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LESSONS

ON

HIGHER GEOMETRY

Taught at the University of Lyon
and edited by ANZEMBERGER

REVISED AND EXPANDED EDITION

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PREFACE

In order to present the first edition of his book (which is currently out of print) to the public, Vessiot had no need for anyone else. That first edition sufficed to establish the reputation of the book, and the second one, by reason of the improvements that had been introduced, served only to further insure it.

Despite the fact that here one is dealing with a formal presentation to the public (which are presentations that are notoriously useless), I did not feel compelled to decline the friendly offer that my fellow scholar and friend kindly made that I might write some lines at the head of his book. At the same time, as a testament to my profound respect for the knowledge and expository talent of the author, I saw an opportunity to highlight the special character that the geometric work has taken on in latter years and felt the necessity of disseminating a book such as this one, which opens the doors to progress for those diligent students (who are, unfortunately, too rare) whose tastes lean towards geometry.

In the first part of the last century, which one can bound approximately with the year 1870, geometry made inestimable conquests along the most diverse paths and on the most varied terrains. That heroic period saw the birth of the most essential notions, and it also saw them develop and lead to great problems whose solution, which demanded analysis, demanded a profound movement in that sister science that amplified daily. It suffices to cite the great names of Monge, Dupin, Gauss, Serret, Lamé, Ossian Bonnet, and Bour for one to evoke the large inroads that were opened up in the theory of surfaces and in the theory of curvilinear coordinates. Those of Poncellet and Chasles recall the first renderings of the great laws of correspondence and transformations that would find a magnificent inflorescence in the work of Sophus Lie some time later. Towards the end of the same period, one found the coronation of the principle of duality in the duality of the geometry of points with the geometry of planes. Plücker established the geometry of the line and systematically introduced the notions of congruence and complex into that science, in which one must nonetheless recognize that long before him, Malus, Dupin, and Transon had produced some interesting considerations and important results in the context of optics. In the same era, Chasles showed how the kinematics of solid bodies put the principle of duality to work, and concurrently with Plücker, he grasped what the latter called a "linear complex."

Therefore, during that illustrious period, materials of the greatest value were accumulated in the most diverse order. Those great ideas, which were born apart from each other, and whose creators were often ignorant of each other, seemed destined to follow their path in isolation and to constitute just as many specialized categories for future geometers.

It belonged to the last part of the Nineteenth Century to belie those delusions and to bring about the marvelous fusion of all those elements.

The great authors of that fusion, using different methods, were Sophus Lie and Darboux. It would be unfair to forget Ribaucour, who contributed considerably in his own right to the realization of that very fertile interpenetration of the various branches of geometry and introduced the kinematical method into geometry by means of some unforgettable examples.

Today, one cannot deal with surfaces without introducing conjugate systems, nor address conjugate systems without introducing some congruence of lines, either the ones that cut out the conjugate net on the surface or the ones that are composed of the tangents to one of the lines of the net. On the other hand, conjugate nets are dependent upon equations that were envisioned by Laplace in a different era, and he found that the transformations that one knows from those equations correspond to the transition from one of the focal surfaces of the congruence to the other one.

The spherical representation of a surface, which seems to be a clever tool that is due to Gauss for extending the notion of curvature to surfaces by analogy with curves, is found to enter into the ideas of Chasles on duality that lead one to define a surface by its tangent planes. As Darboux has shown, it will suffice to introduce the distance from a fixed point to the tangent plane, along with the direction cosines of the normal that Gauss considered in order to define the spherical representation of the surface.

The hard problem of the deformation of surfaces remains an example that eternally bumps into unforeseen and fruitful links. If, as Bour once believed for a time, one has succeeded in integrating the partial differential equation that the problem depends upon analytically, then, without a doubt, it would have lost much of its interest in the eyes of a number of mathematicians that were too inclined to appreciate only the analytical aspects of those questions. That did not prevent either Ribaucour or Darboux from putting their cyclic systems into play, nor did it prevent Darboux and an entire clique of geometers from discovering the singular circumstances that accompany the deformation of quadrics.

Here, we shall not multiply the analogous examples that abound in transcendental geometry. The few that we shall give will suffice to show us that in our present era, the immortal work of the old geometers of the last century must no longer appear to us as isolated monuments, but rather, as the superb arches of a unique and grandiose edifice whose parts are all unified, and in which it is no longer permissible for the geometer to remain confined in one corner. The questions today are linked with each other in such a way that research that pursues a problem that is taken from a particular terrain can boast that it preserves the same horizon, because it often happens that the clear solution and the full blossoming of the problem must take place on a terrain that is very different from the one that one started out from.

It is always inexpert to pretend that a question that one is studying is isolated, and that is why one is advised to approach it head on with analysis. A question of structure will be found to be posed at each instant that does not coincide with the calculations of the project architect and the gravel and lime that his workers pile up. In any question of geometric structure, nothing can replace a deep knowledge of the geometric topic itself, the application of dogged reflection, and finally, the spontaneous exercise of one's intuition. It is not that often that the result obtained will be clothed in a simple form that calculation would recover with no effort later. However, what is really difficult is to first ponder it.

The proper and independent existence of the geometric viewpoint has always given rise to some divergent opinions, and it would be undoubtedly premature to foretell the conclusion. The *Eloges Académiques* by Joseph Bertrand have evoked some of the illustrious episodes in the past. One of the more striking ones is the iciness by which Cauchy received the appearance of Poncelet's celebrated *Traité des propriétés projectives des figures* in his own time. Later on, resistance to it was manifested by the small degree of favor that was accorded to the discoveries of Chasles that are immortal today. It took the papers on the attraction of ellipsoids, which are remarkable, moreover, to clear up certain prejudices.

In support of the general thesis, it would not be out of place to quote some lines about Poncelet by Joseph Bertrand from *Eloge*:

> "Descartes (said Bertrand) believed that by some uniform process of calculation, one could abolish the right to contrive in geometry. Believing that he had prepared and predicted everything, while leaving the pleasure of making any progress to his successors, he had usurped all of the credit and glory in advance at more than one point. I hope, he said, that our nephews will thank us for not just the things that I have explained, but also the ones that I have omitted voluntarily in order to leave the pleasure of inventing them to those nephews."
>
> "The ones (Joseph Bertrand continued) who, from their faith in impressive wizardry, believed that the original age of discoveries had concluded with the study of curves, naturally looked for a more fruitful use of their efforts and a greater degree of progress that was ever made in that beautiful theory in the other branches of science, with no difficulty, and thus had the opportunity and the cue that it was time to stop its advance."
>
> "Descartes had forgotten that, according to the very fortunate expression of a contemporary geometer, geometry is an art, as well as a science: *Mathesis ars et scientia dicenda*, and that it is sometimes possible for a science to mark the end of its efforts and the term in which it progressed with a definitive formula, while art is inexhaustible and infinite, always young, and always fertile with new ideas."

Please allow us to introduce another viewpoint here. In our time, the intense development of analytic theories and the great place that they occupy in the programs that realize the result that it is, above all, by rigorous and abstract logical gymnastics that we exercise the intelligence of our young minds. Now, the experience of testing, which is old already, says quite eloquently how incomplete and inoperative that development will be if it does not find a counterweight in the practice of more concrete realities in geometry or mechanics. It should then be greatly desired that the tastes and cult of geometry should be favored in our education more than they are today.

By its nature, Vessiot's book contributes, at the highest level, to the dissemination of those beneficent studies in which the French mentality is manifested so harmoniously in the form of elegance and grandeur. Vessiot, who is also an informed analyst, has given his exposition a form that is impeccable in its precision, and which carefully facilitates the handling of quantitative notions and the general formulas in which they intervene.

In his book, he has encompassed everything that is essential if one is to understand, or even read fruitfully, the original work of the inventors. The friends of geometry can only rejoice at the assistance that this book gives to their favorite science and must express the best wishes for the continuation of its success.

G. Koenigs

PREFACE TO THE FIRST EDITION

These lectures were taught by the science faculty at Lyons in 1905-1906, in response to the special program of mathematical analysis of the process of nomination to Associate. They were written up at the demand of my students and were edited by one of them.

Perhaps it might be useful to the students that are desirous of being initiated into higher geometry, and it might give them a good preparation into the study of the books of Darboux and the original papers.

I have assumed that only the simplest principles of the theory of contact are known. I have reviewed the essential points of the theory of skew curves and the theory of surfaces, while emphasizing the essential role of the Frenet formulas, and Gauss's two quadratic differential forms.

