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On the theory of continuous groups

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This paper (1) has the objective of facilitating the application of the theory of continuous groups to the integration of partial differential equations by the systematic exposition of the known results, which we have sought to complete at various points.

Although these results apply just as well to finite continuous groups, it is, above all, the infinite groups that we have in mind, as well as the groups that enter into that study that are always defined by their defining equations. Except for some propositions on the theory of finite continuous groups, we assume as known only some general propositions that were contained in the memoir of S. Lie, *Die Grundlagen für die Theorie der unendlichen Gruppen.* (*Leipziger Berichte*, 1891.)

It was Engel (2) that originated the very fertile idea of giving a common form to the equations of definition of all the point groups that belong to the same type – i.e., such that one passes from one to the other by a simple change of variables.

Engel was occupied only with the equations of definition of infinitesimal transformations. Medolaghi (3) gave the corresponding form for the equations of definition of finite transformations. We have reprised the results of these two authors by a new and simple method, by developing the auxiliary problems that the effective separation of the various types of groups depends upon. We have shown how the theory of similarity of infinite groups follows from that theory by the study of the types of point groups.

We then extend the method to the study of the types of subgroups of a given group; to our knowledge, this subject has not been treated. In particular, we have sharpened the nature of the various operations that are necessary for the determination of the invariant subgroups, which is important in the theories of integration that are founded upon the consideration of groups.

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(1) This paper constitutes part of a memoir that was awarded a prize by the Academy of Sciences. The two essential parts of the same memoir will published sequentially: the one (*Sur la Théorie de Galois et ses généralisations*), in this issue, and the other one (*Sur l’intégration des systèmes différentiels qui admettant des groupes de transformations*), in the *Acta Mathematica.*

(2) *Ueber die Definitionsgleichungen der continuierlichen Transformationsgruppen* (*Math. Annalen,* t. XXVII). – *Kleinere Beiträge zur Gruppentheorie* (*Leipziger Berichte*, t. IX, 1894).

(3) *Sulla teori dei gruppi infinite continui* (*Annali de Matematica,* 1897). *Contributo alla determinazione dei gruppi continui,* etc. (*Rendicont della R. Accademia dei Lincei,* 1899).

We conclude by giving a definition of the isomorphism of infinite groups, which allows the applications of that fundamental notion to the theories of integration ([1]).

I. – Two modes of prolonging point groups.

1. First, consider the infinitesimal transformation:

$$XF = \sum_{i=1}^{n} \xi_i(x_1, \cdots, x_n)\frac{\partial F}{\partial x_i}, \tag{1}$$

and prolong it by introducing the derivatives with respect to the x_i of n new non-transformed variables $y_1, \ldots, y_n$ that we consider to be functions of the x_i .

If we set, to abbreviate:

$$\frac{\partial^{\alpha_1+\cdots+\alpha_n} f}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}} = f^{(\alpha_1\cdots\alpha_n)}, \qquad f = f^{(0,\ldots,0)},$$

then, according to the well-known procedure, we start with the identity:

$$dy_k^{(\beta_1,\cdots,\beta_n)} - \sum_{i=1}^{n} y_k^{(\beta_1,\cdots,\beta_i+1,\cdots\beta_n)}dx_i = 0.$$

We apply the operation X to it, which gives:

$$dXy_k^{(\beta_1,\cdots,\beta_n)} - \sum_{i=1}^{n} Xy_k^{(\beta_1,\cdots,\beta_i+1,\cdots\beta_n)}dx_i - \sum_{i=1}^{n} y_k^{(\beta_1,\cdots,\beta_i+1,\cdots\beta_n)}d\xi_i = 0,$$

and we infer from this that:

$$X\, y_k^{(\beta_1,\cdots,\beta_i+1,\cdots,\beta_n)} = \frac{d}{dx_i} X\, y_k^{(\beta_1,\cdots,\beta_i+1,\cdots\beta_n)} - \sum_{i=1}^{n} y_k^{(\beta_1,\cdots,\beta_i+1,\cdots\beta_n)} \frac{\partial \xi_i}{\partial x_i}.$$

The right-hand side may be written:

$$\frac{d}{dx_i}\left(Xy_k^{(\beta_1,\cdots,\beta_i+1,\cdots\beta_n)} - \sum_{s=1}^{n} \xi_s \frac{\partial y_k^{(\beta_1,\cdots,\beta_n)}}{\partial x_s}\right) + \sum_{i=1}^{n} \xi_s\, y_k^{(\beta_1,\cdots,\beta_i+1,\cdots\beta_n)} \frac{\partial y_k^{(\beta_1,\cdots,\beta_i+1,\cdots,\beta_n)}}{\partial x_s},$$

and upon setting:

([1]) These pages were being written when Cartan published (*Comptes rendus*, 17 November 1902) a definition of the isomorphism that agrees with our own.

(2) $$\mathcal{X}y_k^{(\alpha_1,\cdots,\alpha_n)} = \sum_{s=1}^n \xi_s \frac{\partial y_k^{(\alpha_1,\cdots,\alpha_n)}}{\partial x_s},$$

one obtains the recurrence formula:

$$Xy_k^{(\beta_1,\cdots,\beta_i+1,\cdots,\beta_n)} - \mathcal{X}y_k^{(\beta_1,\cdots,\beta_i+1,\cdots,\beta_n)} = \frac{d}{dx_i}[Xy_k^{(\beta_1,\cdots,\beta_i+1,\beta_n)} - \mathcal{X}y_k^{(\beta_1,\cdots,\beta_i+1,\beta_n)}].$$

One then painlessly obtains the definitive formula:

(2′) $$Xy_k^{(\beta_1,\cdots,\beta_n)} = \mathcal{X}y_k^{(\beta_1,\cdots,\beta_n)} - \frac{dy^{\beta_1+\cdots+\beta_n}(\mathcal{X}y_k)}{dx_1^{\beta_1}\cdots dx_n^{\beta_n}},$$

where the characteristic d denotes the total derivation:

$$\frac{dF(x_1,\cdots,x_n\,|\cdots,y_k^{(\alpha_1,\cdots,\alpha_n)},\cdots)}{dx_i} = \frac{\partial F}{\partial x_i} + \sum_{k|\alpha_1,\cdots,\alpha_n} \frac{\partial F}{\partial y_k^{(\alpha_a,\cdots,\alpha_n)}} \frac{\partial y_k^{(\alpha_a,\cdots,\alpha_n)}}{\partial x_i}.$$

The right-hand side of this formula is linear and homogeneous with respect to the derivatives of x_i taken up to the order of derivative $y_k^{(\beta_1,\cdots,\beta_n)}$ considered, as well as with respect to the derivatives of just the function y_k taken up to the same order, because the $\frac{\partial y_k^{(\beta_1,\cdots,\beta_n)}}{\partial x_s}$ disappear after reductions.

The prolonged transformation (1) will then be of the form:

(3) $$XF = \sum_{i=1}^n \xi_i \frac{\partial F}{\partial x_i} + \sum_{i|\alpha_1,\cdots,\alpha_n} \frac{\xi_i^{(\alpha_1,\cdots,\alpha_n)}}{\alpha_1!\cdots\alpha_n!} A_{i|\alpha_1,\cdots,\alpha_n}F,$$

where the $A_{i|\alpha_1,\cdots,\alpha_n}F$ are certain infinitesimal transformations that act on the derivatives of the y_k, and which are linear and homogeneous. From formula (3), $A_{i|\alpha_1,\cdots,\alpha_n}F$ is, moreover, what the infinitesimal transformation $x_1^{\alpha_1}\ldots x_n^{\alpha_n}\frac{\partial F}{\partial x_i}$, becomes when, after having been prolonged, one sets:

$$x_1 = \ldots = x_n = 0.$$

Formulas (2), (2′) then give:

(4) $$A_{i|\alpha_1,\cdots,\alpha_n} y_k^{(\beta_1,\cdots,\beta_n)} = -\left[\frac{d^{\beta_1+\cdots+\beta_n}\left(x_1^{\alpha_1}\cdots x_n^{\alpha_n}\frac{\partial y_k}{\partial x_i}\right)}{dx_1^{\beta_1}\cdots x_n^{\beta_n}}\right]_{x_1=\cdots=x_n=0}$$

If we then set, as is customary:

$$C_p^q = \frac{p(p-1)\cdots(p-q+1)}{1\cdot 2\cdots q}$$

then we will have the entirely explicit formulas:

$$(5)\qquad A_{i|\alpha_1,\cdots,\alpha_n} y_k^{(\beta_1,\cdots,\beta_n)} = -\, C_{\beta_1}^{\alpha_1}\cdots C_{\beta_n}^{\alpha_n}\,\frac{\partial y_k^{(\beta_1-\alpha_1,\cdots,\beta_n-\alpha_n)}}{\partial x_i} \qquad (\beta_j \geq \alpha_j)$$

and:

$$(6)\qquad A_{i|\alpha_1,\cdots,\alpha_n} F = -\sum_{(\beta_1,\cdots,\beta_n)} C_{\beta_1}^{\alpha_1}\cdots C_{\beta_n}^{\alpha_n}\,\frac{\partial y_k^{(\beta_1-\alpha_1,\cdots,\beta_n-\alpha_n)}}{\partial x_i}\,\frac{\partial F}{\partial y_k^{(\beta_1,\cdots,\beta_n)}},$$

where the summation must be taken over all systems of values of the $\beta_1, \ldots, \beta_n$ for which one has $\beta_j \geq \alpha_j$ for every j.

2. Now suppose that one limits the prolongation to the derivatives of arbitrary order m. One must then introduce the conditions:

$$0 < \beta_1 + \ldots + \beta_n \leq m, \qquad 0 < \alpha_1 + \ldots + \alpha_n \leq m$$

into the formulas (5), (6).

The corresponding $A_{i|\alpha_1,\cdots,\alpha_n} F$ constitute a system of M linear homogeneous transformations between the M variables:

$$y_k^{(\beta_1,\cdots,\beta_n)}, \qquad 0 < \beta_1 + \ldots + \beta_n \leq m, \qquad M = n\left[\frac{(n+1)\cdots(n+m)}{1\cdot 2\cdots m} - 1\right].$$

These transformations, which we denote by $A_{i|\alpha_1\cdots\alpha_n}^{(m)}$, form a simply transitive group that we call $\mathcal{A}_m$.

Indeed, one has:

$$\left(x_1^{\alpha_1}\cdots x_n^{\alpha_n}\frac{\partial F}{\partial x_i},\, x_1^{\gamma_1}\cdots x_n^{\gamma_n}\frac{\partial F}{\partial x_j}\right)$$
$$= \gamma_i x_1^{\alpha_1+\gamma_1}\cdots x_i^{\alpha_i+\gamma_i-1}\cdots x_n^{\alpha_n+\gamma_n}\frac{\partial F}{\partial x_j} - \alpha_j x_1^{\alpha_1+\gamma_1}\cdots x_j^{\alpha_j+\gamma_j-1}\cdots x_n^{\alpha_n+\gamma_n}\frac{\partial F}{\partial x_i};$$

from this, one deduces, by prolonging up to order m and then annulling all of the x_i, that:

$$(7)\qquad (A_{i|\alpha_1,\cdots,\alpha_n}^{(m)} F, A_{j|\gamma_1,\cdots,\gamma_n}^{(m)} F) = \gamma_i A_{j|\alpha_1+\gamma_1,\cdots,\alpha_i+\gamma_i-1,\cdots,\alpha_n+\gamma_n}^{(m)} - \alpha_j A_{i|\alpha_1+\gamma_1,\cdots,\alpha_j+\gamma_j-1,\cdots,\alpha_n+\gamma_n}^{(m)} F$$

where one must replace every expression $A_{h|\sigma_1\cdots\sigma_n}^{(m)}$ for which the sum $\sigma_1 + \ldots + \sigma_n$ exceeds m with zero.

These formulas (7) show precisely that one is dealing with a group, and give its structure. In order to see that it is transitive, it suffices to remark that an invariant of this group will be a differential invariant of the general point group. Everything comes down to proving that such an invariant $\Omega(\cdots, y_k^{(\beta_1\cdots\beta_n)}, \cdots)$ cannot exist, and, in fact, upon letting $\omega(x_1, \ldots, x_n)$ denote the value that it will take for n arbitrary particular functions y_k , one sees that the partial differential equation:

$$\Omega(\cdots, y_k^{(\beta_1\cdots\beta_n)}, \cdots) = \omega(x_1, \ldots, x_n),$$

which admits all of the point transformations in $x_1, \ldots, x_n$ will admit any arbitrary system of functions as a solution, which is impossible.

