CONTRIBUTION TO THE THEORY

OF

HUYGENS'S ENVELOPING-WAVE PRINCIPLE

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INTRODUCTION

Chapter I begins with a review of the partial differential equation for *waves.* We then study the integration of that equation by utilizing a method that is different from the classical one.

Chapter II is dedicated to the *enveloping-wave principle of Chr. Huygens*. We show that this principle results from the rule for integrating the partial differential equation for waves. We will then define the *ray* to be the locus of the point of contact of an *elementary Huygens wave* with its envelope.

In Chapter III, we simultaneously study waves and rays. They are nothing but the *bicharacteristics of J. Hadamard*, which are envisioned as one-dimensional manifolds, either in geometric space or in space-time.

In Chapter IV, we will define the *transport of a wave in geometric space* with the aid of the differential system of the bicharacteristics. The introduction of the notion of *transport velocity of a wave* obliges us to define the metric on geometric space. That velocity will then be given by an *invariant*; consequently, it will have an intrinsic meaning.

In Chapter V, we indicate how one can write the partial differential equation for waves in the form of a *Jacobi equation.* The Hamiltonian form of the ray equations will result from them immediately. When those equations are applied to the case of homogeneous physical media, that will permit us to establish some remarkable results. In particular, we will show that in the case of a Euclidian metric, the equation for the elementary Huygens wave will be the *point-like* equation that correlates with the partial differential equation for waves, when it is considered to be a tangential equation.

Chapter VI is dedicated to the *reflection and refraction of waves.* We show that with the aid of Huygens's principle, one can construct reflected and refracted waves that are produced by an incident wave, which is assumed to be given. We then establish the general geometric laws of reflection and refraction in the case where the separation surface is *moving.* Those laws are nothing but the *compatibility conditions* that relate to wave functions, which are conditions that must be verified at the separation surface in space-time.

We conclude this monograph with two applications of Huygens's enveloping-wave principle. First, in Chapter VII, we *apply it to the equations of electromagnetism*. After recalling some general equations, among which, one finds Maxwell's equations and the differential equations of transport for electromagnetic energy, we look for the partial differential equation for electromagnetic waves in the case of an *inhomogeneous* medium. That equation will lead to *a generalization of Fresnel's equation.* We then show that *the rays that are associated with the electromagnetic wave front are the trajectories of the electromagnetic energy.* The partial differential equation of electromagnetic *waves* and the differential system of electromagnetic *rays* then represent the *wave-like* and *corpuscular* aspects of the equations of electromagnetism. More generally, we have seen (Chap. I) how one can associate a differential system of mathematical physics with a partial differential equation for *waves* that is compatible with the equations of that system. On the other hand (Chaps. II and III), we have seen how the problem of integrating that wave equation will necessarily introduce *rays* along which the transport of the waves takes place. Consequently, the analysis that leads us to state *Huygens's* *enveloping-wave principle* will show both the *wave-like* and *corpuscular aspect of the equations of physics* (wave-corpuscle duality). We conclude that chapter with the search for the elementary electromagnetic wave equation in the case of a homogeneous medium. That equation will include the equation of the Fresnel wave surface as a special case.

Finally, Chapter VIII is dedicated to the study of the waves and rays that are compatible with a second-order, linear, partial differential equation, when it is considered to be *the fundamental equation of mathematical physics.* That will lead us to mention some general results that relate to the geodesics of a quadratic differential form that is associated with a second-order, linear, partial differential equation. We have obtained those results by systematically utilizing Jacobi's direct and inverse theorems, which will result when a variational principle is applied to a quadratic differential form.

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FIRST CHAPTER

INTEGRATING THE WAVE EQUATION

1. The wave equation. – Consider a physical phenomenon whose mathematical law is expressed by the differential system of order c:

$$\boxed{H_s(x^\alpha, z^r, z^r_{\alpha_1}, z^r_{\alpha_1\alpha_2}, \cdots, z^r_{\alpha_1\cdots\alpha_C}) = 0} \tag{1}$$

$$\begin{pmatrix} r, s = 1, 2, \ldots, m \\ \alpha_1, \alpha_2, \ldots, \alpha_r = 1, 2, \ldots, n \end{pmatrix}$$

that is composed of m partial differential equations in m unknown functions $z^1, \ldots, z^m$ of n independent variables $x^1, \ldots, x^n$, and in which we have set:

$$z^r_{\alpha_1\cdots\alpha_a} = \frac{\partial^a z^r}{\partial x^{\alpha_1} \cdots \partial x^{\alpha_a}} \qquad \begin{pmatrix} \alpha_1, \ldots, \alpha_a = 1, 2, \ldots, n \\ r = 1, 2, \ldots, m \end{pmatrix}. \tag{2}$$

The n^{th} variable x^n will represent time t; we can then set:

$$t \equiv x^n. \tag{3}$$

Suppose that the phenomena envisioned in space-time ($x^1, \ldots, x^{n-1}$; $x^n \equiv t$) are bounded by the hypersurface Ω whose equation is:

$$(\Omega) \qquad \Omega\,(x^1, \ldots, x^{n-1}; t) = 0. \tag{4}$$

That amounts to assuming that *at the instant t*, all points that are found on one side of the n – 2-dimensional surface of equation (4) in the n – 1-dimensional *geometric space* ($x^1, \ldots, x^{n-1}$) are under the influence of the phenomena in question; all of the points that are found on the other side of that surface can have no influence on that instant.

Let I and II represent the regions thus-defined at *the instant t* in the geometric space ($x^1, \ldots, x^{n-1}$); let I denote the region that is swept out (Fig. 1) Then, let:

$$z^r_{\mathrm{I}} \equiv z^r_{\mathrm{I}}(x^1, \ldots, x^{n-1}; t) \tag{5}$$

be the solution to equation (1) that represents the physical state of the medium I at the *instant t*. The physical state of the medium II at *the same* instant is defined by the solution:

$$z^r_{\mathrm{II}} \equiv z^r_{\mathrm{II}}(x^1, \ldots, x^{n-1}; t). \tag{6}$$

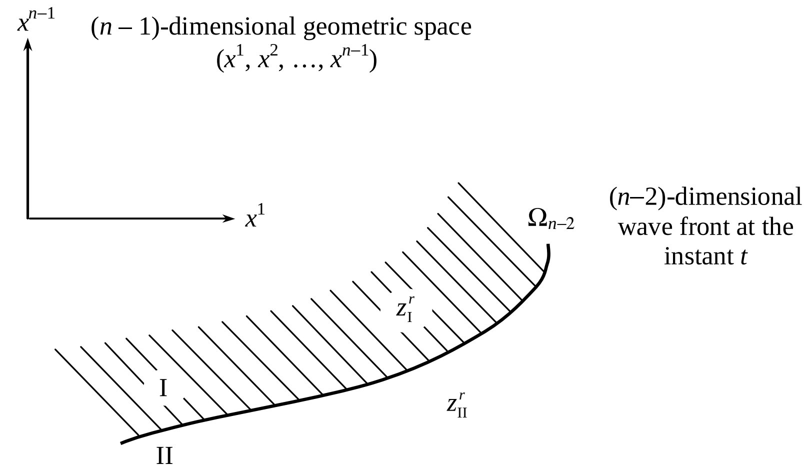

Figure 1.

Assume that the region I expands at the expense of region II; the $(n - 2)$-dimensional variety of equation (4) that bounds the region I that is swept out in geometric space $(x^1, \ldots, x^{n-1})$ *at the instant t* is what one calls the *wave front* $\Omega_{(n-2)}$ in physics. The $(n - 1)$-dimensional surface $\Omega_{(n-1)}$ in *space-time* that is defined by equation (4) constitutes a synthesis of the progress of the wave front $\Omega_{(n-2)}$ in the *geometric space* $(x^1, \ldots, x^{n-1})$ when time $t \equiv x^n$ varies.

That being the case, suppose that the functions (5) and (6) and their spatio-temporal derivatives up to order c inclusive vary continuously when one crosses the wave surface $\Omega_{(n-1)}$. In that case, one will have:

$$z_{\mathrm{II}}^r = z_{\mathrm{I}}^r, \qquad \frac{\partial z_{\mathrm{II}}^r}{\partial x^{\alpha_1}} = \frac{\partial z_{\mathrm{I}}^r}{\partial x^{\alpha_1}}, \qquad \frac{\partial^c z_{\mathrm{II}}^r}{\partial x^{\alpha_1} \cdots \partial x^{\alpha_c}} = \frac{\partial^c z_{\mathrm{I}}^r}{\partial x^{\alpha_1} \cdots \partial x^{\alpha_c}} \qquad [\text{on } \Omega_{(n-1)}]; \tag{7}$$

i.e., *by virtue of* $\Omega(x, t) = 0$. As a result, the two solutions $z_{\mathrm{I}}^1, \ldots, z_{\mathrm{I}}^m$ and $z_{\mathrm{II}}^1, \ldots, z_{\mathrm{II}}^m$ to equations (1) have contact of order c along the wave surface $\Omega_{(n-1)}$. It then results that the wave $\Omega_{(n-1)}$ is *a characteristic Cauchy variety* of system (1) of partial differential equation in mathematical physics.

Consequently, *the partial differential equation of waves that is compatible with the differential system* (1) *of physics is nothing but the equation of the characteristic varieties:*

$$\boxed{O(x^\alpha, \Omega_\alpha) \equiv \left\| \sum_{\alpha_1=1}^{n} \sum_{\alpha_2=1}^{n} \cdots \sum_{\alpha_c=1}^{n} \left(\frac{\partial H_s}{\partial z_{\alpha_1 \cdots \alpha_c}^r} \right) \Omega_{\alpha_1} \cdots \Omega_{\alpha_c} \right\| = 0} \tag{8}$$

$$(r, s = 1, \ldots, m)$$

whose left-hand side is a determinant of order m. The notation:

$$\left(\frac{\partial H_s}{\partial z^r_{\alpha_1\cdots\alpha_c}}\right)$$

represents what one will obtain after substituting the:

$$z^r,\ z^r_{\alpha_1},\ \ldots,\ z^r_{\alpha_1\cdots\alpha_c} \text{ in } \frac{\partial H_s}{\partial z^r_{\alpha_1\cdots\alpha_c}}$$

with their values in (5) and (6) when one takes (7) into account. The expressions:

$$\left(\frac{\partial H_s}{\partial z^r_{\alpha_1\cdots\alpha_c}}\right)$$

are functions of only the independent variables $x^1, \ldots, x^n \equiv t$ then. We have set:

$$\Omega_\alpha \equiv \frac{\partial \Omega}{\partial x^\alpha} \qquad (\alpha = 1, 2, \ldots, n). \tag{9}$$

When the derivatives:

$$\frac{\partial H_s}{\partial z^r_{\alpha_1\cdots\alpha_c}}$$

are independent functions z^r of their derivatives, the wave equation (8) will remain the same for any solution of the differential system (1). The same thing will then be true for the equations of mathematical physics that are linear with respect to the higher-order derivatives whose coefficients depend upon only the x.

By definition, any function $\Omega\ (x^\alpha)$ of the spatio-temporal variables such that $\Omega\ (x^\alpha, \Omega_\alpha) = 0$ is a *relative solution* of the wave equation (8). One says that the function $\Omega\ (x^\alpha)$ is *an absolute solution* of the wave equation (8) if one obtains an identity in $x^1, \ldots, x^n$ when one substitutes that function in the equation.

Any relative or absolute solution of the wave equation (8) *will define a wave surface* $\Omega_{(n-1)}$ *in space-time when it is equated to zero, as well as a wave surface* $\Omega_{(n-1)}$ *in geometric space.*

We remark that the left-hand side of the wave equation (8) is a homogeneous polynomial with respect to the partial derivatives $\Omega_1, \Omega_2, \ldots, \Omega_n$; let μ be its degree of homogeneity. That being the case, solve equation (4) with respect to t; hence:

$$(\Omega) \qquad t = t\ (x^1, \ldots, x^{n-1}) \tag{10}$$

and

$$t_i \equiv \frac{\partial t}{\partial x^i} = -\frac{\Omega_i}{\Omega_n} \qquad (i = 1, 2, \ldots, n-1). \tag{11}$$

It will then result from the homogeneity of (8) and the aforementioned definition of a *relative* solution of the wave equation that the partial differential equation:

$$\boxed{O^*\left(x,t,\frac{\partial t}{\partial x}\right)=0} \tag{12}$$

that one obtains by dividing the left-hand side of (8) by $(\Omega_n)^\mu$ will admit the function $t(x)$ that is defined by (10) as an (absolute) solution. In equation (12), x represents the spatial variables $x^1, \ldots, x^{n-1}$.

Any relative solution to the wave equation (8) *will define an absolute solution of the wave equation* (12) *when it is equated to zero.*

2. Integrating the wave equation. – That amounts to finding a function:

$$t\,(x^1, \ldots, x^{n-1}) \tag{13}$$

of the $(n - 1)$ geometric variables $(x^1, \ldots, x^{n-1})$ such that when one substitutes it into the wave equation (12), one will obtain an identity in $x^1, \ldots, x^{n-1}$ (viz., an *absolute solution*). As in (11), we set:

$$t_i \equiv \frac{\partial t}{\partial x^i} \qquad (i = 1, 2, \ldots, n-1) \tag{11′}$$

here.

In the $(2n - 1)$-dimensional space of *elements* $(x^1, \ldots, x^{n-1}, t\,;\, t_1, \ldots, t_{n-1})$, we let:

$$x^1, \ldots, x^{n-1}, t, t_1, \ldots, t_{n-1} \tag{14}$$

be the coordinates of a point-element in the $(2n - 2)$-dimensional variety $E_{(2n-2)}$ of the equation (12).

Now, let:

$$\delta x^1, \ldots, \delta x^{n-1}, \delta t, \delta t_1, \ldots, \delta t_{n-1} \tag{15}$$

be the components of any elementary displacement that is tangent to that variety E_{2n-2} at the point whose coordinates are (14); one will then have:

$$\delta O^* \equiv \sum_{i=1}^{n-1}\frac{\partial O^*}{\partial x^i}\delta x^i + \frac{\partial O^*}{\partial t}\delta t + \sum_{i=1}^{n-1}\frac{\partial O^*}{\partial t_i}\delta t_i = 0. \tag{16}$$

By virtue of (11′), one will have the relation that is called "contact" or "united elements":

$$\delta t - \sum_{i=1}^{n-1} t_i\,\delta x^i = 0. \tag{17}$$

We then say that (14) represents an *element* of an integral of the proposed equation (12). An element of the integral surface – or wave:

$$t\,(x^1, \ldots, x^{n-1}) \tag{18}$$

of the equation is then defined by a point on that $(n - 1)$-dimensional surface in space-time $(x^1, \ldots, x^{n-1}, t \equiv x^n)$ whose coordinates are $x^1, \ldots, x^{n-1}, t \equiv x^n$ and the tangent plane at that point that is defined by $t_1, \ldots, t_{n-1}$.

That being the case, we subject the elements of an arbitrary integral of (12) to a transport along the lines that are determined in the geometric space by the differential equations $(x^1, \ldots, x^{n-1})$:

$$\frac{dx^i}{X^i} = du \qquad (i = 1, 2, \ldots, n-1), \tag{19}$$

in which the denominators X^i will be specified later, and u refers to a parameter. Now, totally differentiate the function (13), conforming to equations (19); hence:

$$\frac{dt}{du} = \sum_{i=1}^{n-1} X^i\, t_i\,. \tag{20}$$

Finally, prolong the differential equations (19); hence, from (20):

$$\frac{dx^i}{X^i} = \frac{dt}{\sum_{k=1}^{n-1} X^k t_k} = \frac{dt_j}{T_j} = du \qquad (i, j = 1, 2, \ldots, n-1). \tag{21}$$

The functions X^i and T_i:

$$\left\{\begin{aligned} X^i &\equiv X^i(x^i, \ldots, x^{n-1}, t, t_1, \ldots, t_{n-1}) \\ T_i &\equiv T_i(x^i, \ldots, x^{n-1}, t, t_1, \ldots, t_{n-1}) \end{aligned}\right. \qquad (i = 1, 2, \ldots, n-1) \tag{22}$$

are defined by the following condition: *The* X^i *and* T_i *must be such that if the element* (14) *of an integral surface* (18) *varies according to* (21) *then the relations* (16) *and* (17) *will persist.*

The element envisioned (14) will then continue to verify the proposed equation and, at the same, remain a contact element when it displaces along a line (19) of the geometric space $(x^1, \ldots, x^{n-1})$. In order for that to be true, is it necessary and sufficient that one should have the transport relation:

$$\frac{d}{du}\left(\delta t - \sum_{i=1}^{n-1} t_i\, \delta x^i\right) \equiv A\ \delta O^* + B\left(\delta t - \sum_{i=1}^{n-1} t_i\, \delta x^i\right), \tag{23}$$

in which A and B are undetermined. That identity relation in $\delta x^1, \ldots, \delta x^{n-1}, \delta t, \delta t_1, \ldots, \delta t_{n-1}$ shows that if the contact condition (17) for an element of an integral of (12) is verified at a point of a line (19) then that will persist all along that line. Replace δO^* in (23) with its value and invert the order of the operators d / du and δ; after some easy calculations, one will get the identity:

$$\sum_{i=1}^{n-1}\left(T_i - Bt_i + A\frac{\partial O^*}{\partial x}\right)\delta x^i + \left(A\frac{\partial O^*}{\partial t} + B\right)\delta t + \sum_{i=1}^{n-1}\left(A\frac{\partial O^*}{\partial t_i} - X^i\right)\delta t_i \equiv 0 \tag{24}$$

in $\delta x^1, \ldots, \delta x^{n-1}, \delta t, \delta t_1, \ldots, \delta t_{n-1}$. Hence:

$$\begin{cases} B \equiv -A\dfrac{\partial O^*}{\partial t}, \\ X^i \equiv \quad A\dfrac{\partial O^*}{\partial t_i}, \qquad\qquad (i = 1, 2, \ldots, n-1). \\ T_i \equiv -A\left(\dfrac{\partial O^*}{\partial x^i} + \dfrac{\partial O^*}{\partial t}t_i\right) \equiv -A\dfrac{dO^*}{dx^i} \end{cases} \tag{25}$$

Finally, replace the X^i and T_i in (21) with their values (25); we will then get the differential equations:

$$\boxed{\frac{dx^i}{\dfrac{\partial O^*}{\partial t_i}} = \frac{dt}{\displaystyle\sum_{k=1}^{n-1}\frac{\partial O^*}{\partial t_k}t_k} = \frac{dt_i}{-\dfrac{dO^*}{dx^i}} = dv} \qquad (i = 1, 2, \ldots, n-1), \tag{26}$$

in which we have set:

$$dv \equiv A\,du. \tag{27}$$

The differential system (26) defines the *characteristic Cauchy varieties* ([1]) *of the wave equation* (12).

Let:

$$\begin{cases} x^i = x^i(v; x_0^1, \ldots, x_0^{n-1}, t_0, t_1^0, \ldots, t_{n-1}^0), \\ t = t(v; x_0^1, \ldots, x_0^{n-1}, t_0, t_1^0, \ldots, t_{n-1}^0), \qquad (i = 1, 2, \ldots, n-1) \\ t_i = t_i(v; x_0^1, \ldots, x_0^{n-1}, t_0, t_1^0, \ldots, t_{n-1}^0), \end{cases} \tag{28}$$

represent the general solution of the differential equation (26). The equations (28) are the ones of the simple infinitude of characteristic elements that contains the initial element $x_0^1, x_0^2, \ldots, x_0^{n-1}, t_0, t_1^0, \ldots, t_{n-1}^0$. The first n equations in (28) are those of the

([1]) E. GOURSAT, *Cours d'Analyse mathématique*, t. II (Paris, Gauthier-Villars, 1925); see pp. 629 and 634, and especially equation (101).

characteristic Cauchy line that issues from the point $x_0^1, x_0^2, \ldots, x_0^{n-1}, t_0$ (in space-time) to which one associates the contact element $x_0^1, x_0^2, \ldots, x_0^{n-1}, t_0, t_1^0, \ldots, t_{n-1}^0$.

Now, consider an $(n-2)$-dimensional variety $V_{(n-2)}$ in the $2n-1$-dimensional space $(x^1, x^2, \ldots, x^{n-1}, t, t_1, \ldots, t_{n-1})$ that is defined by the parametric equations:

$$(V_{(n-2)}) \qquad \begin{cases} x_0^i = x_0^i(u_1,\ldots,u_{n-2}), \\ t_0 = t_0(u_1,\ldots,u_{n-2}), \\ t_i^0 = t_i^0(u_1,\ldots,u_{n-2}), \end{cases} \tag{29}$$

which one supposes is such that:

$$O^*(x_0^i, t_0, t_i^0) \equiv 0 \tag{30}$$

and

$$\sum_{k=1}^{n-2}\left(\frac{\partial t_0}{\partial u_k} - \sum_{i=1}^{n-1} t_i^0 \frac{\partial x_0^i}{\partial u_k}\right)\delta u_k \equiv 0 \tag{31}$$

are two identities, the first of which is in $u_1, \ldots, u_{n-2}$, and the second of which is in $\delta u_1, \ldots, \delta u_{n-2}$. Hence:

$$\frac{\partial t_0}{\partial u_k} = \sum_{i=1}^{n-1} t_i^0 \frac{\partial x_0^i}{\partial u_k} \qquad (k = 1, 2, \ldots, n-2). \tag{32}$$

The transport of the variety $V_{(n-2)}$ whose parametric equations are (29) in conformity with equations (26) generates an $n-1$-dimensional integral surface – or wave – $\Omega_{(n-1)}$ in the space-time $(x^1, x^2, \ldots, x^{n-1}, t)$. In order to obtain the parametric equations:

$$\begin{cases} x^i = x_*^i(u_1,\ldots,u_{n-2}), \\ t = t_*(u_1,\ldots,u_{n-2}), \end{cases} \qquad (i = 1, 2, \ldots, n-1) \tag{33}$$

for the wave $\Omega_{(n-1)}$, it will suffice to replace the x_0^i, t_0, and t_i^0 in the first n equations (28) with their values in (29). We have set $u_{n-1} \equiv v$ in (33).

Finally, upon solving [1] the first $n-1$ equations in (33) for $u_1, \ldots, u_{n-1}$ and substituting the values for the u thus-obtained as functions of x in the last equation of (33), one will obtain the required integral in the form (13). The corresponding integral surface (or wave) will then be represented by equation (18) [2].

Theorem. – *If two integral surfaces* $\Omega'_{(n-1)}$ *and* $\Omega''_{(n-1)}$ *of the equations:*

$$t = t'(x^1, x^2, \ldots, x^{n-1}) \quad \text{and} \quad t = t''(x^1, x^2, \ldots, x^{n-1}) \tag{34}$$

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(1) That solution will be possible if the Jacobians of the x_0^i with respect to the u_k ($i = 1, 2, \ldots, n-1$; $k = 1, 2, \ldots, n-2$) are not all zero.

(2) The integration method that was developed above is applicable to any first-order partial differential equation.

have the ***element*** $x_0^1, \ldots, x_0^{n-1}, t_0, t_1^0, \ldots, t_{n-1}^0$ *in common in space-time* $(x^1, x^2, \ldots, x^{n-1}, t)$ *then they will agree all along the characteristic Cauchy line that issues from that element.* That line is defined by the first n equations (28).

Indeed, the differential equations (26) are independent of any integral of the proposed equation (12).

3. Special cases. –

1. Let $S_{(n-2)}$ represent the $(n - 2)$-dimensional variety in space-time $(x^1, x^2, \ldots, x^{n-1}, t)$ that is defined by the first n parametric equations in (29). The last $(n - 1)$ equations in (29) are then deduced from (32) and (30). It will then result from the considerations above that the integral surface of the wave equation (12) that contains the variety $S_{(n-2)}$ is the locus of characteristic Cauchy lines that issue from the points of that variety; hence:

Theorem II. – *The hypersurface in space-time that is generated by the characteristic Cauchy lines that issue from the points of any given (n – 2)-dimensional variety will be a wave surface* $\Omega_{(n-1)}$.

2. We now propose to determine the integral surface $\Omega_{(n-1)}$ of (12) that passes through an $(n - 2)$-dimensional variety $S_{(n-2)}^0$ that is taken at the initial instant $t_0 = 0$ in the geometric space $(x^1, x^2, \ldots, x^{n-1})$. Let:

$$S_{(n-2)}^0 \qquad \begin{cases} t = 0 \\ x^{n-1} = \psi(x^1, \ldots, x^{n-2}) \end{cases} \tag{35}$$

be the equations of that variety, in which the function ψ is assumed to have been given explicitly. Adopt $x_0^1, \ldots, x_0^{n-2}$ as independent variables and set:

$$u_k \equiv x_0^k \qquad (k = 1, 2, \ldots, n - 2). \tag{36}$$

Hence, it will result from (32) that:

$$t_k^0 + t_{n-1}^0 \frac{\partial \psi(x_0^1, \ldots, x_0^{n-2})}{\partial x_0^k} = 0 \qquad (k = 1, 2, \ldots, n - 2). \tag{37}$$

Upon replacing the $t_1^0, \ldots, t_{n-2}^0$ in (30) with their values in (37), one will find t_{n-1}^0 as a function of $x_0^1, \ldots, x_0^{n-2}$; hence, from (37), one will finally have:

$$t_i^0 = t_i^0(x_0^1, \ldots, x_0^{n-2}) \qquad (i = 1, 2, \ldots, n - 1). \tag{38}$$

Now, replace the t_i^0 that figure in the first n relations in (28) with their values (38), and then replace x_0^{n-1} with $\psi(x_0^1, \dots, x_0^{n-2})$ and replace t_0 with zero; hence:

$$\left.\begin{array}{r} x^i = x_*^i(v; x_0^1, \dots, x_0^{n-2}), \\ t = t_*(v; x_0^1, \dots, x_0^{n-2}). \end{array}\right\} \tag{39}$$

Now, solve the first $(n - 1)$ equations in (39) for v, x_0^1, x_0^2, …, x_0^{n-2}, and then substitute the values thus-obtained for v, x_0^1, x_0^2, …, x_0^{n-2} into the last relation in (39); from (18), one will get the *equation*:

$$(\Omega_{(n-1)}) \qquad \boxed{t = t(x^1, \dots, x^{n-1})} \tag{40}$$

of the integral surface $\Omega_{(n-1)}$ of (12) – i.e., of the wave surface $\Omega_{(n-1)}$ that is compatible with (1) – *that passes through the initial variety* $S_{(n-2)}^0$ *that was given in advance* ([1]).

Theorem III. – *The wave surface* $\Omega_{(n-1)}$ *of equation* (40) *is determined completely by the following conditions:*

1. *It satisfies the wave equation* (12).
2. *It passes through the initial variety* $S_{(n-2)}^0$ *in equations* (35).