The principal objective of my lectures was the study of systems of lines and their application to the theory of surfaces. It was natural to combine that with the study of systems of spheres, which I have continued up to the very attractive elementary properties of Ribaucour's cyclic systems. I have insisted upon the correspondence between lines and spheres. I have clarified it by the use of the notions of contact elements and multiplicities, which is likewise useful in the theory of ray congruences. I have shown how it translates into Lie's celebrated contact transformation.

I have sought to develop the various subjects along the most natural and the most analytical path. I wished to show my students how methodical research and the deep discussion of questions, even the simplest ones, as well as the attentive study and interpretation of the results of calculation, can lead to the most varied and the most interesting consequences.

Lyon, 1 June 1906
E. VESSIOT.

FOREWORD

The first edition of these lectures, as it was written, was rapidly exhausted, so I accepted the offer to reprint them that Hermann made to me.

The printing errors were corrected by Anzemberger, in view of that reprinting. I have reviewed and improved the editing, and I have made some important additions. Grévy has kindly assisted me in the revision of the text and the correction of the proofs. I would like to acknowledge my gratitude to him here. I would also like to address my thanks to Hermann for the care that they afforded to the printing.

I shall forego giving bibliographic citations. This is an introductory book, and the readers that are desirous of pursuing geometric research must always refer to Darboux's admirable books, in which they will find the necessary documentation.

30 September 1919.

E. VESSIOT

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FIRST CHAPTER

REVIEW OF THE ESSENTIAL POINTS IN THE THEORY OF SKEW CURVES AND DEVELOPABLE SURFACES

I. – SKEW CURVES

Frenet-Serret trihedron

1. – Let (C) be a skew curve whose coordinates we assume to be expressed as functions of a parameter t:

$$x = f(t), \qquad y = g(t), \qquad z = h(t).$$

We consider the *tangent* to such a curve, which has dx / dt, dy / dt, dz / dt for its direction parameters, and the *osculating plane*, which contains the tangent $\left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$ and the acceleration $\left(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}\right)$, and whose coefficients are, in turn, the second-degree determinants that are deduced from the matrix:

$$\begin{matrix} \frac{dx}{dt} & \frac{dy}{dt} & \frac{dz}{dt} \\ \\ \frac{d^2x}{dt^2} & \frac{d^2y}{dt^2} & \frac{d^2z}{dt^2} \end{matrix}.$$

Remark. – If one changes the parameters by setting $t = \varphi(u)$ then the new acceleration $\left(\frac{d^2x}{du^2}, \frac{d^2y}{du^2}, \frac{d^2z}{du^2}\right)$ will always be in the osculating plane.

Consider the tangent MT at a point M of the curve, the normal that is situated in the osculating plane – or *principal normal MN* – and the normal MB that is perpendicular to the osculating plane – or *binormal.* Those three lines will form a tri-rectangular trihedron that we call the *Serret* or *Frenet trihedron.* One of its faces – namely, the one that is determined by the tangent and the principal normal – is the osculating plane. The one that is determined by the principal normal and the binormal is the *normal plane*. Finally, the one that is determined by the tangent and the binormal is called the *rectifying plane.*

Take an origin of an arbitrary arc on the curve and an increasing sense along the arc, which is likewise arbitrary. The differential of the arc-length s is given by the formula:

$$ds^2 = dx^2 + dy^2 + dz^2,$$

so:

$$\frac{ds}{dt} = \varepsilon\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \qquad (\varepsilon = \pm 1),$$

and:

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = 1.$$

dx / ds, dy / ds, dz / ds will then be the direction cosines of the direction of the tangent that corresponds to the sense of increasing arc-length; let α, β, γ be those direction cosines:

(1) $$\alpha = \frac{dx}{ds}, \qquad \beta = \frac{dy}{ds}, \qquad \gamma = \frac{dz}{ds}.$$

We take an arbitrary positive direction along the principal normal with direction cosines α', β', γ' and a positive direction along the binormal whose direction cosines are α'', β'', γ'', such that the trihedron that is composed of those three directions will have the same disposition as the coordinate trihedron. Hence:

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix} = 1,$$

and each element of that determinant will be equal to its coefficient in the development of the determinant.

Serret-Frenet formulas

2. – Some important relations exist between those direction cosines and their differentials. Indeed, upon taking the derivatives of both sides of the relation:

$$\alpha^2 + \beta^2 + \gamma^2 = 1$$

with respect to s, one will infer that:

$$\sum \alpha \frac{d\alpha}{ds} = 0.$$

However, from the relations (1), one will have:

$$\frac{d\alpha}{ds} = \frac{d^2x}{ds^2}, \qquad \frac{d\beta}{ds} = \frac{d^2y}{ds^2}, \qquad \frac{d\beta}{ds} = \frac{d^2z}{ds^2},$$

and the preceding can be written:

$$\sum \alpha \frac{d^2 x}{ds^2} = 0.$$

The direction that has the direction coefficients:

$$\frac{d^2 x}{ds^2}, \frac{d^2 y}{ds^2}, \frac{d^2 z}{ds^2} \qquad \text{or} \qquad \frac{d\alpha}{ds}, \frac{d\beta}{ds}, \frac{d\gamma}{ds}$$

is perpendicular to the tangent then. On the other hand, it is in the osculating plane, since it is the acceleration that corresponds to the parameter s. It is therefore the principal normal, and consequently there will exist a number R such that:

(2) $$\frac{d\alpha}{ds} = \frac{\alpha'}{R}, \qquad \frac{d\beta}{ds} = \frac{\beta'}{R}, \qquad \frac{d\gamma}{ds} = \frac{\gamma'}{R}.$$

Upon multiplying these equations by α', β', γ', respectively, and adding corresponding sides, one will deduce that:

(3) $$\frac{1}{R} = \sum \alpha' \frac{d\alpha}{ds}.$$

Upon now multiplying them by α'', β'', γ'', respectively, and adding corresponding sides, one will get:

$$\sum \alpha'' \frac{d\alpha}{ds} = 0.$$

On the other hand:

$$\sum \alpha \alpha'' = 0.$$

Hence, upon taking derivatives with respect to s:

$$\sum \alpha \frac{d\alpha''}{ds} + \sum \alpha'' \frac{d\alpha}{ds} = 0,$$

and as a result:

$$\sum \alpha \frac{d\alpha''}{ds} = 0.$$

Moreover:

$$\sum \alpha''^2 = 1,$$

so:

$$\sum \alpha'' \frac{d\alpha''}{ds} = 0,$$

and the preceding two relations show that the direction:

$$\frac{d\alpha''}{ds}, \frac{d\beta''}{ds}, \frac{d\gamma''}{ds}$$

is perpendicular to the tangent and the binormal. It is once again the principal normal, and there exists a number T such that:

(4)
$$\frac{d\alpha''}{ds} = \frac{\alpha'}{T}, \qquad \frac{d\beta''}{ds} = \frac{\beta'}{T}, \qquad \frac{d\gamma''}{ds} = \frac{\gamma'}{T}.$$

Upon multiplying those equations by α', β', γ', respectively, and adding corresponding sides, one will get:

(5)
$$\frac{1}{T} = \sum \alpha' \frac{d\alpha''}{ds}.$$

One likewise infers from the relation:

$$\sum \alpha' \alpha = 0$$

that:

$$\sum \alpha \frac{d\alpha'}{ds} = -\sum \alpha' \frac{d\alpha}{ds} = -\frac{1}{R},$$

and finally one infers from the relation:

$$\sum \alpha'^2 = 1$$

that:

$$\sum \alpha' \frac{d\alpha'}{ds} = 0.$$

One then has three equations in $\frac{d\alpha'}{ds}, \frac{d\beta'}{ds}, \frac{d\gamma'}{ds}$:

$$\sum \alpha \frac{d\alpha'}{ds} = -\frac{1}{R}, \qquad \sum \alpha' \frac{d\alpha'}{ds} = 0, \qquad \sum \alpha'' \frac{d\alpha'}{ds} = -\frac{1}{T},$$

whose solution will give:

(6)
$$\frac{d\alpha'}{ds} = -\frac{\alpha}{R} - \frac{\alpha''}{T}, \qquad \frac{d\beta'}{ds} = -\frac{\beta}{R} - \frac{\beta''}{T}, \qquad \frac{d\gamma'}{ds} = -\frac{\gamma}{R} - \frac{\gamma''}{T}.$$