3. One easily obtains the finite equations of the group $\mathcal{A}_m$. Indeed, if XF is an arbitrary infinitesimal point transformation then one only needs to start with an arbitrary finite point transformation:

$$x_i' = \varphi_i(x_1, \ldots, x_n) \qquad (i = 1, 2, \ldots, n), \tag{8}$$

and perform the same operations on it. To that effect, here, we consider, at the same time as (8), the inverse transformation:

$$x_i = \Phi_i(x_1', \cdots, x_n') \qquad (i = 1, 2, \ldots, n). \tag{9}$$

If we then set, to abbreviate:

$$\frac{\partial^{\alpha_1+\cdots+\alpha_n} y_k}{\partial x_1'^{\alpha_1} \cdots \partial x_n'^{\alpha_n}} = y_k'^{(\alpha_1\cdots\alpha_n)}$$

then it suffices to apply the rules of differential calculus to obtain the formulas that give the $y_k'^{(\alpha_1\cdots\alpha_n)}$ as functions of the $y_k^{(\alpha_1\cdots\alpha_n)}$, and they present themselves in the form:

$$y_k'^{(\beta_1\cdots\beta_n)} = b_{\beta_1\cdots\beta_n}(\cdots, y_k^{(\gamma_1\cdots\gamma_n)}, \cdots | \cdots, \Phi_i^{(\alpha_1,\cdots,\alpha_n)}, \cdots) \qquad (k = 1, 2, \ldots, n). \tag{10}$$

These are the finite equations for $\mathcal{A}_m$, if one supposes that one limits oneself to the derivatives that are taken up to order m, and considers the derivatives of Φ to be arbitrary constants (the parameters of the group). One sees that this group transforms the derivatives of each function y_k amongst themselves: It is therefore isomorphic to a homogeneous linear group in M / n variables:

$$y'^{(\beta_1,\cdots,\beta_n)} = b_{(\beta_1\cdots\beta_n)}(\cdots, y_k^{(\gamma_1\cdots\gamma_n)}, \cdots | \cdots, \Phi_i^{(\alpha_1,\cdots,\alpha_n)}, \cdots) \quad (0 < \beta_1 + \ldots + \beta_n \leq m). \tag{11}$$

Finally, one may modify formulas (10) by a change of parameters. Indeed, by the use of the rules of differential calculus, the equivalence of formulas (8) and (9) leads to:

$$\text{(12)}\qquad \left\{\begin{aligned} \Phi_i^{(\alpha_1,\cdots,\alpha_n)} &= \varpi_{i|\alpha_1,\cdots,\alpha_n}(\cdots,\varphi_j^{(\delta_1,\cdots,\delta_n)},\cdots) \\ \varphi_i^{(\alpha_1,\cdots,\alpha_n)} &= \varpi_{i|\alpha_1,\cdots,\alpha_n}(\cdots,\Phi_j^{(\delta_1,\cdots,\delta_n)},\cdots) \end{aligned}\right\} \qquad (i = 1, 2, \ldots, n),$$

by means of which one will give formulas (10) the new form:

$$\text{(13)}\qquad y_k'^{(\beta_1,\cdots,\beta_n)} = a_{(\beta_1\cdots\beta_n)}(\cdots,y_k^{(\gamma_1\cdots\gamma_n)},\cdots|\cdots,\varphi_j^{(\delta_1,\cdots,\delta_n)},\cdots) \qquad (k = 1, 2, \ldots, n).$$

and formulas (11) get the corresponding form:

$$\text{(14)}\qquad y'^{(\beta_1,\cdots,\beta_n)} = a_{(\beta_1\cdots\beta_n)}(\cdots,y^{(\gamma_1\cdots\gamma_n)},\cdots|\cdots,\varphi_j^{(\delta_1,\cdots,\delta_n)},\cdots).$$

One may again remark that one will have formulas that define the inverse transformations by the simple exchange of Φ and φ; i.e.:

$$\text{(15)}\qquad y_k^{(\beta_1,\cdots,\beta_n)} = a_{(\beta_1\cdots\beta_n)}(\cdots,y_k^{(\gamma_1\cdots\gamma_n)},\cdots|\cdots,\Phi_j^{(\delta_1,\cdots,\delta_n)},\cdots).$$

and:

$$\text{(16)}\qquad y_k'^{(\beta_1,\cdots,\beta_n)} = b_{(\beta_1\cdots\beta_n)}(\cdots,y_k'^{(\gamma_1\cdots\gamma_n)},\cdots|\cdots,\varphi_i^{(\alpha_1,\cdots,\alpha_n)},\cdots).$$

4. We now consider the transformation:

$$\text{(17)}\qquad YF = \sum_{k=1}^{n}\eta_k(y_1,\cdots,y_n)\frac{\partial F}{\partial y_k},$$

and we prolong it by adding n untransformed variables $x_1, \ldots, x_n$ to the y_k, and considering the y_k to be functions of these new variables. Here, the method that was employed in no. 1 gives immediately:

$$Yy_k^{(\beta_1,\cdots,\beta_{i+1},\cdots,\beta_n)} = \frac{dYy_k^{(\beta_1,\cdots,\beta_i,\cdots\beta_n)}}{dx_i},$$

and one then deduces that:

$$\text{(18)}\qquad Yy_k^{(\beta_1,\cdots,\beta_n)} = \frac{d^{\beta_1+\cdots+\beta_n}\eta_k}{dx_1^{\beta_1}\cdots dx_n^{\beta_n}},$$

d always denoting a total derivation. If one then sets:

$$\eta_k^{(\alpha_1,\cdots,\alpha_n)} = \frac{\partial^{\alpha_1+\cdots+\alpha_n}\eta_k}{\partial y_1^{\alpha_1}\cdots\partial y_n^{\alpha_n}}$$

then one obtains the desired prolongation in the form:

(19) $$YF = \sum_{i=1}^{n} \eta_i \frac{\partial F}{\partial y_i} + \sum_{i|\alpha_1\cdots\alpha_n} \frac{\eta_i^{(\alpha_1,\cdots,\alpha_n)}}{\alpha_1!\cdots\alpha_n!} B_{i|\alpha_1,\cdots,\alpha_n} F ,$$

where the $B_{i|\alpha_1\cdots\alpha_n}$ are infinitesimal transformations whose coefficients do not depend upon the choice of η_k .

One further sees in this formula that $B_{i|\alpha_1\cdots\alpha_n} F$ is what the infinitesimal transformation $y_1^{\alpha_1}\cdots y_n^{\alpha_n} \frac{\partial F}{\partial y_i}$ becomes when one makes $y_1 = \ldots = y_n = 0$ after one has prolonged it.

One easily deduces from this that if, as in the preceding, one limits the prolongation to order m then these infinitesimal transformations become M transformations $B_{i|\alpha_1\cdots\alpha_n}^{(m)} F$ that define a group $\mathcal{B}_m$ that is isomorphic to the group $\mathcal{A}_m$, and also simply transitive, like the latter group. The theorems of Lie prove that they are, in turn, similar.

One verifies this by remarking that if, in the transformation:

$$Y^{(m)}F = \sum_{i=1}^{n} \eta_i \frac{\partial F}{\partial y_i} + \sum_{i|\alpha_1\cdots\alpha_n} \frac{\eta_i^{(\alpha_1\cdots\alpha_n)}}{\alpha_1!\cdots\alpha_n!} B_{i|\alpha_1\cdots\alpha_n}^{(m)} F$$

one performs the change of variables on the derivatives of y_k:

(20) $$y_i^{(\alpha_1\cdots\alpha_n)} = \varpi_{i|\alpha_1,\cdots,\alpha_n}(\cdots, x_j^{(\delta_1,\cdots,\delta_n)}, \cdots) ,$$

which gives (*see* no. 3) the expression for these derivatives as functions of the derivatives:

$$x_j^{(\delta_1\cdots\delta_n)} = \frac{\partial x_j^{\delta_1+\cdots+\delta_n}}{\partial y_1^{\delta_1}\cdots\partial y_n^{\delta_n}}$$

of the $x_1, \ldots, x_n$, when considered as the system of functions of $y_1, \ldots, y_n$ that comes about by inverting the system of functions $y_1, \ldots, y_n$ of $x_1, \ldots, x_n$ that has been under consideration up to now, one will obtain precisely what YF becomes when one prolongs it by considering the n untransformed variables $x_1, \ldots, x_n$ to be functions of $y_1, \ldots, y_n$; i.e.:

$$\overline{Y^{(m)}}F = \sum_{i} \eta_i \frac{\partial F}{\partial y_i} + \sum_{i|\alpha_1\cdots\alpha_n} \frac{\eta_i^{(\alpha_1\cdots\alpha_n)}}{\alpha_1!\cdots\alpha_n!} \overline{A_{i|\alpha_1\cdots\alpha_n}^{(n)}} F ,$$

if we let:

$$\overline{A_{i|\alpha_1\cdots\alpha_n}^{(m)}}$$

denote the transformations $A_{i|\alpha_1\cdots\alpha_n}^{(m)}$, where one puts the letters $x_k^{(\beta_1,\cdots,\beta_n)}$ in place of the letters $y_k^{(\beta_1,\cdots,\beta_n)}$. One concludes from this that the change of variables (20) changes each

$B^{(m)}_{i|\alpha_1\cdots\alpha_n}$ into the $A^{(m)}_{i|\alpha_1\cdots\alpha_n}$ with the same indices (up to notation), which proves the results that were stated above.

5. We further seek the finite transformations of $\mathcal{B}_m$. This time, we start with a transformation:

(21) $$\bar{y}_k = \psi_k(y_1, \ldots, y_n) \qquad (k = 1, 2, \ldots, n),$$

or the inverse transformation:

(22) $$y_k = \Psi_k(\bar{y}_1, \cdots, \bar{y}_n) \qquad (k = 1, 2, \ldots, n),$$

and we differentiate with respect to $x_1, \ldots, x_n$. The calculation is the same as the one in no. 3, up to notations: Here, the ψ_k play the role of the y_k in the calculations of no. 3, the y_k play the role of the Φ_k, and the x_i play the role of the x'_i. One will thus obtain, upon setting:

$$\frac{\partial^{\alpha_1+\cdots+\alpha_n}\bar{y}_k}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}} = \bar{y}_k^{(\alpha_1,\cdots,\alpha_n)},$$

the new group of formulas:

(23) $$\begin{cases} \bar{y}_k^{(\beta_1,\cdots,\beta_n)} = b_{\beta_1,\cdots,\beta_n}(\cdots, \psi_k^{(\gamma_1,\cdots,\gamma_k)}, \cdots | \cdots, y_i^{(\alpha_1,\cdots,\alpha_n)}, \cdots), \\ y_k^{(\beta_1,\cdots,\beta_n)} = b_{\beta_1,\cdots,\beta_n}(\cdots, \Psi_k^{(\gamma_1,\cdots,\gamma_k)}, \cdots | \cdots, \bar{y}_i^{(\alpha_1,\cdots,\alpha_n)}, \cdots), \\ \bar{y}_k^{(\beta_1,\cdots,\beta_n)} = a_{\beta_1,\cdots,\beta_n}(\cdots, \Psi_k^{(\gamma_1,\cdots,\gamma_k)}, \cdots | \cdots, y_i^{(\alpha_1,\cdots,\alpha_n)}, \cdots), \\ y_k^{(\beta_1,\cdots,\beta_n)} = a_{\beta_1,\cdots,\beta_n}(\cdots, \psi_k^{(\gamma_1,\cdots,\gamma_k)}, \cdots | \cdots, \bar{y}_i^{(\alpha_1,\cdots,\alpha_n)}, \cdots). \end{cases}$$

One then passes from the finite equations for $\mathcal{A}_m$ to those of $\mathcal{B}_m$ by exchanging the roles of the variables and the parameters. This leads one to think that, from a theorem of Lie ([1]), these two groups are both reciprocal simply transitive groups, each of which serves to parameterize the other one.