Theorem IV. – *The wave surface* $\Omega_{(n-1)}$ *of equation* (40) *is the locus of characteristic Cauchy lines that issue from the points of the initial variety* $S_{(n-2)}^0$ *of equations* (35).

3. Consider a fixed point P_0 in space-time $(x^1, x^2, \dots, x^{n-1}, t)$ with coordinates $(x_0^1, x_0^2, \dots, x_0^{n-1}, t_0)$ and a direction at that point that is defined by $t_1^0, \dots, t_{n-1}^0$. Now, express the idea that the coordinates of the point P_0 and the direction coefficients $t_1^0, \dots, t_{n-1}^0$ verify the proposed equation (12); hence, one will have the identity relation (30). One can then suppose that:

$$t_i^0 = t_i^0(u_1, \dots, u_{n-2}) \qquad (i = 1, 2, \dots, n-1), \tag{41}$$

in which the functions t_i^0 are such that when one substitutes them in (30), one will get an identity in $u_1, \dots, u_{n-2}$. Next, replace the t_i^0 in the first n equations (28) with their aforementioned values. Hence, upon setting $u_{n-1} \equiv v_*$, one will get *the parametric equations:*

([1]) The variety $S_{(n-2)}^0$ is the initial position $\Omega_{(n-2)}^0$ of the wave surface $\Omega_{(n-2)}$ in geometric space. Recall that, in *space-time*, the wave $\Omega_{(n-1)}$ will constitute a synthesis of the advance of the wave $\Omega_{(n-2)}$ in *geometric space.*

$$\left.\begin{array}{r} x^i = x_*^i(u_1,\ldots,u_{n-1},x_0^1,\ldots,x_0^{n-1},t_0), \\ t = t_*(u_1,\ldots,u_{n-1},x_0^1,\ldots,x_0^{n-1},t_0) \end{array}\right\} \qquad (i = 1, 2, \ldots, n-1) \tag{42}$$

of the wave surface $\Omega_{(n-1)}$ that is the integral of (12) that is generated by the characteristic Cauchy lines that issue from the point P_0 ($x_0^1, \ldots, x_0^{n-1}, t_0$). Upon solving the first $(n - 1)$ equations (42) for $u_1, \ldots, u_{n-1}$ and replacing the u in the last of equations (42) with their values thus-calculated, one will get the equation of that integral surface in the *explicit* form:

$$\boxed{t = t(t_0; x_0^1, \ldots, x_0^{n-1}; x^1, x^2, \ldots, x^{n-1}).} \tag{43}$$

That wave surface $\Omega_{(n-1)}$ admits the point P_0 as a multiple point.

4. Bicharacteristics. – J. Hadamard's bicharacteristics of the physical differential system (1) are, by definition, the Cauchy characteristics of the corresponding wave equation (12). Those bicharacteristics are then determined by the differential equations (26) or by their general integral (28). Hence, one has the **theorems**:

a) *The surface in space-time that is generated by the bicharacteristics of the differential system* (1) *that issue of the points of any (n – 2)-dimensional variety that is given in advance is a wave* $\Omega_{(n-1)}$ *that is compatible with that system.*

b) *Two wave surfaces* $\Omega'_{(n-1)}$ *and* $\Omega''_{(n-1)}$ *in space-time that have a common contact element will coincide all along the bicharacteristic that issues from that element.*

The locus of bicharacteristic lines that issue from the point P_0 ($x_0^1, \ldots, x_0^n \equiv t_0$) is, by definition, J. Hadamard's *characteristic conoid* whose summit is P_0. That surface is given by equation (43). The following proposition will then result from Theorem I of § 2:

c) *The wave* $\Omega_{(n-1)}$ *of equation* (40) *is the envelope of the characteristic conoids whose summits are found on the initial variety* $S^0_{(n-2)}$ *of equations* (35).

d) *More generally, the envelope in space-time of the characteristic conoids whose summits are the points of any (n – 2)-dimensional variety that is given in advance is a wave* $\Omega_{(n-1)}$.

Remark. – Instead of considering the explicit form (10) for the equation of a wave surface, we return to the implicit form (4). The wave equation (12) will then take the homogeneous form (8), namely:

$$\mathbf{O}\,(x^\alpha, \Omega_\alpha) = 0. \tag{44}$$

Now, thanks to (3) and (11), *the differential equation* (26) *of the bicharacteristics* will likewise take a homogeneous form, namely:

$$\frac{dx^\alpha}{\dfrac{\partial \mathbf{O}(x^\beta, \Omega_\beta)}{\partial \Omega_\alpha}} = \frac{d\Omega}{\sum_{\gamma=1}^{n} \dfrac{\partial \mathbf{O}(x^\beta, \Omega_\beta)}{\partial \Omega_\gamma}\Omega_\gamma} = \frac{-d\Omega_\alpha}{\dfrac{\partial \mathbf{O}(x^\beta, \Omega_\beta)}{\partial x^\alpha}} \qquad (\alpha, \beta = 1, 2, \ldots, n). \tag{45}$$

Note that, thanks to Euler's theorem on homogeneous functions, one will have:

$$\sum_{\gamma=1}^{n} \frac{\partial \mathbf{O}(x^\beta, \Omega_\beta)}{\partial \Omega_\gamma}\Omega_\gamma \equiv \mu\, \mathbf{O}\,(x^\beta, \Omega_\beta); \tag{46}$$

hence, (45) will become:

$$\boxed{\frac{dx^\alpha}{\dfrac{\partial \mathbf{O}}{\partial \Omega_\alpha}} = \frac{d\Omega}{0} = \frac{-d\Omega_\alpha}{\dfrac{\partial \mathbf{O}}{\partial x^\alpha}}} \qquad (\alpha = 1, 2, \ldots, n), \tag{47}$$

when one takes (44) into account. As a result, *any bicharacteristic that issues from a point P that is taken on the wave* $\Omega_{(n-1)}$ *whose equation is* (4) *will be contained entirely on that wave* (see Fig. 2).

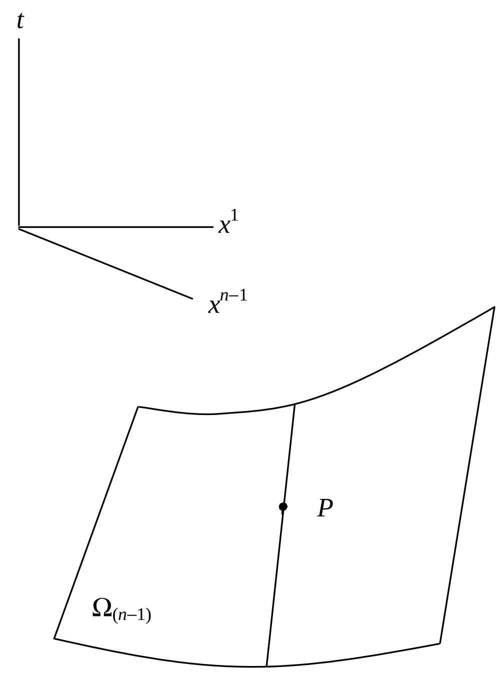

Figure 2.

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CHAPTER II

PRINCIPLE OF ENVELOPING WAVES

5. Geometric construction of an integral surface for the wave equation. – Consider the (n – 2)-dimensional variety in space-time ($x^1, x^2, \ldots, x^{n-1}, t$) whose equations are:

$$(\Omega^0_{(n-2)}) \qquad \begin{cases} t = t_0 \\ \Omega^0(x^1, \ldots, x^{n-1}) = 0. \end{cases} \tag{47}$$

We propose to construct the integral surface to the wave equation (12) that passes through the variety $(\Omega^0_{(n-2)})$, which is assumed to be known. We just saw that only one wave surface in space-time passes through that variety; let:

$$(\Omega_{(n-2)}) \qquad \Omega\,(x^1, x^2, \ldots, x^{n-1}, t) = 0 \tag{48}$$

be its equation.

The variety $(\Omega^0_{(n-2)})$ is the initial position ($t = t_0$) of the wave $\Omega_{(n-2)}$ in the geometric space ($x^1, x^2, \ldots, x^{n-1}$). At the instant t_1, the wave $\Omega_{(n-2)}$ will occupy the position in geometric space of the (n – 2)-dimensional variety that is defined by the equation:

$$(\Omega^0_{(n-2)}) \qquad \Omega\,(x^1, x^2, \ldots, x^{n-1}, t_1) = 0. \tag{49}$$

Equation (48) represents an (n – 1)-dimensional variety in space-time ($x^1, x^2, \ldots, x^{n-1}$, t) that constitutes the synthesis of the advance of the wave $(\Omega^0_{(n-2)})$ in the geometric space when t_1 varies (see Fig. 3).

The intersection $\bar{\Omega}^1_{(n-2)}$ of the wave surface $\Omega_{(n-1)}$ with the plane $\varpi^1_{(n-1)}$ whose equation is:

$$(\varpi^1_{(n-1)}) \qquad t = t_1 \tag{50}$$

determines the position $\Omega^1_{(n-2)}$ in geometric space of the wave at the instant t_1. Indeed, in order to obtain $\Omega^1_{(n-2)}$, it will suffice to project the variety $\bar{\Omega}^1_{(n-2)}$ onto the plane $\varpi^0_{(n-1)}$ whose equation is:

$$(\varpi^0_{(n-1)}) \qquad t = t_0 \tag{51}$$

and is parallel to the time axis (see Fig. 3) ([1]). Let P_0 be a point on $\Omega^0_{(n-2)}$ whose coordinates are $x_0^1, \ldots, x_0^{n-1}, t_0$, and let $\bar{\Gamma}^1_{(n-2)}$ denote the intersection of the conoid $\Gamma_{(n-1)}$ whose summit is P_0 with the plane $\varpi^1_{(n-1)}$. The conoid $\Gamma_{(n-1)}$ is tangent to the wave surface $\Omega_{(n-1)}$ along the bicharacteristic c_0 that issues from P_0. As a result, $\Gamma^1_{(n-2)}$ will be tangent to the variety $\bar{\Omega}^1_{(n-2)}$ at the point $\bar{P}_1$ where c_0 pierces the plane $\varpi^1_{(n-1)}$. *When P_0 describes $\Omega^0_{(n-2)}$, the bicharacteristic c_0 will generate the desired integral surface (or wave) $\Omega_{(n-1)}$*; furthermore, at the same time, the conoid whose summit is P_0 will envelope the surface $\Omega_{(n-1)}$, and the point $\bar{P}_1$ will describe the variety $\bar{\Omega}^1_{(n-2)}$ that envelopes the variety $\Gamma^1_{(n-2)}$.

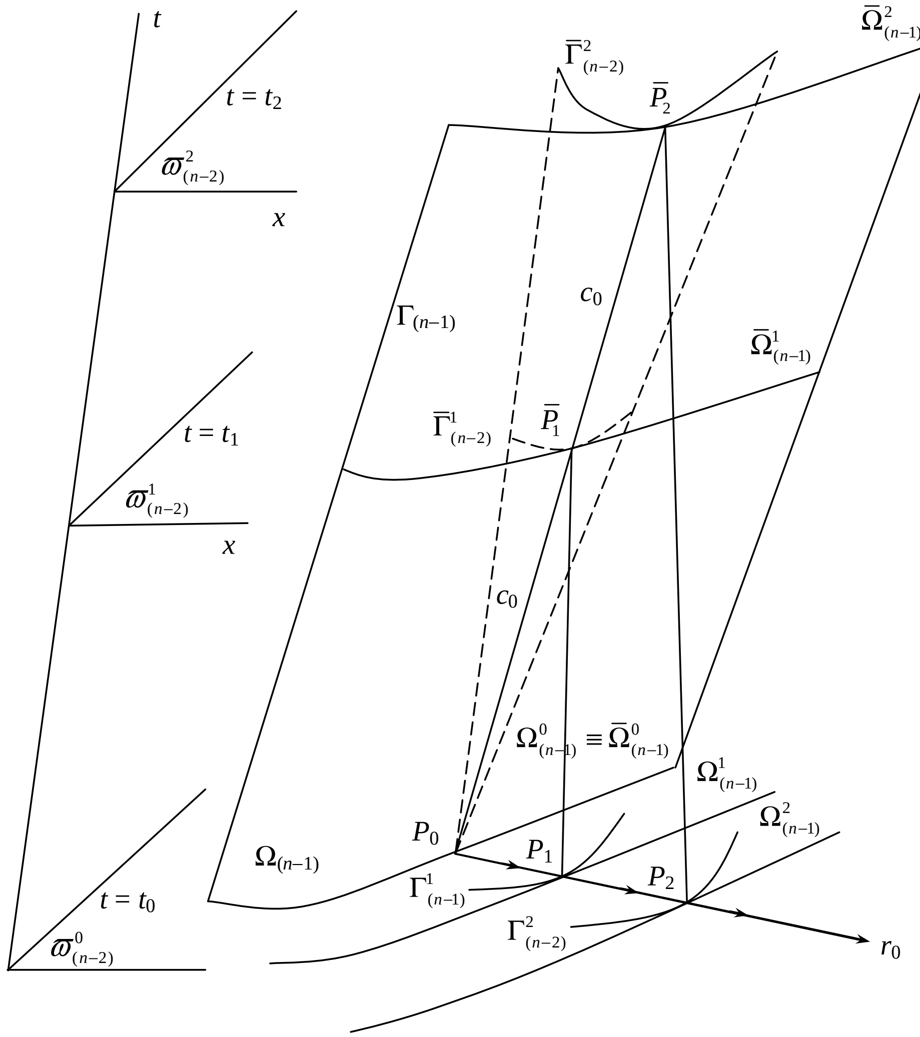

Figure 3.

([1]) In that figure, geometric space $(x^1, \ldots, x^{n-1})$ is represented by the plane $\varpi^0_{(n-1)}$ in space-time. The time t is reckoned along an axis perpendicular to that plane. The figure is therefore only schematic.

Similarly, project the variety $\bar{\Gamma}^1_{(n-2)}$ onto the plane $\varpi^0_{(n-1)}$ parallel to the t-axis. One will then obtain a variety $\Gamma^1_{(n-2)}$ that is tangent to $\Omega^1_{(n-2)}$ at the point P_1 that is the projection of $\bar{P}_1$ onto the plane $\varpi^0_{(n-1)}$. As a result, *the position of the wave at the instant* t_1 *is nothing but the envelope of the varieties* $\Gamma^1_{(n-2)}$. We have thus constructed the position of the wave at any later instant when we start from its initial position (*which is assumed to be known*); that will succeed in integrating the wave equation.

6. Principle of enveloping waves. – Consider the surface $\Omega^0_{(n-2)}$ in geometric space $(x^1, .., x^{n-1})$ to be the initial position of the front $\Omega_{(n-2)}$ of a disturbance that is governed by partial differential equations (1) of order c. In § **1**, we recalled how that disturbance propagates in geometric space in such a fashion that the front of the disturbance occupies the position of $\Omega^1_{(n-2)}$ at the current instant t_1. We likewise note that in the space-time $(x^1, .., x^{n-1}, t)$, the surface $\Omega_{(n-1)}$ will give a synthesis of the advance of the front of the disturbance in geometric space.

In particular, when $\Omega^0_{(n-2)}$ reduces to the point P_0, the front of the disturbance in geometric space at the instant t_1 will be the surface $\Gamma^1_{(n-2)}$. The advance of the front is exhibited in space-time by the characteristic conoid whose summit is at P_0. Consequently, one will see that the variety $\Gamma^1_{(n-2)}$ is nothing but the position at the instant t_1 of the *elementary Huygens wave* that issues from the geometric point P_0 $(x_0^1, \dots, x_0^{n-1})$ at the initial instant t_0. The propagation of the elementary Huygens wave that issues from the geometric point P_0 is then represented in space-time by the characteristic conoid whose summit is P_0 $(x_0^1, \dots, x_0^{n-1}, t_0)$. We say that the *Hadamard conoid is the elementary Huygens wave in space-time.*

Now, assume, with Huygens ([1]), that each point P_0 with coordinates $x_0^1, \dots, x_0^{n-1}$ of the position $\Omega^0_{(n-2)}$ of the wave $\Omega_{(n-1)}$ at the initial instant t_0 in geometric space communicates its disturbance to the ambient medium. We just saw that the propagation of the disturbance is represented in space-time by the characteristic conoid whose summit is the point $x_0^1, \dots, x_0^{n-1}, t_0$, and that at any instant $t_1 > t_0$, the front of that disturbance will occupy the position of the surface $\Gamma^1_{(n-2)}$ in geometric space. The *principle of enveloping waves* is expressed by:

([1]) C. HUYGENS, *Traité de la Lumière* (coll. "Les Maîtres de la Pensèe scientifique"), Paris, Gauthier-Villars, 1920; see pp. 21, second line: "…in such a way that there is a wave around each particle whose center is at that particle…" See also, pp. 22, Fig. 6.

The wave $\Omega^1_{(n-2)}$ *in geometric space, when taken at the instant* t_1, *is the envelope of the elementary Huygens waves* $\Gamma^1_{(n-2)}$ *that issue from the various points of the initial position* $\Omega^0_{(n-2)}$ *of the wave envisioned at the instant* t_0.

Now, we have seen that this is true (§ **5**); that principle is therefore only a theorem.

More generally, consider an arbitrary $(n-2)$-dimensional variety $S_{(n-2)}$ in space-time $(x^1, .., x^{n-1}, t)$. We know that such a variety will define *one* integral surface of the wave equation (12); i.e., a wave surface $\Omega_{(n-1)}$ in space-time. Thanks to the last two propositions in § **4**, one can determine the position of the wave in geometric space $(x^1, \dots, x^{n-1})$ at any instant with the aid of elementary Huygens waves. Indeed, let P be a point on the variety $S_{(n-2)}$ whose spatio-temporal coordinates are $\xi^1, .., \xi^{n-1}, \tau$. Let $\Gamma_{(n-2)}$ represent the point in the geometric space at the instant t_1 of the elementary Huygens wave that issues from the geometric point $(\xi^1, .., \xi^{n-1})$ at the instant τ_1. Hence, it will result immediately from the propositions that were mentioned in § **4** that *the position of the elementary Huygens wave in geometric space at the instant t is the envelope of the elementary Huygens waves* $\Gamma_{(n-2)}$ *that are taken at the same instant t.* We remark that *this principle is nothing but the rule for integrating first-order partial differential equations when stated from the viewpoint of wave theory.*

7. Rays of the wave $\Omega_{(n-2)}$. – We return to the initial position $\Omega^0_{(n-2)}$ of the wave $\Omega_{(n-2)}$ (Fig. 3). At the instant t_1, the elementary Huygens wave $\Gamma^1_{(n-2)}$ that issues from the point P_0 $(x_0^1, \dots, x_0^{n-1})$ of $\Omega^0_{(n-2)}$ at the initial instant $t_0 < t_1$ will be tangent to the wave $\Omega^1_{(n-2)}$ at the point P_1. By definition, *the ray* r_0 *of the wave that issues from the point* P_0 *is the locus of the point* P_1 *when* t_1 *varies.* One then sees that the projections onto the plane $\varpi^0_{(n-2)}$ of the Hadamard characteristics that issue from the points $(x_0^1, \dots, x_0^{n-1}, t_0)$ of $\Omega^0_{(n-2)}$ are nothing but the things that physicists call the *rays* of the wave $\Omega_{(n-2)}$. They are then the Hadamard bicharacteristics, when considered to be *one*-dimensional varieties in geometric space. Those considerations show that the ray r_0 that issues from the geometric point P_0 is the trajectory of the front of the disturbance that issues from that point at the initial instant t_0.

Theorem. – *Two wave surfaces in geometric space* $\Omega'_{(n-2)}$ *and* $\Omega''_{(n-2)}$ *that are tangent at a geometric point P at a certain instant t will remain tangent at that point when it traverses a ray.*

That being the case, let $\Omega^0_{(n-2)}$ be the position of the wave $\Omega_{(n-2)}$ at the instant t_2, let $\Gamma^2_{(n-2)}$ be the position at the same instant of the elementary Huygens wave that issues from the geometric point P_0 at the instant t_0, and let $\Gamma'^2_{(n-2)}$ be the position at the instant t_2

of the elementary Huygens wave that issues from the geometric point P_1 at the instant t_1 (see Fig. 4). The surfaces $\Gamma^2_{(n-2)}$ and $\Gamma'^2_{(n-2)}$ are tangent to the surface $\Omega^2_{(n-2)}$ at the point P_2 of the ray r_0 that issues from P_0 and passes through P_1 . The wave surface $\Omega^2_{(n-2)}$ is the envelope of *both* the elementary waves $\Gamma^2_{(n-2)}$ and $\Gamma'^2_{(n-2)}$. In a general fashion, one will then see that *in order to deduce the position* $\Omega^0_{(n-2)}$ *of a wave front* $\Omega_{(n-2)}$ *that is known at the instant* t_0 *when it is at a later instant* $t_2 > t_0$ *, one can first seek the position* $\Omega^1_{(n-2)}$ *at an intermediate instant* t_1 ($t_0 < t_1 < t_2$), *and the construct the final position* $\Omega^2_{(n-2)}$ *at the final instant* t_2 *with the aid of that intermediary position* $\Omega^1_{(n-2)}$. That proposition (which is an obvious consequence of the principle of enveloping waves) makes the mathematical notion of *group* appear in its most elementary form.

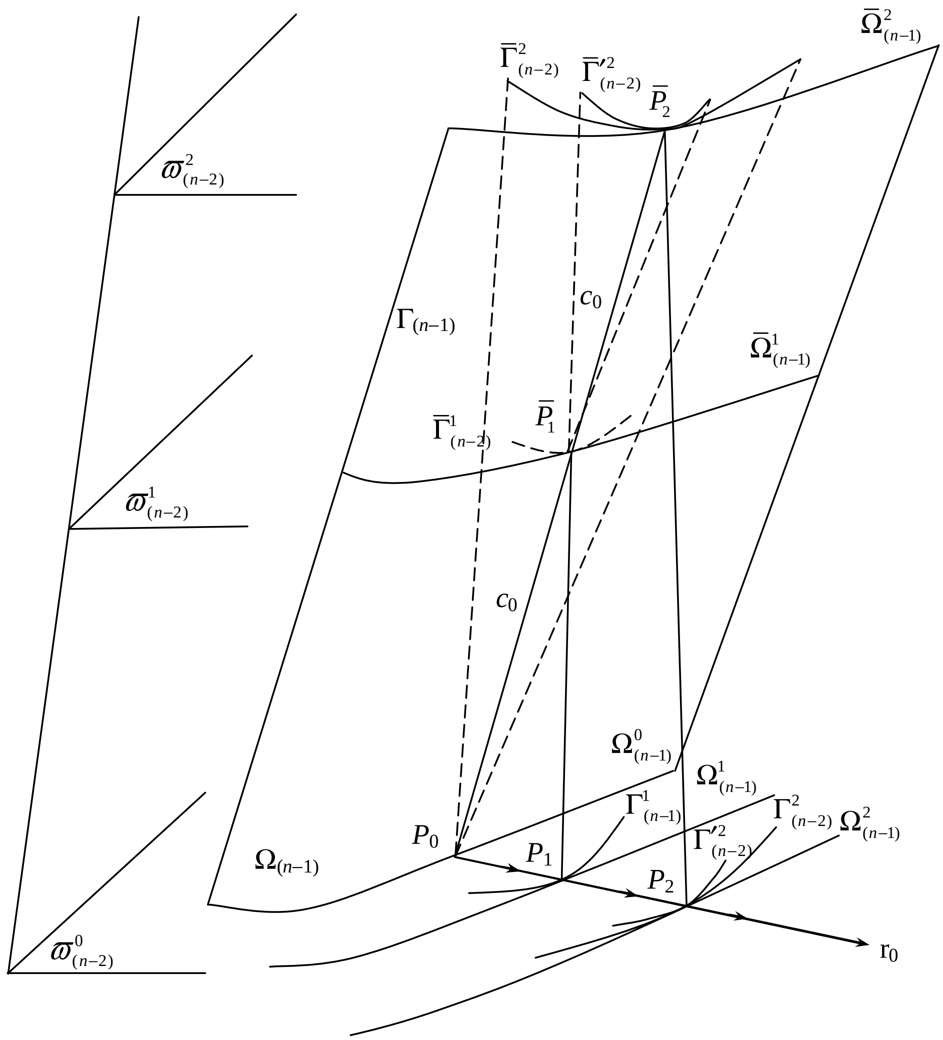

Figure 4.

CHAPTER III

WAVES AND RAYS

8. Return to the homogeneous wave equation. – We saw in Chapter I that any relative solution to the wave equation (8) will correspond to a wave surface $\Omega_{(n-1)}$.

Theorem. – *Any relative solution to the wave equation* (8) *can be associated with an absolute solution to that equation.*

Proof. – Indeed, let $\Omega\ (x^\alpha)$ be a relative solution; one will then have:

$$\mathbf{O}\ (x^\alpha, \Omega_\alpha) = 0, \tag{52}$$

by virtue of $\Omega\ (x^\alpha) = 0$. Solve the equation of the wave surface $\Omega_{(n-1)}$:

$$\Omega\ (x^1, \ldots, x^n) = 0 \tag{53}$$

for one of the variables x – for example, the variable $x^n \equiv t$; that will give the equation of the wave surface Ω in its *explicit* form:

$$\mathbf{O}^*(x^i, t, t_i) = 0. \tag{54}$$

That being the case, set:

$$\Omega^* \equiv t - t\ (x^1, \ldots, x^{n-1}); \tag{55}$$

hence, one will get the equation:

$$\Omega^*(x, t) = 0 \tag{53″}$$

for the wave surface $\Omega_{(n-1)}$ envisioned. One deduces from (55) that:

$$\Omega_i^* \equiv \frac{\partial \Omega^*}{\partial x^i} = -\frac{\partial t}{\partial x^i} \equiv -t_i \qquad \text{and} \qquad \Omega_n^* \equiv \frac{\partial \Omega^*}{\partial x^n} = 1 \qquad (i = 1, 2, \ldots, n-1). \tag{56}$$

However, it will then result from (54) that:

$$\mathbf{O}^*\left(x^i, t, -\frac{\Omega_i^*}{\Omega_n^*}\right) = 0; \tag{57}$$

hence:

$$\mathbf{O}\ (x^\alpha,\ \Omega_i^*) = 0 \qquad (\alpha = 1, 2, \ldots, n). \tag{58}$$

The function Ω^* that is defined in (55) is an *absolute* solution to the wave equation (8) in its homogeneous form. Q. E. D.

Conclusion. – One can then put the equation of a wave surface into a form such that the left-hand side of the equation is an absolute solution to the wave equation (8) in its homogeneous form. We give the name of *wave function* to the left-hand side of the equation for the wave surface that is defined in that way.

