of $\mathfrak{T}_n$ will correspond to the decompositions of the number n into only distinct summands:

$$n = \nu_1 + \nu_2 + \ldots + \nu_m\,; \qquad (\nu_1 > \nu_2 > \ldots > \nu_m > 0),$$

and their degrees will be:

$$f_{\nu_1 \cdots \nu_m} = 2^{\left[\frac{n-m}{2}\right]} \frac{n!}{\nu_1!\nu_2!\cdots\nu_m!} \prod_{\alpha<\beta} \frac{\nu_\alpha - \nu_\beta}{\nu_\alpha + \nu_\beta}.$$

([230]) R. FRUCHT: J. reine angew. Math. **166** (1931), 16-29.
([231]) I. SCHUR: J. reine angew. Math. **139** (1911), 155-250.

In the event that the schemata that belong to two decompositions $n = \nu_1 + \nu_2 + \ldots + \nu_m$ go to each other under reflection (i.e., switching the rows and columns), the associated irreducible representations of $\mathfrak{T}_n$ will differ only by the signs of their matrices; they are then associated with each other (i.e., projectively equivalent).

If $n - m$ is odd then the representations of $\mathfrak{T}_n$ will also yield irreducible representations of the subgroups $\mathfrak{B}_n$. If $n - m$ is even then the representations of $\mathfrak{B}_n$ will split into two representations of equal degrees. The degrees of the projective representations of $\mathfrak{S}_n$ ($n \leq 7$) that are mediated by $\mathfrak{T}_n$ are given in the following table. The ones that split $\mathfrak{A}_n$ into two representations are underlined. The full representations stand before the separating line:

$n = 4$:	1, $\underline{2}$, 3,	2, $\underline{4}$
$n = 5$:	1, 4, 5, $\underline{6}$,	$\underline{4}$, 4, 6
$n = 6$:	1, 5, 5, 9, 10, $\underline{16}$,	4, 4, $\underline{16}$, $\underline{20}$
$n = 7$:	1, 6, 14, 14, 15, $\underline{20}$, 21, 35,	$\underline{8}$, 20, 20, $\underline{28}$, 36.

The groups $\mathfrak{C}_6$ and $\mathfrak{C}_7$ possess 31 (40, resp.) essentially different irreducible representations, and among them are 9 (12, resp.) pairs of complex-conjugate representations that do not appear already in the representations of $\mathfrak{T}_6$ ($\mathfrak{T}_7$, resp.). The degrees of the latter are:

$n = 6$:	3, 3, 6, 9, 15,	6, 6, 12, 12
$n = 7$:	6, 15, 15, 21, 21, 24, 24,	6, 6, 24, 24, 36.

The representations that are already mediated by a three-to-one homomorphic covering of $\mathfrak{A}_6$ are in front of the line.

The two projective representations of degree three of the group $\mathfrak{A}_6$ produce both of the ternary Valentiner groups (cf., § 9). The remaining groups that were mentioned in § 9, which are isomorphic to $\mathfrak{A}_5$, $\mathfrak{S}_5$, $\mathfrak{A}_6$, $\mathfrak{S}_6$, and $\mathfrak{A}_7$, are naturally represented in our table; in addition, one infers that there can be no other quaternary projective groups $\mathfrak{A}_n$ or $\mathfrak{S}_n$ ([232]). I. SCHUR gave the three projective representations of degree six of $\mathfrak{A}_7$ explicitly. The characters of the other ones can be achieved by composition and reduction.

K. ASANO ([232a]) has examined the representations of a finite group by *real* projective transformations.

([232]) Cf., H. MASCHKE: Math. Ann. **51** (1899), 253-294.
([232a]) K. ASANO: Proc. Imp. Acad. Jap. **9** (1933), 574-576.

§ 22. The rational representations of the general linear group.

The theory of representations of the full linear group $GL(\mathbb{K})$, where $\mathbb{K}$ is a field of characteristic zero, can be achieved completely by using algebraic methods, as long as one restricts oneself to the representations for which the elements of the representative matrices $T(\mathsf{A})$ are *entire rational functions* of the matrix elements of a transformation A in $GL(\mathbb{K})$ ([233]).