Indeed, one has identically:

$$(XF, YF) = 0,$$

and, in particular:

$$\left(x_1^{\alpha_1}\cdots x_n^{\alpha_n}\frac{\partial F}{\partial x_i}, y_1^{\beta_1}\cdots y_n^{\beta_n}\frac{\partial F}{\partial y_k}\right) = 0.$$

Prolonging up to order m, while considering the $y_1, \ldots, y_n$ to be functions of the $x_1, \ldots, x_n$, and annulling the $x_1, \ldots, x_n$; $y_1, \ldots, y_n$ in the result, one obtains:

$$(A^{(m)}_{i|\alpha_1,\cdots,\alpha_n}F, B^{(m)}_{k|\beta_1,\cdots,\beta_n}F) = 0,$$

([1]) From an idea that seems to be due to Engel (*see* LIE and ENGEL, *Th. der Transf. gr.*, Bd. 1, pp. 428).

which proves precisely that the groups $\mathcal{A}_m$ and $\mathcal{B}_m$ are two reciprocal simply transitive groups, in the sense of Lie.

II. – Diverse forms of the equations of definition of a group.

6. Let (G) be a point group on the space of n dimensions. One knows that it may be defined by either the equations of definition of its finite transformations, which we denote by ($\mathcal{E}$) or by the equations of definition of its infinitesimal transformations, which we denote by (E). These systems of equations are susceptible to various remarkable forms, which we shall recall.

We let:

$$x_i' = y_i(x_1, \ldots, x_n) \qquad (i = 1, 2, \ldots, n) \tag{24}$$

denote an arbitrary finite transformation of (G) and let:

$$XF = \sum_{i=1}^{n} \xi_i(x_1, \cdots, x_n) \frac{\partial F}{\partial x_i} \tag{25}$$

denote any of these infinitesimal transformations.

According to Lie [1], the equations ($\mathcal{E}$) may be put into the form:

$$U_s(y_1, \ldots, y_n, \ldots, y_k^{(\beta_1, \cdots, \beta_n)}, \ldots) = \omega_s(x_1, \ldots, x_n) \qquad (s = 1, 2, \ldots, p), \tag{26}$$

where the U_s constitute a complete system of differential invariants of (G) when one considers the y_k to be the transformed variables and the x_i to be untransformed, and where the ω_s are deduced from the U_s by replacing the functions $y_1, \ldots, y_n$ with $x_1, \ldots, x_n$, respectively, and their derivatives by the values that result.

For the applications of the theory of groups, it is important to show that the reduction of equations ($\mathcal{E}$), which are assumed to be given in an arbitrary form, to the form (26), involves only rational calculations.

For the sake of conciseness, we assume that the given equations ($\mathcal{E}$) are algebraic, rational, and entire with respect to the derivatives $y_k^{(\beta_1, \cdots, \beta_n)}$. One then deduces, by a sequence of eliminations, the differential relations that link a system of n functions $x_1, \ldots, x_n$ of n independent variables $t_1, \ldots, t_n$ to the functions $x_1', \ldots, x_n'$ of the same variables that result from them by an arbitrary transformation (24) of the group, while the independent variables $t_1, \ldots, t_n$ are not transformed. Let:

$$f_s(x_1, \ldots, x_n, \ldots, x_i^{(\alpha_1, \cdots, \alpha_n)}, \ldots \mid x_1', \ldots, x_n', \ldots, x_i'^{(\alpha_1, \cdots, \alpha_n)}, \ldots) = 0, \quad (s = 1, 2, \ldots, p) \tag{27}$$

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[1] *Leipziger Berichte*, 1891, pp. 391.

be these relations. Upon specifying that they also link the functions $x_1', \ldots, x_n'$ to a new system of functions $x_1'', \ldots, x_n''$, it then results from another transformation of the group that one obtains:

$$(28) \qquad f_s(x_1'', \ldots, x_n'', \ldots, x_i''^{(\alpha_1,\cdots,\alpha_n)}, \ldots \mid x_1', \ldots, x_n', \ldots, x_i'^{(\alpha_1,\cdots,\alpha_n)}, \ldots) = 0, \quad (s = 1, 2, \ldots, q).$$

One will then specify that the systems (27) and (28) in:

$$x_1', \ldots, x_n', \ldots, x_i'^{(\alpha_1,\cdots,\alpha_n)}, \ldots$$

must be equivalent.

If the group (G) is transitive then $x_1', \ldots, x_n'$ will play only the role of parameters in this calculation, and one will rationally obtain relations of the form:

$$U_s(x_1, \ldots, x_n, \ldots, x_i^{(\alpha_1,\cdots,\alpha_n)}, \ldots) = U_s(x_1'', \ldots, x_n'', \ldots, x_i^{(\alpha_1,\cdots,\alpha_n)}, \ldots) \qquad (s = 1, 2, \ldots, p),$$

where the functions U_s will be the required functions. If $x_1', \ldots, x_n'$ figure in these functions U_s then this will be true by virtue of arbitrary constants, and one successively gives them various systems of values, until one no longer has any further equations that are independent of the equations that were already written down.

If the group (G) is intransitive then $x_1', \ldots, x_n'$ intervene effectively in the identification of the systems (27) and (28), upon leaving aside the greatest possible number of equations (27) that imply no relation between the $x_1', \ldots, x_n'$ in which any of their derivatives figure, and upon comparing the corresponding equations (28), one will again rationally obtain *differential invariants*, properly speaking. As for the invariants of order zero, one will seek them directly by means of the given equations ($\mathcal{E}$): One will eliminate all of the derivatives, which will give a certain number of relations of the form:

$$g_k(y_1, \ldots, y_n; x_1, \ldots, x_n) = 0 \qquad (k = 1, 2, \ldots, r),$$

and it is upon identifying the system:

$$g_k(y_1', \ldots, y_n'; x_1, \ldots, x_n) = 0 \qquad (k = 1, 2, \ldots, r),$$

with respect to the $x_1, \ldots, x_n$ that one will find the relations:

$$U_k(y_1, \ldots, y_n) = U_k(y_1', \ldots, y_n') \qquad (k = 1, 2, \ldots, r),$$

whose left-hand sides will be the desired invariants.

One may thus say nothing precise about the nature – rational or algebraic – of this part of the calculation if one does not assume that the given equations ($\mathcal{E}$) are also rational with respect to the $x_1, \ldots, x_n$, or, at the very least, that the combinations $g_k = 0$ are rational with respect to $x_1, \ldots, x_n$.

7. Therefore, suppose that the equations ($\mathcal{E}$) are known in the form (26), which we call the *Lie form.* According to Lie, the U_s constitute a system of independent integrals of a certain complete system, where one may assume that the equations are obtained by equating to zero a certain number of infinitesimal transformations:

$$Y_k F = \sum_{i=1}^{n} \eta_{ki}(y_1, \cdots, y_n) \frac{\partial F}{\partial y_i}$$

of the group (G), when prolonged up to order m of the system (26), conforming to formula (19). This complete system thus admits the transformations of the group $\mathcal{A}_m$. One will have, in turn, some identities of the form:

(29) $$A^{(m)}_{i|\alpha_1,\cdots,\alpha_n} U_s = \lambda_{i|\alpha_1,\cdots,\alpha_n|s}(U_1, \cdots, U_p),$$

and the infinitesimal transformations:

(30) $$A_{i|\alpha_1,\cdots,\alpha_n} F = \sum_{s=1}^{p} \lambda_{i|\alpha_1,\cdots,\alpha_n|s}(u_1, \cdots, u_p) \frac{\partial F}{\partial u_s}$$

define a group $\mathcal{L}$ that is isomorphic to the group $\mathcal{A}_m$.

If one takes into account the finite equations of the group $\mathcal{A}_m$ then one sees that by means of formulas (8) and (13), one has identically relations of the form:

(31) $$U_s(y_1, \ldots, y_n, \ldots, y'^{(\beta_1,\cdots,\beta_n)}_k, \ldots)$$
$$= L_s[\ldots, U_h(y_1, \ldots, y_n, \ldots, y^{(\beta_1,\cdots,\beta_n)}_k, \ldots), \ldots | \ldots, \varphi^{(\delta_1,\cdots,\delta_n)}_j, \ldots] \qquad (s = 1, 2, \ldots, p),$$

in such a way that the group $\mathcal{L}$ has the finite equations:

(32) $$u'_s = L_s(u_1, \ldots, u_p \mid \ldots, \varphi^{(\delta_1,\cdots,\delta_n)}_j, \ldots) \quad (s = 1, 2, \ldots, p),$$

where the $\varphi^{(\delta_1,\cdots,\delta_n)}_j$ are parameters.

One may further say that the infinitesimal transformation:

(33) $$\Xi F = \sum_{i=1}^{n} \xi_i \frac{\partial F}{\partial x_i} + \sum_{i|\alpha_1,\cdots,\alpha_n} \frac{\xi_i^{(\alpha_1,\cdots,\alpha_n)}}{\alpha_1!, \cdots, \alpha_n!} \Lambda_{i|\alpha_1,\cdots,\alpha_n} F,$$

where the ξ_i are arbitrary functions of $x_1, \ldots, x_n$, is the infinitesimal transformation of an infinite group whose finite equations are:

$$(34)\qquad \begin{cases} x_i' = \varphi_i(x_1, \cdots, x_n) & (i = 1, 2, \cdots, n), \\ u_s' = L_s(u_1, \cdots, u_p \mid \cdots, \varphi_j^{(\delta_1, \cdots, \delta_n)}, \cdots) & (s = 1, 2, \cdots, p). \end{cases}$$

Several times, we have needed to solve equation (32) with respect to $u_1, \ldots, u_p$; this obviously gives:

$$(35)\qquad u_s = L_s(u_1', \ldots, u_p' \mid \ldots, \Phi_j^{(\delta_1, \cdots, \delta_n)}, \ldots) \quad (s = 1, 2, \ldots, p),$$

and, upon introducing the φ by means of formulas (12), one will obtain equations of the form:

$$(36)\qquad u_s = \mathcal{L}_s(u_1', \ldots, u_p' \mid \ldots, \varphi_j^{(\delta_1, \cdots, \delta_n)}, \ldots) \quad (s = 1, 2, \ldots, p).$$

We finally remark that the transformations of the group $\mathcal{A}_m$ [i.e., the general point transformations that were performed on $x_1, \ldots, x_n$ in the invariants U_s of (G), since the variables $x_1, \ldots, x_n$ do not explicitly enter into the U_s] leave invariant any invariant of (G) of order zero if it is intransitive, and transform all of the differential invariants that are of order less than or equal to an arbitrary order (less than or equal to m) amongst themselves.

8. The preceding results lead to a new *form for the equations* $(\mathcal{E})$ *that is due to Medolaghi* ([1]). Lie has shown that in order for a transformation:

$$(37)\qquad x_i' = \varphi_i(x_1, \ldots, x_n) \qquad (i = 1, 2, \ldots, n)$$

to belong to (G), it is necessary and sufficient that it leave invariant the system $(\mathcal{E})$, and consequently that the equations:

$$(38)\qquad U_s(y_1, \ldots, y_n, \ldots, y_k'^{(\beta_1, \cdots, \beta_n)}, \ldots) = \omega_s(x_1', \ldots, x_n') \qquad (s = 1, 2, \ldots, p)$$

are consequences of equations (26) and (27). Now, from (31), the equations (38) are written:

$$L_s[\ldots, U_h(y_1, \ldots, y_n, \ldots, y_k^{(\beta_1, \cdots, \beta_n)}, \ldots), \ldots \mid \ldots, \varphi_j^{(\delta_1, \cdots, \delta_n)}, \ldots] = \omega_s(\varphi_1, \ldots, \varphi_n)$$
$$(s = 1, 2, \ldots, p),$$

and as a result, the stated condition reduces to:

$$L_s[\omega_1(x_1, \ldots, x_n), \ldots, \omega_p(x_1, \ldots, x_n) \mid \ldots y_j^{(\delta_1, \cdots, \delta_n)} \ldots] = \omega_s(y_1, \ldots, y_n)$$
$$(s = 1, 2, \ldots, p).$$

([1]) *Annali di Matematica*, 1897, pp. 179-218.