9. Homogeneous differential system for the bicharacteristics (or space-time rays). – Let Ω be an absolute solution of the wave equation (8); hence, thanks to (46), one will have the identity in $x^1, \ldots, x^n \equiv t$:

$$\sum_{\gamma=1}^{n} \frac{\partial \mathbf{O}(x^\beta, \Omega_\beta)}{\partial x^\gamma} \Omega_\gamma \equiv 0. \tag{59}$$

Hence, the homogeneous differential system of the bicharacteristics (or rays in space-time) is deduced from (45), when one takes (59) into account; it has the form:

$$\boxed{\frac{dx^\alpha}{\dfrac{\partial \mathbf{O}}{\partial \Omega_\alpha}} = \frac{d\Omega}{0} = \frac{-d\Omega_\alpha}{\dfrac{\partial \mathbf{O}}{\partial x^\alpha}} = d\theta} \qquad (\alpha = 1, 2, \ldots, n), \tag{60}$$

in which θ represents an arbitrary parameter.

One can now repeat the considerations that were developed in Chapter I when we started with the differential system (26). That is what we shall now do briefly ([1]).

The general integral of (60) has the form:

$$\left\{\begin{aligned} x^\alpha &= \xi^\alpha(\theta; x_0^1, \ldots, x_0^n, \Omega_1^0, \ldots, \Omega_n^0), \\ \Omega &= \text{invariant}, \\ \Omega_\alpha &= W_\alpha(\theta; x_0^1, \ldots, x_0^n, \Omega_1^0, \ldots, \Omega_n^0) \end{aligned}\right. \qquad (\alpha = 1, 2, \ldots, n). \tag{61}$$

As before, set:

$$x^n \equiv t \quad \text{and} \qquad x_0^n \equiv t_0\,. \tag{62}$$

One can then infer θ as a function of t, $x_0^1, \ldots, x_0^{n-1}, t_0, \Omega_1^0, \ldots, \Omega_{n-1}^0, \Omega_n^0 \equiv \Omega_t^0$ from the n^{th} equation in (61), and upon replacing θ with its value thus-obtained in the other equations (61), one will get:

$$x^i = \xi_*^i(t; x_0^1, \ldots, x_0^{n-1}, t_0, \Omega_1^0, \ldots, \Omega_{n-1}^0, \Omega_t^0)\,, \tag{63}$$

$$\Omega = \text{Invariant}, \tag{64}$$

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([1]) Note that here the space of elements $(x^1, x^2, \ldots, x^{n-1}, t, t_1, \ldots, t_{n-1})$ in § **2** is replaced with the space of elements $(x^1, \ldots, x^n, \Omega, \Omega_1, \ldots, \Omega_n)$.

$$\left.\begin{aligned}\Omega_i &= W_i^*(t; x_0^1, \ldots, x_0^{n-1}, t_0, \Omega_1^0, \ldots, \Omega_{n-1}^0, \Omega_t^0), \\ \Omega_t &= W_n^*(t; x_0^1, \ldots, x_0^{n-1}, t_0, \Omega_1^0, \ldots, \Omega_{n-1}^0, \Omega_t^0)\end{aligned}\right\} \qquad (i = 1, 2, \ldots, n-1). \tag{65}$$

That being the case, we propose to look for a solution Ω to the wave equation (8):

$$\mathbf{O}(x^\alpha, \Omega_\alpha) = 0 \qquad \text{or} \qquad \mathbf{O}\left(x, t, \frac{\partial\Omega}{\partial x}, \frac{\partial\Omega}{\partial t}\right) = 0, \tag{66}$$

which will reduce to a function $\Omega^0(x^1, \ldots, x^{n-1})$ of the spatial variables for:

$$t = t_0 . \tag{67}$$

We shall utilize the variables x_0^1, x_0^2, …, x_0^{n-1} for the independent variables. The solution Ω at the point (x_0^1, …, x_0^{n-1}) in geometric space at the instant t_0 will then have a value Ω^0 that is given by:

$$\Omega^0 \equiv \Omega^0(x_0^1, \ldots, x_0^{n-1}). \tag{68}$$

Consequently:

$$\Omega_i^0 \equiv \frac{\partial\Omega^0(x_0)}{\partial x_0^i} \qquad (i = 1, 2, \ldots, n-1). \tag{69}$$

Now, replace the Ω_i^0 in:

$$\mathbf{O}(x_0^1, \ldots, x_0^{n-1}, t_0, \Omega_1^0, \ldots, \Omega_{n-1}^0, \Omega_t^0) \tag{70}$$

with their values (69) that are deduced from (68); hence:

$$\Omega_t^0 = \Omega_t^0(x_0^1, \ldots, x_0^{n-1}, t_0). \tag{71}$$

Now return to formulas (63) and (64), in which one must replace Ω_i^0 and Ω_t^0 with their values that were calculated above in (69) and (71), resp.; one will find that:

$$x^i = x^i(t; t_0, x_0^1, \ldots, x_0^{n-1}), \tag{72}$$

$$\Omega = \Omega^0(x_0^1, \ldots, x_0^{n-1}). \tag{73}$$

The relation (73) is an immediate consequence of the invariance of Ω with respect to the differential system (60). As for equations (72), they represent the *ray* in geometric space that issues from the initial point (x_0^1, …, x_0^{n-1}) at the initial instant t_0. Upon solving the relations (72) for x_0^i, one will find the equations:

$$x_0^i = x^i(t_0, t, x^1, \ldots, x^{n-1}), \tag{74}$$

in which the functions x^i are identical to the ones that figured in (72).

Finally, replace the initial coordinates x_0^i in (73) with their values in (74); that will imply the required integral:

$$\Omega = \Omega\,(x^1, \ldots, x^{n-1}, t). \tag{75}$$

It results from the invariant character of Ω with respect to the differential system (60) that the surface whose equation is:

$$\Omega\,(x^1, \ldots, x^{n-1}, t) = 0 \tag{76}$$

will be the position at the *instant t* in geometric space $(x^1, \ldots, x^{n-1})$ of the wave $\Omega_{(n-2)}$ that coincides with the surface whose equation is:

$$\Omega^0\,(x^1, \ldots, x^{n-1}, t) = 0 \tag{77}$$

at the *initial instant* t_0 .

In order to geometrically obtain the wave surface $\Omega_{(n-2)}$ at an arbitrary instant t when one starts from its initial position $\Omega^0_{(n-2)}$, it will suffice to consider the point $P(x^1, \ldots, x^{n-1})$ on the ray (72) that issues from the point $P_0\,(x_0^1, \ldots, x_0^{n-1})$ on $\Omega^0_{(n-2)}$ at the instant $t_0 < t$ that corresponds to the instant t. *The wave surface* $\Omega_{(n-2)}$ *will then be the locus of the point P* at the instant t when P_0 describes the surface $\Omega^0_{(n-2)}$.

One can recall Theorems I, II, III, and IV of §§ **2** and **3** here; it will suffice to replace the "Cauchy characteristic line" with "bicharacteristic (or space-time ray)." We confine ourselves to recalling the following two propositions here:

1. *Two wave surfaces* $\Omega'_{(n-1)}$ *and* $\Omega''_{(n-1)}$ *whose equations are:*

$$\Omega'\,(x^1, \ldots, x^n) = 0 \quad \text{and} \quad \Omega''\,(x^1, \ldots, x^n) = 0, \quad \text{resp.,} \tag{78}$$

and have an element $(x^1, \ldots, x^n, \Omega_1, \ldots, \Omega_n)$ *in common in space-time* $(x^1, \ldots, x^n)$ *will agree along the bicharacteristic (or space-time ray) that issues from that element.*

2. *Two wave surfaces* $\Omega'_{(n-2)}$ *and* $\Omega''_{(n-2)}$ *whose equations are:*

$$\Omega'\,(x^1, \ldots, x^{n-1}, t) = 0 \quad \text{and} \quad \Omega''\,(x^1, \ldots, x^{n-1}, t) = 0, \quad \text{resp.,} \tag{78}$$

and are tangent at a point in geometric space $(x^1, \ldots, x^{n-1})$ *at a certain instant will remain tangent at that point if they traverse a ray in geometric space* ([1]).

([1]) See § **7**.

10. Equation for the elementary Huygens wave. – Consider a fixed point P_0 in geometric space ($x^1, \ldots, x^{n-1}$) whose coordinates are ($x_0^1, \ldots, x_0^{n-1}$) and an arbitrary direction at that point that is defined by the $n - 2$ numbers $p_1, p_1, \ldots, p_{n-2}$. One will then have:

$$\Omega_i^0 = \Omega_i^0 (p_1, \ldots, p_{n-2}); \tag{80}$$

hence, from (70) and (71):

$$\Omega_t^0 = \Omega_t^0 (p_1, \ldots, p_{n-2}). \tag{80′}$$

Finally, replace the Ω_i^0 and Ω_t^0 in (63) with their values in (80) and (80′), resp.; that will give the parametric equations:

$$x^i = \chi^i (t\,;\ x_0^1, \ldots, x_0^{n-1}, t_0; p_1, \ldots, p_{n-2}) \qquad (i = 1, 2, \ldots, n-1) \tag{81}$$

for the position that is occupied at the instant t by the elementary Huygens wave that issues from the point P_0 whose coordinates are ($x_0^1, \ldots, x_0^{n-1}$) at the instant t_0 . In order to obtain the elementary wave (81) geometrically, it will suffice to consider the point P ($x^1, \ldots, x^{n-1}$) on *one* ray (63) that issues from the point P_0 ($x_0^1, \ldots, x_0^{n-1}$) at the instant t_0 that corresponds to the instant $t > t_0$. The elementary wave will be the locus of the point P when the tangent to the ray at P_0 varies.

The elimination of the parameters $p_1, p_2, \ldots, p_{n-2}$ from the ($n - 1$) equations (81) will lead to the finite equation of the elementary Huygens wave.

Recall that *an elementary Huygens wave* will generate *a characteristic Hadamard conoid* in space-time when time t varies. Finally, it will result from the propositions that were mentioned at the end of § **9** that in order to obtain the wave $\Omega_{(n-2)}$ in equation (76) geometrically when starting from its initial position $\Omega_{(n-2)}^0$ in equation (77), *one can always take the envelope at the instant t of the elementary Huygens waves that issue from the points of* $\Omega_{(n-2)}^0$ *at the initial instant* $t_0 < t$ (recall the principle of enveloping waves).

CHAPTER IV

WAVE TRANSPORT

11. Metric on geometric space. – By definition, the metric on geometric space (x^1, …, x^{n-1}) at the instant t – i.e., when:

$$\delta t \equiv 0, \tag{82}$$

is determined by the differential quadratic form:

$$(\delta\sigma)^2 \equiv \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} B_{ij}\,\delta x^i\,\delta x^j \qquad (i, j = 1, 2, \ldots, n-1), \tag{83}$$

which is assumed to be invariant under an arbitrary change of spatial variables x^1, …, x^{n-1}. By hypothesis, one will have:

$$B_{ij}(x, t) \equiv B_{ji}(x, t), \tag{84}$$

in which x represents the geometric or spatial variables x^1, …, x^{n-1}.

Let B^{ij} the minor of the $(n-1)$-order determinant $\| B_{kl} \|$ that corresponds to the element B_{ij}, divided by $\| B_{ij} \|$:

$$B^{ij} \equiv \frac{\text{minor of } B_{ij}}{\| B_{kl} \|}. \tag{85}$$

Hence, the $B_{ij}(x, t)$ and the $B^{ij}(x, t)$ are the covariant and contravariant components, resp. – *under an arbitrary change of variables* x^1, …, x^{n-1} – of second-order symmetric tensor.

Set:

$$\Delta_1\,\Omega = \sum_{i=1}^{n-1}\sum_{j=1}^{n-1} B^{ij}\,\Omega_i\,\Omega_j\,. \tag{80}$$

The covariant and contravariant components of the unit vector **N** that is normal to the wave $\Omega_{(n-2)}$ are, in turn, given by:

$$N_i \equiv \frac{\Omega_i}{\sqrt{\Delta_i\,\Omega}}, \qquad N^i \equiv \sum_{j=1}^{n-1} B^{ij} N_j \qquad (i = 1, 2, \ldots, n-1), \tag{81}$$

respectively, when the equation for the wave $\Omega_{(n-2)}$ is given by (4).

Recall that the Lamé differential parameter $\Delta_1\Omega$ is invariant under an arbitrary change of variables x^1, …, x^{n-1}, in which the wave function $\Omega(x, t)$ is considered to be an invariant in the broader sense.

12. Differential equations of the rays of a wave. – Let:

$$\Omega \equiv \Omega\,(x, t) \tag{88}$$

be a wave function (see § **8**). One will then have the identity in x and t:

$$\mathbf{O}\left(x, t, \frac{\partial\Omega}{\partial x}, \frac{\partial\Omega}{\partial t}\right) \equiv 0. \tag{89}$$

It will then result from the differential system (60) that the rays of the wave equation:

$$\Omega\,(x, t) = 0 \tag{90}$$

are given by the differential equations ([1]):

$$\boxed{\frac{dx^i}{\left(\dfrac{\partial\mathbf{O}}{\partial\Omega_i}\right)} = \frac{dt}{\left(\dfrac{\partial\mathbf{O}}{\partial\Omega_t}\right)} = d\theta} \qquad (i = 1, 2, \ldots, n-1) \tag{91}$$

with:

$$\frac{d\Omega}{d\theta} \equiv 0. \tag{92}$$

The notations $\left(\dfrac{\partial\mathbf{O}}{\partial\Omega_i}\right)$ and $\left(\dfrac{\partial\mathbf{O}}{\partial\Omega_t}\right)$ signify that one has replaced the Ω_i and Ω_t in $\dfrac{\partial\mathbf{O}}{\partial\Omega_i}$ and $\dfrac{\partial\mathbf{O}}{\partial\Omega_t}$ with the first partial derivatives of the wave function (88). Recall that:

$$\Omega_i \equiv \frac{\partial\Omega}{\partial x^i}, \qquad \Omega_t \equiv \frac{\partial\Omega}{\partial t}. \tag{93}$$

The identity relation (92) expresses the idea that *any wave function* $\Omega(x, t)$ *is an invariant of the differential system of the rays.*

13. Wave transport in geometric space ([2]). – Now, set:

$$w^i \equiv w^i\,(x, t) \equiv \left(\frac{\partial\mathbf{O}}{\partial\Omega_i}\right) : \left(\frac{\partial\mathbf{O}}{\partial\Omega_t}\right) \qquad (i = 1, 2, \ldots, n-1). \tag{94}$$

([1]) Upon replacing Ω in $\partial\Omega\,/\,\partial x^\alpha$ with the wave function (88), the last n equations of the differential system (60) will be satisfied identically.

([2]) J. VAN MIEGHEM, *Étude sur la théorie des ondes* (Paris, Gauthier-Villars, 1934); see pp. 99, *et seq.*

The transport of the wave $\Omega_{(n-2)}$ of equation (90) in geometric space $(x^1, \ldots, x^{n-1})$ is defined by the differential system:

$$\boxed{\frac{dx^i}{w^i} = dt} \qquad (i = 1, 2, \ldots, n-1). \tag{95}$$

The identity (92) becomes:

$$\boxed{\frac{d\Omega}{dt} = \frac{\partial\Omega}{\partial t} + \sum_{i=1}^{n-1} \frac{\partial\Omega}{\partial x^i} w^i \equiv 0,} \tag{96}$$

which is an identity in $x^1, \ldots, x^{n-1}$ and t.

Theorem. – *Any wave function is an invariant of the differential system of wave transport that is defined by that function.*

Definition. – *The velocity of transport T of the wave $\Omega_{(n-2)}$ is the component w_N normal to the wave $\Omega_{(n-2)}$ of the velocity vector w whose components are $w^1, \ldots, w^{n-1}$.* One will then have:

$$\boxed{T \equiv w_N .} \tag{97}$$

It will then result from the identity (96) and formulas (87) that:

$$\sum_{i=1}^{n-1} w^i N_i \equiv \sum_{i=1}^{n-1} w_i N^i \equiv -\frac{\Omega_i}{\sqrt{\Delta_1\Omega}}, \tag{98}$$

in which $w^1, \ldots, w^{n-1}$ are the contravariant components of the velocity vector w whose covariant components are represented by $w_1, \ldots, w_{n-1}$; hence:

$$w_i \equiv \sum_{i=1}^{n-1} B_{ij} w^j, \qquad w^i \equiv \sum_{i=1}^{n-1} B^{ij} w_j . \tag{99}$$

However, one obviously has:

$$w_N \equiv \sum_{i=1}^{n-1} w^i N_i \equiv \sum_{i=1}^{n-1} w_i N^i . \tag{100}$$

It then results from (98) that *the component w_N of the velocity w that is normal to the wave Ω is an invariant.*

Finally, by virtue of the definition (97), that invariant is nothing but *the transport velocity T of the wave $\Omega_{(n-2)}$* ; hence:

$$\boxed{T(x,t) \equiv \frac{-\partial\Omega(x,t)/\partial t}{\sqrt{\Delta_1\Omega(x,t)}}.} \tag{101}$$

CHAPTER V

JACOBIAN FORM OF THE WAVE EQUATION AND HAMILTONIAN FORM OF THE RAY EQUATIONS

14. Jacobian form of the wave equation. – The homogeneity of the wave equation (8) permits one to write it in the form:

$$\mathbf{O}(x^\alpha, \Omega_\alpha) \equiv P_0\,(\Omega_n)^\mu + P_1\,(\Omega_n)^{\mu-1} + \ldots + P_k\,(\Omega_n)^{\mu-k} + \ldots + P_\mu = 0, \tag{102}$$

in which $P_0, P_1, \ldots, P_k, \ldots, P_\mu$, are homogeneous polynomials in $\Omega_1, \ldots, \Omega_{n-1}$ that have degree $0, 1, \ldots, k, \ldots, \mu$, respectively; μ is the degree of the homogeneity of $\mathbf{O}(x^\alpha, \Omega_\alpha)$.

We shall always denote the temporal variable by x^n or t. Let:

$$-H_1\left(x, t, \frac{\partial\Omega}{\partial x}\right), \ldots, \ -H_r\left(x, t, \frac{\partial\Omega}{\partial t}\right) \tag{103}$$

be the r ($\leq \mu$) distinct roots of equation (102) in Ω_n or $\Omega_t \equiv \partial\Omega\,(x, t) / \partial t$. The functions (103) are homogeneous and of degree *one* in $\Omega_1, \ldots, \Omega_{n-1}$. Recall that $\partial\Omega / \partial x$ represents the partial derivatives of Ω with respect to the spatial variables $x^1, \ldots, x^{n-1}$.

Each of the r *Jacobian equations:*

$$\boxed{\frac{\partial\Omega}{\partial t} + H_p\left(x, t, \frac{\partial\Omega}{\partial x}\right) = 0} \qquad (p = 1, 2, \ldots, r) \tag{104}$$

is the equation of a *family of wave surfaces* then. We let $\Omega^{(p)}$ denote a wave of that family.

15. Hamiltonian form of the ray equations. – The Cauchy characteristics of the wave equation (104) are defined by the *Hamiltonian equations:*

$$\frac{dx^i}{dt} = \frac{\partial H_p}{\partial \Omega_i}, \tag{105}$$

$$\frac{d\Omega_i}{dt} = -\frac{\partial H_p}{\partial x^i} \tag{106}$$

$$(i = 1, 2, \ldots, n-1).$$

Equations (105) admit any solution:

$$\Omega^{(p)}\,(x, t)$$

of the wave equation (104) as an *invariant*, because if one takes the homogeneity of H_μ into account then one will have, in turn:

$$\frac{d\Omega^m}{dt} \equiv \frac{\partial \Omega^{(p)}}{\partial t} + \sum_{i=1}^{n-1} \frac{\partial \Omega^{(p)}}{\partial x^i} \frac{\partial H_p}{\partial \Omega_i^{(p)}} \equiv \frac{\partial \Omega^{(p)}}{\partial t} + H_p \left(x, t, \frac{\partial \Omega^{(p)}}{\partial x} \right) \equiv 0. \tag{107}$$

The differential equations (105) indeed define the transport of a wave $\Omega^{(p)}$ then; *the lines along which that transport takes place are* (as we saw above) *the rays of the wave whose equation in geometric space* ($x^1, \ldots, x^{n-1}$) *is:*

$$\Omega^{(p)} (x, t) = 0. \tag{108}$$

Theorem I. – *I say that the Pfaff form:*

$$\sum_{i=1}^{n-1} \Omega_i^{(p)} \, \delta x^i \tag{109}$$

*is an **absolute** differential invariant of the Hamiltonian system* (105) *and* (106). The symbol δ represents a "truncated" variation; i.e., $\delta t \equiv 0$.

Proof. – Indeed, thanks to (105) and (106), one will have:

$$\frac{d}{dt} \sum_{i=1}^{n-1} \Omega_i^{(p)} \, \delta x^i \equiv - \sum_{i=1}^{n-1} \frac{\partial H_p}{\partial x^i} \delta x^i + \sum_{i=1}^{n-1} \Omega_i^{(p)} \delta \frac{\partial H_p}{\partial \Omega_i^{(p)}} \; . \tag{110}$$

However, due to the homogeneity of H_p, it will result that:

$$\sum_{i=1}^{n-1} \Omega_i^{(p)} \cdot \frac{\partial H_p}{\partial \Omega_i^{(p)}} \equiv H_p \left(x, t, \frac{\partial \Omega^{(p)}}{\partial x} \right); \tag{111}$$

hence, from (110):

$$\frac{d}{dt} \sum_{i=1}^{n-1} \Omega_i^{(p)} \, \delta x^i \equiv - \sum_{i=1}^{n-1} \frac{\partial H_p}{\partial x^i} \delta x^i + \delta \left(\sum_{i=1}^{n-1} \Omega_i^{(p)} \frac{\partial H_p}{\partial \Omega_i^{(p)}} \right) - \sum_{i=1}^{n-1} \frac{\partial H_p}{\partial \Omega_i^{(p)}} \delta \Omega_i^{(p)} \, , \tag{112}$$

or rather:

$$\frac{d}{dt} \sum_{i=1}^{n-1} \Omega_i^{(p)} \, \delta x^i \equiv - \delta H_p + \delta \left(\sum_{i=1}^{n-1} \Omega_i^{(p)} \frac{\partial H_p}{\partial \Omega_i^{(p)}} \right). \tag{113}$$

Hence, upon taking (111) into account, one will finally have:

$$\boxed{\frac{d}{dt}\left(\sum_{i=1}^{n-1}\Omega_i^{(p)}\,\delta x^i\right)=0.} \tag{114}$$

Q. E. D.

Theorem II. – *When the Hamiltonian H_p is independent of time t, that function will be an invariant of the Hamiltonian equations* (105) *and* (106).

Proof. – Indeed, it is easy to verify that the identity:

$$\frac{dH_p}{dt}\equiv\frac{\partial H_p}{\partial t} \tag{115}$$

is a consequence of the differential equations (105) and (106). As a result, the relation:

$$\frac{\partial H_p}{\partial t}\equiv 0 \tag{116}$$

will imply that:

$$\frac{dH_p}{dt}\equiv 0. \tag{117}$$

Q. E. D.

Remark. – When H_p does not depend explicitly upon time t, the wave function $\Omega^{(p)}(x, t)$ will have the form:

$$\Omega^{(p)}\equiv -\,ht+\varphi^{(p)}(x), \tag{118}$$

in which h is a constant, and $\varphi^{(p)}$ is a function of the spatial variables that is a solution to the partial differential equation:

$$\boxed{H_p\left(x,\frac{\partial\varphi}{\partial x}\right)=h.} \tag{119}$$

16. Case of a homogeneous medium. – When the coefficients of the wave equation (104) are constant, one will have:

$$\frac{\partial H_p}{\partial x^i}\equiv 0 \qquad \text{and} \qquad \frac{\partial H_p}{\partial t}\equiv 0. \tag{120}$$

In that case, the Hamiltonian differential system (105) and (106) is integrated immediately; one will have:

$$\begin{cases} \Omega_i = \Omega_i^0, & (121) \\ x^i = x_0^i + \left(\dfrac{\partial H_p}{\partial \Omega_i}\right)_0 (t - t_0), & (122) \end{cases}$$

in which Ω_i^0 and x_0^i are integration constants, and the notation $\left(\dfrac{\partial H_p}{\partial \Omega_i}\right)_0$ represents what one will get after replacing Ω_i with Ω_i^0 in $\dfrac{\partial H_p}{\partial \Omega_i}$.

We then see that *the rays are straight lines* here. That situation will be produced whenever the coefficients of the highest-order derivatives that appear in the physical equations (1) reduce to constants. In that case, one says ([1]) that the physical medium that is governed by those equations is *homogeneous.*

Theorem I. – *In a homogeneous medium, the space-time rays* ([2])*, as well as the rays in geometric space, will be straight lines.*

Corollaries:

1) *The Hadamard conoid in a homogeneous medium has rectilinear generators.*

2) *The elementary Huygens waves that issue from a point of a homogeneous medium are mutually homothetic; their locus in space-time is J. Hadamard's characteristic cone.*

We remark that $\left(\dfrac{\partial H_p}{\partial \Omega_i}\right)_0$ is homogeneous of degree zero in $\Omega_1^0, \ldots, \Omega_{n-1}^0$. It will then results that the coefficients $\left(\dfrac{\partial H_p}{\partial \Omega_i}\right)_0$ of equations (122) depend upon only $(n - 2)$ parameters, namely, the $(n - 2)$ relationships between the $(n - 1)$ constants $\Omega_1^0, \ldots, \Omega_{n-1}^0$ and any one of them. The elimination of those $(n - 2)$ parameters from equations (122) leads to the equation of the surface that occupies the position of the elementary Huygens wave that issues from the point $(x_0^1, \ldots, x_0^{n-1})$ at the initial instant $t_0 < t$ when the current instant is t.

In order to obtain the present position Ω of a wave when one starts from its initial position Ω_0, draw the line through the point $P_0(x_0^1, \ldots, x_0^{n-1})$ of Ω_0 that is defined by equations (122) and consider the point $P\ (x^1, \ldots, x^{n-1})$ on it that is taken at the current instant t. Ω will then be the locus of the point P when P_0 describes Ω_0. Recall that Ω is

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([1]) T. LEVI-CIVITA, *Caratteristiche dei sistemi differenziali e propagazione ondose* (Bologna, N. Zanichelli, 1931); see pps. 54 and 55.

([2]) Or J. Hadamard's bicharacteristics.

also the envelope at the current instant t of the elementary Huygens waves that issue from the points of Ω_0 at the initial instant $t_0 < t$.

We just saw that the points of Ω are deduced from the points of Ω_0 by *translation*; we remark that this translation is not the same for every point of Ω_0, *except when* Ω_0 *is a plane* in geometric space.