The three groups of relations (2), (4), (6) constitute the *Serret formulas* or *Frenet formulas.*

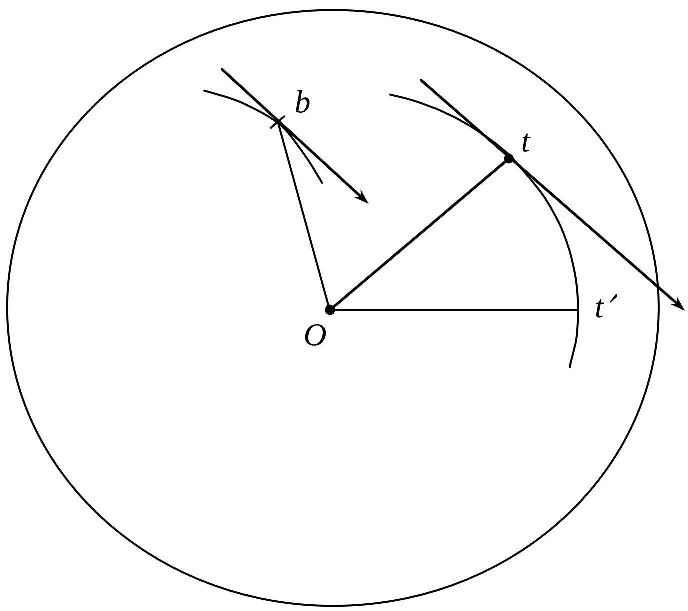
Curvature and torsion

3. *Interpretation of R*. – Consider the point t whose coordinates are α, β, γ. Formulas (2) express a property of the curve that is the locus of those points. That curve is traced upon a sphere of radius 1 that one calls the *spherical indicatrix* of the curve (C), and the formulas (2) show that *the tangent to the spherical indicatrix at t is parallel to the principal normal to the curve* (C) *at M*. Let σ be the arc-length of that indicatrix, when measured from an arbitrary origin in an arbitrary sense:

$$\frac{d\alpha}{d\sigma} = \varepsilon\alpha', \qquad \frac{d\beta}{d\sigma} = \varepsilon\beta', \qquad \frac{d\gamma}{d\sigma} = \varepsilon\gamma'.$$

Hence, upon taking formulas (2) into account:

$$\frac{1}{R} = \varepsilon\frac{d\sigma}{ds}.$$



Now consider the points t, t' that correspond to the points M, M', resp. $d\sigma / ds$ is, up to sign, the limit of the ratio $\dfrac{\text{arc } tt'}{\text{arc } MM'}$ when M' tends to M. The arc tt' is an infinitely small equivalent to the arc-length of the great circle tt', which has the same measure as the angle tOt' between the two infinitely-close tangents or the *angle of contingency*. $\left|\dfrac{d\sigma}{ds}\right|$ is then the limit of the ratio $\dfrac{\text{arc } \widehat{tOt'}}{\text{arc } \widehat{MM'}}$, which is called the *curvature* of the curve at the point M; R is *radius of curvature* at the point M.

Interpretation of T. – In order to interpret T, one must likewise consider the locus of the point b whose coordinates are $\alpha'', \beta'', \gamma''$, or the *second spherical indicatrix.* From formulas (2), (4), one must remark that *the tangents at t, b to the two indicatrices are parallel to the principal normal at M.* If τ is the arc-length of that second spherical indicatrix then one will find, as before, that:

$$\frac{1}{T} = \varepsilon' \frac{d\tau}{ds} \qquad (\varepsilon' = \pm 1),$$

and $| 1 / T |$ will be the limit of the ratio of the angle between the osculating planes at M, M' to the arc-length MM' when M' tends to M ; it is the *torsion* at M, and T is the *radius of torsion.*

The two indicatrices are both polars to the sphere.

Discussion. Center of curvature

4. – The direction cosines that we introduced depend upon three arbitrary hypotheses, namely, the sense of increasing arc-length, how the positive sense is chosen along the principal normal, and the disposition of the coordinate trihedron. If we change that hypothesis and let $\varepsilon_1, \varepsilon_2, \varepsilon_3$ denote numbers that are equal to ± 1 then s will be replaced by $\varepsilon_i\, s$. α, β, γ will become $\varepsilon_i\,\alpha, \varepsilon_i\,\beta, \varepsilon_i\,\gamma$, α', β', γ' will become $\varepsilon_i\,\alpha', \varepsilon_i\,\beta', \varepsilon_i\,\gamma'$, and finally, from the relations:

$$\alpha'' = \varepsilon_3\,(\beta\gamma' - \gamma\beta'), \quad \beta'' = \varepsilon_3\,(\gamma\alpha' - \alpha\gamma'), \quad \gamma'' = \varepsilon_3\,(\alpha\beta' - \beta\alpha'),$$

$\alpha'', \beta'', \gamma''$ will be replaced with $\varepsilon_1\,\varepsilon_2\,\varepsilon_3\,\alpha''$, $\varepsilon_1\,\varepsilon_2\,\varepsilon_3\,\beta''$, $\varepsilon_1\,\varepsilon_2\,\varepsilon_3\,\gamma''$. Formulas (2) will then give:

$$\frac{\varepsilon_1\,d\alpha}{\varepsilon_2\,ds} = \frac{\varepsilon_2\,\alpha'}{R}, \ \ldots;$$

i.e., R will change into $\varepsilon_2\,R$, and its sign will depend upon only the choice of positive direction along the principal normal.

Hence, the point C along the principal normal such that $MC = R$ (R being defined algebraically as before) is a geometric element that is attached to the given curve. That point C is called the *center of curvature at M*.

Now look at T. Formulas (4) then give:

$$\frac{\varepsilon_1\varepsilon_2\varepsilon_3\,d\alpha''}{\varepsilon_1\,ds} = \frac{\varepsilon_2\alpha'}{T}, \quad \ldots,$$

or:

$$\varepsilon_3\,\frac{d\alpha''}{ds} = \frac{\alpha'}{T}, \ \ldots,$$

Hence, T will change into $\varepsilon_3\, T$, and the sign of T will depend upon the disposition of the coordinate trihedron uniquely. There is no reason to define a center of torsion then.

Sign of torsion. Form of the curve

5. – In order to interpret the sign of T, we shall study the rotation of a plane that passes through the tangent MT and a point M' along the curve that is infinitely close to M. Refer the curve to the Serret trihedron, so the tangent is OX, the principal normal is OY, and the binormal is OZ. We will then have $\alpha = 1$, $\beta = 0$, $\gamma = 0$; $\alpha' = 0$, $\beta' = 1$, $\gamma' = 0$; $\alpha'' = 0$, $\beta'' = 0$, $\gamma'' = 1$. We shall seek the developments of the coordinates of a point of the curve that is infinitely-close to M in increasing powers of ds (viz., the arc-length of the curve when measured from the point M).

We have:

$$\begin{cases} X = \dfrac{ds}{1}\dfrac{dx}{ds} + \dfrac{ds^2}{2}\dfrac{d^2x}{ds^2} + \dfrac{ds^3}{6}\dfrac{d^3x}{ds^3} + \cdots, \\ Y = \cdots, \\ Z = \cdots \end{cases}$$

Now, from the Frenet formulas:

$$\frac{dx}{ds} = \alpha = 1,$$

$$\frac{d^2x}{ds^2} = \frac{d\alpha}{ds} = \frac{\alpha'}{R} = 0,$$

$$\frac{d^3x}{ds^3} = \frac{d^2\alpha}{ds^2} = \frac{1}{R}\frac{d\alpha'}{ds} + \alpha'\frac{d\left(\frac{1}{R}\right)}{ds} = \frac{1}{R}\left(-\frac{\alpha}{R} - \frac{\alpha''}{T}\right) - \frac{\alpha'}{R^2}\frac{dR}{ds} = -\frac{1}{R^2},$$

and similarly for the other coordinates. One will then find that:

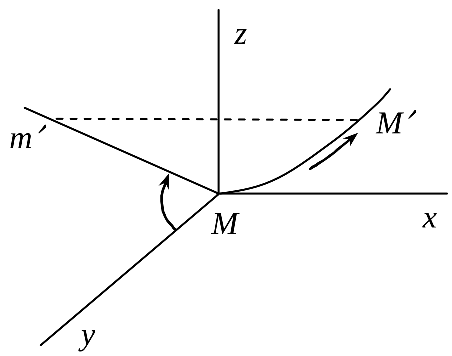
$$(7) \qquad \begin{cases} X = ds & & -\dfrac{1}{6R^2}ds^3 & + \cdots \\ Y = & \dfrac{1}{2R}ds^2 & -\dfrac{1}{6R^2}\dfrac{dR}{ds}ds^3 & + \cdots \\ Z = & & -\dfrac{1}{6RT}ds^3 & + \cdots \end{cases}$$

These are the developments of the coordinates of a point M' that is infinitely close to M.