Upon substituting the letters y for the letters φ, we thus have the form of the stated equations ($\mathcal{E}$):

$$L_s[\omega_1(x_1, \dots, x_n), \dots, \omega_s(x_1, \dots, x_n) \mid \dots, y_j^{(\delta_1,\cdots,\delta_n)}, \dots] = \omega_s(y_1, \dots, y_n), \quad (s = 1, 2, \dots, p), \tag{39}$$

which, from (36), may also be written:

$$\mathcal{L}_s[\omega_1(y_1, \dots, y_n), \dots, \omega_p(y_1, \dots, y_n) \mid \dots, y_j^{(\delta_1,\cdots,\delta_n)}, \dots] = \omega_s(x_1, \dots, x_n) \quad (s = 1, 2, \dots, p). \tag{40}$$

If one likewise seeks the condition for the infinitesimal transformation (25) to leave invariant the system (26) then one obtains the equations of definition (E) of the infinitesimal transformations of (G). Now, due to the formulas (3) and (29), one finds for that condition:

$$\sum_{i|\alpha_1,\cdots,\alpha_n} \frac{\xi_i^{(\alpha_1,\cdots,\alpha_n)}}{\alpha_1!\cdots\alpha_n!} \lambda_{i|\alpha_1,\cdots,\alpha_n|s}[\omega_1(x_1,\cdots,x_n),\cdots,\omega_p(x_1,\cdots,x_n)] - \sum_{i=1}^n \xi_i \frac{\partial\omega_s}{\partial x_i} = 0 \quad (s = 1, 2, \dots, p). \tag{41}$$

This remarkable form of equations (E) *is due to Engel* ([1]).

S. Lie made the important remark ([2]) that one arrives at equations (41) [and, in turn, equations (39) or (40)] by seeking those transformations of the group (33) [or (34)] that leave invariant the system or, in geometric language, the multiplicity:

$$u_s - \omega_s(x_1, \dots, x_n) = 0 \qquad (s = 1, 2, \dots, p). \tag{42}$$

III. – Groups that are similar to a given group.

9. We start with the group (G) that was considered in the preceding paragraph and seek to deduce from its equation of definition those groups (G') that one obtains by transforming it by a point transformation.

In order to better pose the question, we denote an arbitrary transformation of (G) by:

$$y_i = g_i(x_1, \dots, x_n) = \Theta\, x_i \qquad (i = 1, 2, \dots, n) \tag{Θ}$$

and the transformation by which we shall transform (Θ) by:

$$x'_i = f_i(x_1, \dots, x_n) = T x_i \qquad (i = 1, 2, \dots, n). \tag{T}$$

([1]) *Math. Annalen*, Bd. XXVII. *Ueber die Definitionsgleichungen*, etc., § 3.
([2]) *Ibid.*

Furthermore, set:

$$\bar{y}_i = f_i(y_1, \ldots, y_n) = Ty_i \qquad (i = 1, 2, \ldots, n);$$

by definition, the transform Θ' of Θ will be represented by the equations:

$$f_i(y_1, \ldots, y_n) = g_i[f_1(x_1, \ldots, x_n), \ldots, f_n(x_1, \ldots, x_n)] \qquad (i = 1, 2, \ldots, n),$$

i.e.:

$$T\,y_i = \Theta\, T\, x_i \qquad (i = 1, 2, \ldots, n),$$

or:

$$y_i = T^{-1}\Theta T\, x_i \qquad (i = 1, 2, \ldots, n).$$

We then write, symbolically:

$$\Theta' = T^{-1}\Theta T, \quad (G') = T^{-1}(G)T.$$

Having said this, we have to replace the $\bar{y}$ and the x' with their values in the equations of definition ($\mathcal{E}$), written with these variables; i.e., in the equations:

$$U_s(\bar{y}_1, \cdots, \bar{y}_n, \cdots, \bar{y}_k^{(\beta_1, \ldots, \beta_n)}, \cdots) = \omega_s(x'_1, \cdots, x'_n) \qquad (s = 1, 2, \ldots, p).$$

First, replace the x'_i with their expressions Tx_i ; this is accomplished by means of formulas (31) and gives:

$$L_s[\cdots, U_h(\bar{y}_1, \cdots, \bar{y}_n, \cdots, \bar{y}_k^{(\beta_1, \ldots, \beta_n)}, \cdots), \cdots | \cdots, f_j^{(\delta_1, \ldots, \delta_n)}, \cdots] = \omega_s(f_1, \ldots, f_n),$$

which one may write, by virtue of formulas (36):

$$U_s(\bar{y}_1, \cdots, \bar{y}_n, \cdots, \bar{y}_k^{(\beta_1, \ldots, \beta_n)}, \cdots) = \mathcal{L}_s[\ldots, \omega_h(f_1, \ldots, f_n), \ldots | \ldots, f_j^{(\delta_1, \ldots, \delta_n)}, \ldots].$$

We set:

$$\bar{\omega}_s(x_1, \cdots, x_n) = \mathcal{L}_s\{\ldots, \omega_h[f_1(x), \ldots, f_n(x)], \ldots | \ldots, f_j^{(\delta_1, \ldots, \delta_n)}(x), \ldots\} \qquad (s = 1, 2, \ldots, p). \tag{43}$$

On the other hand, upon replacing the $\bar{y}_i$ with their expressions Ty_i , one obtains identities of the form:

$$\bar{U}_s(y_1, \cdots, y_n, \cdots, y_k^{(\beta_1, \ldots, \beta_n)}, \cdots) = U_s[f_1(y), \ldots, f_n(y), \ldots, f_k^{(\beta_1, \ldots, \beta_n)}(y), \ldots]. \tag{44}$$

With these notations, the equations of definition of (G') will be:

$$\bar{U}_s(y_1, \cdots, y_n, \cdots, y_k^{(\beta_1, \ldots, \beta_n)}, \cdots) = \bar{\omega}_s(x_1, \cdots, x_n) \qquad (s = 1, 2, \ldots, p). \tag{45}$$

Therefore, it suffices to make the change of variables on the y only in the left-hand sides of the equalities (26), and to replace the right-hand sides with the functions (43).

10. However, one may arrive at a result that is even simpler. Indeed, from identities (44), i.e.:

$$\bar{U}_s(y_1, \cdots, y_n, \cdots, y_k^{(\beta_1,\dots,\beta_n)}, \cdots) = U_s(\bar{y}_1, \cdots, \bar{y}_n, \cdots, \bar{y}_k^{(\beta_1,\dots,\beta_n)}, \cdots),$$

one concludes that if one performs an arbitrary transformation on the $x_1, \dots, x_n$ in the $\bar{U}_s$:

$$x_i' = \varphi_i(x_1, \dots, x_n) \qquad (i = 1, 2, \dots, n)$$

then the $\bar{U}_s$ transform according to the group $\mathcal{L}$; i.e., that one will have:

$$\bar{U}_s(y_1, \cdots, y_n, \cdots, y_k^{(\beta_1,\dots,\beta_n)}, \cdots) = L_s[\cdots, \bar{U}_h(y_1, \cdots, y_n, \cdots, y_k^{(\beta_1,\dots,\beta_n)}, \cdots), \cdots \mid \cdots, \varphi_j^{(\beta_1,\dots,\beta_n)}, \cdots]$$
$$(s = 1, 2, \dots, p),$$

and consequently *the equations of definition of* (G') *are deduced from the equations of definition* (39), (40), *or* (41) *of* (G) *by replacing the functions* ω_s *everywhere with the given functions* $\bar{\omega}_s$ *using formulas* (43).

It is this fact that makes the canonical form of Engel and Medolaghi so interesting.

It is, moreover, clear that, conversely, the equations that are deduced from equations (39), (40), or (41) by the indicated rule always define a group that is similar to (G).

One may interpret this result by saying that every group that is similar to (G) is obtained by seeking those transformations of the group (33) that leave invariant one of the multiplicities:

$$u_s - \bar{\omega}_s(x_1, \cdots, x_n) = 0 \qquad (s = 1, 2, \dots, p)$$

that is homologous to the multiplicity (42) with respect to the group:

$$x_i' = f_i(x_1, \dots, x_n) \qquad (i = 1, 2, \dots, n),$$
$$u_s' = L_s(u_1, \dots, u_p \mid \dots, f_j^{(\delta_1,\dots,\delta_n)}, \dots) \qquad (s = 1, 2, \dots, p).$$

IV. – On the determination of point groups.

11. One must give credit to Engel ([1]) for the original idea of a general method for determining the types of point groups in the space of n dimensions by their equations of definition. Engel was occupied only with the equations of definition of infinitesimal transformations. Medolaghi ([2]) has also considered the equations of definition of finite

([1]) *Loc. cit., Math. Annalen*, Bd. XXV.
([2]) *Loc. cit., Annali di Matematica*, 1897.

transformations, and has completed the method at various points. We shall exhibit how we can complete the results of the preceding paragraph by first starting with two reciprocal propositions.

For the first one, suppose that one knows q differential functions of the form:

$$V_s(y_1, \ldots, y_n, \ldots, y_k^{(\beta_1,\ldots,\beta_n)}, \ldots) \qquad (s = 1, 2, \ldots, q)$$

that satisfy identities of the form:

$$A_{i|\alpha_1,\ldots,\alpha_n} V_s = \mu_{i|\alpha_1,\ldots,\alpha_n|s}(V_1,\ldots,V_q) \qquad (s = 1, 2, \ldots, q) \qquad (i = 1, 2, \ldots, n)$$

for all systems of values of $\alpha_1, \ldots, \alpha_n$ such that one has:

$$0 < \alpha_1 + \ldots + \alpha_n \leq m,$$

m being the maximum order of the V_s. This amounts to saying that under the transformation (8), one obtains identities:

$$(46) \qquad V_s(y_1, \ldots, y_n, \ldots, y_k^{(\beta_1,\ldots,\beta_n)}, \ldots)$$
$$= M_s[\ldots, V_s(y_1, \ldots, y_n, \ldots, y_k^{(\beta_1,\ldots,\beta_n)}, \ldots), \ldots | \ldots \varphi_j^{(\delta_1,\ldots,\delta_n)} \ldots]$$
$$(s = 1, 2, \ldots, q).$$

Now consider the differential system:

$$(47) \qquad V_s(y_1, \ldots, y_n, \ldots, y_k^{(\beta_1,\ldots,\beta_n)}, \ldots) = \zeta_s(x_1, \ldots, x_n) \qquad (s = 1, 2, \ldots, q),$$

where the right-hand sides are derived from the left-hand sides by the substitutions $y_1 = x_1, \ldots, y_n = x_n$. This system is not necessarily found to be in a completely integrable form, however, this is not impossible, since it admits the solution $y_1 = x_1, \ldots, y_n = x_n$. Moreover, by differentiations, one deduces a completely integrable system of differential equations from this whose left-hand sides enjoy some properties relative to the group $\mathcal{A}_m$ that are similar to the ones that assumed for the V_s, as one sees upon differentiating the identities (46). However, it will suffice for us to retain equations (47), which suffice to define all of their solutions. We shall show that if this system has more than one solution then the set of its solutions constitutes a group when one considers the equations:

$$(48) \qquad y_i = \chi_i(x_1, \ldots, x_n) \qquad (i = 1, 2, \ldots, n),$$

which define any of these solutions as the equations of a point transformation. We must remark that we suppose implicitly that the equations (47) are satisfied by the independent systems of functions (48), which is effectively true at least for the solution $y_1 = x_1, \ldots, y_n = x_n$, and we reserve the name of *solution* for the systems of functions (48) that satisfy the differential system considered, which replaced this condition. The other ones are improper solutions, because they satisfy the equation:

$$\frac{D(y_1,\ldots,y_n)}{D(x_1,\ldots,x_n)} = 0,$$

which is not a consequence of equations (47).

We now express the idea that the system (47) admits the solution (48). We may do this by first performing the change of variables:

$$x'_i = \chi_i(x_1, \ldots, x_n) \qquad (i = 1, 2, \ldots, n) \tag{49}$$

in the system and specifying that the transformed system must admit the solution $y_1 = x'_1$, …, $y_n = x'_n$.