Theorem II. – *A plane wave in a homogeneous medium will remain a plane wave as time varies.*

Corollary. – *No matter what phenomenon is envisioned in a homogeneous medium, the propagation of plane waves will always be possible.*

Theorem III. – *The transport of a point on a wave in a homogeneous medium is a uniform, rectilinear motion.*

That transport will be the same for all points of a wave only when it is a plane.

Corollary. – *In a homogeneous medium, the transport velocity of a wave depends upon only the direction of propagation.*

17. Transport velocity. – The metric on geometric space $(x^1, \ldots, x^{n-1})$ is defined by the invariant differential quadratic form (83). The covariant components $N_i^{(p)}$ and contravariant ones $N^i_{(p)}$ of the unit vector $\mathbf{N}^{(p)}$ that is normal to the wave $\mathbf{\Omega}^{(p)}$ whose equation is:

$$\Omega^{(p)}(x, t) = 0 \tag{123}$$

are given by formulas (87), and the transport velocity T_p of the wave $\Omega^{(p)}$ is given by formula (101). Having recalled that, divide equation (104) by $\sqrt{\Delta_1 \Omega^{(p)}}$, in which the Lamé parameter $\Delta_1\Omega^{(p)}$ is defined by (86); hence, the *transport velocity* T_p *of the wave* $\Omega^{(p)}$ will be:

$$\boxed{T_p \equiv H_p(x, t, N^{(p)}).} \tag{124}$$

Remark. – Suppose that the Hamiltonian functions H_p and H_q are independent of time t ; i.e., that:

$$\frac{\partial H_p}{\partial t} \equiv 0, \qquad \frac{\partial H_q}{\partial t} \equiv 0, \tag{125}$$

and that one has the identity:

$$H_p\left(x, \frac{\partial \Omega}{\partial x}\right) + H_q\left(x, \frac{\partial \Omega}{\partial x}\right) \equiv 0 \tag{126}$$

in x^i and Ω_i, moreover. Hence, the two Jacobian equations:

$$\frac{\partial\Omega}{\partial t} + H_p\left(x, \frac{\partial\Omega}{\partial x}\right) = 0 \tag{127}$$

and

$$\frac{\partial\Omega}{\partial t} + H_q\left(x, \frac{\partial\Omega}{\partial x}\right) = 0 \tag{128}$$

will represent the *same* family of waves that propagate in the contrary sense. Indeed, any solution:

$$\Omega^{(p)} \equiv -ht + \varphi(x) \tag{129}$$

to the wave equation (127) can be associated with the solution:

$$\Omega^{(p)} \equiv +ht + \varphi(x) \tag{130}$$

to the wave equation (128). Recall that h represents an arbitrary constant. Now, thanks to (125) and (126), the differential equations for the transport of the waves $\Omega^{(p)}$ and $\Omega^{(q)}$ can be written:

$$\frac{dx^i}{\dfrac{\partial H\left(x, \dfrac{\partial\varphi}{\partial x}\right)}{\partial\varphi_i}} = \frac{dt}{\pm 1} \qquad (i = 1, 2, \ldots, n-1), \tag{131}$$

in which:

$$H \equiv H_p \equiv -H_q\,. \tag{132}$$

Equations (131) exhibit the fact that the transport of the waves $\Omega^{(p)}$ and $\Omega^{(q)}$ takes place in the contrary sense along the same rays, namely, along the lines:

$$\frac{dx^i}{\dfrac{\partial H}{\partial\varphi_1}} = \ldots = \frac{dx^i}{\dfrac{\partial H}{\partial\varphi_{n-1}}} \tag{133}$$

in geometric space ($x^1, \ldots, x^{n-1}$). Now, note that by virtue of (124) and (126), the transport velocities T_p and T_q of that wave are coupled by the relation:

$$T_p + T_q = 0. \tag{134}$$

The families of waves $\Omega^{(p)}$ and $\Omega^{(q)}$ are identical, but they propagate in the opposite sense along the same lines or rays with the same velocity.

We shall assume in what follows that this situation presents itself ρ times; i.e., that among the r families of waves (104), there are ρ pairs of wave families such that the two families of the same pair will propagate in opposite senses along the same lines or space-time rays.

18. Hyper-refringence. – Let:

$$\Omega_t + H_1\left(x, t, \frac{\partial \Omega}{\partial x}\right) = 0, \ \ldots, \ \Omega_t + H_{r-\rho}\left(x, t, \frac{\partial \Omega}{\partial x}\right) = 0 \tag{135}$$

be the Jacobian equations of the $r - \rho$ distinct families of waves that propagate along rays that are all different and given by the differential equations:

$$\left\{\begin{aligned} \frac{dx^i}{dt} &= \frac{\partial H_1}{\partial \Omega_i} \\ \frac{d\Omega_i}{dt} &= -\frac{\partial H_1}{\partial x^i} \end{aligned}\right. \qquad \left\{\begin{aligned} \frac{dx^i}{dt} &= \frac{\partial H_{r-\rho}}{\partial \Omega_i} \\ \frac{d\Omega_i}{dt} &= -\frac{\partial H_{r-\rho}}{\partial x^i} \end{aligned}\right. \qquad (i = 1, 2, \ldots, n-1), \qquad \begin{aligned} &(136) \\ &(137) \end{aligned}$$

respectively.

It will then result from equations (135) and (136) that at any point $(x^1, \ldots, x^{n-1})$ of geometric space there will be $(r - \rho)$ propagation velocities for the wave and $(r - \rho)$ distinct rays that correspond to each direction $(N_1, \ldots, N_{n-1})$ that is given in advance and issues from the point $(x^1, \ldots, x^{n-1})$.

We will then say that *the physical medium that occupies the geometric space* $(x^1, \ldots, x^{n-1})$ *is* $(r - \rho)$*-times refringent.*

19. Elementary Huygens wave in a homogeneous medium when the metric is Euclidian. – In that case, the wave equation has the form:

$$\frac{\partial \Omega}{\partial t} + H\left(\frac{\partial \Omega}{\partial x}\right) = 0, \tag{138}$$

in which H is a homogeneous function of degree one with respect to the partial derivatives $\partial\Omega / \partial x^1, \ldots, \partial\Omega / \partial x^{n-1}$, and the notation $\partial\Omega / \partial x$ represents all of those derivatives.

The direction cosines $N_1, \ldots, N_{n-1}$ of the normal to the wave are given by (87):

$$N_i \equiv N^i \equiv \frac{\Omega_i}{\sqrt{\Delta_1 \Omega}}, \tag{139}$$

and the transport velocity T of the wave is given by (101):

$$T \equiv -\frac{\Omega_i}{\sqrt{\Delta_1 \Omega}}, \tag{140}$$

with

$$\Delta_1\Omega \equiv \sum_{i=1}^{n-1} (\Omega_i)^2 . \tag{141}$$

It then results from equation (138) and the homogeneity of H, from (124):

$$\boxed{T \equiv H(N),} \tag{142}$$

in which N takes the place of the direction cosines $N_1, \ldots, N_{n-1}$.

The rays that issue from the origin O of the coordinates at the initial instant $t_0 \equiv 0$ are defined by equations (122):

$$x^i = \frac{\partial H\left(\frac{\partial \Omega}{\partial x}\right)}{\partial \Omega_i} \cdot t, \tag{143}$$

or rather, thanks to the homogeneity of H, by the equations:

$$\boxed{x^i = \frac{\partial H(N)}{\partial N_i} \cdot t,} \tag{144}$$

in which, by virtue of (121), the Ω_i and the N_i are constants, respectively. Note that the constants N_i are normalized by:

$$\sum_{i=1}^{n-1} (N_i)^2 \equiv 1. \tag{145}$$

That being the case, let P be a point that is taken at the instant $t > 0$ on the elementary Huygens wave that issues from the origin O at the instant $t_0 \equiv 0$. The distance Δ from the point O to the tangent plane at P to the elementary wave that is envisioned is defined by the well-known formula:

$$\Delta = \sum_{i=1}^{n-1} N_i \, x^i , \tag{146}$$

so, thanks to (144):

$$\Delta = \sum_{i=1}^{n-1} N_i \frac{\partial H(N)}{\partial N_i} \cdot t , \tag{147}$$

or rather, if one takes into account the homogeneity of H and the relation (142):

$$\boxed{\Delta \equiv T \cdot t.} \tag{148}$$

As a result, *the elementary Huygens wave that issues from the origin O at the instant* $t_0 = 0$ *is, at the instant* $t > 0$, *the envelope of the plane waves of the equation:*

$$\sum_{i=1}^{n-1} N_i \, x^i - T \, t = 0, \tag{149}$$

whose coefficients $N_1, N_2, \ldots, N_{n-1}, T$ *verify the relation* (142).

The elementary Huygens wave is then the envelope in geometric space $(x^1, \ldots, x^{n-1})$ of an $(n - 2)$-parameter family of planes.

We finally remark that *the equation of the elementary Huygens wave is the point-like equation in* $x^1, \ldots, x^{n-1}$ *that correlates with the tangent equation* (142) *in* $N_1, N_2, \ldots, N_{n-1}$.

CHAPTER VI

REFLECTION AND REFRACTION OF WAVES

20. Incident, reflected, and refracted waves. – Consider two physical media **1** and **2** in geometric space ($x^1, \ldots, x^{n-1}$) that are separated at each instant t by a moving surface S whose equation is:

$$S(x, t) = 0. \tag{150}$$

Let:

$$\begin{cases} \mathbf{O}^1\left(x,t,\dfrac{\partial\Omega}{\partial x},\dfrac{\partial\Omega}{\partial t}\right)=0, \\ \mathbf{O}^2\left(x,t,\dfrac{\partial\Omega}{\partial x},\dfrac{\partial\Omega}{\partial t}\right)=0 \end{cases} \tag{151}$$

be the partial differential equations of waves that are compatible with the equations of physics that govern the media **1** and **2**, resp. Let μ_1 and μ_2 be the degrees of homogeneity of $\mathbf{O}^1$ and $\mathbf{O}^2$, resp., with respect to the partial derivatives of Ω. From (103), let:

$$\begin{cases} -H_{s_1}^1\left(x,t,\dfrac{\partial\Omega}{\partial x}\right) & (152) \\ -H_{s_2}^2\left(x,t,\dfrac{\partial\Omega}{\partial x}\right) & (152') \end{cases} \qquad \begin{pmatrix} s_1 = 1,2,\ldots,\nu_1 \leq \mu_1 \\ s_2 = 1,2,\ldots,\nu_2 \leq \mu_2 \end{pmatrix}$$

denote the v_1 (v_2, resp.) roots of equation (151) [(151′), resp.] (which have degrees μ_1 and μ_2 in $\partial\Omega / \partial t$, respectively), which correspond to the families of waves that propagate along the various rays (§ **17**).

The Jacobian equations:

$$\begin{cases} \dfrac{\partial\Omega}{\partial t}+H_{s_1}^1\left(x,t,\dfrac{\partial\Omega}{\partial x}\right)=0, & (153) \\ \dfrac{\partial\Omega}{\partial t}+H_{s_2}^2\left(x,t,\dfrac{\partial\Omega}{\partial x}\right)=0, & (153') \end{cases} \qquad (s_1 = 1, 2, \ldots, v_1,\ s_2 = 1, 2, \ldots, v_2)$$

are then those of the v_1 and v_2 families of distinct waves that are compatible with the physical equations that relate to the media **1** and **2**, resp. One then sees that medium **1** is v_1-times refringent, and medium **2** is v_2-times refringent.

Let $\Omega^{s_1}(x,t)$ denote a solution to one of the Jacobian equations (153), so:

$$\Omega^{s_1}(x,t) = 0 \tag{154}$$

will be the equation of a wave in medium **1**; by definition, it will be a *incident wave.* The surfaces S and $\Omega^{s_1}(x,t)$ intersect in space-time ($x^1, \ldots, x^{n-1}, x^n \equiv t$) along an ($n$ – 2)-dimensional variety $V_{(n-2)}$. It results from Theorem III of § **3** that the *incident wave* will give rise to v_1 *reflected waves* and v_2 *refracted waves* that pass through $V_{(n-2)}$. Let:

$$\Omega^{r_1}(x,t) = 0 \qquad \text{and} \qquad \Omega^{s_2}(x,t) = 0 \tag{155}$$

be the equations of a *reflected wave* Ω^{r_1}, which is a solution of one of equations (153), and a *refracted wave* Ω^{s_2}, which is a solution of one of equations (153′), respectively. The waves Ω^{r_1} and Ω^{s_2} are determined completely in space-time by the demand that they must be contained in the variety $V_{(n-2)}$.

21. Applying the principle of enveloping waves. – Let $\xi^1, \ldots, \xi^{n-1}, \tau$ be the coordinates of a point on the variety $V_{(n-2)}$ in space-time. In geometric space ($x^1, \ldots, x^{n-1}$), let:

$$\Gamma^{(r_1)}_{(n-2)} \qquad \text{and} \qquad \Gamma^{(s_1)}_{(n-2)}$$

represent the position *at the instant t* in media **1** and **2**, respectively, of the elementary Huygens waves that issue from the point whose coordinates are ($\xi^1, \ldots, \xi^{n-1}$) at the instant τ. The elementary waves $\Gamma^{(r_1)}_{(n-2)}$ and $\Gamma^{(s_1)}_{(n-2)}$ are integral surfaces of the equations:

$$\frac{\partial \Omega}{\partial t} + H^1_{r_1}\left(x, t, \frac{\partial \Omega}{\partial x}\right) = 0 \qquad \text{and} \qquad \frac{\partial \Omega}{\partial t} + H^2_{s_2}\left(x, t, \frac{\partial \Omega}{\partial x}\right) = 0, \tag{156}$$

respectively.

Hence, it results from the principle of enveloping waves that ([1]) *the reflected wave* $\Omega^{(r_1)}$ *in geometric space at the instant t is the envelope of the elementary wave* $\Gamma^{(r_1)}_{(n-2)}$ *when the point* ($\xi^1, \ldots, \xi^{n-1}, \tau$) *describes the variety* $V_{(n-2)}$ *in space-time.* Similarly, the *refracted wave* $\Omega^{(s_2)}$ *at the same instant t is the envelope of the elementary wave* $\Gamma^{(s_1)}_{(n-2)}$.

22. Geometric laws of reflection and refraction. ([2]). – By hypothesis, the wave functions $\Omega^{(s_1)}(x, t)$, $\Omega^{(r_1)}(x, t)$, and $\Omega^{(s_2)}(x, t)$ are differentiable on the surface S in space-time; hence, from (154) and (155):

([1]) J. HADAMARD, *Leçons sur la Propagation des ondes et les équations de l'Hydrodynamique* (Paris, Hermann, 1903); see, pp. 295.

([2]) J. VAN MIEGHEM, *Wis- en Naturkundig Tijdschrift* (Gent); deel VII, 1934; see pp. 15 and 18.

$$\left.\begin{aligned}
&\sum_{i=1}^{n-1}\frac{\partial\Omega^{(s_1)}}{\partial x^i}\delta x^i+\frac{\partial\Omega^{(s_1)}}{\partial t}\delta t=0,\\
&\sum_{i=1}^{n-1}\frac{\partial\Omega^{(r_1)}}{\partial x^i}\delta x^i+\frac{\partial\Omega^{(r_1)}}{\partial t}\delta t=0,\\
&\sum_{i=1}^{n-1}\frac{\partial\Omega^{(s_2)}}{\partial x^i}\delta x^i+\frac{\partial\Omega^{(s_2)}}{\partial t}\delta t=0,
\end{aligned}\right\} \tag{157}$$

in which the δx^i and δt verify the *single* condition:

$$\sum_{i=1}^{n-1}\frac{\partial S}{\partial x^i}\delta x^i+\frac{\partial S}{\partial t}\delta t=0. \tag{158}$$

The differential relations (157) and (158) say that in space-time the wave surfaces $\Omega^{(s_1)}$, $\Omega^{(r_1)}$, and $\Omega^{(s_2)}$, and the surface S have an $(n-2)$-dimensional variety in common, namely, the variety $V_{(n-2)}$.

One deduces immediately from the relations (157) and (158) that:

$$\boxed{\frac{\partial\Omega^{(r_1)}}{\partial x^i}-\frac{\partial\Omega^{(s_1)}}{\partial x^i}=\lambda_1\frac{\partial S}{\partial x^i},\quad \frac{\partial\Omega^{(r_1)}}{\partial t}-\frac{\partial\Omega^{(s_1)}}{\partial t}=\lambda_1\frac{\partial S}{\partial t}} \tag{159}$$

and

$$\boxed{\frac{\partial\Omega^{(s_2)}}{\partial x^i}-\frac{\partial\Omega^{(s_1)}}{\partial x^i}=\lambda_2\frac{\partial S}{\partial x^i},\quad \frac{\partial\Omega^{(s_2)}}{\partial t}-\frac{\partial\Omega^{(s_1)}}{\partial t}=\lambda_2\frac{\partial S}{\partial t},} \tag{159′}$$

in which λ_1 and λ_2 are two arbitrary functions:

$$\lambda_1\equiv\lambda_1(x,t)\qquad\text{and}\qquad\lambda_2\equiv\lambda_2(x,t) \tag{160}$$

of the spatio-temporal variables $x^1, \ldots, x^{n-1}$, and t.

The compatibility conditions (159) *and* (159′) *that relate to the functions* $\Omega^{(s_1)}$, $\Omega^{(r_1)}$, and $\Omega^{(s_2)}$ *on the separation surface S express the geometric laws of reflection and refraction.*

We propose to put the relations (159) and (159′) into a form that is close to the classical form of the laws of reflection and refraction (viz., *Descartes's laws*). To fix ideas, suppose that $n \equiv 4$, and set:

$$N_i\equiv\frac{\dfrac{\partial S}{\partial x^i}}{\sqrt{\Delta_1 S}},\quad N_i^{(s_1)}\equiv\frac{\dfrac{\partial\Omega^{(s_1)}}{\partial x^i}}{\sqrt{\Delta_1\Omega^{(s_1)}}},\quad N_i^{(r_1)}\equiv\frac{\dfrac{\partial\Omega^{(r_1)}}{\partial x^i}}{\sqrt{\Delta_1\Omega^{(r_1)}}},\quad N_i^{(s_2)}\equiv\frac{\dfrac{\partial\Omega^{(s_2)}}{\partial x^i}}{\sqrt{\Delta_1\Omega^{(s_2)}}}, \tag{161}$$

$$(i = 1, 2, 3),$$

in which N_i, $N_i^{(s_1)}$, $N_i^{(r_1)}$, $N_i^{(s_2)}$ are the covariant components of the unit vectors that are normal to the surfaces S, $\Omega^{(s_1)}$, $\Omega^{(r_1)}$, $\Omega^{(s_2)}$, resp. Their transport velocities T, T^{s_1}, T^{r_1}, T^{s_2} in geometric space (x^1, x^2, x^3) are given by:

$$T \equiv \frac{\dfrac{\partial S}{\partial t}}{\sqrt{\Delta_1 S}}, \quad T^{s_1} \equiv \frac{\dfrac{\partial \Omega^{(s_1)}}{\partial t}}{\sqrt{\Delta_1 \Omega^{(s_1)}}}, \qquad T^{r_1} \equiv \frac{\dfrac{\partial \Omega^{(r_1)}}{\partial x^i}}{\sqrt{\Delta_1 \Omega^{(r_1)}}}, \qquad T^{s_2} \equiv \frac{\dfrac{\partial \Omega^{(s_2)}}{\partial t}}{\sqrt{\Delta_1 \Omega^{(s_2)}}}, \tag{162}$$

respectively.

By convention, the positive normal to S is directed from the medium **1** towards medium **2**. As for the positive normals to the wave surfaces $\Omega^{(s_1)}$, $\Omega^{(r_1)}$, $\Omega^{(s_2)}$, they point in the direction of their propagation.

By virtue of (162) and (124), one will have:

$$\left.\begin{aligned} T^{s_1} &\equiv H^1_{s_1}(x, t, N^{(s_1)}), \\ T^{r_1} &\equiv H^1_{r_1}(x, t, N^{(r_1)}), \\ T^{s_2} &\equiv H^1_{s_2}(x, t, N^{(s_2)}). \end{aligned}\right\} \tag{163}$$

Thanks to the definitions (161) and (162), it is easy to transform formulas (159) and (159′); one will find that:

$$\left.\begin{aligned} N_i^{(r_1)}\sqrt{\Delta_1\Omega^{(r_1)}} - N_i^{(s_1)}\sqrt{\Delta_1\Omega^{(s_1)}} &= \lambda_1 N_i \sqrt{\Delta_1 S}, \\ T^{(r_1)}\sqrt{\Delta_1\Omega^{(r_1)}} - T^{(s_1)}\sqrt{\Delta_1\Omega^{(s_1)}} &= \lambda_1 T \sqrt{\Delta_1 S}, \end{aligned}\right\} \qquad (i = 1, 2, 3), \tag{164}$$

and

$$\left.\begin{aligned} N_i^{(s_2)}\sqrt{\Delta_1\Omega^{(s_2)}} - N_i^{(s_1)}\sqrt{\Delta_1\Omega^{(s_1)}} &= \lambda_2 N_i \sqrt{\Delta_1 S}, \\ T^{(s_2)}\sqrt{\Delta_1\Omega^{(s_2)}} - T^{(s_1)}\sqrt{\Delta_1\Omega^{(s_1)}} &= \lambda_2 T \sqrt{\Delta_1 S}. \end{aligned}\right\} \tag{164′}$$

Eliminate λ_1 and λ_2 from relations (164) and (164′), respectively; that will give:

$$N_i^{(r_1)}\sqrt{\Delta_1\Omega^{(r_1)}} - N_i^{(s_1)}\sqrt{\Delta_1\Omega^{(s_1)}} = \frac{T^{r_1}\sqrt{\Delta_1\Omega^{(r_1)}} - T^{s_1}\sqrt{\Delta_1\Omega^{(s_1)}}}{T} N_i \tag{165}$$

and

$$N_i^{(s_2)}\sqrt{\Delta_1\Omega^{(s_2)}} - N_i^{(s_1)}\sqrt{\Delta_1\Omega^{(s_1)}} = \frac{T^{s_2}\sqrt{\Delta_1\Omega^{(s_2)}} - T^{s_1}\sqrt{\Delta_1\Omega^{(s_1)}}}{T} N_i \qquad (i = 1, 2, 3). \tag{165′}$$

Now multiply formulas (165) and (165′) by the contravariant components N^i of the unit vector **N** that is normal to S; hence:

$$\left(\sum_{i=1}^{3} N^i N_i^{(r_1)} - \frac{T^{r_1}}{T}\right)\sqrt{\Delta_1 \Omega^{(r_1)}} - \left(\sum_{i=1}^{3} N^i N_i^{(s_1)} - \frac{T^{s_1}}{T}\right)\sqrt{\Delta_1 \Omega^{(s_1)}} = 0 \tag{166}$$

and

$$\left(\sum_{i=1}^{3} N^i N_i^{(s_2)} - \frac{T^{s_2}}{T}\right)\sqrt{\Delta_1 \Omega^{(s_2)}} - \left(\sum_{i=1}^{3} N^i N_i^{(s_1)} - \frac{T^{s_1}}{T}\right)\sqrt{\Delta_1 \Omega^{(s_1)}} = 0. \tag{166′}$$

However, one has:

$$\sum_{i=1}^{3} N^i N_i^{(s_1)} \equiv \cos\,\varphi^{s_1}, \qquad \sum_{i=1}^{3} N^i N_i^{(r_1)} \equiv \cos\,\varphi^{r_1}, \qquad \sum_{i=1}^{3} N^i N_i^{(s_2)} \equiv \cos\,\varphi^{s_2}, \tag{167}$$

in which φ^{s_1}, φ^{r_1}, φ^{s_2} represent the angles in geometric space (x^1, x^2, x^3) between the pairs of surfaces (S, $\Omega^{(s_1)}$), (S, $\Omega^{(r_1)}$), and (S, $\Omega^{(s_2)}$) (see Fig. 5).

Hence, (166) and (166′) will give:

$$\frac{T^{r_1} - T\cos\varphi^{r_1}}{T^{s_1} - T\cos\varphi^{s_1}} = \sqrt{\frac{\Delta_1\Omega^{(s_1)}}{\Delta_1\Omega^{(r_1)}}} \tag{168}$$

and

$$\frac{T^{s_2} - T\cos\varphi^{s_2}}{T^{s_1} - T\cos\varphi^{s_1}} = \sqrt{\frac{\Delta_1\Omega^{(s_1)}}{\Delta_1\Omega^{(s_2)}}}\,. \tag{168′}$$

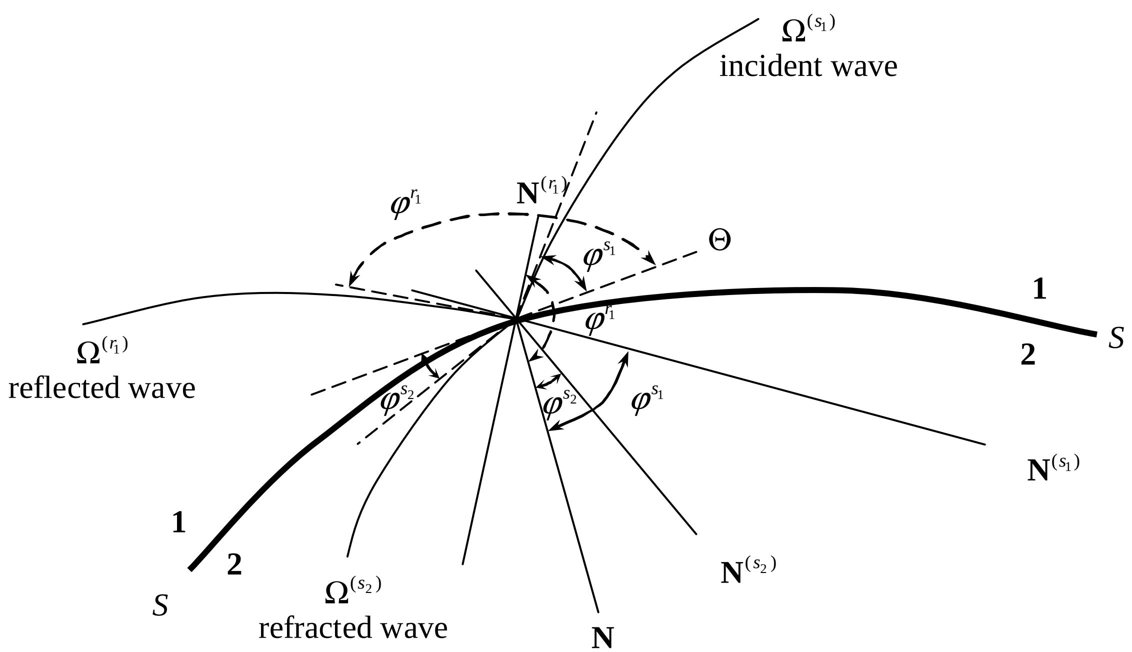

Figure 5.