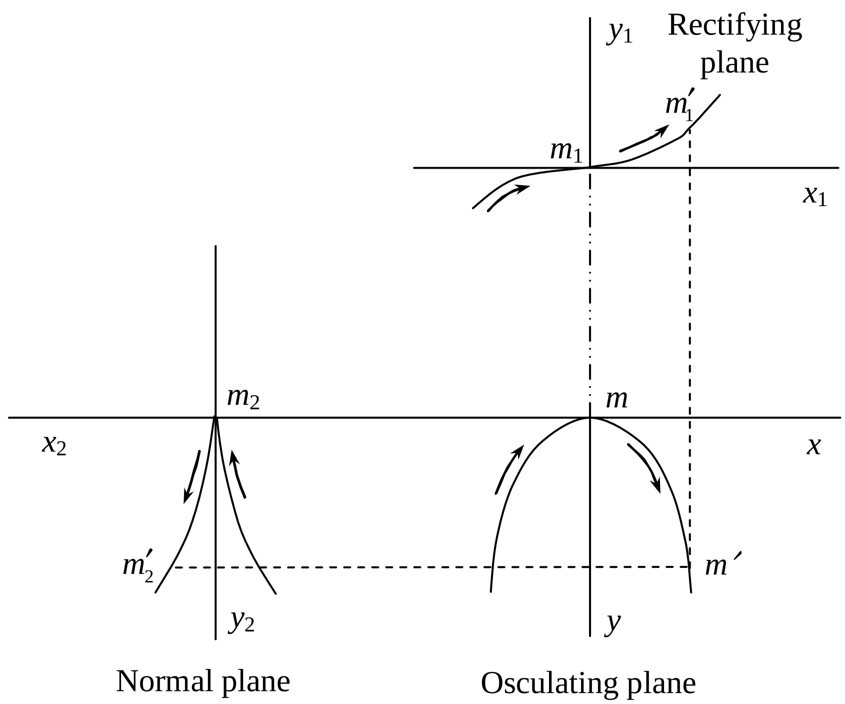
The plane that we consider passes through the tangent. The initial sense of its rotation when *ds* varies upon starting with zero is given by the sign of *Z* / *Y*, which is the angular coefficient of its trace in the plane of the *YZ*. Now:

$$\frac{Z}{Y} = -\frac{ds}{3T}[1 + ds\,(\ldots)].$$

That angular coefficient will be positive when $T < 0$ for increasing *s*; i.e., when the point displaces in the positive direction along the tangent. The plane will then turn in the positive sense. Moreover, if one supposes, for example, that $R > 0$ then the point *M* will be above the *XY*-plane, and if $T < 0$ then the arc MM' of the curve will be in front of the *XZ*-plane; on the contrary, when $T > 0$, that point will be behind that plane.



Formulas (7) permit us to represent the projections of the curve onto the three faces of the Serret trihedron in the neighborhood of the point *M*. In order to draw those projections, we shall suppose that $R > 0$ and $T < 0$.



A consideration of formulas (7), when taken two at a time, will show that the projection onto the rectifying plane (*XZ*) will have a point of inflection at the point m_1, where the inflectional tangent will be *OX*. The projection onto the osculating plane will have an ordinary point *m* whose tangent is *OX*. Finally, the projection onto the normal plane (*YZ*) will have a point of regression at m_2 whose tangent of regression will be *OY*.

Motion of the Serret-Frenet trihedron

6. – *Remark.* – Consider a point *P* that is invariably linked with the Serret trihedron, and let *X*, *Y*, *Z* be its coordinates, which are constant with respect to that trihedron; let ξ, η, ζ be the coordinates of that point with respect to a system of fixed axes. Upon remarking that:

$$\begin{cases} \xi = x + \alpha X + \alpha' Y + \alpha'' Z, \\ \eta = y + \beta X + \beta' Y + \beta'' Z, \\ \zeta = z + \gamma X + \gamma' Y + \gamma'' Z, \end{cases}$$

when the summit of the Serret trihedron describes the given curve, the projections of the velocity of the point *P* onto the fixed axes will be:

$$\begin{cases} \dfrac{d\xi}{dt} = \dfrac{dx}{dt} + X\dfrac{d\alpha}{dt} + Y\dfrac{d\alpha'}{dt} + Z\dfrac{d\alpha''}{dt}, \\ \dfrac{d\eta}{dt} = \dfrac{dy}{dt} + X\dfrac{d\beta}{dt} + Y\dfrac{d\beta'}{dt} + Z\dfrac{d\beta''}{dt}, \\ \dfrac{d\zeta}{dt} = \dfrac{dz}{dt} + X\dfrac{d\gamma}{dt} + Y\dfrac{d\gamma'}{dt} + Z\dfrac{d\gamma''}{dt}, \end{cases}$$

or rather:

$$\begin{cases} \dfrac{d\xi}{dt} = \left[\alpha + X\dfrac{\alpha'}{R} - Y\left(\dfrac{\alpha}{R} + \dfrac{\alpha''}{T}\right) + Z\dfrac{\alpha'}{T}\right]\dfrac{ds}{dt}, \\ \dfrac{d\eta}{dt} = \cdots \\ \dfrac{d\zeta}{dt} = \cdots \end{cases}$$

The projections of the velocity onto the moving axes will then be:

$$\left.\begin{aligned} V_X &= \alpha\frac{d\xi}{dt} + \beta\frac{d\eta}{dt} + \gamma\frac{d\zeta}{dt} = \left(1 - \frac{Y}{R}\right)\frac{ds}{dt}, \\ V_Y &= \alpha'\frac{d\xi}{dt} + \beta'\frac{d\eta}{dt} + \gamma'\frac{d\zeta}{dt} = \left(\frac{X}{R} + \frac{Y}{R}\right)\frac{ds}{dt}, \\ V_Z &= \alpha''\frac{d\xi}{dt} + \beta''\frac{d\eta}{dt} + \gamma''\frac{d\zeta}{dt} = -\frac{Y}{T}\frac{ds}{dt}. \end{aligned}\right\}$$

ds / dt is the velocity of the summit of the trihedron. If we consider only the rotational velocity then we will know that if p, q, r are the components of the instantaneous rotation along the moving axes then:

$$V_X = qZ - rY, \qquad V_Y = rX - pZ, \qquad V_Z = pY - qX,$$

and upon identifying those expressions with the preceding ones, we will then find that:

$$p = -\frac{1}{T}\frac{ds}{dt}, \quad q = 0, \quad r = \frac{1}{R}\frac{ds}{dt},$$

which shows that:

The instantaneous rotation at each instant is in the rectifying plane, and if one supposes that $t = s$ then it will have the torsion and curvature for its components along the tangent and binormal, resp.

If one supposes that the Serret trihedron has been transported to the origin then it will turn around its summit, so the instantaneous axis of rotation will be in the rectifying plane, and the motion of the trihedron will be obtained by rolling a certain cone on that plane.