Now, by virtue of the identities (46), the transformed system will take the form:

$$V_s(y_1, \ldots, y_n, \ldots, y_k^{(\beta_1,\ldots,\beta_n)}, \ldots) = \bar{\zeta}_s(x'_1, \cdots, x'_n) \qquad (s = 1, 2, \ldots, q),$$

and the desired condition then amounts to:

$$\bar{\zeta}_s(x'_1, \cdots, x'_n) \equiv \zeta_s(x'_1, \cdots, x'_n) \qquad (s = 1, 2, \ldots, q).$$

The condition is then that for the transformation (49) the system (47) must take the form:

$$V_s(y_1, \ldots, y_n, \ldots, y_k^{(\beta_1,\ldots,\beta_n)}, \ldots) = \zeta_s(x'_1, \cdots, x'_n) \qquad (s = 1, 2, \ldots, q),$$

i.e., that it must admit the transformation (49). It then results that the set of solutions (48) correspond to the set of transformations (49) that leave the system (47) invariant: It then defines precisely the set of finite transformations of a point group.

12. Suppose, in the second place, that one has a group $\mathcal{L}$ that is isomorphic to the group $\mathcal{A}_m$. One may suppose that its infinitesimal transformations, when referred isomorphically to those of $\mathcal{A}_m$, are the transformations (30), and that its finite equations have been presented in the form (32), while the correspondence of this group with the group (13) is defined by the equality of the values of the parameters $\varphi_j^{(\delta_1,\ldots,\delta_n)}$.

If one then considers the systems (39) [or (40)] and (41), where *one will suppose that the ω_s are chosen in such a manner that* (39) *admits other solutions than the identity solution* $y_1 = x_1, \ldots, y_n = x_n$, then their transformations – either finite or infinitesimal – define the same group. One may show this for equations (39) when one appeals to the theorem of the preceding number.

It is, moreover, simple to appeal to the remark of Lie that was recalled at the end of no. **8**. Under the hypotheses made, there exists an infinite group that is presented by equations (34), and whose general infinitesimal transformation is given by the formula (33).

Furthermore, equations (39) or (41) define the subgroup of this group that leaves invariant the multiplicity:

$$u_s - \omega_s(x_1, \ldots, x_n) = 0, \qquad (s = 1, 2, \ldots, p).$$

They thus indeed define a group.

One sees from this that the determination of the point groups in n variables decomposes into *two problems*, which we shall examine more closely:

1. *Determine the equations of the various groups* $\mathcal{L}$ of the form (32).

2. *Determine the functions* ω_s that figure in the equations (39), (40), and (41), in such a manner that they satisfy the condition that was stated above.

V. – Study of the transitive groups.

13. For the sake of neatness, we first limit ourselves to the study of transitive groups. What are the corresponding groups $\mathcal{L}$?

The response is simple: For a transitive group (G), all of the invariants U_s and the functions of these invariants are differential invariants, properly speaking; i.e., it effectively contains any derivative of the y_k . It then results that any of the functions U_s may remain invariant under any transformation that is performed on the x_i, and consequently, the corresponding group $\mathcal{L}$ is transitive.

Conversely, if the group $\mathcal{L}$ that corresponds to a group (G) is transitive then the same is true for (G), because any function of the U_s – i.e., any differential invariant of (G) – in the $y_1, \ldots, y_n$ and their derivatives may not be of order zero.

Thus, one limits oneself to transitive groups $\mathcal{L}$. One must remark, moreover (which will also be true for the intransitive groups (G)), that for a given group (G) the group $\mathcal{L}$ is not defined entirely, since one may replace the U_s with p functions of these U_s, that are subject only to being independent. Therefore, the group $\mathcal{L}$ may be replaced with any of the similar groups in $u_1, \ldots, u_p$.

One thus limits oneself to the *search for a representative for each of the types of transitive groups that are isomorphic to* $\mathcal{A}_m$, and for this, one only needs to apply the method that was given by Lie [1]: One will look for the various types of subgroups of $\mathcal{B}_m$. For each of them, one will calculate the invariants that are exchanged between them by the transformations of $\mathcal{A}_m$, according to a group $\mathcal{L}$ that answers the question, and which is immediately found to be presented in the form (32).

[1] *Theorie der Transf.,* Bd. I, pp. 430, *et seq.*

One may further remark that since $\mathcal{A}_m$ and $\mathcal{B}_m$ are similar, one may perform the inverse – i.e., search for the subgroups of $\mathcal{A}_m$, calculate their invariants, and transform them by the group $\mathcal{B}_m$.

14. If one finds a group $\mathcal{L}$ by the preceding process then all that remains to be done is to write the corresponding equations (39) or (40), and to search for the integrability conditions: They must give one or more systems of partial differential equations that determine the functions ω_s. We shall see that one may operate in such a manner as to obtain distinct differential systems for the various types of groups (G) – i.e., such that the functions ω_s that are provided by the various solutions to one of these systems will correspond to the various groups that are similar to the same group (G).

To that effect [1], consider at the same time as the system (40) – i.e.:

$$(50)\qquad \mathcal{L}_s\,[\omega_1(y_1, \ldots, y_n), \ldots, \omega_p\,(y_1, \ldots, y_n) \mid \ldots, y_j^{(\delta_1,\ldots,\delta_n)}, \ldots] = \omega_s\,(x_1, \ldots, x_n) \qquad (s = 1, 2, \ldots, p),$$

the slightly more general system:

$$(51)\qquad \mathcal{L}_s\,[\omega_1(y_1, \ldots, y_n), \ldots, \omega_p\,(y_1, \ldots, y_n) \mid \ldots, y_j^{(\delta_1,\ldots,\delta_n)}, \ldots] = \theta_s\,(x_1, \ldots, x_n) \qquad (s = 1, 2, \ldots, p).$$

We first remark that no matter what the functions ω_s are, if one performs the transformation (37) on the $x_1, \ldots, x_n$ in the left-hand side of equations (50) then one will obtain the identities:

$$\mathcal{L}_s\,(\omega_1, \ldots, \omega_p \mid \ldots, y_j^{(\delta_1,\ldots,\delta_n)}) = \mathcal{L}_s\,[\ldots, \mathcal{L}_h\,(\omega_1, \ldots, \omega_p \mid \ldots, y_j^{(\delta_1,\ldots,\delta_n)}), \ldots \mid \ldots, \varphi_j^{(\delta_1,\ldots,\delta_n)}] \qquad (s = 1, 2, \ldots, p),$$

because, upon considering the ω_s to be independent variables, it results from this that equations (34) are the equations of a group. Indeed, in order to convince oneself of this, it suffices to consider the two transformations that correspond, by isomorphism, to:

$$x_i' = \varphi_i\,(x_1, \ldots, x_n), \qquad (i = 1, 2, \ldots, n),$$
$$x_i'' = y_i'(x_1', \ldots, x_n'), \qquad (i = 1, 2, \ldots, n),$$

and whose product corresponds to:

$$x_i'' = y_i'(\varphi_1, \ldots, \varphi_n) = y_i(x_1, \ldots, x_n), \qquad (i = 1, 2, \ldots, n).$$

Having said this, if one supposes that the system (51) admits a solution:

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[1] The principle of that method is due to P. Medolaghi (*Rendiconti del R. dei Lincei*, pp. 291, 1899.)

(52) $$y_i = f_i\,(x_1, \ldots, x_n), \qquad (i = 1, 2, \ldots, n)$$

that is comprised of independent functions by performing the transformation:

(53) $$x'_i = f_i\,(x_1, \ldots, x_n), \qquad (i = 1, 2, \ldots, n)$$

then it will take the form:

$$\mathcal{L}_s\,[\omega_1\,(y_1, \ldots, y_n), \ldots, \omega_p\,(y_1, \ldots, y_n) \mid \ldots, y_j'^{(\delta_1,\ldots,\delta_n)}\,] = \bar{\theta}_s(x'_1, \ldots, x'_n)\,, \; (s = 1, 2, \ldots, p),$$

and since it must admit the solution $y_i = x'_i$, the functions $\bar{\theta}_s$ are identical to the ω_s. Therefore, under the transformation (53), the system (51) reduces to the system (50), which already proves that the general integrals of the two systems have the same degree of generality.

Moreover, for each system of functions ω_s that make the equations (50) compatible, the general values of the functions θ_s such that the equations (51) admit a general integral that has the same degree of generality are obtained, from the preceding, by specifying that they admit an arbitrary solution (52); i.e., they are furnished by the formulas:

$$\theta_s\,(x_1, \ldots, x_n) = \mathcal{L}_s\,[\omega_1\,(f_1, \ldots, f_n), \ldots, \omega_p\,(f_1, \ldots, f_n) \mid \ldots, f_j^{(\delta_1,\ldots,\delta_n)}\,] \qquad (s = 1, 2, \ldots, p).$$

Comparing these formulas with formulas (43) of no. 9, one sees that they are precisely the various systems of functions that one must substitute for the ω_s in equations (50) in order to obtain the various groups that belong to the same type as the group (G) that is assumed to be defined by these equations (50).

The study of integrability conditions of the system (51) must then permit us to separate the various types of groups (G) that have equations of the form (50).

15. We thus study these integrability conditions more closely.

We may assume that the group $\mathcal{L}$ from which we start is the one that corresponds to the equations of the desired group (G), presented in completely integrable form, and consequently, we limit ourselves to specifying that the system (51) is one that is completely integrable. The system of conditions that is thus obtained, if it has a solution that gives two systems of functions for $\omega_1, \ldots, \omega_p$ and $\theta_1, \ldots, \theta_p$ that correspond to two similar groups, will have other ones that correspond to arbitrary pairs of groups that are similar to those two; i.e., it must be invariant under the infinite group that has the general transformation:

$$\begin{aligned}
&y'_i = \psi_i(y_1, \ldots, y_n), && (i = 1, 2, \ldots, n),\\
&\omega_s = \mathcal{L}_s(\omega'_1, \cdots, \omega'_p \mid \cdots, \psi_j^{(\delta_1,\ldots,\delta_n)}) && (s = 1, 2, \ldots, p),\\
&x'_i = \varphi_i(x_1, \ldots, x_n), && (i = 1, 2, \ldots, n),\\
&\theta_s = \mathcal{L}_s(\theta'_1, \cdots, \theta'_p \mid \cdots, \varphi_j^{(\delta_1,\ldots,\delta_n)}) && (s = 1, 2, \ldots, p),
\end{aligned}$$

where the ψ_i and the φ_i are arbitrary functions.

We shall conclude by first saying that the integrability conditions may not contain relations of order zero between just the $\omega_s(y)$ and the $\theta_s(x)$. Indeed, if they contain them then the set of these relations will constitute a system that is invariant under the finite group that is composed of the transformations:

$$\begin{aligned} \omega_s &= \mathcal{L}_s(\omega'_1, \cdots, \omega'_p \mid \cdots, \psi_j^{(\delta_1, \cdots, \delta_n)}, \cdots) \\ \theta_s &= \mathcal{L}_s(\theta'_1, \cdots, \theta'_p \mid \cdots, \varphi_j^{(\delta_1, \cdots, \delta_n)}, \cdots) \end{aligned} \qquad (s = 1, 2, \ldots, p),$$

where the $\psi_j^{(\delta_1, \ldots, \delta_n)}$ and the $\varphi_j^{(\delta_1, \ldots, \delta_n)}$ are considered to be arbitrary constants. Now, since this group is transitive, any invariant system includes the relations that are obtained by specifying that its equations may not be solved with respect to $2p$ of the parameters. In turn, equations (50) of the group (G), or the analogous equations of the similar group that corresponds to the functions θ_s, or even neither of them, may be solved with respect to p of the derivatives of the y_k. For example, if this is true for equation (50) then one may deduce relations between just the x_k and the y_k; i.e., the group may not be transitive, as we assumed. The stated impossibility results from this.

Having established this point, how does one reduce the calculation of the condition that express that the system (51) is completely integrable? We follow the lead of Delassus ([1]). We first take those equations in (51) that are of maximum order m and independent with respect to the derivatives of that order; let their number be p'. One may solve them with respect to p' derivatives of order m and n' unknown functions. Let ν_m be the number of derivatives of order m of an arbitrary function y_k. By differentiating the group of equations considered, it will be necessary that the equations obtained may be solved only with respect to p'' derivatives of order $(m + 1)$ of the same n' unknown functions, p'' being determined by the condition:

$$n' \nu_{m+1} - p'' = n' \nu_m - p'.$$

Since they may never be solved with respect to less than p'' of these derivatives, and they are linear with respect to the derivatives of order $(m + 1)$, one will have to equate to zero certain determinants whose elements obviously depend upon only the $\omega_s(y_1, \ldots, y_n)$, and not their derivatives. These determinants must be identically zero, or else the group $\mathcal{L}$ will have to be rejected; then, upon specifying that these determinants must be annulled as a consequence of equations (51), one will obtain relations between the ω_s and the θ_s alone which, as one has seen, will not exist.