That being the case, let Θ^1, Θ^2, Θ^3 be the contravariant components of the unit vector that is tangent to the line of intersection of the surfaces S and $\Omega^{(s_1)}$ in space (x^1, x^2, x^3) (i.e., the intersection of the separation surface S and the wave surface $\Omega^{(s_1)}$, both of which are taken at the same instant t). One will then have the identity:

$$\sum_{i=1}^{3} \Theta^i N_i \equiv 0. \tag{169}$$

Now multiply (165) and (165′) by Θ^i; that will give:

$$\frac{\sum_{i=1}^{3} \Theta^i N_i^{(r_1)}}{\sum_{i=1}^{3} \Theta^i N_i^{(s_1)}} = \sqrt{\frac{\Delta_1 \Omega^{(s_1)}}{\Delta_1 \Omega^{(r_1)}}}, \qquad \frac{\sum_{i=1}^{3} \Theta^i N_i^{(s_2)}}{\sum_{i=1}^{3} \Theta^i N_i^{(s_1)}} = \sqrt{\frac{\Delta_1 \Omega^{(s_1)}}{\Delta_1 \Omega^{(s_2)}}}. \tag{170}$$

However, one has:

$$\sum_{i=1}^{3} \Theta^i N_i^{(s_1)} \equiv \sin\ \varphi^{s_1}, \quad \sum_{i=1}^{3} \Theta^i N_i^{(r_1)} \equiv \sin\ \varphi^{r_1}, \quad \sum_{i=1}^{3} \Theta^i N_i^{(s_2)} \equiv \sin\ \varphi^{s_2}; \tag{171}$$

hence, from (170):

$$\sqrt{\frac{\Delta_1 \Omega^{(s_1)}}{\Delta_1 \Omega^{(r_1)}}} = \frac{\sin \varphi^{r_1}}{\sin^{s_1}}, \qquad \sqrt{\frac{\Delta_1 \Omega^{(s_1)}}{\Delta_1 \Omega^{(s_2)}}} = \frac{\sin \varphi^{s_2}}{\sin^{s_1}}. \tag{172}$$

Finally, one deduces *the laws of reflection and refraction in a remarkable and general form* from formulas (168), (168), and (172):

$$\boxed{\frac{\sin \varphi^{r_1}}{T^{r_1} - T\cos \varphi^{r_1}} = \frac{\sin \varphi^{s_1}}{T^{s_1} - T\cos \varphi^{s_1}} = \frac{\sin \varphi^{s_2}}{T^{s_2} - T\cos \varphi^{s_2}},} \tag{173}$$

or rather, from (163):

$$\frac{\sin \varphi^{r_1}}{H^1_{r_1}(x, t, N^{r_1}) - T\cos \varphi^{r_1}} = \frac{\sin \varphi^{s_1}}{H^1_{s_1}(x, t, N^{s_1}) - T\cos \varphi^{s_1}} = \frac{\sin \varphi^{s_2}}{H^1_{s_2}(x, t, N^{s_2}) - T\cos \varphi^{s_2}}. \tag{174}$$

Upon setting $T \equiv 0$ in (173) – i.e., upon supposing that the separation surface S between the physical media **1** and **2** is fixed – one will, in fact, recover Descartes's classical laws:

$$\frac{\sin\varphi^{r_1}}{T^{r_1}} = \frac{\sin\varphi^{s_1}}{T^{s_1}} = \frac{\sin\varphi^{s_2}}{T^{s_2}}. \tag{175}$$

Finally, when medium **1** is isotropic, in addition, one will have:

$$T^{s_1} \equiv T^{r_1}, \tag{176}$$

and as a result, from (175)

$$\sin\ \varphi^{s_1} = \sin\ \varphi^{r_1},$$

or

$$\boxed{\varphi^{r_1} + \varphi^{s_1} = \pi} \tag{177}$$

and

$$\boxed{\frac{\sin\varphi^{s_2}}{\sin\varphi^{s_1}} = \frac{T^{s_2}}{T^{s_1}}.} \tag{178}$$

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CHAPTER VII

APPLICATION OF HUYGENS'S PRINCIPLE TO THE PROPAGATION OF ELECTROMAGNETIC WAVES

23. Review of the general equations. – The most general electromagnetic field is determined by four vectors, namely:

The electrical force:	$\mathbf{H}$ (H_1, H_2, H_3),
The magnetic force:	$\boldsymbol{\mathcal{H}}$ $(\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3)$,
The electrical induction:	$\mathbf{B}$ (B_1, B_2, B_3),
The magnetic induction:	$\boldsymbol{\mathcal{B}}$ $(\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3)$.

The components H_i , $\mathcal{H}_i$, B_i , $\mathcal{B}_i$ (i = 1, 2, 3) generally depend upon the spatio-temporal variables x^1, x^2, x^3, and t.

At any point of the geometric space (x^1, x^2, x^3) that is the seat of the electromagnetic field that one imagines, and at every instant t, there exist the following vectorial Maxwell equations between the aforementioned physical quantities ([1]):

$$\boxed{\begin{aligned} \operatorname{div} \mathbf{B} &= 4\pi\rho, \\ \operatorname{div} \boldsymbol{\mathcal{B}} &= 0, \end{aligned}} \tag{179}$$

and

$$\boxed{\begin{aligned} \operatorname{rot} H &= -\frac{1}{c}\frac{\partial \boldsymbol{\mathcal{B}}}{\partial t}, \\ \operatorname{rot} \boldsymbol{\mathcal{H}} &= +\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} + \frac{4\pi}{c}\mathbf{C}, \end{aligned}} \tag{180}$$

in which the scalar:

$$\rho \equiv \rho\,(x, t)$$

represents the *realized density of electricity*, the vector:

$$\mathbf{C} \equiv \mathbf{C}\,(x, t)$$

is the electric current of convection and conduction, and c is nothing but the well-known constant 3×10^{10} cm / sec in the Gaussian system of units.

One knows ([2]) that the electromagnetic energy is given by:

([1]) TH. DE DONDER, *Théorie mathématique de l'Électricité*, Introduction aux Équations du Maxwell (Paris, Gauthier-Villars, 1925), see Book III.

([2]) *Ibidem*; see pp. 170 and 171, form. (608) and (609′).

$$W \equiv \frac{1}{8\pi}\{(\mathbf{B}\cdot\mathbf{H}) + (\boldsymbol{\mathcal{B}}\cdot\boldsymbol{\mathcal{H}})\}, \tag{181}$$

and *Poynting's radiation vector* by:

$$\mathbf{S} \equiv \frac{1}{8\pi}(\mathbf{H}\cdot\boldsymbol{\mathcal{H}}). \tag{182}$$

Recall here that $(\mathbf{A}\cdot\mathbf{B})$ and $(\mathbf{A}\times\mathbf{B})$ represent the scalar and vectorial product, respectively, of the vectors **A** and **B**.

Thanks to the definitions (181) and (182), *the balance of electromagnetic energy* ([1]) can be written:

$$\boxed{-\frac{\partial}{\partial t}\int_v W\,\delta v = \int_v A\,\delta v + \oint_S \mathcal{S}_N\,\delta S,} \tag{183}$$

in which v is an arbitrary volume of the geometric space that is bounded by the closed surface S, and in which:

$$A \equiv (\mathbf{H}\cdot\mathbf{C}) + \frac{1}{8\pi}\left\{\left(\frac{\partial\mathbf{B}}{\partial t}\cdot\mathbf{H}\right) - \left(\mathbf{H}\cdot\frac{\partial\mathbf{H}}{\partial t}\right)\right\} + \frac{1}{8\pi}\left\{\left(\frac{\partial\boldsymbol{\mathcal{B}}}{\partial t}\cdot\boldsymbol{\mathcal{H}}\right) - \left(\boldsymbol{\mathcal{B}}\cdot\frac{\partial\boldsymbol{\mathcal{H}}}{\partial t}\right)\right\}. \tag{184}$$

The symbol $\mathcal{S}_N$ represents the component of the Poynting vector $\boldsymbol{\mathcal{S}}$ that is evaluated on along the exterior normal to the surface S.

The balance equation (183) expresses the idea that *the reduction per unit time in the electromagnetic energy that is contained in the volume v is equal to the power that is dissipated by the Joule effect and electromagnetic hysteresis inside the volume v, augmented by power that is radiated across the boundary surface S of that volume.*

From TH. DE DONDER ([2]), *the transport of electromagnetic energy* is defined by the vector:

$$u \equiv \frac{\boldsymbol{\mathcal{S}}}{W} \equiv c\,\frac{(\mathbf{H}\times\boldsymbol{\mathcal{H}})}{\tfrac{1}{2}(\mathbf{B}\cdot\mathbf{H}) + \tfrac{1}{2}(\boldsymbol{\mathcal{B}}\cdot\boldsymbol{\mathcal{H}})}. \tag{185}$$

The differential equations of that transport are then:

$$\boxed{\frac{dx^i}{u^i} = dt,} \tag{186}$$

with:

([1]) *Ibidem:* See, pp. 171, form. (609).

([2]) TH. DE DONDER, *Interprétation cinématique du théorème de Poynting*, C. R. Acad. Sci. **158** (1914), pp. 687, see equation (2).

$$\boxed{u^i \equiv c\frac{H_{i+1}\mathcal{H}_{i+2} - H_{i+2}\mathcal{H}_{i+1}}{\frac{1}{2}\sum_{k=1}^{3}(B_k H_k + \mathcal{B}_k\mathcal{H}_k)}} \qquad (i = 1, 2, 3). \tag{187}$$

By convention, the indices i, $i + 1$, $i + 2$ (for $i = 1, 2, 3$) that differ by three units must be considered to be identical. Along the energy trajectories (186), one will have, in turn:

$$\left.\begin{aligned} \frac{d}{dt}\int_v W\,\delta v &\equiv \int_v\left(\frac{dW}{dt} + W\operatorname{div}\mathbf{u}\right)\delta v \equiv \frac{\partial}{\partial t}\int_v W\,\delta v + \int_v(\operatorname{div} W\mathbf{u})\,\delta v \\ &\equiv \frac{\partial}{\partial t}\int_v W\,\delta v + \int_v(\operatorname{div}\boldsymbol{\mathcal{S}})\,\delta v, \end{aligned}\right\} \tag{188}$$

or rather:

$$\frac{d}{dt}\int_v W\,\delta v \equiv \frac{\partial}{\partial t}\int_v W\,\delta v + \oint_S \mathcal{S}_N\,\delta S\,; \tag{189}$$

hence, when one takes the balance equation (183) into account, one will finally have:

$$\frac{d}{dt}\int_v W\,\delta v \equiv -\int_v A\,\delta v\,, \tag{190}$$

or rather:

$$\frac{dW}{dt} + W\sum_{i=1}^{3}\frac{\partial u^i}{\partial x^i} = -A. \tag{190$'$}$$

Theorem. – *When there is neither Joule effect nor electromagnetic hysteresis in the physical medium that is envisioned:*

$$\int_v W\,\delta v \tag{191}$$

will be an integral invariant of the differential equations (186) *of the transport of electromagnetic energy.*

24. Case of an inhomogeneous, anisotropic, non-absorbent medium. – Consider, *for example*, a transparent crystal referred to its principal axes x^1, x^2, x^3; one will then have:

$$\left.\begin{aligned} B_k &= \Phi_k(x, t; H_k), \\ \mathcal{B}_k &= \mu(x,t)\mathcal{H}_k, \end{aligned}\right\} \tag{192}$$

and

$$\left.\begin{aligned} C &\equiv 0, \\ \rho &\equiv 0. \end{aligned}\right\} \tag{193}$$

Set:

$$\frac{\partial \Phi_k}{\partial H_k} \equiv \varepsilon_k\,(x,\,t;\,H_k). \tag{194}$$

In this case, Maxwell's equations (179) and (180) will become:

$$\left.\begin{aligned} &\sum_{k=1}^{3}\left(\frac{\partial \Phi_k}{\partial x^k} + \varepsilon_k \frac{\partial H_k}{\partial x^k}\right) = 0, \\ &\sum_{k=1}^{3}\left(\frac{\partial \mu}{\partial x^k}\mathcal{H}_k + \mu \frac{\partial \mathcal{H}_k}{\partial x^k}\right) = 0, \end{aligned}\right\} \tag{195}$$

and

$$\left.\begin{aligned} &\frac{\partial \mathcal{H}_{i+2}}{\partial x^{i+1}} - \frac{\partial \mathcal{H}_{i+1}}{\partial x^{i+2}} = +\frac{1}{c}\left(\frac{\partial \Phi_i}{\partial t} + \varepsilon_i \frac{\partial H_i}{\partial t}\right), \\ &\frac{\partial H_{i+2}}{\partial x^{i+1}} - \frac{\partial H_{i+1}}{\partial x^{i+2}} = -\frac{1}{c}\left(\frac{\partial \mu}{\partial t}\mathcal{H}_i + \mu \frac{\partial \mathcal{H}_i}{\partial t}\right). \end{aligned}\right\} \tag{196}$$

Remarks. –

1. It results from the first equation (193) that there is no Joule effect in a transparent crystalline medium.

2. Set:

$$\Phi_k\,(x,\,t;\,H_k) \equiv \varphi_k\,(x,\,t\,;\,H_k)\cdot H_k\,. \tag{197}$$

There will be no electromagnetic hysteresis in the crystal considered when φ_k is independent of the electric field **H**, and φ_k and μ are independent of time t.

25. Definition of an electromagnetic wave. – Suppose that the first derivatives of the components of the electric force **H** and the magnetic force $\boldsymbol{\mathcal{H}}$ with respect to the spatio-temporal variables x^1, x^2, x^3, and t are continuous functions at any point of the geometric space (x^1, x^2, x^3), except at any instant t and every point of the surface **Ω** whose equation is:

$$\Omega\,(x,\,t) = 0. \tag{198}$$

That discontinuity surface displaces and deforms with time t in the geometric space (x^1, x^2, x^3); it constitutes a *wave,* the Hugoniot sense. We say that *the surface is an electromagnetic wave.*

26. Compatibility conditions. – *The geometric and kinematic compatibility conditions* of J. Hadamard ([1]) have the following form here:

$$\left[\frac{\partial H_k}{\partial x^i}\right] = \lambda_k^{(e)}\,\Omega_i\,, \qquad \left[\frac{\partial \mathcal{H}_k}{\partial x^i}\right] = \lambda_k^{(m)}\,\Omega_i\,, \tag{199}$$

and

$$\left[\frac{\partial H_k}{\partial t}\right] = \lambda_k^{(e)}\,\Omega_t\,, \qquad \left[\frac{\partial \mathcal{H}_k}{\partial t}\right] = \lambda_k^{(m)}\,\Omega_t\,, \tag{200}$$

in which:

$$\Omega_i \equiv \frac{\partial \Omega}{\partial x^i} \qquad \text{and} \qquad \Omega_t \equiv \frac{\partial \Omega}{\partial t}\,, \tag{201}$$

and in which the symbol $[F]$ represents the jump of the function F upon crossing the surface Ω.

One knows ([2]) that the quantities $(\lambda_1^{(e)}, \lambda_2^{(e)}, \lambda_3^{(e)})$ and $(\lambda_1^{(m)}, \lambda_2^{(m)}, \lambda_3^{(m)})$ can be considered to be the components of two vectors $\lambda^{(e)}$ and $\lambda^{(m)}$ in geometric space.

It will then result from the relations (192) and (194) that:

$$\left[\frac{\partial B_k}{\partial x^i}\right] = \varepsilon_k\left[\frac{\partial H_k}{\partial x^i}\right], \qquad \left[\frac{\partial B_k}{\partial t}\right] = \varepsilon_k\left[\frac{\partial H_k}{\partial t}\right], \tag{202}$$

and

$$\left[\frac{\partial \mathcal{B}_k}{\partial x^i}\right] = \mu\left[\frac{\partial \mathcal{H}_k}{\partial x^i}\right], \qquad \left[\frac{\partial \mathcal{B}_k}{\partial t}\right] = \mu\left[\frac{\partial H_k}{\partial t}\right] \qquad (i, k = 1, 2, 3), \tag{203}$$

while assuming the continuity of the derivatives:

$$\frac{\partial \Phi_k}{\partial x^i}, \quad \frac{\partial \Phi_k}{\partial t}, \quad \frac{\partial \mu}{\partial x^i}, \quad \frac{\partial \mu}{\partial t}$$

when one crosses Ω; hence:

$$\left[\frac{\partial B_k}{\partial x^i}\right] = \varpi_k^{(e)}\,\Omega_i\,, \qquad \left[\frac{\partial \mathcal{B}_k}{\partial x^i}\right] = \varpi_k^{(m)}\,\Omega_i\,, \tag{204}$$

and

$$\left[\frac{\partial B_k}{\partial t}\right] = \varpi_k^{(e)}\,\Omega_t\,, \qquad \left[\frac{\partial \mathcal{B}_k}{\partial t}\right] = \varpi_k^{(m)}\,\Omega_t\,, \tag{205}$$

with [(199) and (200)]:

$$\left.\begin{aligned} \varpi_k^{(e)} &\equiv \varepsilon_k \lambda_k^{(e)}, \\ \varpi_k^{(m)} &\equiv \mu \lambda_k^{(m)}. \end{aligned}\right\} \tag{206}$$

([1]) J. HADAMARD, *Leçons sur la Propagation des ondes et les équations de l'Hydrodynamique* (Paris, Hermann, 1903); see pp. 85, *et seq.*

([2]) J. VAN MIEGHEM, *Étude sur la théorie des ondes* (*loc. cit.*); see pp. 31.

Similarly, $(\varpi_1^{(e)}, \varpi_2^{(e)}, \varpi_3^{(e)})$ and $(\varpi_1^{(m)}, \varpi_2^{(m)}, \varpi_3^{(m)})$ are the components of the two vectors $\varpi^{(e)}$ and $\varpi^{(m)}$.

That being the case, one will have:

$$\left.\begin{aligned} H_{\text{II}} &= H_{\text{I}} + \lambda^{(e)}\Omega + \cdots, \\ \mathcal{H}_{\text{II}} &= \mathcal{H}_{\text{I}} + \lambda^{(m)}\Omega + \cdots, \end{aligned}\right\} \qquad \left.\begin{aligned} B_{\text{II}} &= B_{\text{I}} + \varpi^{(e)}\Omega + \cdots, \\ \mathcal{B}_{\text{II}} &= \mathcal{B}_{\text{I}} + \varpi^{(m)}\Omega + \cdots, \end{aligned}\right\} \tag{207}$$

in which H_{I}, $\mathcal{H}_{\text{I}}$, B_{I}, $\mathcal{B}_{\text{I}}$, and H_{II}, $\mathcal{H}_{\text{II}}$, B_{II}, $\mathcal{B}_{\text{II}}$ are the values of the electric force, the magnetic force, the electric induction, and the magnetic induction, respectively, in the two well-defined media I and II at the instant t in the geometric space (x^1, x^2, x^3) ([1]) on the discontinuity surface – or wave – Ω. If Ω is a front then it will propagate into the region I, and one will have:

$$H_{\text{I}} \equiv \mathcal{H}_{\text{I}} \equiv B_{\text{I}} \equiv \mathcal{B}_{\text{I}} \equiv 0. \tag{208}$$

We now return to equations (195) and (196) of electromagnetism; we will deduce immediately that:

$$\left.\begin{aligned} &\sum_{k=1}^{3} \varepsilon_k \left[\frac{\partial H_k}{\partial x^k}\right] = 0, \\ &\sum_{k=1}^{3} \left[\frac{\partial \mathcal{H}_k}{\partial x^k}\right] = 0, \end{aligned}\right\} \tag{209}$$

and

$$\left.\begin{aligned} \left[\frac{\partial H_{i+2}}{\partial x^{i+1}}\right] - \left[\frac{\partial H_{i+1}}{\partial x^{i+2}}\right] &= -\frac{\varepsilon_i}{c}\left[\frac{\partial H_t}{\partial t}\right], \\ \left[\frac{\partial \mathcal{H}_{i+2}}{\partial x^{i+1}}\right] - \left[\frac{\partial \mathcal{H}_{i+1}}{\partial x^{i+2}}\right] &= -\frac{\mu}{c}\left[\frac{\partial \mathcal{H}_t}{\partial t}\right], \end{aligned}\right\} \qquad (i = 1, 2, 3) \tag{210}$$

on Ω. We then replace the brackets in (209) and (210) with their values in (199), (200), (204), and (205); hence, when we take (206) into account, we will have the *dynamical compatibility conditions:*

$$\left.\begin{aligned} &\sum_{i=1}^{3} \varpi_i^{(e)}\,\Omega_i = 0, \\ &\sum_{i=1}^{3} \varpi_i^{(m)}\,\Omega_i = 0, \end{aligned}\right\} \tag{211}$$

and

([1]) Here, one assumes implicitly that the vectorial fields H_{I}, $\mathcal{H}_{\text{I}}$, B_{I}, $\mathcal{B}_{\text{I}}$, and H_{II}, $\mathcal{H}_{\text{II}}$, B_{II}, $\mathcal{B}_{\text{II}}$ exist analytically in all of space, but have a physical sense only in the media I and II, *respectively*.

$$\left.\begin{aligned} \lambda_{i+2}^{(e)}\,\Omega_{i+1} - \lambda_{i+2}^{(e)}\,\Omega_{i+2} &= -\varpi_i^{(m)}\,\frac{\Omega_i}{c}, \\ \lambda_{i+2}^{(m)}\,\Omega_{i+1} - \lambda_{i+2}^{(m)}\,\Omega_{i+2} &= -\varpi_i^{(e)}\,\frac{\Omega_i}{c}. \end{aligned}\right\} \tag{212}$$

Now, introduce the unit vector **N** that is normal to the wave Ω and the transport velocity T of that wave; in the case of a Euclidian metric, (87) and (101), one will have:

$$N_i \equiv N^i \equiv \frac{\Omega_i}{\sqrt{\Delta_i\,\Omega}} \tag{213}$$

and

$$T \equiv -\frac{\Omega_t}{\sqrt{\Delta_i\,\Omega}}, \tag{214}$$

with

$$\Delta_i\,\Omega = \sum_{i=1}^{3}(\Omega_i)^2. \tag{215}$$

Thanks to the definitions (213) and (214), the dynamical compatibility conditions (211) and (212) will take the following *vectorial form:*

$$\left.\begin{aligned} (\varpi^{(e)}\cdot\mathbf{N}) &= 0, \\ (\varpi^{(m)}\cdot\mathbf{N}) &= 0, \end{aligned}\right\} \tag{216}$$

and ([1]):

$$\boxed{\begin{aligned} (\mathbf{N}\times\lambda^{(e)}) &= +\frac{T}{c}\varpi^{(m)}, \\ (\mathbf{N}\times\lambda^{(m)}) &= -\frac{T}{c}\varpi^{(e)}. \end{aligned}} \tag{217}$$

Upon scalar-multiplying the conditions (217) by **N**, one will recover the conditions (216); as a result, *only the conditions* (216) *are distinct.*

Finally, we remark that in the case envisioned, we will have:

$$\varpi^{(m)} \equiv \mu\,\lambda^{(m)}. \tag{218}$$

Replace $\varpi^{(m)}$ with its value (218) in the first relation (217); hence:

$$\lambda^{(m)} = +\,\frac{c}{T}\cdot\frac{1}{\mu}(\mathbf{N}\times\lambda^{(e)}); \tag{219}$$

([1]) The relations (216) and (217) will persist no matter what relations exist between the electric induction and magnetic induction, on the one hand, and the electric force and magnetic force, on the other. They are therefore *completely general.*

and then replace $\lambda^{(m)}$ in the second relation (217) with its aforementioned value. Hence, one will have the single vectorial condition:

$$\boxed{(\mathbf{N}\times(\mathbf{N}\times\lambda^{(e)})) = -\mu\left(\frac{T}{c}\right)^2 \varpi^{(e)},} \tag{220}$$

or rather:

$$\boxed{\mathbf{N}(\mathbf{N}\cdot\lambda^{(e)}) - \lambda^{(e)} = -\mu\left(\frac{T}{c}\right)^2 \varpi^{(e)}.} \tag{221}$$

27. Consequences of the compatibility conditions. –

1. *The vectors $\varpi^{(e)}$ and $\varpi^{(m)}$ that determine the sharp variations in the first derivatives of the electric induction* **B** *and magnetic induction* $\mathcal{B}$ *at any point of the discontinuity surface – or wave –* Ω *is found in the tangent plane to* Ω *at that point.*

That proposition is a consequence of the relations (216).

2. *The vectors $\varpi^{(m)}$ and $\lambda^{(m)}$ that determine the sharp variations in the first derivatives of the electric induction $\mathcal{B}$ and magnetic force $\mathcal{H}$ have the same line of application.*

That proposition is a consequence of (218).

3. *The vectors $\lambda^{(m)}$ and $\lambda^{(m)}$ that determine the sharp variations in the first derivatives of the electric force and magnetic force, resp., are perpendicular to the vectors $\varpi^{(m)}$ and $\lambda^{(m)}$ that determine the sharp variations in the first derivatives of the magnetic induction* **B** *and the electric induction* $\mathcal{B}$ ([1])*, respectively.*

That proposition is a consequence of the relations (217).

4. CONCLUSION. – *The trihedron that is composed of the three vectors $\varpi^{(e)}$, $\varpi^{(m)}$,* **N** – or the vectors $\varpi^{(e)}$, $\lambda^{(m)}$, **N** – *is right-handed and tri-rectangular* ([2]); *moreover, the vectors $\lambda^{(e)}$, $\varpi^{(e)}$, and* **N** *are coplanar (see fig. 6).*

5. As a result of the relations (207) and (208), and the aforementioned propositions, one will have the *fundamental theorem:*

([1]) The propositions that are stated in 1 and 3 are completely general. (See the footnote on the previous page.)

([2]) Here, one implicitly assumes that $\mu(x, t) > 0$.

At a point of the electromagnetic field that immediately follows the passage of the electromagnetic wave front, the electric induction **B**, *the magnetic induction* $\mathcal{B}$, *and the unit vector* **N** *that is normal to the wave front form a right-handed, tri-rectangular trihedron; moreover, the magnetic induction* $\mathcal{B}$ *and the magnetic force* $\mathcal{H}$ *have the same line of application. The electric force* **H**, *the electric induction* **B**, *and the normal vector* **N** *are three coplanar vectors.*

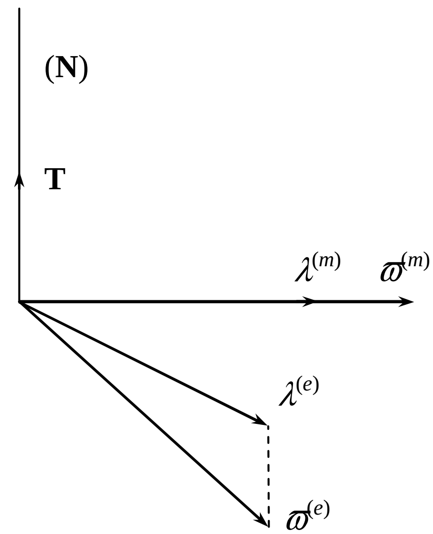

Figure 6.