Calculation of *R*

7. – Recall formula (3):

$$\frac{1}{R} = \sum \alpha' \frac{d\alpha}{ds}.$$

From the relation:

$$\alpha = \frac{dx}{ds},$$

one infers that:

$$\frac{d\alpha}{ds} = \frac{ds\, d^2x - dx\, d^2s}{ds^3}.$$

Now set:

$$A = dy\, d^2z - dz\, d^2y, \quad B = dz\, d^2x - dx\, d^2z, \quad C = dx\, d^2y - dy\, d^2x,$$

and

$$\pm\sqrt{A^2+B^2+C^2} = D.$$

A, *B*, *C* are the coefficients of the osculating plane. As a result, the sign of *D* can be chosen arbitrarily, so the direction cosines of the binormal will be:

$$\alpha'' = \frac{A}{D}, \qquad \beta'' = \frac{B}{D}, \qquad \gamma'' = \frac{C}{D},$$

and the direction cosines of the principal normal will be:

$$\alpha' = \gamma\beta'' - \beta\gamma'' = \frac{B\,dz - C\,dy}{D\,ds} = \frac{d^2x\,(dx^2+dz^2) - dx\,(dy\,d^2y + dz\,d^2z)}{D\,ds}$$
$$= \frac{d^2x\,ds^2 - dx\,ds\,d^2s}{D\,ds} = \frac{ds\,d^2x - dx\,d^2s}{D},$$

and similarly:

$$\beta' = \frac{ds\,d^2y - dy\,d^2s}{D},$$
$$\gamma' = \frac{ds\,d^2z - dz\,d^2s}{D},$$

so:

$$\frac{1}{R} = \sum \alpha' \frac{d\alpha}{ds} = \sum \frac{B\,dz - C\,dy}{D\,ds}\,\frac{ds\,d^2x - dx\,d^2s}{ds^2},$$

which can be written:

$$\frac{1}{R} = \frac{1}{D\,ds^2}\sum d^2x\,(B\,dz - C\,dy) - \frac{d^2s}{D\,ds^2}\sum dx\,(B\,dz - C\,dy).$$

The second sum is zero, and:

$$\frac{1}{R} = \frac{1}{D\,ds^2}\sum d^2x\,(B\,dz - C\,dy) = \frac{1}{D\,ds^2}\begin{vmatrix} dx & dy & dz \\ d^2x & d^2y & d^2z \\ A & B & C \end{vmatrix} = \frac{D}{ds^2},$$

so finally:

$$\left|\frac{1}{R}\right| = \frac{\sqrt{\sum (dy\,d^2z - dz\,d^2y)^2}}{(dx^2+dy^2+dz^2)^{3/2}}.$$

Calculation of T

8. – Similarly:

$$\frac{1}{T} = \sum \alpha' \frac{d\alpha''}{ds} = \sum \frac{B\,dz - C\,dy}{D \cdot ds} \frac{D \cdot dA - A\,dD}{D^2 ds},$$

which can be written:

$$\frac{1}{T} = \frac{1}{D^2\,ds^2} \sum dA(B\,dz - C\,dy) - \frac{dD}{D^2\,ds^2} \sum A(B\,dz - C\,dy).$$

The second sum is zero, and:

$$\frac{1}{T} = \frac{1}{D^2\,ds^2} \sum dA(B\,dz - C\,dy) = \frac{1}{D^2\,ds} \sum (dy\,d^3z - dz\,d^3y)(ds\,d^2x - dx\,d^2s),$$

or:

$$\frac{1}{T} = \frac{1}{D^2} \sum d^2x\,(dy\,d^3z - dz\,d^3y) - \frac{d^2s}{D^2 ds} \sum dx\,(dy\,d^3z - dz\,d^3y).$$

The second sum is zero, and:

$$\frac{1}{T} = \frac{1}{D^2} \sum d^2x\,(dy\,d^3z - dz\,d^3y) = -\frac{1}{D^2} \begin{vmatrix} dx & dy & dz \\ d^2x & d^2y & d^2z \\ d^3x & d^3x & d^3x \end{vmatrix},$$

in which:

$$D^2 = \sum (dy\,d^2z - dz\,d^2y)^2 .$$

Remark. – In order for the torsion of a curve to be constantly zero, it is necessary and sufficient that one must constantly have:

$$\begin{vmatrix} dx & dy & dz \\ d^2x & d^2y & d^2z \\ d^3x & d^3x & d^3x \end{vmatrix} = 0,$$

which demands that x, y, z must be coupled by a linear relation with constant coefficients; i.e., the curve must be planar. Hence:

The curves with torsion that is constantly zero will be plane curves.

Osculating sphere

9. – We look for the spheres that have second-order contact with the curve considered at M. From the theory of contact, the center (x_0, y_0, z_0) and the radius R_0 of such a sphere are determined by the following equations, which we develop by means of the Serret-Frenet formulas:

$$\sum (x - x_0)^2 - R_0^2 = 0,$$

$$\frac{d}{ds}[\sum (x - x_0)^2 - R_0^2] = 0, \qquad \text{or} \qquad \sum \alpha(x - x_0) = 0,$$

$$\frac{d^2}{ds^2}[\sum (x - x_0)^2 - R_0^2] = 0, \qquad \text{or} \qquad 1 + \frac{1}{R}\sum \alpha'(x - x_0) = 0.$$

If we take the Serret-Frenet trihedron to be the coordinate trihedron, as we did above, then those equations will reduce to:

$$\sum x_0^2 - R_0^2 = 0, \qquad x_0 = 0, \qquad y_0 = R_0^2,$$

and if Z_0 remains arbitrary then the general equation of the desired spheres will be:

$$X^2 + Y^2 + Z^2 - 2RY - 2Z_0 Z = 0.$$

That is a sheaf of spheres that includes the osculating plane $Z = 0$. One then verifies the property of contact with the osculating plane.

From the theory of contact for curves, the circle that is common to all of those spheres is, moreover, the one that has second-order contact with the curve; i.e., the *osculating circle.* Its equations are:

$$Z = 0, \qquad X^2 + Y^2 + Z^2 - 2RY = 0,$$

so it will be in the osculating plane, its center will be the center of curvature C ($X = 0$, $Y = R$), and it will pass through M. The locus of centers of the spheres considered is the axis of the osculating circle.

Among all of those spheres, there is one of them that has third-order contact with the curve. One obtains it by introducing the new condition:

$$\frac{d^3}{ds^3}[\sum (x - x_0)^2 - R_0^2] = 0;$$

i.e.:

$$-\frac{1}{R^2}\frac{dR}{ds}\sum \alpha'(x - x_0) - \frac{1}{R}\left[\frac{1}{R}\sum \alpha(x - x_0) + \frac{1}{T}\sum \alpha''(x - x_0)\right] = 0.$$

With the particular axes that are being employed and the values for x_0, y_0 that were found before, that will reduce to:

$$z_0 = -T\,\frac{dR}{ds}.$$

The center of that sphere, which is the *osculating sphere*, will then be defined by the formulas:

$$X_0 = 0, \qquad Y_0 = +R, \qquad Z_0 = -T\frac{dR}{ds},$$

and its radius will be given by the formula:

$$R_0^2 = R^2 + T^2\left(\frac{dR}{ds}\right)^2.$$

II. – DEVELOPABLE SURFACES

General properties

10. – A skew curve is the locus of ∞^1 points. Correlatively, we consider a *developable surface*, which is the envelope of ∞^1 planes. The *characteristic* of one of those planes corresponds to the tangent to the curve at a point, since it is the intersection of two infinitely-close planes.

Let:

(1) $$u X + v Y + w Z + h = 0$$

be the general equation of the planes considered, in such a way that u, v, w, h denote given functions of a parameter t.

From the theory of envelopes, the characteristics have the general equations:

(2) $$\begin{cases} uX + vY + wZ + h = 0, \\ Xdu + Ydu + Zdw + dh = 0. \end{cases}$$

From the theory of envelopes, the developable surface that is the envelope of the planes (1) is the locus of the lines (2), which will consequently be rectilinear generators. Moreover, again from the theory of envelopes, each of the planes (1) will be tangent to the surface along the generator (2) that corresponds to the same value of t.

Consider the curve (C) then, which is the locus of points (x, y, z) that are defined by the equations:

(3) $$\begin{cases} ux + vy + wz + h = 0, \\ x\,du + y\,dv + w\,dz + dh = 0, \\ x\,d^2u + y\,d^2v + w\,d^2z + d^2h = 0. \end{cases}$$

Any of its points M will be on the line (2) and correspond to the same value of t, and consequently, it will be in the corresponding plane (1). We seek the tangent to (C) at M. In order to do that, differentiate equations (3). If we differentiate each of the first two then upon taking into account the following one we will find that:

$$(4) \qquad \begin{cases} u\cdot dx + v\cdot dy + w\cdot dz = 0, \\ du\cdot dx + dv\cdot dy + dw\cdot dz = 0, \end{cases}$$

which expresses the idea that the direction of the tangent is the same as that of the line (2). Hence, the tangents to (*C*) are the generators of the developable.

We again seek the osculating plane to (*C*) at *M*. It must pass through the tangent and be parallel to the direction (d^2x, d^2y, d^2z). Now, if we differentiate the first of equations (4) then upon taking the second one into account, we will find that:

$$u \cdot d^2x + v \cdot d^2y + w \cdot d^2z = 0,$$

which shows that the plane (1) satisfies the preceding conditions. Hence, the osculating plane to (*C*) will be the plane that envelopes the developable.