If this preliminary condition is assumed to be satisfied then the equations of order $(m + 1)$ that are obtained are solved with respect to p'' of the derivatives $(m + 1)$, and upon substituting their values into the other equations of order $(m + 1)$, the derivatives of order $(m + 1)$ are eliminated from them.

([1]) *Annales de l'École Normale supérieure*, 1896, pp. 445, *et seq.*

One will thus only have to specify that the equations thus obtained, as well as the ones that result from the differentiation of equations (51) of order less than m, are consequences of the equations (51).

Now, this is, up to notations, the calculation that one will have to carry out in order to determine a certain class of differential systems of first order that are invariant under the group:

(54) $$\begin{cases} u_s = \mathcal{L}_s(u_1', \cdots, u_p' \mid \cdots, \varphi_j^{(\delta_1, \ldots, \delta_n)}, \cdots) & (s = 1, 2, \cdots, p), \\ x_s' = \varphi_j(x_1, \cdots, x_n) & (i = 1, 2, \cdots, n). \end{cases}$$

As Lie showed in his general theory, it will lead to systems that may be composed of:

1. Relations of entirely well-defined form that are the same for the ω_s and θ_s :

(55) $$\Omega_h\left(\omega_1, \cdots, \omega_p \mid \cdots, \frac{\partial \omega_s}{\partial y_j}, \cdots\right) = 0,$$

(56) $$\Omega_h\left(\theta_1, \cdots, \theta_p \mid \cdots, \frac{\partial \theta_s}{\partial x_j}, \cdots\right) = 0;$$

2. Relations of the form:

(57) $$J_k\left(\omega_1, \cdots, \omega_p \mid \cdots, \frac{\partial \omega_s}{\partial y_j}, \cdots\right) = J_k\left(\theta_1, \cdots, \theta_p \mid \cdots, \frac{\partial \theta_s}{\partial x_j}, \cdots\right).$$

If it contains equations of the form (57) then the separation of the types is immediate, since the equations (57) show precisely that for the same type the invariants J_k must be equal to constants that are always the same.

16. In summation, one obtains systems of the general form:

(58) $$\Omega_h\left(u_1, \cdots, u_p \mid \cdots, \frac{\partial u_s}{\partial x_i}, \cdots\right) = 0 \qquad (h = 1, 2, \ldots, \rho),$$

(59) $$J_k\left(u_1, \cdots, u_p \mid \cdots, \frac{\partial u_s}{\partial x_i}, \cdots\right) = c_k \qquad (k = 1, 2, \ldots, r),$$

such that any of these systems will furnish the set of all group (G) of the same type; in other words, the most general solution to any of these systems is deduced from any particular solution:

(60) $$u_s = \omega_s(x_1, \ldots, x_n) \qquad (s = 1, 2, .., p)$$

by a general transformation of the group (54). The solution (60) of the system (58), (59) gives the group (G) as the one that is composed of transformations:

$$x'_i = \varphi_i(x_1, \ldots, x_n) \qquad (i = 1, 2, \ldots, n)$$

that correspond to the transformations (54) that leave the multiplicity (60) invariant. The latter form a group that is isomorphic to (G), and which we call ($\mathcal{G}$).

The various solutions (60) thus give all of the groups (G) of the same type; however, it might happen that two of these solutions give the same group. The condition for this is that the transformation (54) that lets one pass from the one to the other leaves invariant the group ($\mathcal{G}$) that corresponds to the first one. This fact will thus present itself when the group ($\mathcal{G}$) is invariant under a subgroup ($\mathcal{G}_0$) that is much larger than the group (54); i.e., when (G) is invariant under a more extended point group (G_0) that corresponds to ($\mathcal{G}_0$) in the same way that (G) corresponds to ($\mathcal{G}$). To each solution (60) there will then correspond a group ($\mathcal{G}_0$) of transformations (54) that transform them into all of the other solutions that give the same group (G). We shall seek to separate the families of solutions that are associated in that manner.

To that effect, we consider the equations of definition (41) of the infinitesimal transformations, and we specify that two systems of values for the ω_s change into two equivalent systems. We thus obtain a certain number of distinct relations of the form:

$$P_k\left(\omega_1, \ldots, \omega_p \mid \ldots, \frac{\partial \omega_s}{\partial x_i}, \ldots\right) = P_k\left(\bar{\omega}_1, \ldots, \bar{\omega}_p \mid \ldots, \frac{\partial \bar{\omega}_s}{\partial x_i}, \ldots\right) \qquad (k = 1, 2, \ldots, \mu).$$

Moreover, any of these families of solutions considered will be defined by a system of the form:

$$P_k\left(u_1, \ldots, u_p \mid \ldots, \frac{\partial u_s}{\partial x_i}, \ldots\right) = \pi_k(x_1, \ldots, x_n) \qquad (k = 1, 2, \ldots, \mu), \tag{61}$$

and everything comes down to determining the functions π_k and then looking for a solution of the system [(58), (59), (61)] that corresponds to one of the solutions of the auxiliary system to which these functions π_k must satisfy. This auxiliary system will be, moreover, deduced from the system [(58), (59)] by means of formulas (61), which define a transformation of this known system.

We further remark that one may replace this auxiliary system with the system that is the analogue of the system [(58), (59)], which will serve to determine the groups of the same type as ($\mathcal{G}_0$), and which one may always obtain by appealing to equations (41), because if the functions π_k are known then one will obtain the equations of definition of ($\mathcal{G}_0$) by specifying that the transformation (54) must leave the system (61) invariant. Conversely, if ($\mathcal{G}_0$) is known then one may, without integration, form the equations of definition of the infinitesimal transformations of the various transitive invariant subgroups, as we will verify later on; i.e., the ones that one must have in order to deduce the functions π_k.

As for the nature of the system [(58), (59), (60)], which defines the finite transformations of an invariant subgroup of ($\mathcal{G}_0$), we shall also return to this later.

We finally remark that it will be possible that two systems [(58), (59)] provide only one and the same type; the consideration of functions P_k will further permit us to confirm this.

17. *Example 1: n* = 2, *m* = 1. – One takes $\mathcal{L}$ to be following group:

$$(\mathcal{L})\qquad \begin{cases} x_i' = \varphi_i(x_1, x_2), \qquad (i = 1, 2), \\ u_1 = u_1' \dfrac{\partial \varphi_1}{\partial x_1} + u_2' \dfrac{\partial \varphi_2}{\partial x_1}, \\ u_2 = u_1' \dfrac{\partial \varphi_1}{\partial x_2} + u_2' \dfrac{\partial \varphi_2}{\partial x_2}, \\ u_3 = u_3' \dfrac{D(\varphi_1, \varphi_2)}{D(x_1, x_2)}. \end{cases}$$

The equations of definition of the corresponding group (G) are thus:

$$(\mathcal{E})\qquad \begin{cases} \omega_1(y_1, y_2)\dfrac{\partial y_1}{\partial x_1} + \omega_2(y_1, y_2)\dfrac{\partial y_2}{\partial x_1} = \omega_1(x_1, x_2), \\ \omega_1(y_1, y_2)\dfrac{\partial y_1}{\partial x_2} + \omega_2(y_1, y_2)\dfrac{\partial y_2}{\partial x_2} = \omega_2(x_1, x_2), \\ \omega_3(y_1, y_2)\dfrac{D(y_1, y_2)}{D(x_1, x_2)} = \omega_3(x_1, x_2). \end{cases}$$

We thus consider the most general system:

$$\omega_1(y_1, y_2)\frac{\partial y_1}{\partial x_1} + \omega_2(y_1, y_2)\frac{\partial y_2}{\partial x_1} = \theta_1(x_1, x_2),$$
$$\omega_1(y_1, y_2)\frac{\partial y_1}{\partial x_2} + \omega_2(y_1, y_2)\frac{\partial y_2}{\partial x_2} = \theta_2(x_1, x_2),$$
$$\omega_3(x_1, x_2)\frac{D(y_1, y_2)}{D(x_1, x_2)} = \theta_3(x_1, x_2).$$

The conditions for it to be completely integrable reduce to:

$$\frac{\dfrac{\partial \omega_1(y_1, y_2)}{\partial y_2} - \dfrac{\partial \omega_2(y_1, y_2)}{\partial y_1}}{\omega_3(y_1, y_2)} = \frac{\dfrac{\partial \theta_1(y_1, y_2)}{\partial x_2} - \dfrac{\partial \theta_2(y_1, y_2)}{\partial x_1}}{\theta_3},$$

in such a way that for each type one has a unique condition of the form (59):

$$\frac{\partial \omega_1(x_1, x_2)}{\partial x_2} - \frac{\partial \omega_2(x_1, x_2)}{\partial x_1} = c\ \omega_3(x_1, x_2),$$

where c is an entirely arbitrary constant.

For $c = 0$, one may take:

$$\omega_1 = 1, \qquad \omega_2 = 1, \qquad \omega_3 = 1,$$

and one obtains the group type:

$$(G) \qquad y_1 = x_1 + a, \qquad y_2 = x_2 + \varphi(x_1).$$

One then sees that all of the non-zero values of c give the same type, for which, one may take:

$$c = 1, \qquad \omega_1 = x_2, \qquad \omega_2 = 0, \qquad \omega_3 = 1,$$

which gives the group type:

$$(G) \qquad y_1 = \varphi(x_1), \qquad y_2 = \frac{x_2}{\varphi'(x_1)}.$$

Example II: $n = 3$, $m = 1$. – One thus starts with:

$$x_i' = \varphi_i(x_1, x_2, x_3) \qquad (i = 1, 2, 3);$$

and the group $\mathcal{L}$:

$$(\mathcal{L}) \qquad u_1 = \frac{\dfrac{\partial \varphi_1}{\partial x_2} + u_1' \dfrac{\partial \varphi_2}{\partial x_2} + u_2' \dfrac{\partial \varphi_3}{\partial x_2}}{\dfrac{\partial \varphi_1}{\partial x_1} + u_1' \dfrac{\partial \varphi_2}{\partial x_1} + u_2' \dfrac{\partial \varphi_3}{\partial x_1}}, \qquad u_2 = \frac{\dfrac{\partial \varphi_1}{\partial x_3} + u_1' \dfrac{\partial \varphi_2}{\partial x_3} + u_2' \dfrac{\partial \varphi_3}{\partial x_3}}{\dfrac{\partial \varphi_1}{\partial x_1} + u_1' \dfrac{\partial \varphi_2}{\partial x_1} + u_2' \dfrac{\partial \varphi_3}{\partial x_1}}.$$

This gives the equations:

$$(\mathcal{E}) \qquad \begin{cases} \dfrac{\dfrac{\partial y_1}{\partial x_2} + \omega_1(y_1, y_2, y_3)\dfrac{\partial y_2}{\partial x_2} + \omega_2(y_1, y_2, y_3)\dfrac{\partial y_3}{\partial x_2}}{\dfrac{\partial y_1}{\partial x_1} + \omega_1(y_1, y_2, y_3)\dfrac{\partial y_2}{\partial x_1} + \omega_2(y_1, y_2, y_3)\dfrac{\partial y_3}{\partial x_1}} = \omega_1(x_1, x_2, x_3), \\[2ex] \dfrac{\dfrac{\partial y_1}{\partial x_3} + \omega_1(y_1, y_2, y_3)\dfrac{\partial y_2}{\partial x_3} + \omega_2(y_1, y_2, y_3)\dfrac{\partial y_3}{\partial x_3}}{\dfrac{\partial y_1}{\partial x_1} + \omega_1(y_1, y_2, y_3)\dfrac{\partial y_2}{\partial x_1} + \omega_2(y_1, y_2, y_3)\dfrac{\partial y_3}{\partial x_1}} = \omega_2(x_1, x_2, x_3), \end{cases}$$

and the application of the method leads to just one integrability condition of the form (58):

$$0 = \frac{\partial\omega_1(x_1,x_2,x_3)}{\partial x_3} - \frac{\partial\omega_2(x_1,x_2,x_3)}{\partial x_2} + \omega_1(x_1,x_2,x_3)\frac{\partial\omega_2(x_1,x_2,x_3)}{\partial x_1} - \omega_2(x_1,x_2,x_3)\frac{\partial\omega_1(x_1,x_2,x_3)}{\partial x_1};$$

one obtains just one type that has the representative:

$$(G) \qquad y_1 = \varphi_1(x_1), \qquad y_2 = \varphi_1(x_1, x_2, x_3), \qquad y_1 = \varphi_3(x_1, x_2, x_3).$$

Remark. – one sees from this example that, contrary to an assertion of Medolaghi, the conditions that define a type of group might not be comprised uniquely of equations of the form (59) and that, in turn, the theorem that Medolaghi concluded with – namely, that any Engel type includes a Picard group, i.e., contains all translations – remains to be proved.