6. Finally, the components (187) of the vector **u** at any point of the *front* of the electromagnetic wave Ω are given by:

$$u^i \equiv c\,\frac{\lambda_{i+1}^{(e)}\,\lambda_{i+2}^{(m)} - \lambda_{+2}^{(e)}\,\lambda_{i+1}^{(m)}}{\frac{1}{2}\sum_{k=1}^{3}(\varpi_k^{(e)}\,\lambda_k^{(m)} + \varpi_k^{(m)}\lambda_k^{(m)})} \qquad \text{(on } \Omega). \tag{222}$$

Hence, one has the *theorems:*

a) *The vectors* $\lambda^{(e)}$, $\varpi^{(e)}$, **u**, *and* **N** *are found in a plane that is perpendicular to the line of application of the vectors* $\lambda^{(m)}$, $\varpi^{(m)}$.

b) *The trihedron that is defined by the vectors* $\lambda^{(e)}$, $\lambda^{(m)}$, **u** *– or* {$\lambda^{(e)}$, $\varpi^{(m)}$, **u**} *– is right-handed and tri-rectangular.*

c) *At a point in the electromagnetic field that is immediately behind the electromagnetic wave front, the electric force* **H**, *the electric induction* **B**, *the velocity* **u** *of the transport of the electromagnetic energy, and the unit vector* **N** *that is normal to the wave front are found in a plane that perpendicular to the line of application of the magnetic force* $\mathcal{H}$ *and the magnetic induction* $\mathcal{B}$.

d) *At a point in an electromagnetic field that is immediately behind the electromagnetic wave front, the electric force* **H**, *the magnetic force* $\boldsymbol{\mathcal{H}}$, *and the velocity* **u** *of the transport of electromagnetic energy constitute a right-handed tri-rectangular trihedron.*

28. Partial differential equations of electromagnetic waves. – We have reduced the dynamical compatibility conditions to the single vectorial relation (221); hence:

$$(\mathbf{N}\cdot\lambda^{(e)})\,N_i-\lambda_i^{(e)}=-\mu\left(\frac{T}{c}\right)^2\varpi_i^{(e)}, \tag{223}$$

with (206):

$$\varpi_i^{(e)}\equiv\varepsilon_i\,\lambda_i^{(e)}\quad(i=1,2,3). \tag{223′}$$

Upon replacing $\varpi_i^{(e)}$ with its value (223′) in (223), one will get a system of three linear homogeneous equations in the three unknowns $\lambda_1^{(e)}$, $\lambda_2^{(e)}$, $\lambda_3^{(e)}$. In order for that system to be compatible, it is necessary and sufficient that its determinant should be zero. Upon annulling the determinant of that system, one will obtain a relation between the derivatives Ω_1, Ω_2, Ω_3, and Ω_t that is nothing but the wave equation that is compatible with Maxwell's equations (195) and (196).

One then eliminates the components $\lambda_1^{(e)}$, $\lambda_2^{(e)}$, $\lambda_3^{(e)}$ of the vector $\lambda^{(e)}$ from (223) and (223′). To that effect, one first sets [1]:

$$c_i^2\equiv\frac{c^2}{\varepsilon_i\,\mu}\qquad(i=1,2,3), \tag{224}$$

so

$$(\mathbf{N}\cdot\lambda^{(e)})\,N_i=\lambda_i^{(e)}\left(1-\frac{T^2}{c_i^2}\right), \tag{225}$$

or rather:

$$\boxed{\lambda_i^{(e)}=\frac{(\mathbf{N}\cdot\lambda^{(e)})}{1-\left(\dfrac{T}{c_i}\right)^2}N_i.} \tag{226}$$

But then, one will deduce from (223′) and (226) that:

$$\lambda_i^{(e)}=\varepsilon_i\frac{(\mathbf{N}\cdot\lambda^{(e)})}{1-\left(\dfrac{T}{c_i}\right)^2}N_i, \tag{227}$$

---

[1] The c_i are functions of the x and t and the components of the electric field [see form. (194)].

or rather (224):

$$\boxed{\varpi_i^{(e)} = -\frac{c^2}{\mu} \cdot \frac{(\mathbf{N} \cdot \lambda^{(e)})}{T^2 - c_i^2} N_i ,} \tag{228}$$

so (216):

$$\boxed{\sum_{i=1}^{3} \frac{N_i^2}{T^2 - c_i^2} = 0.} \tag{229}$$

Finally, upon replacing N_i and T with their values (213) and (214), resp., in (229), one will get *the partial differential equation for electromagnetic waves:*

$$\boxed{\sum_{i=1}^{3} \frac{\Omega_i^2}{\Omega_t^2 - c_i^2\, \Delta_1\, \Omega} = 0.} \tag{230}$$

The following *theorem* results from equation (229):

At any point in the electromagnetic field, any direction (N_1, N_2, N_3) *in space will correspond to two propagation speeds T that are defined by equation* (229). Consequently, the crystalline medium envisioned will be *birefringent.*

29. Electromagnetic rays. – Electromagnetic rays are defined by the differential system:

$$\frac{dx^i}{w^i} = dt \qquad (i = 1, 2, 3), \tag{231}$$

in which:

$$w^i \equiv \left(\frac{\partial \mathbf{O}}{\partial \Omega_i}\right) : \left(\frac{\partial \mathbf{O}}{\partial \Omega_t}\right), \tag{232}$$

with

$$\mathbf{O} \equiv \sum_{i=1}^{3} \frac{\Omega_i^2}{\Omega_t^2 - c_i^2\, \Delta_1\, \Omega}. \tag{233}$$

Recall that $\Omega(x, t)$ represents a wave function (§ **8**).

One easily obtains:

$$w^i \equiv \frac{\dfrac{\Omega_i}{\Omega_t^2 - c_i^2\, \Delta_1\, \Omega} + \displaystyle\sum_{k=1}^{3} \frac{\Omega_k^2\, c_k^2}{(\Omega_t^2 - c_k^2\, \Delta_1\, \Omega)^2}\, \Omega_i}{-\displaystyle\sum_{k=1}^{3} \frac{\Omega_k^2}{(\Omega_t^2 - c_k^2\, \Delta_1\, \Omega)^2}\, \Omega_t}. \tag{234}$$

However, one deduces from (230) that:

$$\sum_{i=1}^{3}\frac{\Omega_i^2(\Omega_t^2-c_i^2\Delta_1\Omega)}{(\Omega_t^2-c_i^2\,\Delta_1\,\Omega)^2}\equiv\Omega_t^2\sum_{i=1}^{3}\frac{\Omega_i^2}{(\Omega_t^2-c_i^2\,\Delta_1\,\Omega)^2}-\Delta_1\Omega\sum_{i=1}^{3}\frac{\Omega_i^2\,c_i^2}{(\Omega_t^2-c_i^2\,\Delta_1\,\Omega)^2}\equiv 0, \tag{235}$$

when one takes into account that $\Omega \equiv \Omega\,(x, t)$ is a wave function; hence (234):

$$\boxed{w^i\equiv-\frac{\Omega_i}{\Omega_t}\left(\frac{\Omega_i^2}{\Delta_1\Omega}+\frac{\dfrac{1}{\Omega_t^2-c_i^2\,\Delta_1\,\Omega}}{\displaystyle\sum_{k=1}^{3}\left(\frac{\Omega_k}{\Omega_t^2-c_i^2\,\Delta_1\,\Omega}\right)^2}\right)}\qquad (i = 1, 2, 3), \tag{236}$$

or rather, thanks to (213) and (214):

$$w^i\equiv\frac{N^i}{T}\left(T^2+\frac{A^2}{T^2-c_i^2}\right), \tag{237}$$

in which one has set:

$$A^2\equiv\frac{1}{\displaystyle\sum_{k=1}^{3}\left(\frac{N_k}{T^2-c_k^2}\right)^2}. \tag{238}$$

Formula (237) permits one to complete the proposition that was stated at the end of § **27**:

At any point in the electromagnetic field, any spatial direction (N_1, N_2, N_3) *will correspond to two velocities of wave propagation T and two rays w.*

Theorem I. – *The normal component of the wave vector* **w** = (w_1, w_2, w_3) *is equal to the velocity of transport T of the wave.*

Indeed, if results from (237) that:

$$(\mathbf{N}\cdot\mathbf{w})\equiv\sum_{i=1}^{3}N_i w^i\equiv T, \tag{239}$$

when one takes (229) into account.

Theorem II. – *The vectors* $\lambda^{(e)}$ *and* $\lambda^{(m)}$ *are perpendicular to the vector* **w**.

Indeed, one has, from (237):

$$(\mathbf{w}\cdot\lambda^{(e)})\equiv\sum_{i=1}^{3}w^i\lambda_i^{(e)}\equiv T\,(\mathbf{N}\cdot\lambda^{(e)})+\sum_{i=1}^{3}\frac{N_i\,\lambda_i^{(e)}}{T^2-c_i^2}, \tag{240}$$

or rather:

$$\sum_{i=1}^{3} w^i \lambda_i^{(e)} \equiv \left(T - \frac{A^2}{T} \sum_{i=1}^{3} \frac{N_i^2\, c_i^2}{(T^2 - c_i^2)^2} \right) (\mathbf{N} \cdot \lambda^{(e)}) \,. \tag{241}$$

However, it results from (229) that:

$$\sum_{i=1}^{3} \frac{N_i\, \lambda_i^{(e)}}{T^2 - c_i^2} \equiv \sum_{i=1}^{3} \frac{N_i^2\, (T^2 - c_i^2)}{(T^2 - c_i^2)} \equiv T^2 \sum_{i=1}^{3} \frac{N_i^2}{(T^2 - c_i^2)} - \sum_{i=1}^{3} \frac{N_i^2\, c_i^2}{(T^2 - c_i^2)} \equiv 0; \tag{242}$$

hence, from (238) and (241):

$$(\mathbf{w} \cdot \lambda^{(e)}) \equiv \sum_{i=1}^{3} w^i \lambda_i^{(e)} \equiv \left(T - \frac{A^2}{T} \cdot \frac{T^2}{A^2} \right) (\mathbf{N} \cdot \lambda^{(e)}) \equiv 0. \tag{243}$$

Q. E. D.

If one takes into account the first relation (217) and formula (218) then one will have:

$$(\mathbf{w} \cdot \lambda^{(m)}) \equiv + \frac{c}{T} \cdot \frac{1}{\mu} (\mathbf{w} \cdot (\mathbf{N} \times \lambda^{(e)})). \tag{244}$$

Now, calculate the components of the vector $(\mathbf{N} \times \lambda^{(e)})$; one easily finds that from (226):

$$(\mathbf{N} \times \lambda^{(e)})_i \equiv N_{i+1}\, \lambda_{i+2}^{(e)} - N_{i+2}\, \lambda_{i+1}^{(e)} \equiv (\mathbf{N} \cdot \lambda^{(e)})\, N_{i+1}\, N_{i+2} \left(\frac{1}{1 - \dfrac{T^2}{c_{i+2}^2}} - \frac{1}{1 - \dfrac{T^2}{c_{i+1}^2}} \right), \tag{245}$$

or rather:

$$(\mathbf{N} \times \lambda^{(e)})_i \equiv (\mathbf{N} \cdot \lambda^{(e)})\, N_{i+1}\, N_{i+2} \frac{c_{i+1}^2 - c_{i+2}^2}{(T^2 - c_{i+1}^2)(T^2 - c_{i+2}^2)}, \tag{246}$$

and in turn, from (244) and (237):

$$\left. \begin{aligned} &(\mathbf{w} \cdot (\mathbf{N} \times \lambda^{(e)})) \equiv \sum_{i=1}^{3} w^i (\mathbf{N} \times \lambda^{(e)})_i \\ &\equiv T(\mathbf{N} \cdot \lambda^{(e)}) \sum_{i=1}^{3} N_i N_{i+1} N_{i+2} \frac{c_{i+1}^2 - c_{i+2}^2}{(T^2 - c_{i+1}^2)(T^2 - c_{i+2}^2)} \left(T^2 + \frac{A^2}{T^2 - c_i^2} \right), \end{aligned} \right\} \tag{247}$$

or rather:

$$\left. \begin{aligned} &(\mathbf{w} \cdot (\mathbf{N} \times \lambda^{(e)})) \equiv \\ &T(\mathbf{N} \cdot \lambda^{(e)}) N_1 N_2 N_3 \left\{ \sum_{i=1}^{3} \frac{c_{i+1}^2 - c_{i+2}^2}{(T^2 - c_{i+1}^2)(T^2 - c_{i+2}^2)} + \frac{A^2}{(T^2 - c_1^2)(T^2 - c_2^2)(T^2 - c_3^2)} \sum_{i=1}^{3} (c_{i+1}^2 - c_{i+2}^2) \right\}. \end{aligned} \right\} \tag{248}$$

However:

$$\sum_{i=1}^{3}\frac{c_{i+1}^2-c_{i+2}^2}{(T^2-c_{i+1}^2)(T^2-c_{i+2}^2)}=\frac{1}{(T^2-c_1^2)(T^2-c_2^2)(T^2-c_3^2)}\left\{T^2\sum_{i=1}^{3}(c_{i+1}^2-c_{i+2}^2)-\sum_{i=1}^{3}(c_{i+1}^2-c_{i+2}^2)c_i^2\right\}. \tag{249}$$

One easily sees that the sums that appear in the right-hand sides of (248) and (249) are identically zero; hence, from (244):

$$(\mathbf{w}\cdot\lambda^{(m)})\equiv 0. \tag{250}$$

Q. E. D.

Corollaries:

1. **u** *and* **w** *will have the same line of application and the same axis at any point of the wave front.*

2. *The trihedron that is defined by the vectors* $\lambda^{(m)}$, $\lambda^{(m)}$, *and* **w** *is a right-hand tri-rectangular trihedron.*

3. *At any point of a electromagnetic field that immediately follows the passage of a wave front, the electric force* **H**, *the magnetic force* $\mathcal{H}$, *and the velocity vector* **w** *that defines the transport of the front along the rays forms a right-handed tri-rectangular trihedron.*

The angle that the vectors $\lambda^{(e)}$ and $\varpi^{(e)}$ subtend is equal to the angle between **w** and **N**. Set:

$$\alpha\equiv\text{angle }(\lambda^{(e)},\varpi^{(e)})\equiv\text{angle }(\mathbf{w},\mathbf{N}); \tag{251}$$

hence:

$$\boxed{T\equiv|\mathbf{w}|\cos\alpha,} \tag{252}$$

in which | **w** | represents the algebraic value of the vector **w**.

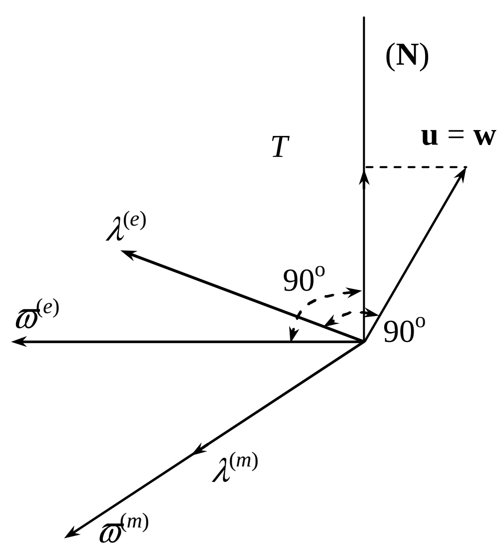

Figure 7.

Corollary. – *The vectors* $\lambda^{(e)}$, $\varpi^{(e)}$, $\mathbf{w}$, $\mathbf{u}$, $\mathbf{N}$ *are in a plane that is perpendicular to the line of application of the vectors* $\lambda^{(m)}$ *and* $\varpi^{(m)}$; moreover, *the vectors* $\mathbf{w}$ *and* $\mathbf{u}$ *have the same axis* (Fig. 7).

Theorem III. – *I say that at any point of the wave front, one will have the equipollence:*

$$\boxed{\mathbf{u} \equiv \mathbf{w}.} \tag{253}$$

Indeed, if we return to the definition (222):

$$\mathbf{u} = c\,\frac{(\lambda^{(e)} \times \lambda^{(m)})}{\frac{1}{2}[(\lambda^{(e)} \cdot \varpi^{(m)}) + (\lambda^{(m)} \cdot \varpi^{(m)})]}; \tag{254}$$

hence, thanks to (217):

$$\mathbf{u} = T\,\frac{(\lambda^{(e)} \times \lambda^{(m)})}{-\frac{1}{2}[(\lambda^{(e)} \cdot (\mathbf{N} \times \lambda^{(m)})) + (\lambda^{(m)} \cdot (\mathbf{N} \times \lambda^{(e)}))]}, \tag{255}$$

or rather ([1]):

$$\boxed{\mathbf{u} = T\,\frac{(\lambda^{(e)} \times \lambda^{(m)})}{(\mathbf{N} \cdot (\lambda^{(e)} \times \lambda^{(m)}))}.} \tag{256}$$

However, one has:

$$(\mathbf{N} \cdot (\lambda^{(e)} \times \lambda^{(m)})) \equiv |(\lambda^{(e)} \times \lambda^{(m)})| \cos \alpha. \tag{257}$$

On the other hand:

$$\frac{(\lambda^{(e)} \times \lambda^{(m)})}{|(\lambda^{(e)} \times \lambda^{(m)})|}$$

is, by virtue of the preceding theorem, the unit vector on the axis that coincides with the line of application of the vector $\mathbf{w}$; hence:

$$\frac{(\lambda^{(e)} \times \lambda^{(m)})}{|(\lambda^{(e)} \times \lambda^{(m)})|} \equiv \frac{\mathbf{w}}{|\mathbf{w}|}. \tag{258}$$

Finally, one has:

$$\mathbf{u} = T\,\frac{\mathbf{w}}{|\mathbf{w}|} \cdot \frac{1}{\cos \alpha}, \tag{259}$$

and consequently, from (252):

$$\mathbf{u} \equiv \mathbf{w}.$$

Q. E. D.

Remarks. –

1. One deduces from (237) that:

([1]) The relation (256) is general (see the remark at the bottom of page 7). In any case, one will have:

$$\boxed{u_N \equiv w_N \equiv T.}$$

$$|\,\mathbf{w}\,|^2 \equiv w^2 \equiv \sum_{i=1}^{3} (w^i)^2 = \sum_{i=1}^{3} \left(T\,N_i + \frac{A^2}{T} \frac{N_i}{T^2 - c_i^2} \right)^2 , \tag{260}$$

or rather, from (229):

$$w^2 \equiv T^2 + \frac{A^2}{T} \sum_{i=1}^{3} \left(\frac{N_i}{T^2 - c_i^2} \right)^2 , \tag{261}$$

and as a result, from (238):

$$w^2 = T^2 + \frac{A^2}{T} . \tag{262}$$

Thanks to (252), one will then have:

$$\boxed{A^2 = T^2 (w^2 - T^2) = T^4 \tan^2 \alpha.} \tag{263}$$

2. Return to formula (237); if one takes (262) into account then one will get:

$$w^i \equiv \frac{N_i}{T} \left(T^2 + \frac{T^2 (w^2 - T^2)}{T^2 - c_i^2} \right) = TN_i \frac{w^2 - c_i^2}{T^2 - c_i^2} . \tag{264}$$

Hence, upon setting:

$$T^i \equiv T\,N^i \equiv T\,N_i , \tag{265}$$

one will find the following *remarkable relation:*

$$\boxed{\frac{w^i}{w^2 - c_i^2} = \frac{T^i}{T^2 - c_i^2} .} \tag{266}$$

Conclusion. – *The rays that are associated with the electromagnetic wave front* Ω *are the trajectories of the electromagnetic energy*. – As a result, the analogy between the rays *of the mathematicians*, to which one is led by the problem of integrating the wave equation, on the one hand, and the rays that are considered *by the physicists*, on the other hand, is complete.

The partial differential equation:

$$\boxed{\sum_{i=1}^{3} \frac{\Omega_i^2}{\Omega_i^2 - c_i^2\, \Delta_1 \Omega} = 0} \tag{267}$$

represents the *wave aspect*, and the differential system:

$$\boxed{\frac{dx^i}{dt} = T \frac{w^2 - c_i^2}{T^2 - c_i^2} N^i ,} \tag{268}$$

with

$$w^2 \equiv \sum_{i=1}^{3} \left(\frac{dx^i}{dt} \right)^2 ,$$

represents the *corpuscular aspect* of Maxwell's system of equations (195) and (196) for the electromagnetic field in a crystalline medium.

Particular case. – When the medium envisioned is *isotropic*, one will have:

$$\varepsilon_1 \equiv \varepsilon_2 \equiv \varepsilon_3 \equiv \varepsilon . \tag{269}$$

In this case, the partial differential equation for the wave and the differential system for the rays reduce to:

$$\boxed{\Omega_t^2 - c_*^2 \Delta_1 \Omega = 0} \tag{267′}$$

and

$$\boxed{\frac{dx^i}{dt} = TN^i} \qquad (i = 1, 2, 3), \tag{268′}$$

respectively, in which:

$$c_*^2 \equiv \frac{c^2}{\varepsilon\mu} \tag{270}$$

and

$$T \equiv u \equiv w \equiv c_* . \tag{271}$$

Equations (268′) show that the rays will be normal to the wave in an isotropic medium.

30. Elementary Huygens wave. – We saw in (§ **10**) that the equation for an elementary Huygens wave is deduced from the general integral (61):

$$\left.\begin{aligned} x^i &= x^i(\tau; x_0, t_0, \Omega_x^0, \Omega_t^0), \\ t &= t(\tau; x_0, t_0, \Omega_x^0, \Omega_t^0), \end{aligned}\right\} \tag{272}$$

$$\Omega = \Omega_0 = \text{invariant},$$

$$\left.\begin{aligned} \Omega_i &= \Omega_i(\tau; x_0, t_0, \Omega_x^0, \Omega_t^0), \\ \Omega_t &= \Omega_t(\tau; x_0, t_0, \Omega_x^0, \Omega_t^0) \end{aligned}\right\} \tag{273}$$

of the differential system for the bicharacteristics (60):

$$\frac{dx^i}{\dfrac{\partial \mathbf{O}}{\partial \Omega_i}} = \frac{dt}{\dfrac{\partial \mathbf{O}}{\partial \Omega_t}} = \frac{d\Omega}{0} = \frac{-d\Omega_i}{\dfrac{\partial \mathbf{O}}{\partial x^i}} = \frac{-d\Omega_t}{\dfrac{\partial \mathbf{O}}{\partial t}} = d\tau \qquad (i = 1, 2, 3) \tag{274}$$

with

$$\mathsf{O} \equiv \sum_{i=1}^{3} \frac{\Omega_i^2}{\Omega_i^2 - c_i^2\,\Delta_1\,\Omega}. \tag{274′}$$

Conforming to what we said in § **10**, consider all of the rays that issue from the point x_0 at the instant t_0 . These rays are defined by equations (272), in which the Ω_x^0 represent any direction that emanates from the point P, and thanks to the wave equation (267), Ω_t^0 is a known function of the Ω_x^0. One can then assume that the Ω_x^0 are normal. Then set:

$$\Omega_i^0 \equiv \omega_i\,, \qquad \Omega_t^0 \equiv \theta, \tag{275}$$

with

$$\sum_{i=1}^{3} \omega_i^2 \equiv 1. \tag{276}$$

Furthermore, the problem of the search for the equation of Huygens's elementary wave reduces to the problem of eliminating τ, ω_1, ω_2, ω_3 , and θ from the relations (272), (276), and:

$$\sum_{i=1}^{3} \frac{\omega_i^2}{\theta^2 - c_{0i}^2} = 0, \tag{277}$$

in which we have set:

$$c_{0i} \equiv c_i\,(x_0, t_0;\ H_i^0) \qquad \text{with} \qquad H_i^0 \equiv H\,(x_0, t_0). \tag{278}$$

31. Case of a homogeneous medium. Fresnel's wave surface. – The crystalline medium envisioned is homogeneous when ε_1, ε_2, ε_3 , and μ are constants. In that case, one will have:

$$B_i = \varepsilon_i\,H_i \qquad \text{and} \qquad \mathcal{B}_i = \mu\,\mathcal{H}_i\,; \tag{279}$$

hence, from (194) and (224):

$$c_i = \text{constant} \qquad (i = 1, 2, 3). \tag{280}$$

As a result, the wave equation (267) will have constant coefficients, and from (235), the differential system of the bicharacteristics (274) will reduce to:

$$\frac{dx^i}{\left\{\dfrac{\dfrac{1}{\Omega_t^2 - c_i^2\Delta_1\,\Omega}}{\displaystyle\sum_{k=1}^{3}\frac{\Omega_k^2}{\Omega_t^2 - c_k^2\Delta_1\,\Omega}} + \dfrac{\Omega_t^2}{\Delta_1\,\Omega}\right\}\Omega_i^2} = \frac{dt}{-\Omega_t} = \frac{d\Omega}{0} = \frac{d\Omega_i}{0} = \frac{d\Omega_t}{0} \qquad (i = 1, 2, 3). \tag{281}$$

Hence, one will have the invariants:

$$\Omega = \Omega^0, \qquad \Omega_i = \Omega_i^0, \qquad \Omega_t = \Omega_t^0, \tag{282}$$

and in turn:

$$N_i = N_i^0 \qquad \text{and} \qquad T = T^{\,0}. \tag{283}$$

That will lead us to set:

$$N_i \equiv \alpha_i\,, \qquad T \equiv \theta, \tag{284}$$

with

$$\sum_{i=1}^{3} \alpha_i^2 \equiv 1, \tag{285}$$

in which the α_1 , α_2 , α_3 , and θ are *constants*. Furthermore, it is natural (§ **16**) to envision the propagation of *plane waves*. Therefore, consider the wave function:

$$\Omega\,(x, t) \equiv \sum_{i=1}^{3} \alpha_i\, x^i - \theta\, t, \tag{286}$$

whose coefficients are coupled by the relation (229), namely:

$$\sum_{i=1}^{3} \frac{\alpha_i^2}{\theta^2 - c_i^2} = 0. \tag{287}$$

It will then result from (281) and the definition (234) that the equations of the rays are:

$$\boxed{x^i = x_0^i + w^i (t - t_0)} \qquad (i = 1, 2, 3) \tag{288}$$

in which the constant coefficients w^i are given by (237):

$$w^{\,i} \equiv \frac{\alpha^i}{\theta} \left(\theta^2 + \frac{\dfrac{1}{\sum_{k=1}^{3} \left(\dfrac{\alpha_k}{\theta^2 - c_k^2} \right)^2}}{\theta^2 - c_i^2} \right) \qquad (i = 1, 2, 3). \tag{289}$$

That being the case, in order to obtain *the equation of elementary Huygens wave*, as we just saw (§ **30**), it will suffice to eliminate α_1, α_2, α_3, and θ from the five equations (288), (287), and (285).