(*C*) is called the *edge of regression* of the developable.

Hence:

Any developable is the envelope of the osculating planes to its edge of regression and is generated by the tangents to its edge of regression.

Remarks. – We have implicitly made various hypotheses. First of all, that equations (3) define *x*, *y*, *z*; i.e., that their determinant is not identically zero. If it were then one would have:

$$\begin{vmatrix} u & v & w \\ du & dv & dw \\ d^2u & d^2v & d^2w \end{vmatrix} = 0$$

for any *t*, which would express the idea that *u*, *v*, *w* are coupled by a homogeneous linear relation with constant coefficients; i.e., that the planes (1) are parallel to a fixed line. In that case, the lines (2) would be parallel to that same direction, and the surface would be a *cylinder*. In that case, the singular case would occur in which all of the planes (1) pass through a fixed line, which would then be their envelope.

If we discard that case then we will have assumed that there is a locus of points *M*. That supposes that *M* is not fixed. If that were true then since equations (3) are verified by the coordinates of that fixed point, the planes (1) would pass through that fixed point, as well as the lines (2). The envelope would be a *cone.*

We again discard that case. We assume, moreover, that the lines (2) generate a surface. Now, that will break down only if they coincide, which is the singular case that was examined already.

Finally, we remark that the curve (*C*) is unavoidably skew, since it is plane, and its plane is its unique osculating plane. Our arguments will not cease to apply, so all of the planes (1) will coincide. There will not be ∞^1 planes (1) then.

Converses

11. – *Conversely, the osculating planes at all points of a skew curve will envelop a developable.* – Indeed, if we recall the notations of § 1 then the osculating plane at a point x, y, z of a curve will have the equation:

$$\sum \alpha''(X - x) = 0.$$

Its characteristic is represented by the preceding equations, and:

$$\sum \frac{d\alpha''}{ds}(X - x) - \sum \alpha'' \frac{dx}{ds} = 0.$$

Now:

$$\sum \alpha'' \frac{dx}{ds} = \sum \alpha\alpha'' = 0, \qquad \frac{d\alpha''}{ds} = \frac{\alpha'}{T}.$$

The equations of the characteristic will then be:

$$\sum \alpha''(X - x) = 0, \qquad \sum \alpha'(X - x) = 0.$$

If one takes the Serret-Frenet trihedron to be the coordinate trihedron then they will reduce to:

$$Z = 0, \quad Y = 0.$$

Hence, the characteristic of the osculating plane at a point of a skew curve is the tangent to that curve, and the envelope of that plane will indeed be a developable surface. The edge of regression is defined by the equations:

$$\begin{cases} \sum \alpha''(X - x) = 0, \\ \sum \alpha'(X - x) = 0, \\ \sum \dfrac{d\alpha'}{ds}(X - x) - \sum \alpha' \dfrac{dx}{ds} = 0. \end{cases}$$

Consider the third equation; we remark that:

$$\sum \alpha' \frac{dx}{ds} = \sum \alpha\alpha' = 0$$

and

$$\frac{d\alpha'}{ds} = -\frac{\alpha}{R} - \frac{\alpha''}{T}.$$

That equation will then become:

$$\sum\left(\frac{\alpha}{R}+\frac{\alpha''}{T}\right)(X-x) = 0,$$

or furthermore, upon taking the first equation into account:

$$\sum\alpha\,(X-x) = 0.$$

We then obtain three linear homogeneous equations in $X - x$, $Y - y$, $Z - z$ whose determinant is 1; hence:

$$X - x = 0, \qquad Y - y = 0, \qquad Z - z = 0;$$

i.e., the edge of regression is the curve itself.

Remark. – The name "edge of regression" comes from the fact that the *section of the developable by the normal plane to the edge of regression at M will present a point of regression at the point M*. Indeed, refer the curve to the Serret trihedron that relates to the point M: From the formulas that were established in § 5, the coordinates of a point on the curve that is close to the point M will be:

$$\begin{cases} x = ds - \dfrac{1}{6R^2}ds^3 + \cdots \\ y = \dfrac{1}{2R}ds^2 - \dfrac{1}{6R^2}\dfrac{dR}{ds}ds^3 + \cdots \\ z = -\dfrac{1}{6RT}ds^3 + \cdots \end{cases}$$

The coordinates of a point on the tangent to the point x, y, z are:

$$X = x + \lambda\,\frac{dx}{ds} = \left(ds - \frac{1}{6R^2}ds^3 + \cdots\right) + \lambda\left(1 - \frac{1}{2R^2}ds^2 + \cdots\right),$$

$$Y = y + \lambda\,\frac{dy}{ds} = \left(\frac{1}{2R}ds^2 - \frac{1}{6R^2}\frac{dR}{ds}ds^3 + \cdots\right) + \lambda\left(\frac{1}{R}ds - \frac{1}{2R^2}\frac{dR}{ds}ds^2 + \cdots\right),$$

$$Z = z + \lambda\,\frac{dz}{ds} = \left(-\frac{1}{6RT}ds^3 + \cdots\right) + \lambda\left(-\frac{1}{2RT}ds^2 + \cdots\right).$$

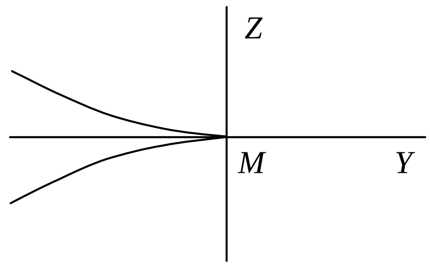
Take the intersection of that tangent with the normal plane $X = 0$, which will give:

$$\lambda = -\frac{ds + \cdots}{1 + \cdots} = -\,ds + \ldots,$$

and the curve of intersection will have the equations:

$$Y = -\frac{1}{2R^2}ds^2 + \ldots, \quad Z = \frac{1}{3RT}ds^3 + \ldots$$

One sees that it has a point of regression at the point M, and the tangent of regression will be the principal normal.



Rectifying surface. Polar surface

12. – *Remarks.* – We seek the developable surfaces that are enveloped by the faces of the Serret trihedron on a skew curve (C). We just saw that *the osculating plane envelopes the developable surface that admits* (C) *for its edge of regression.*

Now consider the rectifying plane:

$$\sum \alpha' (X - x) = 0,$$

whose characteristic is represented by the preceding equation and the equation:

$$\frac{1}{R}\sum \alpha(X - x) + \frac{1}{T}\sum \alpha''(X - x) = 0.$$

If one takes the Serret equations then those equations will become:

$$Y = 0, \qquad \frac{1}{R}X + \frac{1}{T}Z = 0,$$

whose characteristic will contain the point $Y = 0$, $X = -1 / T$, $Z = 1 / R$, which is the extremity of the vector that represents the instantaneous rotation of the trihedron. *It is the instantaneous axis of rotation of the Serret trihedron.* Its locus is called the *rectifying surface.* It contains the curve (C).

Finally, consider the normal plane:

$$\sum \alpha (X - x) = 0,$$

and the other equation of the characteristic is:

$$\sum \frac{d\alpha}{ds}(X - x) - \sum \alpha \frac{dx}{ds} = 0,$$

or:

$$\frac{1}{R}\sum \alpha' (X - x) - 1 = 0.$$

That characteristic is called the *polar line*, and its locus is called the *polar surface.*

Once more, take the Serret axes, so the equations of the polar line will become:

$$X = 0, \qquad Y = R.$$

The polar line is then *the axis of the osculating circle.*

The point of contact of the polar line with the edge of regression of the polar surface is given by the three equations:

$$\sum \alpha (X - x) = 0,$$
$$\sum \alpha' (X - x) - R = 0,$$

$$\sum \frac{d\alpha'}{ds}(X - x) - \sum \alpha' \frac{dx}{ds} - \frac{dR}{ds} = 0.$$

Upon taking the first one into account, the last one will become:

$$\frac{1}{T}\sum \alpha''(X - x) + \frac{dR}{ds} = 0.$$

Upon taking the Serret axes, one will then get:

$$X = 0, \quad Y = R, \quad Z = - T\frac{dR}{ds}.$$

Those are the coordinates of the center of the osculating sphere (see § 9).