VI. – Study of transitive subgroups of a given transitive group.

18. The method given in the preceding paragraph for the determination of groups, may, when conveniently modified, be of service in the study of transitive subgroups of a given transitive group (G). Here, two subgroups belong to the same *type* only if they are transforms of each other by a transformation of (G). In this case, we further say that they are *homologous* in (G).

We seek the subgroups whose equations of definition, when presented in completely integrable form, are of order at most m. We start with the equations of definition of (G) – differentiated, if necessary – up to order m, which will always be assumed to be greater than or equal to the maximum order of the equations of definition (G), when presented in completely integrable form. In the present form, they again form a completely integrable system that enjoys the properties that were established in the preceding paragraph. They correspond to a transitive group $\mathcal{L}$ that expresses how the group $\mathcal{A}_m$ transforms the invariants of a certain subgroup (l) of $\mathcal{A}_m$, and consequently an infinite group of the form:

$$(62) \qquad \begin{cases} x_i' = \varphi_i(x_1,\cdots,x_n), & (i = 1, 2, \cdots, n), \\ u_s = \mathcal{L}_s(u_1',\ldots,u_p' \mid \ldots, \varphi_j^{(\delta_1,\ldots,\delta_n)}, \ldots), & (s = 1, 2, \cdots, p). \end{cases}$$

As a result the equations of definition of (G), in the form considered – i.e., differentiated up to order m – are:

$$(63) \qquad \begin{cases} \mathcal{L}_s[\omega_1(y_1,\cdots,y_n),\cdots,\omega_p(y_1,\cdots,y_n)]\cdots, y_j^{(\delta_1,\ldots,\delta_n)},\ldots] = \omega_s(x_1,\cdots,x_n) \\ \qquad (s = 1, 2, \cdots, p). \end{cases}$$

The equations of any of the desired subgroups – when differentiated, if necessary, up to order m – contain, in addition to the equations (63), certain other equations of the form:

$$W_h(y_1, \ldots, y_n, \ldots, y_k^{(\delta_1,\ldots,\delta_n)}, \ldots) = \pi_h(x_1, \ldots, x_n) \qquad (h = 1, 2, \ldots, q),$$

and as a result the infinite group that is analogous to (62) and which corresponds to the subgroup (Γ) considered is composed of the equations (62) and other equations of the form:

(64) $$w_h = \mathcal{R}_h(u'_1, \cdots, u'_p \mid w'_1, \cdots, w'_q \mid \cdots, \varphi_j^{(\delta_1,\ldots,\delta_n)}, \cdots) \quad (h = 1, 2, \ldots, q).$$

This group [(62), (64)], or at least the finite group (Λ) that one deduces by suppressing the equations $x'_i = \varphi_i(x_1, \ldots, x_n)$ and considering the $\varphi_j^{(\delta_1,\ldots,\delta_n)}$ to be arbitrary constants, may be obtained upon introducing the invariants of a subgroup (λ') of $\mathcal{A}_m$ into the equations of $\mathcal{A}_m$. Now, the invariants of (λ') that correspond to the variables $u_1, \ldots, u_p$ in these calculations, when introduced by themselves, will give the group $\mathcal{L}$ itself. Therefore, there exist invariants of a subgroup of $\mathcal{B}_m$ that is greater than (λ'); let (l') be that subgroup. From a theorem of Lie ([1]), one may refer $\mathcal{A}_m$ to itself isomorphically, in such a manner that (l) and (l') correspond because they give the same group $\mathcal{L}$; however, the subgroup (λ') of (l') will then correspond to a subgroup (λ) of (l) that gives the same subgroup (Λ).

One concludes from this that in order to obtain the groups (Λ) that correspond to the desired subgroups of (G), it suffices to look for the various types of subgroups (λ) of (l), and to introduce their new invariants, at the same time as the ones of (l) that allowed us to find $\mathcal{L}$ in the equations of $\mathcal{A}_m$.

The method is quite precise, on the condition that one gets to know (l) when one knows only the equations of (G) in the Lie form, when differentiated up to order m. Now, from the argument in no. 7, in order to do this it suffices to look for the transformations of $\mathcal{B}_m$ that leave invariant the right-hand sides U_s of the equations of definition, when presented in the form (26), by treating $y_1, \ldots, y_n$ as arbitrary constants, since $\mathcal{L}$ expresses precisely the law of transformation of these U_s under the transformations of $\mathcal{A}_m$.

19. The equations of definition of one of the desired subgroups are thus obtained by adjoining to equations (63) some equations of the form:

(65) $$\begin{cases} \mathcal{R}_h[\omega_1(y_1, \cdots, y_n), \cdots, \omega_p(y_1, \ldots, y_n) \mid \pi_1(y_1, \cdots, y_n), \ldots, \pi_q(y_1, \cdots, y_n)] \ldots, y_j^{(\delta_1,\ldots,\delta_n)}, \ldots] \\ \qquad = \pi_h(x_1, \cdots, x_n) \qquad (h = 1, 2, \ldots, q), \end{cases}$$

([1]) *Theorie der Transf. gr.,* Bd. I, pp. 445.

where the $\mathcal{R}_h$ are known functions of their arguments; i.e., where all that remains is to determine the functions π_h by specifying that the system [(63), (65)] must be completely integrable.

The calculation to be done is identical to the one that was explained in the preceding paragraph. It might provide various systems, and any of them would be of the form:

$$(66)\qquad \begin{cases} \Omega_h\left(u_1,\dots,u_p\,|\dots,\dfrac{\partial u_s}{\partial x_i},\dots\right)=0 & (h=1,2,\dots,\rho), \\ J_k\left(u_1,\dots,u_p\,|\dots,\dfrac{\partial u_s}{\partial x_i},\dots\right)=c_k & (k=1,2,\dots,r); \end{cases}$$

$$(67)\qquad \begin{cases} \Theta_\alpha\left(u_1,\dots,u_p\,|\dots,\dfrac{\partial u_s}{\partial x_i},\dots|\,w_1,\cdots,w_q\,|,\cdots,\dfrac{\partial w_h}{\partial x_i},\cdots\right)=0 & (\alpha=1,2,\dots,\tau), \\ \mathrm{H}_\beta\left(u_1,\dots,u_p\,|\dots,\dfrac{\partial u_s}{\partial x_i},\dots|\,w_1,\cdots,w_q\,|,\cdots,\dfrac{\partial w_h}{\partial x_i},\cdots\right)=b_\beta & (\beta=1,2,\dots,t), \end{cases}$$

where one has exhibited those of the relations in which only the u_s appear, and which are themselves verified for $u_1 = \omega_1, \dots, u_p = \omega_p$.

The general solution of this system is deduced from a particular solution by transformations of the group [(62), (54)], and here one has to search for the solutions for which one constantly has:

$$(68)\qquad u_1 = \omega_1(x_1, \dots, x_n), \qquad \dots, \qquad u_p = \omega_p(x_1, \dots, x_n).$$

They are thus deduced from each other by the transformations in [(62), (64)] that leave this system invariant. In order to express that condition, it suffices to recall equations (62), and it results from what we saw in no. 8 that it consists in saying that one must consider only the transformations:

$$x_i' = \varphi_i(x_1, \dots, x_n) \qquad\qquad (i = 1, 2, \dots, n)$$

that belong to the given group (G).

One will thus obtain, by definition, for each type of subgroup of (G), an entirely well-defined system, of the form:

$$(69)\qquad \begin{cases} \Theta_\alpha\left(\omega_1,\cdots,\omega_p,\cdots,\dfrac{\partial \omega_s}{\partial x_i},\cdots|\,w_1,\cdots,w_q,\cdots,\dfrac{\partial w_h}{\partial x_i},\cdots\right)=0 & (\alpha=1,2,\cdots,\tau), \\ \mathrm{H}_\beta\left(\omega_1,\cdots,\omega_p,\cdots,\dfrac{\partial \omega_s}{\partial x_i},\cdots|\,w_1,\cdots,w_q,\cdots,\dfrac{\partial w_h}{\partial x_i},\cdots\right)=b_\beta & (\beta=1,2,\cdots,t), \end{cases}$$

whose general solution is deduced from a particular solution by the transformations of the infinite group, which is obtained by associating the various transformations:

(70) $$x_i' = \bar{\varphi}_i(x_1, \cdots, x_n) \qquad (i = 1, 2, \ldots, n),$$

of the group (G) with the corresponding transformations:

(71) $$w_h = \mathcal{R}_h(\omega_1, \ldots, \omega_p \mid w_1', \ldots, w_q' \mid \ldots, \bar{\varphi}_j^{(\delta_1, \ldots, \delta_n)}, \ldots) \qquad (h = 1, 2, \ldots, q).$$

This amounts to saying that from the geometric viewpoint the multiplicities:

$$w_h = \pi_h(x_1, \ldots, x_n) \qquad (h = 1, 2, \ldots, q)$$

that are furnished by the various solution to (69) are homologous to each other by respect to the group [(70), (71)], and that each of them admit a subgroup of this group [(70), (71)] that reduces to the corresponding subgroup of (G) when one considers only the manner in which the variables $x_1, \ldots, x_n$ transform into $x_1', \ldots, x_n'$.

VII. – Invariant subgroups.

20. For a given system (69), the preceding method gives all of the subgroups of (G) that belong to the same type, and formula (71) gives the law by which these groups are exchanged when one transforms them by the various transformations of (G).

It then results, in a manner that is completely analogous to the one that we saw in no. 15, that various solutions of (69) give the same subgroup (Γ) of (G) whenever this subgroup is invariant in a larger subgroup of (G), and that formulas (71) couple these diverse solutions that correspond to the transformations (70) that belong to the latter subgroup.

One will verify this fact as in no. 15 by appealing to the infinitesimal transformations of (Γ), whose equations of definition are composed, firstly, of those of (G):

(72) $$\sum_{i \mid \alpha_1, \ldots, \alpha_n} \frac{\xi_i^{(\alpha_1, \ldots, \alpha_n)}}{\alpha_1! \cdots \alpha_n!} \lambda_{i \mid \alpha_1, \ldots, \alpha_n \mid s}(\omega_1, \ldots, \omega_p) = \sum_{i=1}^n \xi_i \frac{\partial \omega_s}{\partial x_i} \qquad (s = 1, 2, \ldots, p),$$

and then equations of the form:

(73) $$\sum_{i \mid \alpha_1, \ldots, \alpha_n} \frac{\xi_i^{(\alpha_1, \ldots, \alpha_n)}}{\alpha_1! \cdots \alpha_n!} \nu_{i \mid \alpha_1, \ldots, \alpha_n \mid h}(\omega_1, \ldots, \omega_p \mid \pi_1, \ldots, \pi_q) = \sum_{i=1}^n \xi_i \frac{\partial \pi_h}{\partial x_i} \qquad (h = 1, 2, \ldots, q).$$

The condition for them to represent the same group for two systems of values that are attributed to the functions π_h is expressed by relations of the form:

$$\mathcal{R}_k\left(\omega_1,\ldots,\omega_p \mid \pi_1,\ldots,\pi_q,\ldots,\frac{\partial \pi_h}{\partial x_i},\ldots\right) = \mathcal{R}_k\left(\omega_1,\ldots,\omega_p \mid \pi'_1,\ldots,\pi'_q,\ldots,\frac{\partial \pi'_h}{\partial x_i},\ldots\right)$$
$$(k = 1, 2, \ldots, \sigma),$$

i.e.:

$$(74) \qquad \mathcal{R}_k\left(\omega_1,\ldots,\omega_p \mid \pi_1,\ldots,\pi_q,\ldots,\frac{\partial \pi_h}{\partial x_i},\ldots\right) = \varepsilon_k(x_1, \ldots, x_n) \qquad (k = 1, 2, \ldots, \sigma).$$

One will thus have to see whether one may determine the ε_k in such a manner as to have more than one common solution to the systems (69) and (74).