To fix ideas, suppose that the point x_0 coincides with the origin O of the coordinates; hence:

$$t_0 \equiv 0. \tag{290}$$

Equations (288) will then become:

$$x^i = w^i\, t. \tag{291}$$

At the instant t, let P be any point of the elementary Huygens wave that issues from O at the initial instant $t = 0$. The coordinates x^1, x^2, x^3 of the point P are given by (291), and the normal to the wave surface at that point is determined by the direction cosines α_1, α_2, α_3. As a result, the distance from the origin O to the tangent plane to the elementary wave at P will be defined by:

$$\sum_{i=1}^{3} \alpha_i\, x^i \equiv \sum_{i=1}^{3} \alpha_i\, w^i\, t\,, \tag{292}$$

so, thanks to (287) and (289):

$$\sum_{i=1}^{3} \alpha_i\, x^i \equiv \theta\, t. \tag{292′}$$

That distance is then equal to $\theta\, t$. As a consequence, at the instant t, *the elementary Huygens wave that issues from the origin O at the initial instant* $t = 0$ *will be the envelope of the wave planes of the equation:*

$$\sum_{i=1}^{3} \alpha_i\, x^i - \theta\, t = 0 \tag{293}$$

whose coefficients α_1, α_2, α_3, and θ verify both the relations (285) and (287). One will then see that the elimination problem that was posed above is equivalent to the problem of searching for the equation of the envelope of a two-parameter family of planes. The equation of that envelope – i.e., the elementary Huygens wave – is nothing but the *point-like equation in* x^1, x^2, x^3 *that corresponds to the tangential equation* (287) in α_1, α_2, α_3.

We are then led to seek that equation of the envelope of the planes (293) that depend upon two parameters. To that effect, set ([1]):

$$\varphi = \frac{1}{2}\sum_{i=1}^{3} \frac{\alpha_i^2}{\theta^2 - c_i^2} \tag{294}$$

and

$$\varphi_i = \frac{\partial \varphi}{\partial \alpha_i} = \frac{\alpha_i}{\theta^2 - c_i^2}\,, \tag{295}$$

$$\varphi_t = \frac{\partial \varphi}{\partial \theta} = -\,\theta \sum_{i=1}^{3} \varphi_i^2\ . \tag{296}$$

We now differentiate the relations (293), (285), and (287); hence:

$$\sum_{i=1}^{3} x^i\, d\alpha_i - t\, d\theta = 0, \tag{297}$$

$$\sum_{i=1}^{3} \alpha_i\, d\alpha_i = 0, \tag{298}$$

([1]) Here, we are inspired by a calculation of LEVI-CIVITA, *loc. cit.*, see, pp. 90, *et seq.*

$$\sum_{i=1}^{3} \varphi_i \, d\alpha_i + \varphi_4 \, d\theta = 0. \tag{299}$$

Replace $d\theta$ in (299) with its value that one infers from (297); one then obtains:

$$\sum_{i=1}^{3} \left(\varphi_i + \frac{x^i}{t} \varphi_4 \right) d\alpha_i = 0, \tag{300}$$

in which the $d\alpha_i$ are linked by the relation (298); as a result:

$$\boxed{\varphi_i + \frac{x^i}{t} \varphi_4 = \chi \alpha_i} \qquad (i = 1, 2, 3), \tag{301}$$

in which χ is a parameter. In order to calculate the value of χ, multiply (301) by α_i and sum over the index i; hence:

$$\sum_{i=1}^{3} \alpha_i \, \varphi_i + \theta \, \varphi_4 = \chi, \tag{302}$$

when one takes (293) and (285) into account. However, thanks to (287), one will have the identity:

$$\sum_{i=1}^{3} \alpha_i \, \varphi_i \equiv 0, \tag{303}$$

and consequently:

$$\chi \equiv \theta \, \varphi_4 \, , \tag{304}$$

or rather, from (296):

$$\chi \equiv - \, \theta^2 \sum_{i=1}^{3} \varphi_i^2 \; . \tag{305}$$

Having achieved that, we proceed to eliminate the α_1, α_2, α_3 . In order to do that, we return to the relation (301) and replace α_i with its value as a function of φ_i that we infer from (295); we will have:

$$\frac{x^4}{t} \varphi_4 = \varphi_i \left(\chi \left(\theta^2 - c_i^2 \right) - 1 \right) \qquad (i = 1, 2, 3); \tag{306}$$

hence:

$$\varphi_i = - \, \frac{\varphi_4}{t} \frac{x^4}{1 - \chi\theta^2 + \chi c_i^2} \; . \tag{307}$$

We then eliminate θ; to that end, we calculate $1 - \chi \, \theta^2$. It results from the relation (301) that:

$$\frac{x^i}{t}\varphi_4 = \chi\,\alpha_i - \varphi_i\,. \tag{308}$$

Next, square both sides of (308) and sum over the index i ; hence, thanks to (303) and (285):

$$\varphi_4^2\left(\frac{r}{t}\right)^2 = \chi^2 + \sum_{i=1}^{3}\varphi_i^2\,, \tag{309}$$

with:

$$r^2 \equiv \sum_{i=1}^{3}(x^i)^2\,. \tag{310}$$

Now, replace $\sum_{i=1}^{3}\varphi_i^2$ with its value that one infers from (296); if one takes (304) into account then one will get:

$$\varphi_4^2\left(\frac{r}{t}\right)^2 = \chi\,\theta\,\varphi_4 - \frac{\varphi_4}{\theta} = \frac{\varphi_4}{\theta}(\chi\,\theta^2 - 1), \tag{311}$$

or rather, from (304):

$$1 - \chi\,\theta^2 = -\varphi_4\,\theta\left(\frac{r}{t}\right)^2 = -\chi\left(\frac{r}{t}\right)^2. \tag{312}$$

Finally, replace $1 - \chi\,\theta^2$ in (307) with its value (312); hence:

$$\boxed{\varphi_i = \frac{\varphi_4}{\chi}\frac{x^4/t}{(r/t)^2 - c_i^2}} \qquad (i = 1, 2, 3). \tag{313}$$

We now return to the *tangential* equation (287); one first deduces that:

$$0 = \sum_{i=1}^{3}\frac{\alpha_i^2}{\theta^2 - c_i^2} \equiv \sum_{i=1}^{3}\frac{\alpha_i^2(\theta^2 - c_i^2)}{(\theta^2 - c_i^2)^2} \equiv \theta^2\sum_{i=1}^{3}\left(\frac{\alpha_i}{\theta^2 - c_i^2}\right)^2 - \sum_{i=1}^{3}\left(\frac{\alpha_i\,c_i}{\theta^2 - c_i^2}\right)^2, \tag{314}$$

or rather, from (295):

$$\theta^2\sum_{i=1}^{3}\varphi_i^2 - \sum_{i=1}^{3}\varphi_i^2 c_i^2 = 0, \tag{315}$$

and then, from (285):

$$0 = \theta^2\sum_{i=1}^{3}\frac{\alpha_i^2}{\theta^2 - c_i^2} \equiv \sum_{i=1}^{3}\frac{\alpha_i^2\,c_i^2}{\theta^2 - c_i^2} - \sum_{i=1}^{3}\frac{\alpha_i^2(c_i^2 - \theta^2)}{\theta^2 - c_i^2} \equiv \sum_{i=1}^{3}\frac{\alpha_i^2\,c_i^2}{\theta^2 - c_i^2} + 1, \tag{316}$$

or rather, from (295):

$$\boxed{\sum_{i=1}^{3} \alpha_i \, \varphi_i \, c_i^2 + 1 = 0.} \tag{317}$$

Now, replace α_i in (317) with its value in (319); *the equation of the elementary Huygens's wave* will then become:

$$\boxed{\sum_{i=1}^{3} \frac{(x^i / t)^2 c_i^2}{(r / t)^2 - c_i^2} = 0.} \tag{320}$$

Upon setting $t = 1$ in (320), one will recover the classical equation:

$$\boxed{\sum_{i=1}^{3} \frac{(x^i c_i)^2}{r^2 - c_i^2} = 0} \tag{321}$$

of the *Fresnel wave surface.*

CHAPTER VIII

APPLICATION TO SECOND-ORDER LINEAR EQUATIONS

A. – Waves and rays.

32. Second-order equation. – The second-order, linear, partial differential equation:

$$\boxed{\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} B^{\alpha\beta}\frac{\partial^2 z}{\partial x^\alpha \partial x^\beta} + \sum_{\alpha=1}^{n} C^\alpha \frac{\partial z}{\partial x^\alpha} + Dz = E,} \tag{322}$$

whose coefficients $B^{\alpha\beta}$, C^α, D, E depend upon only spatio-temporal variables, plays a very important role in mathematical physics. It also seems indispensible to us to dedicate this chapter to the study of waves and rays that are compatible with that equation.

By hypothesis, one will have:

$$B^{\alpha\beta} \equiv B^{\beta\alpha}. \tag{323}$$

33. Wave equation. – Return to the general equation (8); it reduces to the *Lamé equation:*

$$\boxed{\square_1 \Omega \equiv \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} B^{\alpha\beta}\frac{\partial \Omega}{\partial x^\alpha}\frac{\partial \Omega}{\partial x^\beta} = 0} \tag{324}$$

here, or rather, thanks to (10) and (11):

$$\sum_{i=1}^{n}\sum_{j=1}^{n} B^{ij}\frac{\partial t}{\partial x^i}\frac{\partial t}{\partial x^j} - 2\sum_{\alpha=1}^{n} B^{ni}\frac{\partial z}{\partial x^i} + B^{nn} = 0. \tag{325}$$

Recall that:

$$t \equiv x^n.$$

34. Differential system of the rays. – The Cauchy characteristics of the wave equation (324) – i.e., the Hadamard bicharacteristics of equation (322) of mathematical physics – are defined by the differential system:

$$\frac{dx^\alpha}{\dfrac{\partial \square_1 \Omega}{\partial \Omega_\alpha}} = \frac{d\Omega}{0} = \frac{-d\Omega_\alpha}{\dfrac{\partial \square_1 \Omega}{\partial x^\alpha}} \qquad (\alpha = 1, 2, \ldots, n). \tag{326}$$

As for the *rays* that are associated with the wave function:

$$\Omega \equiv \Omega\,(x, t), \tag{327}$$

they are determined by the differential system:

$$\boxed{\frac{dx^i}{w^i} = dt} \qquad (i = 1, 2, \ldots, n-1), \tag{328}$$

with

$$w^i \equiv \left(\frac{\partial \square_1 \Omega}{\partial \Omega_i}\right) : \left(\frac{\partial \square_1 \Omega}{\partial \Omega_t}\right), \tag{328$'$}$$

in which the parentheses serve to recall that one has replaced Ω with the wave function (327).

We have seen (§ **10**) how one can, in a general fashion, deduce the general integral of the differential system (326) from the *elementary Huygens wave.* However, before we pursue the study of waves and rays that are compatible with the equation (322) of physics, we propose to present some generalities about the geodesics that are associated with an *arbitrary* quadratic differential form (see B). We will then show (see C) how those considerations can be utilized in the theory of waves.

B. – Geodesics of a quadratic differential form.

35. Preliminaries. – Let:

$$\boxed{(ds)^2 \equiv \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} B_{\alpha\beta}\, dx^\alpha dx^\beta} \tag{329}$$

be the quadratic differential equation that defines the metric on space $(x^1, \ldots, x^{n-1})$. By hypothesis, that form is invariant with respect to an arbitrary change of the variables x^α into the variables $\bar{x}^\alpha$:

$$x^\alpha = x^\alpha\,(\bar{x}^1, \ldots, \bar{x}^n) \qquad \text{or} \qquad \bar{x}^\alpha = \bar{x}^\alpha\,(x^1, \ldots, x^{n-1}) \qquad (\alpha = 1, 2, \ldots, n). \tag{330}$$

The $\dfrac{n(n+1)}{1\cdot 2}$ functions $B_{\alpha\beta}$ (*):

$$B_{\alpha\beta} \equiv B_{\alpha\beta}\,(x^1, \ldots, x^{n-1}) \equiv B_{\beta\alpha}\,(x^1, \ldots, x^{n-1}) \tag{331}$$

define the covariant components of a second-order symmetric tensor. Set:

$$B \equiv \| B_{\alpha\beta} \| \tag{332}$$

(*) The functions $B_{\alpha\beta}$ are supposed to satisfy all of the regularity conditions that are necessary to justify the argument and calculations that will follow.

and

$$B^{\alpha\beta} \equiv \frac{\text{minor of } B_{\alpha\beta}}{B} \qquad (\alpha, \beta = 1, 2, \dots, n). \tag{333}$$

Recall that $\sqrt{B}$ has the variance of a multiplier, and that the functions $B^{\alpha\beta}$ are the contravariant components of the tensor that is defined by the coefficients of the invariant differential form (329). One then has the variance relations:

$$B_{\alpha\beta} = \sum_{a=1}^{n}\sum_{b=1}^{n} \frac{\partial \bar{x}^a}{\partial x^\alpha}\frac{\partial \bar{x}^b}{\partial x^\beta}\bar{B}_{ab}, \tag{334}$$

$$B^{\alpha\beta} = \sum_{a=1}^{n}\sum_{b=1}^{n} \frac{\partial x^\alpha}{\partial \bar{x}^a}\frac{\partial x^\beta}{\partial \bar{x}^b}\bar{B}^{ab}, \tag{335}$$

$$\sqrt{B} = \frac{\partial(\bar{x}^1, \dots, \bar{x}^n)}{\partial(x^1, \dots, x^n)}\sqrt{\bar{B}}. \tag{336}$$

Introduce the notation:

$$L^2(x, dx) \equiv \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} B_{\alpha\beta}\, dx^\alpha dx^\beta, \tag{337}$$

and set:

$$u^\alpha \equiv \frac{dx^\alpha}{ds} \qquad (\alpha = 1, 2, \dots, n). \tag{338}$$

One will then get the identity in $x^1, \dots, x^n$:

$$L^2(x, u) \equiv \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} B_{\alpha\beta}\, u^\alpha u^\beta \equiv 1. \tag{339}$$

The covariant components of the vector that is defined by the contravariant components (338) are given by:

$$u_\alpha \equiv \sum_{\beta=1}^{n} B_{\alpha\beta}\, u^\beta \qquad (a = 1, 2, \dots, n). \tag{340}$$

It results from (337), (338), and (339) that:

$$L^2 \equiv \sum_{\alpha=1}^{n} u^\alpha u_\alpha \equiv 1 \tag{341}$$

and

$$u^\alpha \equiv \sum_{\beta=1}^{n} B^{\alpha\beta}\, u_\beta. \tag{342}$$

Hence, one has the identity $x^1, x^2, \dots, x^n$:

$$\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} B^{\alpha\beta}\, u_\alpha\, u_\beta \equiv 1. \tag{343}$$

The quadratic form (343) is the form that is adjoint to (339). We remark that:

$$\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} B_{\alpha\beta}\, u^\alpha u^\beta \equiv \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} B^{\alpha\beta}\, u_\alpha\, u_\beta . \tag{344}$$

36. Definition and Lagrangian equations of the geodesics. – The geodesics of the line element (329) are the extremals of the integral:

$$I \equiv \int_{x_0}^{x_1} L(x, dx) = \int_{x_0}^{x_1} L(x, u)\, ds . \tag{345}$$

They represent the length of an arc of a curve that joins the point x_0 to the point x_1 in the n-dimensional non-Euclidian space whose metric is defined by the quadratic differential form (329). The first variation of I is given by:

$$\delta I = \sum_{\alpha=1}^{n}\left\{\left(\frac{\partial L}{\partial u^\alpha}\right)_1 \delta x_1^\alpha - \left(\frac{\partial L}{\partial u^\alpha}\right)_0 \delta x_0^\alpha\right\} + \int_{x_0}^{x_1}\left\{\sum_{\alpha=1}^{n}\left(\frac{\partial L}{\partial x^\alpha} - \frac{d}{ds}\frac{\partial L}{\partial u^\alpha}\right)\delta x^\alpha\right\} ds . \tag{346}$$

Hence, one has the *Lagrangian differential equations:*

$$\boxed{\frac{\delta L}{\delta x^\alpha} \equiv \frac{\partial L}{\partial x^\alpha} - \frac{d}{ds}\frac{\partial L}{\partial u^\alpha} = 0} \qquad (\alpha = 1, 2, \ldots, n) \tag{347}$$

for the geodesics of the line element (329).

37. Hamiltonian equations of the geodesics. – One knows that equations (347) admit the relative integral invariant:

$$j \equiv \sum_{\alpha=1}^{n} \frac{\partial L}{\partial u^\alpha}\, \delta x^\alpha . \tag{348}$$

Indeed, one easily verifies the identity:

$$\frac{dj}{ds} \equiv \delta L, \tag{349}$$

by virtue of (347).

That being the case, introduce the canonical Poisson variables:

$$u_\alpha \equiv \frac{\partial L}{\partial u^\alpha} \qquad (\alpha = 1, 2, \ldots, n), \tag{350}$$

so, from (339):

$$u_\alpha \equiv \sum_{\beta=1}^{n} B_{\alpha\beta}\, u^\beta \,. \tag{351}$$

The canonical Poisson variables are then the covariant components (340) of the vector that is defined by the contravariant components (338). The vector is nothing but the unit vector that is tangent to the arc of the geodesic that joins the point x_0 to the point x_1 .

From the notation (338) and the identity (349), it will then result that:

$$\sum_{\alpha=1}^{n}\left(\frac{du_\alpha}{ds}\,\delta x^\alpha + u_\alpha \delta u^\alpha\right) \equiv \delta L, \tag{352}$$

or rather:

$$\sum_{\alpha=1}^{n}\left(\frac{du_\alpha}{ds}\,\delta x^\alpha - u^\alpha \delta u_\alpha\right) \equiv \delta\left(L - \sum_{\alpha=1}^{n} u_\alpha\, u^\alpha\right). \tag{353}$$

Now set:

$$H\,(x, u) \equiv - L + \sum_{\alpha=1}^{n} u^\alpha\, u_\alpha \,; \tag{354}$$

hence, thanks to (339) and (350):

$$H\,(x, u) \equiv -1 + \sum_{\alpha=1}^{n} u^\alpha \frac{2\sum_{\beta=1}^{n} B_{\alpha\beta}\, u^\beta}{2\sqrt{\sum_{\beta=1}^{n}\sum_{\beta=1}^{n} B_{\alpha\beta}\, u^\alpha u^\beta}} \equiv \sqrt{\sum_{\beta=1}^{n}\sum_{\beta=1}^{n} B_{\alpha\beta}\, u^\alpha u^\beta} - 1, \tag{355}$$

or rather:

$$H\,(x, u) \equiv \sqrt{\sum_{\beta=1}^{n}\sum_{\beta=1}^{n} B^{\alpha\beta}\, u_\alpha u_\beta} - 1. \tag{356}$$

From (353), (354), and (356), and taking (338) into account, one will then deduce *the canonical Hamiltonian equations of the geodesics,* namely:

$$\boxed{\begin{aligned} \frac{dx^\alpha}{ds} &= \frac{\partial H}{\partial u_\alpha}, \\ \frac{du_\alpha}{ds} &= -\frac{\partial H}{\partial x^\alpha} \end{aligned}} \qquad (\alpha = 1, 2, \ldots, n). \qquad \begin{aligned} (357) \\ (358) \end{aligned}$$

We remark that, thanks to (343), one will have the identity in $x^1, x^2, \ldots, x^n$:

$$H(x, u) \equiv \sqrt{\sum_{\beta=1}^{n}\sum_{\beta=1}^{n} B^{\alpha\beta}\, u_\alpha u_\beta} - 1 \equiv 0. \tag{359}$$

Recall the well-known invariance property of the function H (x, u) with respect to the differential system (357) and (358).

Let:

$$\begin{cases} x^\alpha = x^\alpha(s; x_0^1, \dots, x_0^n, u_1^0, \dots, u_n^0), & (360) \\ u_\alpha = u_\alpha(s; x_0^1, \dots, x_0^n, u_1^0, \dots, u_n^0), & (361) \end{cases} \qquad (\alpha = 1, 2, \dots, n)$$

be the general integral of the Hamiltonian equations (357) and (358). Equations (360) are those of a geodesic that issues from the point x_0 and is tangent at that point to the line whose unit vector is defined by the covariant components $u_1^0, \dots, u_n^0$.

38. Jacobi's theorem. – Now consider ([1]) the function that is defined by:

$$S = \int_{x_0, s_0}^{x, s} L(x, u)\, ds, \tag{362}$$

in which *the integral is taken along a geodesic* (360). Along such a curve, one will then have:

$$\frac{dS}{ds} \equiv L(x, u). \tag{363}$$

That being the case, the identity (349) will give:

$$\frac{d}{ds}\left(\sum_{\alpha=1}^{n} u_\alpha \delta x^\alpha - \delta S\right) = 0; \tag{364}$$

hence:

$$\sum_{\alpha=1}^{n}\left(u_\alpha \delta x^\alpha - \frac{\partial S}{\partial x^\alpha}\delta x^\alpha - \frac{\partial S}{\partial x_0^\alpha}\delta x_0^\alpha\right) - \frac{\partial S}{\partial s}\delta s - \frac{\partial S}{\partial s_0}\delta s_0 \equiv \sum_{\alpha=1}^{n} u_\alpha^0\, \delta x_0^\alpha. \tag{365}$$

Upon identifying the coefficients of δx^α, δx_0^α, δs, and δs_0 in (365), one will get *Jacobi's theorem:*

$$\boxed{\begin{aligned} \frac{\partial S}{\partial x^\alpha} &= u_\alpha, & (366) \\ \frac{\partial S}{\partial x_0^\alpha} &= -u_\alpha^0, & (367) \end{aligned}} \qquad (\alpha = 1, 2, \dots, n)$$

([1]) J. VAN MIEGHEM, *Wis. en Natuurkundig Tijdschrift* (Gent, 1932), deel VI); see pp. 80, no. 4.

as well as the relations:

$$\frac{\partial S}{\partial s} \equiv 0 \quad \text{and} \quad \frac{\partial S}{\partial s_0} \equiv 0. \tag{368}$$

It results from the latter equations that the function S that is defined by (362) is independent of s and s_0 ; it will then depend upon only the coordinates of the extremities of the arc of the geodesic envisioned. We then set:

$$S \equiv S\,(x^1, x^2, \ldots, x^n\,;\; x_0^1,\; x_0^2,\; \ldots,\; x_0^n\,). \tag{369}$$

Finally, note that by virtue of (366) and (359), *the function S (x, x_0) is an integral with n arbitrary constants* $x_0^1,\; x_0^2,\; \ldots,\; x_0^n$ *of the Jacobi partial differential equation:*

$$\boxed{\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} B^{\alpha\beta}\,\frac{\partial S}{\partial x^\alpha}\frac{\partial S}{\partial x^\beta} - 1 = 0.} \tag{370}$$

The function S that is defined by (362) represents the distance between the points x and x_0 in the n-dimensional space whose metric is defined by (329). We say that S is the *geodesic distance* between the points x and x_0 .

Remark. – The arc of the geodesic curve can be considered to be a parametric variety or as a function of the coordinates of the extremities of the arc. Indeed, it will result from (339) and (363) that:

$$\frac{dS}{ds} = 1 \tag{371}$$

along a geodesic (360); hence:

$$S = s, \tag{372}$$

when one sets:

$$s_0 = 0. \tag{373}$$

39. General integral of the differential equations of geodesics. – One knows that the general integral (360) and (361) is given by the Jacobian equations (366) and (367) when one can solve them with respect to the x^α and u_α – i.e., when the integral S has n integration constants that are defined by (362) is a complete integral of the Jacobi equation (370). In the case that we are concerned with here, the relations (366) will give the u_α immediately as functions of the x_0^α and u_α^0 if one can solve equations (367) for the x^α. We shall show that this is impossible.

Theorem I. – *I say that:*

$$\left\| \frac{\partial^2 S}{\partial x_0^\alpha\, \partial x^\beta} \right\| \equiv 0. \tag{374}$$

Proof. – Indeed, since the u_α^0 are invariants of the differential equations (357) and (358), one will have the identities:

$$\frac{d}{ds}\left(\frac{\partial S}{\partial x_0^\alpha}\right) \equiv 0 \qquad (\alpha = 1, 2, \ldots, n) \tag{375}$$

in x^α and x_0^α. Hence:

$$\sum_{\beta=1}^{n} \frac{\partial^2 S}{\partial x_0^\alpha \, \partial x^\beta} u^\beta \equiv 0 \qquad (\alpha = 1, 2, \ldots, n), \tag{376}$$

and as a result:

$$\left\| \frac{\partial^2 S}{\partial x_0^\alpha \, \partial x^\beta} \right\| = 0, \tag{377}$$

since the u^α are not all zero at once.

Q. E. D.

Conclusion. – Since the functional determinant of the functions $\partial S / \partial x_0^1, \ldots, \partial S / \partial x_0^n$ with respect to the variables $x^1, x^2, \ldots, x^n$ is zero, it is *impossible* to solve the equations:

$$\frac{\partial S}{\partial x_0^\alpha} = - u_\alpha^0$$

for the x^α. It will then result that the function S that is defined by (362) is not a *complete* integral of the Jacobi equation (370).

That being the case, set ([1]):

$$\Gamma (x^1, \ldots, x^n ; x_0^1, \ldots, x_0^n) \equiv [S (x^1, \ldots, x^n ; x_0^1, \ldots, x_0^n)]^2, \tag{377}$$

in which $\Gamma (x; x_0)$ represents the square of the geodesic distance from the point x to the point x_0. Hence, (366) and (367) will give:

$$\begin{cases} \dfrac{\partial \frac{1}{2}\Gamma}{\partial x^\alpha} = \quad s u_\alpha, & (378) \\ & \qquad (\alpha = 1, 2, \ldots, n) \\ \dfrac{\partial \frac{1}{2}\Gamma}{\partial x_0^\alpha} = - s u_\alpha^0, & (379) \end{cases}$$

when one takes (372) into account. The relations (378) and (379) shows that $s\, u_\alpha$ and $s\, u_\alpha^0$ are functions of the x and x_0 ; one can then write:

([1]) *Ibidem*, see pp. 83, form. (41).

$$\begin{cases} s\,u_\alpha = U_\alpha(x^1, \ldots, x^n; x_0^1, \ldots, x_0^n), & (380) \\ s\,u_\alpha^0 = U_\alpha^0(x^1, \ldots, x^n; x_0^1, \ldots, x_0^n). & (381) \end{cases}$$

Equations (381) show that *geodesics are straight lines in the system of covariant variables* $(U_1^0, \ldots, U_n^0)$ ([1]).