Therefore:

The point where the polar line touches its envelope is the center of the osculating sphere of the curve (*C*). *The curve* (*C*) *is the orthogonal trajectory of the osculating planes at the locus of the centers of its osculating spheres.*

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CHAPTER II

SURFACES

The ds^2 of the surface and angles

1. – *Curves traced on a surface. Arc-lengths and angles.* – Let (*S*) be a surface, and suppose that the coordinates of a running point are expressed as functions of two parameters *u*, *v*:

$$(S) \qquad x = f(u, v), \qquad y = g(u, v), \qquad z = h(u, v).$$

u, *v* are the *curvilinear coordinates* of a point of the surface (*S*). One defines a curve (*C*) on the surface by establishing a relation between *u*, *v*, or – what amounts to the same thing – by expressing *u*, *v* as functions of the same parameter *t*:

$$(C) \qquad u = \varphi(t), \qquad v = \psi(t).$$

The tangent to that curve will have the direction parameters:

$$(1) \qquad dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv, \quad dy = \frac{\partial y}{\partial u}du + \frac{\partial y}{\partial v}dv, \quad dz = \frac{\partial z}{\partial u}du + \frac{\partial z}{\partial v}dv.$$

The tangent will then be determined by the differentials *du*, *dv*.

The element of arc-length has the expression:

$$(2) \qquad ds^2 = dx^2 + dy^2 + dz^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2 = \Phi(du, dv)$$

upon setting:

$$E = \sum\left(\frac{\partial x}{\partial u}\right)^2, \qquad F = \sum\frac{\partial x}{\partial u}\frac{\partial x}{\partial v}, \qquad G = \sum\left(\frac{\partial x}{\partial v}\right)^2.$$

Consider two curves that pass through the same point (*u*, *v*) on the surface. Let *du*, *dv* be the differentials that correspond to one of them, and let δu, δv be the differentials that correspond to the other, so *ds*, δs are the corresponding differentials of arc-lengths. If *V* is the angle between the two curves then we know that:

$$\cos V = \frac{\sum dx \cdot \delta x}{ds \cdot \delta s}.$$

Now:

$$\sum dx \cdot \delta x = \sum\left(\frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv\right)\left(\frac{\partial x}{\partial u}\delta u + \frac{\partial x}{\partial v}\delta v\right) = E\,du\,du + F\,(du\,\delta v + dv\,\delta u) + G\,dv\,\delta v.$$

That is the polar form of the quadratic form $\Phi(du, dv)$, and:

(3) $$\cos V = \frac{1}{2}\frac{\delta u \dfrac{\partial \Phi(du, dv)}{\partial \cdot du} + \delta v \dfrac{\partial \Phi(du, dv)}{\partial \cdot dv}}{\sqrt{\Phi(du, dv)\Phi(\delta u, \delta v)}}.$$

In order for the two curves to be orthogonal, it is necessary and sufficient that $\cos V = 0$ or:

(4) $$E\, du\, dv + F\,(du \cdot \delta v + dv \cdot \delta u) + G\, dv \cdot \delta v = 0.$$

In particular, we seek the condition for the *coordinate curves* u = const. and v = const. to form an *orthogonal net.* We would then have $dv = 0$, $\delta u = 0$, and the preceding condition would reduce to the identity:

$$F\, du\, \delta v = 0,$$

or since du, δv are not constantly zero, $F = 0$. In that case, the square of the arc-length element would take the characteristic form:

$$ds^2 = E\, du^2 + G\, dv^2.$$

Remark. – If one defines the surface by an equation of the form:

$$z = f(x, y)$$

then upon denoting the partial derivatives of z with respect to x, y by p, q, resp., as usual, one will have:

$$ds^2 = dx^2 + dy^2 + (p\, dx + q\, dy)^2 = (1 + p^2)\, dx^2 + 2pq\, dx\, dy + (1 + q^2)\, dy^2;$$

i.e.:

$$E = 1 + p^2, \qquad F = pq, \qquad G = 1 + q^2.$$

Deformation and conformal representation

2. – *Mappable surfaces. Conformal representations.* – Consider two surfaces (S), (S_1):

(S) $\quad x = f(u, v), \quad y = g(u, v), \quad z = h(u, v),$

(S_1) $\quad x = f_0(u_1, v_1), \quad y = g_0(u_1, v_1), \quad z = h_0(u_1, v_1).$

One can establish a point-to-point correspondence between those two surfaces, and in an infinitude of ways. It suffices to set:

$$u_1 = \varphi(u, v), \quad v_1 = \psi(u, v).$$

The functions are arbitrary φ, ψ, but always under the condition that the preceding equations must be soluble for u, v. The equations of the surface (S_1) will then have the form:

(S_1) $$x = f_1(u, v), \quad y = g_1(u, v), \quad z = h_1(u, v).$$

That amounts to saying that the homologous points correspond to the same systems of values for the parameters.

Now let the elements of arc-length on those two surfaces be:

(1) $$ds^2 = E\ du^2 + 2F\ du \cdot dv + G\ dv^2,$$

(2) $$ds_1^2 = E_1\, du^2 + 2F_1\, du \cdot dv + G_1\, dv^2.$$

Suppose that these elements of arc-length are identical $E \equiv E_1$, $F \equiv F_1$, $G \equiv G_1$. If u, v are expressed as functions of one parameter t then the arc-lengths of the two corresponding curves on the two surfaces that are comprised by the corresponding points will both be expressed by:

$$\int_{t_0}^{t_1} \sqrt{E\,du^2 + 2F du\,dv + C\,dv^2}\ ,$$

in which t_0, t_1 are the values of t that corresponds to the extremities. Conversely, if two arbitrary homologous arcs of two arbitrary homologous curves that are traced on the two surfaces have the same length then the arc-length elements (1) and (2) will be identical when one replaces u and v in them with arbitrary functions of t, and in turn, will be identical in u, v, du, dv. One then says that the two surfaces are *mappable* to each other, or that they can be deduced from each other by *deformation.*

Under that correspondence, the function Φ will be the same for both surfaces, so formula (3) from § 1 will show that the angles are preserved. However, the converse is not true. The expression for $\cos V$ is homogeneous and of degree zero in E, F, G. For the angles between the two arbitrary homologous curves to be equal, it is necessary and sufficient that one must have:

$$\frac{E}{E_1} = \frac{F}{F_1} = \frac{G}{G_1} = \chi(u, v),$$

and that ratio must be independent of du, dv, δu, δv. In that case, one says that there is a *conformal representation* of the two surfaces on each other.

The problem of conformal representation

If one is given two surfaces then it will always be possible to establish a conformal representation between them. That amounts to saying that one can express u_1, v_1 as functions of u, v, in such a way that:

$$E\,du^2 + 2F\,du\,dv + G\,dv^2 \equiv \chi(u, v)\,(E_1\,ds^2 + 2\,F_1\,du\,dv + G_1\,dv^2).$$

Decompose the two ds^2 into first-degree factors. Note that $EG - F^2$ is the sum of the squares of the determinants that are deduced from the matrix:

$$\begin{Vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \end{Vmatrix}.$$

$EG - F^2$ is positive for any real surface. Set:

$$EG - F^2 = H^2,$$

so:

$$ds^2 = E\left(du + \frac{F + iH}{E}dv\right)\left(du + \frac{F - iH}{E}dv\right).$$

Each of the factors on the right-hand side admits an integrating factor, so:

$$du + \frac{F + iH}{E}dv \equiv M(u, v)\, d\alpha(u, v),$$
$$du + \frac{F - iH}{E}dv \equiv N(u, v)\, d\beta(u, v).$$

The functions α, β are independent. Indeed, if $H \neq 0$ then $d\alpha$ and $d\beta$ cannot both be zero, so we assume that condition is fulfilled. We can then take α, β to be curvilinear coordinates on the first surface, and we will have [cf., Chap. III, § 4]:

$$ds^2 = P(u, v)\, d\alpha \cdot d\beta = \Theta(\alpha, \beta)\, d\alpha \cdot d\beta.$$

Likewise, for the second surface:

$$ds_1^2 = P_1(u_1, v_1)\, d\alpha_1 \cdot d\beta_1 = \Theta_1(\alpha_1, \beta_1)\, d\alpha_1 \cdot d\beta_1,$$

in which α_1, β_1 are two independent functions of the u_1, v_1.