21. In particular, in order for the system (69) to furnish just one group (Γ) – i.e., an invariant subgroup of (*G*) – it is necessary and sufficient that all of the solutions belong to the same system (74). Therefore, in this case, the functions ε_k are determined without integration by expressing that equations (74) are consequences of equations (69). Now, these are the values of the only coefficients in equations (72) and (73) that depend upon unknown functions π_h when solved with respect to the largest possible number of ξ_i and their derivatives.

Therefore: *The equations of definition of the infinitesimal transformations of the various (transitive) invariant subgroups of* (*G*) *are obtained in an entirely explicit form.*

Let (Γ) be one of these invariant subgroups. In order to obtain the equations of definition of its finite transformations, one must integrate the system (69). The various solutions of this system correspond to the various systems of fundamental invariants of the group (Γ) that are deduced from each other by the transformations of (*G*). They are thus coupled by relations of the form:

$$(75) \qquad w'_h = f_h(w_1, \ldots, w_q \mid \omega_1, \ldots, \omega_p) \qquad (h = 1, 2, \ldots, q)$$

that define a group that is isomorphic to (*G*). This isomorphism is not holomorphic, because the identity transformation of the group (75) corresponds to the invariant subgroup (Γ) of (*G*), since each of the transformations of (Γ) leave its various fundamental invariants unaltered, and this is characteristic of the transformations of (Γ).

One may observe, moreover, since one knows the equations of definition of the infinitesimal transformations of (Γ), that the search for equations of definition of its finite transformations, which amounts to the determination of its fundamental invariants, depends only upon the integration of the complete system, in order for the integration of the system (69) to reduce to that of ordinary differential equations.

VIII. – On the study of intransitive groups and subgroups.

22. We shall say a few words about the determination of intransitive groups for the space of *n* dimensions. One specifies them by looking for the ones that have a given number $n - r$ of invariants of order zero. Since one only looks for one group of each type, one may assume that these invariants are the variables $x_{i+1}, \ldots, x_n$. The group is then a transitive group on the space of r dimensions with the coordinates $x_1, \ldots, x_r$, into whose transformations the $x_{i+1}, \ldots, x_n$ might enter as parameters.

One must then recall the method of studying the transitive groups, while replacing all of the arbitrary constants that might present themselves, either in the study of the group $\mathcal{L}$, or in the study of the auxiliary systems that were introduced by the integrability conditions, and which serve to separate the types by the arbitrary functions of the variables $x_{i+1}, \ldots, x_n$.

The same idea will serve to find the types of intransitive subgroups of a given group (G). However, here one must first study the systems of functions that might remain invariant under the transformations of a subgroup of (G), which might lead to the integration of the auxiliary differential systems. Having calculated one such system of functions, the simplest one will be to make a change of variables in (G) by taking these functions to be certain new variables. One will then come down to the determination of transitive groups that depend upon arbitrary constants of a given number.

In particular, if one looks for the invariant intransitive subgroups (G) then their invariants of order zero must transform amongst themselves by the transformations of (G). This case will therefore not present itself if (G) is primitive. If it is imprimitive then one must first integrate the various complete systems that are invariant under this group (G). After this, one will arrive at the result with no difficulty.

IX. – On similitude and isomorphism.

23. The systems that are like the system (51) that we introduced in no. 14 present themselves when one seeks to *recognize whether two groups* (G) *and* (G') *that are given by their equations of definition are similar, and to determine the transformations that change these two groups into each other* ([1]). Indeed, the question is the following one:

If the equations of definition of (G) are taken in the form:

$$\mathcal{L}_s[\omega_1(y_1, \ldots, y_n), \ldots, \omega_p(y_1, \ldots, y_n) \mid \ldots, y_j^{(\delta_1, \ldots, \delta_n)}, \ldots] = \omega_s(x_1, \ldots, x_n) \tag{76}$$
$$(s = 1, 2, \ldots, p),$$

then those of (G') may be put into the form:

$$\mathcal{L}_s[\theta_1(y_1, \ldots, y_n), \ldots, \theta_p(y_1, \ldots, y_n) \mid \ldots, y_j^{(\delta_1, \ldots, \delta_n)}, \ldots] = \theta_s(x_1, \ldots, x_n) \tag{77}$$
$$(s = 1, 2, \ldots, p),$$

in such a manner that there exist functions:

$$z_i = f_i(x_1, \ldots, x_n) \qquad (i = 1, 2, \ldots, n) \tag{78}$$

that satisfy the partial differential equations:

([1]) We assume, to simplify, that the two groups are transitive.

$$\mathcal{L}_s\,[\omega_1(x_1, \ldots, x_n), \ldots, \omega_p(x_1, \ldots, x_n) \mid \ldots, z_j^{(\delta_1,\ldots,\delta_n)}, \ldots] = \theta_s(x_1, \ldots, x_n) \qquad (s = 1, 2, \ldots, p). \tag{79}$$

Just which question one must resolve results immediately from the results of no. 10, and one sees that the system (79) differs only by the notations of the system (51).

One thus commences by putting the equations of definition of (G') into the Medolaghi form, as one saw in nos. 6 and 8. They will then be, for example:

$$\bar{\mathcal{L}}_s[\bar{\omega}_1(y_1,\ldots,y_n),\ldots,\bar{\omega}_p(y_1,\ldots,y_n) \mid \ldots, y_j^{(\delta_1,\ldots,\delta_n)},\cdots] = \bar{\omega}_s(x_1,\ldots,x_n) \qquad (s = 1, 2, \ldots, p),$$

because the question presents itself only if they are of the same order as those of (G), and in the same number. One may then modify them only by replacing the fundamental invariants with each other; i.e., by performing the point transformations:

$$\bar{\omega}_1 = f_s(\theta_1, .., \theta_p) \qquad (s = 1, 2, \ldots, p)$$

in these equations.

It thus amounts to first recognizing whether the two *finite* groups (because the $y_j^{(\delta_1,\ldots,\delta_n)}$ play the role of parameters in this calculation):

$$u'_s = \mathcal{L}_s(u_1, \ldots, u_p \mid \ldots, a_j^{(\delta_1,\ldots,\delta_n)}, \ldots) \qquad (s = 1, 2, \ldots, p),$$

and

$$u'_s = \bar{\mathcal{L}}_s(u_1, \ldots, u_p \mid \ldots, a_j^{(\delta_1,\ldots,\delta_n)}, \ldots) \qquad (s = 1, 2, \ldots, p)$$

are similar, and to find all of the transformations that make one pass from one to the other. This is a problem that has been treated completely by S. Lie ([1]), and which requires only performable operations, because one knows the finite equations of these two groups.

One will thus obtain various systems of functions $\theta_1(x_1, \ldots, x_n), \ldots, \theta_p(x_1, \ldots, x_n)$, and one must find out whether there is one of them that satisfies the integrability conditions of the system (79) – i.e., the auxiliary system [(58), (59)] that corresponds to the groups of the same type as (G). In this case, the groups (G) and (G') are essentially similar, and in order to find the transformations that permit one to pass from one to the other, one only has to integrate the various completely integrable systems (79) that one will have thus obtained.

24. We shall attempt to make the notion of *isomorphism* of infinite groups more precise, while limiting ourselves, for the sake of simplicity, to the holomorphic isomorphisms. It is clear that one may take the definition of such a thing to be either that of the theory of substitutions:

([1]) *Theorie der Transf. gr.*, Bd. I, pp. 327 *et seq.*

"Two groups are isomorphic when their finite transformations correspond uniquely in such a manner that the product of two transformations in one of the groups always corresponds to the product of two homologous transformations in the other group"

or the definition in the theory of finite groups:

"Two groups are isomorphic when their infinitesimal transformations correspond uniquely in such a manner that the bracket of two arbitrary infinitesimal transformations of the first group always corresponds to the bracket of two homologous infinitesimal transformations of the other group."

Moreover, it is not difficult to show that these two definitions are, at their basis, equivalent. However, it seems much more difficult to deduce from these conditions an analytical manner of translating the correspondence that was asserted. In order to avoid this difficulty, we propose another definition that is perhaps more restrictive, but is such that the mode of correspondence between the transformations – whether finite or infinitesimal – of the two groups will be determined in an analytically precise manner.

To that effect, we recall what happens for finite groups.

Since the transformations of the two groups depend only upon arbitrary constants, the stated correspondence in the two preceding definitions may only be interpreted by finite equations between the arbitrary constants – or parameters – that are necessary for one to define one of the transformations of either of the two groups. There is then no difficulty in analytically interpreting the isomorphic correspondence.

Always assume that the groups considered (G) and (G_1) are transitive. *Prolong* the first one by considering the variables $x_1, \ldots, x_n$ that it transforms to be functions of one untransformed variable *t;* after performing the prolongation for a sufficiently long time, we conclude by obtaining an intransitive group. We determine its invariants, and if we take new variables in place of certain derivatives $\frac{d^m x_i}{dt^m}$ then we obtain a simply transitive group $(\bar{G})$, which will be imprimitive, and of which one may say that (G) comes about by *shortening;* i.e., by keeping only those of its finite equations that permute the variables $x_1, \ldots, x_n$ amongst themselves.

We likewise obtain a simply transitive group $(\bar{G}_1)$, where (G_1) is deduced by *shortening*, and which will be holomorphically isomorphic to (G_1), and, in turn, to (G) and $(\bar{G})$.

However, since $(\bar{G})$ and $(\bar{G}_1)$ are simply transitive and holomorphically isomorphic they are similar. Therefore, for the sake of specificity, let:

$$x_1, \ldots, x_n, x_{n+1}, \ldots, x_r$$

denote the variables that $(\bar{G})$ transforms and let:

$$y_1, \ldots, y_p, y_{p+1}, \ldots, y_r$$

denote the ones that are transformed by $(\bar{G}_1)$, where the first p of them are the ones that transform under the given group (G_1). One will pass from $(\bar{G})$ to $(\bar{G}_1)$ by the formulas:

$$y_i = \varphi_i(x_1, \ldots, x_n, x_{n+1}, \ldots, x_r) \qquad (i = 1, 2, \ldots, r),$$

and consequently, the passage from $(\bar{G})$ to (G_1) is given by the formulas:

$$y_i = \varphi_i(x_1, \ldots, x_n, x_{n+1}, \ldots, x_r) \qquad (i = 1, 2, \ldots, p). \tag{80}$$

One thus sees that if one overlooks the simple case where (G) and (G_1) are similar then *there correspond two different modes of imprimitivity of the group* $(\bar{G})$. In the first, the functions $x_1, \ldots, x_n$ are permuted amongst themselves, while in the second one, it is the functions of $\varphi_1, \ldots, \varphi_p$, and there is no other transformation of $(\bar{G})$ besides the identity transformation that leaves invariant both the variables $x_1, \ldots, x_n$ and the functions $\varphi_1, \ldots, \varphi_p$ at the same time.

One likewise sees that, conversely, two groups (G) and (G_1) that are associated with a third group $(\bar{G})$ in that manner will be holomorphically isomorphic, without it being necessarily simply transitive.

It is this character that we propose to take as the definition ([1]) of the isomorphism of two infinite groups:

Two infinite (or finite) groups are called *holomorphically isomorphic* if they are either similar or they express two different modes of imprimitivity of the same third group; i.e., they express the law of exchange of two systems of functions of variables that are transformed by the third group, with the condition that the identity transformation is the only transformation of that third group that leaves invariant each of the functions of both of these two systems.

([1]) This definition seems to agree with a very obscure statement made by Lie (*Leipziger Berichte*, 1895, pp. 290, at the bottom of the page).