Theorem II. – *I say that:*

$$\left\| \frac{\partial^2 \frac{1}{2}\Gamma}{\partial x_0^\alpha\, \partial x^\beta} \right\| \neq 0. \tag{382}$$

Proof. – Indeed, one has:

$$\frac{d(s\,u_\alpha^0)}{ds} = u_\alpha^0 \qquad (\alpha = 1, 2, \ldots, n), \tag{383}$$

by virtue of (357) and (358). However, from (379):

$$\frac{d(s\,u_\alpha^0)}{ds} = -\frac{d}{ds}\frac{\partial \frac{1}{2}\Gamma}{\partial x_0^\alpha}, \tag{384}$$

and as a result:

$$\sum_{\beta=1}^{n} \frac{\partial^2 \frac{1}{2}\Gamma}{\partial x_0^\alpha\, \partial x^\beta} u^\beta = -u_\alpha^0; \tag{385}$$

hence:

$$\left\| \frac{\partial^2 \frac{1}{2}\Gamma}{\partial x_0^\alpha\, \partial x^\beta} \right\| \neq 0,$$

because the system (385) admits a solution $u^1, \ldots, u^n$.

Q. E. D.

Conclusion. – The functional determinant of the functions $\partial \frac{1}{2}\Gamma / \partial x_0^1, \ldots, \partial \frac{1}{2}\Gamma / \partial x_0^n$ with respect to the variables $x^1, \ldots, x^n$ is non-zero. Upon substituting the values thus-obtained for the x^α in (378), one will find a remarkable formula for the general integral (360) and (361) of the Hamiltonian differential equations for the geodesics, namely ([2]):

([1]) The contravariant variables:

$$U_0^\alpha \equiv \sum_{\beta=1}^{n} B_0^{\alpha\beta} U_\beta^0 \qquad [\text{in which } B_0^{\alpha\beta} \equiv B^{\beta\alpha}(x_0^1, \ldots, x_0^n)]$$

generalize the *Lipschitz normal variables*; on that subject, see: G. DARBOUX, *Leçons sur la théorie générale des surfaces*, t. II (Paris, Gauthier-Villars, 1915); see pp. 422 and 423.

([2]) J. HADAMARD, *Le problème de Cauchy et les équations aux dérivées partielles linéaires hyperboliques* (Paris, Hermann, 1932); see pp. 120, form. (29).

$$\begin{cases} x^{\alpha} = X^{\alpha}(x_0^1, \ldots, x_0^n; su_1^0, \ldots, su_n^0), & (386) \\ su_{\alpha} = \mathcal{U}_{\alpha}(x_0^1, \ldots, x_0^n; su_1^0, \ldots, su_n^0), & (387) \end{cases} \qquad (\alpha = 1, 2, \ldots, n).$$

The Jacobian relations (366) and (367) show that this integral remains unchanged when one permutes x^{α} with x_0^{α} and u_{α} with $-u_{\alpha}^0$ in it. Furthermore, that amounts to replacing s with $-s$; that is what equations (378) and (379) show clearly.

40. Parametric equations of geodesics. – Introduce the direction coefficients:

$$\lambda_{\alpha} \equiv \frac{u_{\alpha}^0}{u_n^0} \qquad (\alpha = 1, 2, \ldots, n) \tag{388}$$

of the tangent at any point of a geodesic. One then deduces from (388) and (343) that:

$$u_{\alpha}^0 \equiv \frac{\lambda_{\alpha}}{\sqrt{\sum_{\gamma=1}^{n}\sum_{\delta=1}^{n} B_0^{\gamma\delta}\lambda_{\gamma}\lambda_{\delta}}}, \tag{389}$$

with

$$\lambda_n \equiv 1 \tag{390}$$

and

$$B_0^{\alpha\beta} \equiv B^{\alpha\beta}(x_0^1, \ldots x_0^n) \qquad (\alpha, \beta = 1, 2, \ldots, n). \tag{391}$$

We remark that the u_{α}^0 that are defined by (389) verify the identity relation:

$$\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} B_0^{\alpha\beta}\, u_{\alpha}^0\, u_{\beta}^0 \equiv 1. \tag{392}$$

As a result, (359) is verified at the point $s = 0$ (or x_0) of the geodesic considered. It will then result from the invariant character of the Hamiltonian function H that this relation will be verified all along the geodesic that issues from the point x_0 and is tangent at that point to the line whose direction coefficients are $\lambda_1, \ldots, \lambda_{n-1}$.

Replace the u_{α}^0 in (386) with their values (389) as functions of the λ; one will get:

$$\left.\begin{aligned} x^{\alpha} &= X^{\alpha}(x_0; s\,u^0), \\ &= X^{\alpha}\left(x_0^1, \ldots, x_0^n; \frac{s\,\lambda_1}{\sqrt{\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} B_0^{\alpha\beta}\lambda_{\alpha}\lambda_{\beta}}}, \ldots, \frac{s\,\lambda_{n-1}}{\sqrt{\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} B_0^{\alpha\beta}\lambda_{\alpha}\lambda_{\beta}}}, \frac{s}{\sqrt{\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} B_0^{\alpha\beta}\lambda_{\alpha}\lambda_{\beta}}}\right); \end{aligned}\right\} \tag{393}$$

hence, one will have *the parametric equations for geodesics:*

$$\boxed{x^\alpha = \mathcal{K}^\alpha(x_0^1, \dots, x_0^n; \lambda_1, \dots, \lambda_{n-1}, s)} \qquad (\alpha = 1, 2, \dots, 3). \tag{394}$$

Theorem. – *I say that:*

$$J \equiv \frac{\partial(x^1, \dots, x^n)}{\partial(\lambda_1, \dots, \lambda_{n-1}, s)} \neq 0. \tag{395}$$

Proof. – Indeed, in order to show that $J \neq 0$, it will suffice to show that is it possible to solve equations (394) for $\lambda_1, \dots, \lambda_{n-1}$, s. To that end, we return to equations (386), which are deduced from the parametric equations (394). Upon solving equations (386) for the $s\,u_\alpha^0$, one will get equations (381). Hence, thanks to (388), (390), and (392):

$$\lambda_i \equiv \frac{U_i^0(x; x_0)}{U_n^0(x; x_0)}, \quad s \equiv \sqrt{\sum_{\alpha=1}^n \sum_{\beta=1}^n B_0^{\alpha\beta}(x_0) U_\alpha^0(x; x_0) U_\beta^0(x; x_0)} \qquad (i = 1, 2, \dots, n-1). \tag{396}$$

That is the result of solving equations (394) for the $\lambda_1, \dots, \lambda_{n-1}$, s.

Q. E. D.

Remark. – It results from (372), (377), and (396) that the square of the geodesic distance from the point x to the point x_0 is given by:

$$\boxed{\Gamma(x; x_0) \equiv \sum_{\alpha=1}^n \sum_{\beta=1}^n B^{\alpha\beta}(x_0) U_\alpha^0(x; x_0) U_\beta^0(x; x_0).} \tag{397}$$

41. Differential parameters. – Now introduce the Lamé and Beltrami differential parameters, namely:

$$\begin{cases} \square_1 U \equiv \displaystyle\sum_{\alpha=1}^n \sum_{\beta=1}^n B^{\alpha\beta} \frac{\partial U}{\partial x^\alpha} \frac{\partial U}{\partial x^\beta}, & (398) \\[2ex] \square_2 U \equiv \dfrac{1}{\sqrt{B}} \displaystyle\sum_{\alpha=1}^n \frac{\partial \sqrt{B} \sum_{\beta=1}^n B^{\alpha\beta} \dfrac{\partial U}{\partial x^\beta}}{\partial x^\alpha}, & (399) \end{cases}$$

in which U is a function of the x:

$$U \equiv U(x^1, \dots, x^n). \tag{400}$$

Recall that the parameters $\square_1 U$ and $\square_2 U$ are invariant with respect to any change of variables (330); one will then have the invariance relations:

$$\begin{cases} \Box_1 U \equiv \overline{\Box}_1 \bar{U}, \\ \Box_2 U \equiv \overline{\Box}_2 \bar{U}, \end{cases} \tag{401}$$

in which:

$$\overline{\Box}_1 \bar{U} \equiv \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} \bar{B}^{\alpha\beta} \frac{\partial \bar{U}}{\partial \bar{x}^\alpha}\frac{\partial \bar{U}}{\partial \bar{x}^\beta}, \quad \overline{\Box}_2 \bar{U} \equiv \frac{1}{\sqrt{\bar{B}}}\sum_{\alpha=1}^{n} \frac{\partial \sqrt{\bar{B}} \sum_{\beta=1}^{n} \bar{B}^{\alpha\beta} \frac{\partial \bar{U}}{\partial \bar{x}^\beta}}{\partial \bar{x}^\alpha}, \tag{402}$$

and

$$U \equiv U(x^1, \ldots, x^n) = \bar{U}(\bar{x}^1, \ldots, \bar{x}^n) \equiv \bar{U}, \tag{403}$$

by virtue of (330). The symbols $\bar{B}^{\alpha\beta}$ and $\bar{B}$ were defined before [see form. (334), (335), and (336)].

Theorem I. – *I say that* ([1])*:*

$$\boxed{\Box_1 \Gamma \equiv 4\Gamma.} \tag{404}$$

Proof. – Indeed, it results from (370) and (377) that:

$$\Box_1 \Gamma \equiv \sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} B^{\alpha\beta} \frac{\partial \sqrt{\Gamma}}{\partial x^\alpha}\frac{\partial \sqrt{\Gamma}}{\partial x^\beta} \equiv 1. \tag{405}$$

Upon replacing $\dfrac{\partial \sqrt{\Gamma}}{\partial x^\alpha}$ in (405) with its value $\dfrac{1}{2\sqrt{\Gamma}}\dfrac{\partial \Gamma}{\partial x^\alpha}$, one will get (404) immediately.

Theorem II. – *I say that* ([2])*:*

$$\boxed{\Box_2 \Gamma \equiv 2\left(1 + s\frac{d \log\left(\dot{J}\sqrt{B}\right)}{ds}\right).} \tag{406}$$

Proof. – Indeed, by definition, one will have:

$$\Box_2 S \equiv \frac{1}{\sqrt{B}}\sum_{\alpha=1}^{n} \frac{\partial}{\partial x^\alpha}\sqrt{B}\sum_{\beta=1}^{n} B^{\alpha\beta}\frac{\partial S}{\partial x^\beta}, \tag{407}$$

in which:

$$S \equiv \sqrt{\Gamma}. \tag{408}$$

However, thanks to (366) and (342), one will have:

([1]) J. HADAMARD, *loc. cit.*, see pp. 124, form. (32).
([2]) *Ibidem*, see pp. 127, form. (27).

$$u^\alpha \equiv \sum_{\beta=1}^{n} B^{\alpha\beta} \frac{\partial S}{\partial x^\beta} \qquad (\alpha = 1, 2, \ldots, n), \tag{409}$$

so (407) will become:

$$\Box_2 S \equiv \frac{1}{\sqrt{B}} \sum_{\alpha=1}^{n} \frac{\partial \sqrt{B}\, u^\alpha}{\partial x^\alpha}. \tag{410}$$

We now remark that if we take (338) into account then:

$$\frac{d}{ds}\left(\sqrt{B}\, \delta x^1, \ldots, \delta x^n\right) = \left(\sum_{\alpha=1}^{n} \frac{\partial \sqrt{B}\, u^\alpha}{\partial x^\alpha}\right) \delta x^1 \ldots \delta x^n, \tag{411}$$

and as a result, from (410):

$$\frac{d}{ds}\left(\sqrt{B}\, \delta x^1, \ldots, \delta x^n\right) = \sqrt{B}\, \Box_2 S \;\; \delta x^1 \ldots \delta x^n, \tag{412}$$

so

$$\boxed{\Box_2 S \equiv \frac{d}{ds}\left(\log(\sqrt{B}\, \delta x^1 \ldots \delta x^n)\right).} \tag{413}$$

By virtue of the invariance of the Beltrami parameter under an arbitrary change of variables x, one will have:

$$\Box_2 S \equiv \frac{d}{ds}\left(\log(\sqrt{B^*}\, \delta x^1 \ldots \delta x^n \delta s)\right), \tag{414}$$

in which, from (336):

$$\sqrt{B^*} = \frac{\partial(x^1, \ldots, x^n)}{\partial(\lambda_1, \ldots, \lambda_{n-1}, s)} \sqrt{B}. \tag{415}$$

However, by definition, the λ are functions of the invariants $u_1^0, \ldots, u_n^0$ of the Hamiltonian equations for the geodesics; hence:

$$\frac{d\lambda_i}{ds} \equiv 0 \qquad (i = 1, 2, \ldots, n-1), \tag{416}$$

and consequently:

$$\Box_2 S \equiv \frac{d}{ds} \log \sqrt{B^*}, \tag{417}$$

or rather, thanks to (415) and (395):

$$\boxed{\Box_2 S \equiv \frac{d}{ds} \log(J \sqrt{B}).} \tag{418}$$

One will then have, in turn:

$$\left.\begin{aligned}\square_2 S \equiv \square_2\sqrt{\Gamma} &= \frac{1}{\sqrt{B}}\sum_{\alpha=1}^{n}\frac{\partial}{\partial x^\alpha}\sqrt{B}\sum_{\beta=1}^{n}B^{\alpha\beta}\frac{\partial\sqrt{\Gamma}}{\partial x^\alpha}\\ &= \frac{1}{\sqrt{B}}\sum_{\alpha=1}^{n}\frac{\partial}{\partial x^\alpha}\sqrt{B}\sum_{\beta=1}^{n}\frac{B^{\alpha\beta}}{2\sqrt{\Gamma}}\frac{\partial\Gamma}{\partial x^\alpha}\\ &= \frac{1}{2\sqrt{B}\sqrt{\Gamma}}\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\frac{\partial\sqrt{B}\,B^{\alpha\beta}}{\partial x^\alpha}\frac{\partial\Gamma}{\partial x^\alpha}-\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\frac{B^{\alpha\beta}}{4\Gamma\sqrt{\Gamma}}\frac{\partial\Gamma}{\partial x^\alpha}\frac{\partial\Gamma}{\partial x^\beta}+\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n}\frac{B^{\alpha\beta}}{2\sqrt{\Gamma}}\frac{\partial^2\Gamma}{\partial x^\alpha\partial x^\beta},\end{aligned}\right\}\tag{419}$$

or rather, from (404):

$$\square_2\sqrt{\Gamma} = \frac{\square_2\Gamma}{2\sqrt{\Gamma}}-\frac{1}{\sqrt{\Gamma}}.\tag{420}$$

Hence, from (418), (420), and (372), one will finally have:

$$\square_2\Gamma = 2\left(1+s\frac{d}{ds}\log(J\sqrt{B})\right).$$

Corollary. – *I say that:*

$$\boxed{\square_2 S \equiv \square_2\sqrt{\Gamma} \equiv \frac{n-1}{s}+\frac{d}{ds}\log\frac{\sqrt{B}}{\Delta}}\tag{421}$$

and

$$\boxed{\square_2\Gamma \equiv 2\left(n+s\frac{d}{ds}\log\frac{\sqrt{B}}{\Delta}\right),}\tag{422}$$

in which:

$$\Delta \equiv \left\|\frac{\partial^2\,\frac{1}{2}\Gamma}{\partial x^\alpha\,\partial x_0^\beta}\right\| \neq 0.\tag{423}$$

Indeed, the identity:

$$\frac{\partial(\bar{x}^1,\ldots,\bar{x}^n)}{\partial(\lambda_1,\ldots,\lambda_{n-1},s)} \equiv \frac{\partial(\bar{x}^1,\ldots,\bar{x}^n)}{\partial(x^1,\ldots,x^n)}\cdot\frac{\partial(x^1,\ldots,x^n)}{\partial(\lambda_1,\ldots,\lambda_{n-1},s)}\tag{424}$$

and the equality (336) will imply the invariance relation:

$$\bar{J}\sqrt{\bar{B}} \equiv J\sqrt{B},\tag{425}$$

when one takes the notation (395) into account.

In particular, set:

$$\bar{x}^\alpha \equiv U_\alpha^0 \equiv s\,u_\alpha^0 \qquad (\alpha = 1, 2, \ldots, n). \tag{426}$$

Recall that the u_α^0 are then given as functions of the λ_i (i = 1, 2, …, n – 1) by (389). Hence, a simple calculation will show that:

$$\bar{\dot{J}} \equiv s^{n-1}(u_\alpha^0)^n \tag{427}$$

and

$$\frac{\partial(\bar{x}^1,\ldots,\bar{x}^n)}{\partial(x^1,\ldots,x^n)} = \left\| \frac{\partial(s\,u_\alpha^0)}{\partial x^\beta} \right\| = (-1)^n \left\| \frac{\partial^2 \tfrac{1}{2}\Gamma}{\partial x_0^\alpha\,\partial x_0^\beta} \right\|; \tag{428}$$

hence, from (423) and (336):

$$\sqrt{\bar{B}} = (-1)^n \frac{\sqrt{B}}{\Delta}. \tag{429}$$

Finally, upon replacing the product $\dot{J}\sqrt{B}$ in (418) and (406) with its value $\bar{\dot{J}}\sqrt{\bar{B}}$, in which, $\bar{\dot{J}}$ and $\sqrt{\bar{B}}$ are given by (427) and (429), resp., one will get formulas (422) and (421).

C. – Elementary Huygens wave that is associated with a second-order linear equation.

42. Lagrangian form of the bicharacteristic equations. –

Theorem. – *The bicharacteristic lines of the second-order equation* (322) *are the null-length lines of the quadratic differential form* (329) *that is associated with that equation.*

Proof. – It will suffice to show that:

$$\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} B_{\alpha\beta}\,\frac{\partial\Box_1\Omega}{\partial\Omega_\alpha}\frac{\partial\Box_1\Omega}{\partial\Omega_\beta} = 0, \tag{430}$$

in which Ω is a wave function, by hypothesis. Now, one will have, in turn:

$$\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} B_{\alpha\beta}\,\frac{\partial\Box_1\Omega}{\partial\Omega_\alpha}\frac{\partial\Box_1\Omega}{\partial\Omega_\beta} \equiv 4\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} B_{\alpha\beta}\sum_{\gamma=1}^{n}\sum_{\delta=1}^{n} B^{\alpha\gamma}B^{\beta\delta}\Omega_\gamma\,\Omega_\delta \equiv 4\sum_{\gamma=1}^{n}\sum_{\delta=1}^{n} B^{\alpha\gamma}B^{\beta\delta}\Omega_\gamma\,\Omega_\delta \equiv 0,$$

by virtue of (324).

Q. E. D.

The bicharacteristics of J. Hadamard are defined by the canonical Hamiltonian differential system (326), which we write in the following manner:

$$\boxed{\begin{aligned} \frac{dx^\alpha}{d\theta} &= \frac{1}{2}\frac{\partial \square_1 \Omega}{\partial \Omega_\alpha}, \\ \frac{d\Omega_\alpha}{d\theta} &= -\frac{1}{2}\frac{\partial \square_1 \Omega}{\partial x^\alpha} \end{aligned}} \qquad (\alpha = 1, 2, \ldots, n), \tag{431}$$

in which θ is an arbitrary parameter. That system will admit the invariant:

$$\square_2 \Omega \equiv \sum_{\gamma=1}^{n} \sum_{\delta=1}^{n} B^{\alpha\gamma} B^{\beta\delta} \Omega_\gamma \, \Omega_\delta . \tag{432}$$

Set:

$$\Omega^\alpha \equiv \sum_{\beta=1}^{n} B^{\alpha\beta} \Omega_\beta = \frac{1}{2}\frac{\partial \square_1 \Omega}{\partial \Omega_\alpha} \qquad (\alpha = 1, 2, \ldots, n) \tag{433}$$

and

$$\Lambda_1 \Omega \equiv \sum_{\alpha=1}^{n} \sum_{\beta=1}^{n} B_{\alpha\beta} \, \Omega^\alpha \, \Omega^\beta ; \tag{434}$$

hence, thanks to (333) and (433), one will have:

$$\Lambda_1 \Omega \equiv \square_1 \Omega \tag{435}$$

and

$$\Omega_\alpha \equiv \sum_{\beta=1}^{n} B_{\alpha\beta} \, \Omega^\beta = \frac{1}{2}\frac{\partial \Lambda_1 \Omega}{\partial \Omega^\alpha} \qquad (\alpha = 1, 2, \ldots, n). \tag{436}$$

We then remark that:

$$\frac{\partial \Lambda_1 \Omega}{\partial x^\alpha} = \sum_{\gamma=1}^{n} \sum_{\delta=1}^{n} \frac{\partial B_{\gamma\delta}}{\partial x^\alpha} \Omega^\gamma \, \Omega^\delta , \tag{437}$$

but:

$$\frac{\partial B_{\gamma\delta}}{\partial x^\alpha} = -\sum_{\mu=1}^{n} \sum_{\nu=1}^{n} B_{\gamma\mu} B_{\delta\nu} \frac{\partial B^{\mu\nu}}{\partial x^\alpha} . \tag{438}$$

Hence, from (437) and (438):

$$\frac{\partial \Lambda_1 \Omega}{\partial x^\alpha} = -\frac{\partial \square_1 \Omega}{\partial x^\alpha} \qquad (\alpha = 1, 2, \ldots, n). \tag{439}$$

Finally, it results from (433), (435), and (439) that the *Hamiltonian* equations (431) can take the *Lagrangian* form:

$$\frac{\partial \Lambda_1 \Omega}{\partial x^\alpha} - \frac{d}{d\theta}\left(\frac{\partial \Lambda_1 \Omega}{\partial \Omega^\alpha}\right) = 0 \qquad (\alpha = 1, 2, \ldots, n). \tag{440}$$

However, one will have:

$$\frac{\partial \sqrt{\Lambda_1 \Omega}}{\partial x^\alpha} = \frac{d}{d\theta}\frac{\partial \sqrt{\Lambda_1 \Omega}}{\partial \Omega^\alpha} \equiv \frac{1}{2\sqrt{\Lambda_1 \Omega}}\left(\frac{\partial \Lambda_1 \Omega}{\partial x^\alpha} - \frac{d}{d\theta}\left(\frac{\partial \Lambda_1 \Omega}{\partial \Omega^\alpha}\right)\right), \tag{441}$$

when

$$\Lambda_1\, \Omega = \text{constant} \neq 0 \qquad \text{or} \qquad \square_1 \Omega = \text{constant} \neq 0. \tag{442}$$

In that case, equations (440) will be equivalent to the *Lagrangian* equations:

$$\frac{\partial \sqrt{\Lambda_1 \Omega}}{\partial x^\alpha} - \frac{d}{d\theta}\frac{\partial \sqrt{\Lambda_1 \Omega}}{\partial \Omega^\alpha} = 0 \qquad (\alpha = 1, 2, \ldots, n). \tag{443}$$

We note that equations (443) will remain unchanged when one replaces θ in them with an arbitrary function of θ; the choice of the parameter θ will then remain arbitrary here.

Equations (443) are identical to the Lagrangian equations (347) of the geodesics. Indeed, it will suffice to substitute s for θ (s being defined by $ds / d\theta = \sqrt{\Lambda_1 \Omega} \equiv$ constant) and $u^\alpha \sqrt{\Lambda_1 \Omega}$ for Ω^α in (443) in order to obtain equations (347) identically. We remark that we have made a choice of parameter s in the geodesic equations (347) that has a well-defined geometric sense. Finally, equations (443), along with those (347) of the geodesics, will lose all meaning when:

$$\Lambda_1\, \Omega = 0 \qquad \text{or} \qquad \square_1 \Omega = 0. \tag{444}$$

As a result, one can consider the Lagrangian equations (440) to be more general than the equations of the same type (443) and (347).

It will likewise result from those considerations that the null-length lines that are associated with $(ds)^2$ in (329), despite being extremals of the integral (345), do not appear among the geodesics that are define by equations (360) and (361). That essentially comes from the fact that in the course of integrating the differential equations for the geodesics, one has supposed that $\sum_{\alpha=1}^{n}\sum_{\beta=1}^{n} B_{\alpha\beta}\, u^\alpha u^\beta \equiv 1$ (i.e., $\square_1 \Omega \neq 0$ or $\Lambda_1\, \Omega \neq 0$). That restrictive condition is imposed by the nature itself of the geodesic problem. Consequently, the null-length lines of a quadratic differential form are not geodesics of that form, *properly speaking.*

43. Characteristic Hadamard conoid and elementary Huygens wave. – It results from the theorem in § **42** and the general definition in § **4** that *the null-length lines of the* *quadratic differential form* (329) *that is associated with the partial differential equation* (322) *of physics that issue from a point in space-time generate the characteristic Hadamard conoid that has that point as its summit.* Recall that one can consider the characteristic conoid to be the elementary Huygens wave in space-time (see § **6**).

Theorem. – *I say that:*

$$\Gamma \equiv \Gamma(x, t; x_0, t_0) = 0 \tag{445}$$

is the equation of the characteristic conoid whose summit is (x_0, t_0).

Proof. – Indeed, we first remark that the surface whose equation is (445) is indeed a wave surface, because, thanks to (404), one will have:

$$\Box_1 \Gamma = 0,$$

by virtue of $\Gamma = 0$.

Now let P_0 be the point whose coordinates are $(x_0^1, x_0^2, \ldots, x_0^{n-1}, t_0 \equiv x_0^n)$, and let P be an arbitrary point of the characteristic conoid whose summit is P_0. *One* null-length line in space-time will pass through the point P and P_0. That line is an extremal of the integral (345); as a result, the geodesic distance between any two of its points will be zero. Now, we have seen that the square of the geodesic distance between the points P and P_0 is given by Γ [see formulas (362) and (377)]. Consequently, equation (445) indeed defines the characteristic Hadamard conoid.

Q. E. D.

That being the case, solve equation (445) for t; hence:

$$t = t(t_0; x, x_0).$$

Now set:

$$\Omega^*(x, t) \equiv t - t(t_0; x, x_0).$$

It will then result from the theorem in § **8** that the function Ω^* that was just defined is a solution of the wave equation (324), namely, $\Box_1 \Omega = 0$.

The elementary Huygens wave that issues from the point x_0 *in geometric space at the instant* t_0 *occupies the position of the surface whose equation is:*

$$t(t_0; x, x_0) = t$$

or

$$\Gamma(t, x; t_0, x_0) = 0$$

at the instant $t > t_0$, in which time t is considered to be a parameter here.

Finally, it will result from formula (397) that *the characteristic Hadamard conoid in space* $(U_1^0, U_2^0, \ldots, U_n^0)$ *is a cone of degree two.*