When one considers the center of the group $GL(\mathbb{K})$, which consists of the transformations λI, more closely, one easily proves that any entire rational representation will decompose completely into ones for which the elements of the representative matrices are *homogeneous functions* (say, of degree m) of the matrix elements $a_{\kappa\lambda}$ of A:

(1) $$d_{ik} = \sum c_{ik,\kappa_1\cdots\kappa_m,\lambda_1\cdots\lambda_m} a_{\kappa_1\lambda_1} a_{\kappa_2\lambda_2} \cdots a_{\kappa_m\lambda_m}\,.$$

We will call the number m the *rank* of the representation.

A particular representation is the *tensor representation* $\mathfrak{T}_m$ of rank m, which one can define as the product representation $\mathfrak{T}_1 \times \mathfrak{T}_1 \times \ldots \times \mathfrak{T}_1$, where $\mathfrak{T}_1$ is the *vector representation*, under which, the transformation A will be represented by its own matrix A. If $u_1, \ldots, u_n$ are the basis vectors of the n-dimensional vector space, and likewise $v_1, \ldots, v_n$ are those of a second (cogrediently transformed) vector space, etc., then the products $u_\lambda\, v_\mu \ldots w_\nu$ will be the basis tensors of the *tensor space of rank m* in which the tensor representation takes place; tensors will then be expressions of the form:

$$t = \sum t_{\lambda\mu\ldots\nu}\, u_\lambda\, v_\mu \ldots w_\nu\,,$$

which will be determined by n^m arbitrary tensor components $t_{\lambda\mu\ldots\nu}\, u_\lambda$. The matrices of the tensor representation will obviously be:

(2) $$a_{\kappa_1\cdots\kappa_m,\lambda_1\cdots\lambda_m} = a_{\kappa_1\lambda_1} a_{\kappa_2\lambda_2} \cdots a_{\kappa_m\lambda_m}\,.$$

As H. WEYL ([234]) showed quite simply, the linear hull of the set of matrices (2) consists of all *symmetric transformations* – i.e., those transformations of the tensor space into itself whose matrix elements remain invariant under any permutation Q that acts upon the sequence of κ and simultaneously on the sequence of λ. The system of symmetric transformations will be called $\mathfrak{S}$.

Everything that follows will rest upon the almost self-explanatory theorem:

([233]) For the theorems and methods of this paragraph, see I. SCHUR: "Über eine Klasse von Matrices, die sich einer gegebenen Matrix zuordnen lassen," Diss., Berlin, 1901. – H. WEYL: Math. Z. **23** (1925), 271-300. – I. SCHUR: S.-B. preuss. Akad. Wiss. (1927), 58-75. – H. WEYL: *Gruppentheorie und Quantenmechanik*, 2nd ed., Leipzig, 1931, chap. V.

([234]) H. WEYL: Ann. of Math. (2) **30** (1929), 499-516.

Any representation (1) *of rank m of* $GL(\mathbb{K})$ *can be extended in a unique way to a representation of the hypercomplex system* $\mathfrak{S}$; *therefore, equivalent (reducible, decomposable, resp.) representations of* $GL(\mathbb{K})$ *will again yield representations of* $\mathfrak{S}$ *of the same kind.*

The desired representation of $\mathfrak{S}$ will obviously be given by:

$$d_{ik} = \sum c_{ik,\kappa_1\cdots\kappa_m,\lambda_1\cdots\lambda_m} a_{\kappa_1\cdots\kappa_m,\lambda_1\cdots\lambda_m} . \tag{3}$$

We will now characterize the system $\mathfrak{S}$ in yet another way, and then prove that it is semi-simple.

The tensor t might go to Qt under the permutation Q that acts upon the locations of the tensor indices $\lambda_1, \ldots, \lambda_m$. The transformations that are induced in that way by the permutations Q define a system $\mathfrak{Q}$ of linear transformations of the tensor space $\mathfrak{M}$. The definition of $\mathfrak{S}$ can be also turned into: *The system* $\mathfrak{S}$ *consists of the transformations of* $\mathfrak{M}$ *into itself that commute with all transformations of the system* $\mathfrak{Q}$. Now, $\mathfrak{Q}$ will be a representation of the symmetric group $\mathfrak{S}_m$ (and thus completely reducible) in the event that the characteristic of the field $\mathbb{K}$ does not go into the order m! of the group, and thus especially in the case of characteristic zero. From § 11, it will follow immediately from this that the system $\mathfrak{S}$ is a direct sum of full matrix rings, and is thus semi-simple.

From § 12, it follows further from the semi-simplicity of $\mathfrak{S}$ that any representation of $\mathfrak{S}$ is completely reducible and that the irreducible representations are already contained in the regular representation. Now, $\mathfrak{S}$ is given from the outset as a system of linear transformations, and thus in a faithful representation; all irreducible representations appear in this representation at least once (otherwise it would not be faithful). It then follows that: *Any entire rational representation of the general linear group is completely reducible, and the irreducible representations of rank m are already contained in the tensor representation* $\mathfrak{T}_m$ *as components.*

From the theorem of RABINOWITSCH (§ 12), in order to be able to invert the commutation relation between $\mathfrak{Q}$ and $\mathfrak{S}$, we must add all linear combinations to $\mathfrak{Q}$. We achieve this when we extend the representation $\mathfrak{Q}$ of the group $\mathfrak{S}_n$ to a representation $\mathfrak{R}^*$ of the group ring $\mathfrak{R}$ of the group $\mathfrak{S}_n$ (cf., § 13). If $r = \sum \lambda_Q Q$ is an element of $\mathfrak{R}$ then in order to find the transformation in tensor space that induced by t, we must set:

$$r = \sum \lambda_Q \, Q \, t \, ; \tag{4}$$

the transformations thus obtained define the linear hull $\mathfrak{R}^*$ of $\mathfrak{Q}$.

From § 12, $\mathfrak{R}^*$ is also the system of transformations that commute with all transformations of $\mathfrak{S}$. From § 11, the subspaces of the tensor space $\mathfrak{M}$ that are invariant $\mathfrak{S}$ are in one-to-one correspondence with the right ideals $\mathfrak{r}^*$ of the ring $\mathfrak{R}^*$, where the concepts of equivalent, reducible, and decomposable carry over. If $\mathfrak{r}^*$ were generated by the idempotent e^*: $\mathfrak{r}^* = e^* \mathfrak{R}^*$, then one would have $\mathfrak{N} = \mathfrak{r}^*\mathfrak{M}$, i.e.: *An invariant subspace of the tensor space* $\mathfrak{M}$ *consists of all tensors of the form* e^*t, *where* e^* *is an idempotent of the ring* $\mathfrak{R}^*$ *of the operations (4), and where t runs through all tensors, as such.*

In this way, a decomposition of $\mathfrak{R}^*$ into minimal right ideals yields a decomposition of $\mathfrak{M}$ into subspaces that are irreducible under $\mathfrak{S}$.

In the case of $n \geq m$, $\mathfrak{R}^*$ will be a faithful representation of $\mathfrak{R}$; there will then be a tensor t with only one non-zero component $t_{12\ldots m}$ that, along with its permuted tensors Qt, will define a system of linearly-independent tensors, which is a system that is affected with the regular (faithful) representation of $\mathfrak{R}$ precisely. In this case, one can simply replace $\mathfrak{R}^*$ with the group ring $\mathfrak{R}$ in all of the theorems above.

By contrast, in the case of $n < m$, one will have $\mathfrak{R} \cong \mathfrak{R} / \mathfrak{R}_1$, where $\mathfrak{R}_1$ is a two-sided ideal of $\mathfrak{R}$ that is characterized by $\mathfrak{R}_1\mathfrak{M} = 0$. If one sets $\mathfrak{R} = \mathfrak{R}_1 + \mathfrak{R}_2$ then $\mathfrak{R}^*$ will be a faithful image of $\mathfrak{R}_2^*$. Any invariant subspace $\mathfrak{N}$ of $\mathfrak{M}$ can thus be obtained uniquely in the form:

$$\mathfrak{N} = \mathfrak{r}\,\mathfrak{M} = e\,\mathfrak{M},$$

where $\mathfrak{r} = e\,\mathfrak{R} = e\,\mathfrak{R}_2$ is a right ideal that is contained in $\mathfrak{R}_2$. The minimal left ideals of $\mathfrak{R}$ are contained in either $\mathfrak{R}_1$ or $\mathfrak{R}_2$; only the latter will give rise to irreducible spaces $\mathfrak{N} = \mathfrak{r}\,\mathfrak{M}$, while the former will always yield $\mathfrak{r}\,\mathfrak{M} = 0$.

From § 20, the generators of the minimal right ideal $\mathfrak{r}$ of $\mathfrak{R}$ have the form $e_\alpha = \lambda_\alpha P_\alpha N_\alpha$, where e_α is an idempotent $\alpha = (\alpha_1, \ldots, \alpha_h)$ refers to a tableau T_α. One now easily sees that for $h > n$ the operator N_α, and therefore also e_α, will annihilate any tensor t. *One then obtains a decomposition of* $\mathfrak{M}$ *into a sufficient system of irreducible subspaces* $\mathfrak{N} = e_\alpha \mathfrak{M} = P_\alpha N_\alpha \mathfrak{M}$ *when one restricts oneself to those tableaux* T_α *that contain at most n rows.* If one adds possible zeroes to the a_ν then one can assume that $h = n$. Any index combination $\alpha_1, \ldots, \alpha_h$ that satisfies the conditions (1), § 20 then corresponds to an irreducible representation of rank m of the group $GL(\mathbb{K})$. We shall call it $\mathfrak{F}$.

I. SCHUR determined the character Φ_α of $\mathfrak{F}$, which is also called the *characteristic* in this case, by algebraic methods, while H. WEYL determined it by transcendental ones

([233]). If $w_1, \ldots, w_n$ are the characteristic roots of the matrix A, and one sets (as in § 20) $b_\nu = a_\nu + n - n$ then $\Phi_\alpha(A)$ will be a quotient of n-rowed determinants ([235])([236]):

$$\Phi_\alpha(A) = \left| w_j^{\beta_k} \right| : | w_j{}^k | . \tag{5}$$

The following relations exist between the characters $\Phi_\alpha(A)$ of $GL(\mathrm{K})$ and $\chi_\alpha(Q)$ of $\mathfrak{S}_n$:

$$s_{\gamma_1} s_{\gamma_2} \cdots = \sum_\alpha \chi_\alpha(Q) \Phi_\alpha(A) \tag{6}$$

$$\Phi_\alpha(A) = \frac{1}{m!} \sum_\alpha \chi_\alpha(Q^{-1}) s_{\gamma_1} s_{\gamma_2} \cdots \tag{7}$$

In them, the permutation Q is again a product of cycles of lengths $\gamma_1, \gamma_2, \ldots$, and s_γ means $S(A^\gamma) = w_1{}^\gamma + w_2{}^\gamma + \ldots + w_n{}^\gamma$. One proves (6) when one calculates the trace of the transformation $Q \cdot T_m(A)$ in the full tensor space $\mathfrak{M}$ in two ways: First, by starting with the basis $u_\lambda\, v_\mu \ldots w_\nu$ (cf., the beginning of this paragraph), and then by decomposing $\mathfrak{M}$ into irreducible subspaces relative to the two commuting systems $\mathfrak{Q}$ and $\mathfrak{S}$ according to the schema of § 11. (7) follows from (6) on the basis of the orthogonality relations for the characters. According to I. SCHUR, one can employ (7) for the proof of (5), as well as for a new derivation of the generating function of the characters of the symmetric group $\mathfrak{S}_m$ (cf., § 20).

H. WEYL ([236]) carried out investigations into the representations of the complex and rotation groups that were similar to the ones that were presented here.

([235]) An equivalent rational expression was given by I. SCHUR: S.-B. preuss. Akad. Wiss. (1927), 71, formulas (37) and (39).

([236]) H. WEYL: Nachr. Ges. Wiss. Göttingen (1926), 235-243 – Math. Z. **35** (1932), 300-320.