We will then have to satisfy the identity:

$$\Theta(\alpha, \beta)\, d\alpha \cdot d\beta \equiv \Omega(\alpha, \beta)\, \Theta_1(\alpha_1, \beta_1)\, d\alpha_1 \cdot d\beta_1,$$

in which Ω, α_1, β_1 are unknown functions of α, β.

Hence, for $d\alpha = 0$, one must have $d\alpha_1 \cdot d\beta_1 = 0$. If we take $d\alpha_1 = 0$ then α_1 will be a function of α, and similarly β_1 will be a function of β:

$$\alpha_1(u_1, v_1) = \varphi(\alpha(u, v)), \qquad \beta_1(u_1, v_1) = \psi(\beta(u, v)).$$

Upon taking $\beta_1 = 0$, β_1 will be a function of α, and similarly α_1 will be a function of β:

$$\beta_1(u_1, v_1) = \varphi(\alpha(u, v)), \qquad \alpha_1(u_1, v_1) = \psi(\beta(u, v)).$$

One then sees that one can always establish a conformal representation, because in the two cases, for any functions φ and ψ, $\Theta_1(\alpha_1, \beta_1)\, d\alpha_1 \cdot d\beta_1$ will indeed be proportional to $\Theta(\alpha, \beta)\, d\alpha \cdot d\beta$, and we will have the general solution to the problem, moreover, since the functions φ and ψ are arbitrary.

Condition for two surfaces to be mappable

Two given surfaces cannot be mapped to each other, in general. – In other words, if one is given two surfaces then it is generally impossible to establish a correspondence between them such that $ds^2 = ds_1^2$. Indeed, if one repeats the preceding calculation then it will be necessary to satisfy the relation:

$$\Theta(\alpha, \beta)\, d\alpha \cdot d\beta \equiv \Theta_1(\alpha_1, \beta_1)\, d\alpha_1 \cdot d\beta_1 .$$

As before, one must take, for example:

$$\alpha_1 = \varphi(\alpha), \qquad \beta_1 = \psi(\beta),$$

and the relation to be verified will become:

$$\Theta(\alpha, \beta) \equiv \Theta_1(\varphi(\alpha), \psi(\beta))\, \varphi'(\alpha), \psi'(\beta).$$

It is easy to see that if the functions Θ, Θ_1 are given then it will be impossible, in general, to find functions φ, ψ that satisfy that relation. Indeed, consider the particular case in which the second surface is the plane $z = 0$. In that case $ds_1^2 = dx^2 + dy^2 = d\alpha_1 \cdot d\beta_1$, and one must have:

$$\Theta(\alpha, \beta) = \varphi'(\alpha)\, \psi'(\beta) .$$

When the function Θ is arbitrary, it will not be the product of a function of α with a function of β.

In order for that to be true, it is necessary and sufficient that one must have:

$$\log \Theta(\alpha, \beta) = \log \varphi'(\alpha) + \log \varphi'(\beta)$$

or

$$\frac{\partial^2 \log \Theta(\alpha, \beta)}{\partial\alpha \cdot \partial\beta} = 0.$$

One might just as well show that a surface is not, in general, mappable to a plane and find a necessary and sufficient condition for a surface to be mappable to a plane. We shall return to that later on (Chap. IV, § 3).

Conjugate directions and the form $\sum l\, d^2 x$

3. – *Circumscribed developables. Conjugate directions.* – Correlative to the curves that are traced on a surface, which are loci of ∞^1 points on the surface, we consider the circumscribed developables, which are envelopes of ∞^1 planes tangent to the surface. Define the tangent plane to a point of the surface. Let l, m, n be the direction coefficients of the normal, and suppose that the coordinates are rectangular. For any curve on the surface:

$$l\,dx + m\,dy + n\,dz = 0.$$

In particular, for the coordinate curves u = const., v = const., we will have:

$$l\frac{\partial x}{\partial u} + m\frac{\partial y}{\partial u} + n\frac{\partial z}{\partial u} = 0,$$

$$l\frac{\partial x}{\partial v} + m\frac{\partial y}{\partial v} + n\frac{\partial z}{\partial v} = 0,$$

and those relations will show that l, m, n are proportional to the functional determinants A, B, C:

$$A = \frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial u} = \frac{D(y,z)}{D(u,v)}, \qquad B = \frac{D(z,x)}{D(u,v)}, \quad C = \frac{D(x,y)}{D(u,v)}. \tag{1}$$

Moreover, we have that:

$$A^2 + B^2 + C^2 = H^2.$$

Hence, the direction cosines of the normal will be:

$$\lambda = \frac{A}{H}, \qquad \mu = \frac{B}{H}, \qquad \nu = \frac{C}{H}, \tag{2}$$

in which the positive direction thus-defined will depend upon the sign that is adopted for H.

Consider a circumscribed developable. We define that by expressing u, v as a function of one parameter t:

$$u = \varphi(t), \qquad v = \psi(t).$$

The point (u, v) will then describe a curve (C) on the surface, and the planes tangent to the surface at the various points of (C) will envelop the developable in question. If X, Y, Z are the running coordinates then the tangent plane to the surface at the point (x, y, z) will be:

$$l \cdot (X - x) + m \cdot (Y - y) + n \cdot (Z - z) = 0.$$

The *characteristic* is defined by the preceding equation and the equation:

$$dl \cdot (X - x) + dm \cdot (Y - y) + dn \cdot (Z - z) = 0,$$

which is obtained by differentiating the preceding one with respect to t and remarking that:

$$l\,dx + m\,dy + n\,dz = 0.$$

Let us see what the direction of that characteristic is. Let δx, δy, δz be its direction coefficients. It is tangent to the surface, so one can choose δu, δv in such a manner that:

$$\begin{cases} \delta x = \dfrac{\partial x}{\partial u}\delta u + \dfrac{\partial x}{\partial v}\delta v, \\ \delta y = \dfrac{\partial y}{\partial u}\delta u + \dfrac{\partial y}{\partial v}\delta v, \\ \delta z = \dfrac{\partial z}{\partial u}\delta u + \dfrac{\partial z}{\partial v}\delta v, \end{cases}$$

and upon replacing $X - x$, $Y - y$, $Z - z$ with the proportional quantities δx, δy, δz, we will get:

$$dl \cdot dx + dm \cdot dy + dn \cdot dz = 0.$$

Now:

$$\begin{cases} dl = \dfrac{\partial l}{\partial u}du + \dfrac{\partial l}{\partial v}dv, \\ dm = \dfrac{\partial m}{\partial u}du + \dfrac{\partial m}{\partial v}dv, \\ dn = \dfrac{\partial n}{\partial u}du + \dfrac{\partial n}{\partial v}dv, \end{cases}$$

so the preceding relation can be written:

$$\sum\left(\frac{\partial l}{\partial u}du + \frac{\partial l}{\partial v}dv\right)\left(\frac{\partial x}{\partial u}\delta u + \frac{\partial x}{\partial v}\delta v\right) = 0.$$

Arrange this with respect to the du, dv, δu, δv. Note that:

$$\sum l\frac{\partial x}{\partial u} = 0.$$

Hence, upon differentiating with respect to u and v, we will get:

$$\sum l\frac{\partial^2 x}{\partial u^2} + \sum\frac{\partial l}{\partial u}\frac{\partial x}{\partial u} = 0, \qquad \sum l\frac{\partial^2 x}{\partial u\,\partial v} + \sum\frac{\partial l}{\partial v}\frac{\partial x}{\partial u} = 0.$$

Similarly, the relation:

$$\sum l \frac{\partial x}{\partial v} = 0$$

will give:

$$\sum l \frac{\partial^2 x}{\partial v^2} + \sum \frac{\partial l}{\partial v}\frac{\partial x}{\partial v} = 0$$

and:

$$\sum l \frac{\partial^2 x}{\partial u\,\partial v} + \sum \frac{\partial l}{\partial u}\frac{\partial x}{\partial v} = 0,$$

in such a way that the desired relation can be written as:

$$\text{(3)} \qquad \sum l \frac{\partial^2 x}{\partial u^2} du \cdot \delta u + \sum l \frac{\partial^2 x}{\partial u\,\partial v}(du \cdot \delta v + dv \cdot \delta u) + \sum l \frac{\partial^2 x}{\partial v^2} dv \cdot \delta v = 0.$$

That is the relation that exists between the direction coefficients of the characteristic and the tangent to the contact curve. It will obviously be just as clear in oblique coordinates, when *l*, *m*, *n* are then the coefficients of the equation of the tangent plane. Set:

$$\text{(4)} \qquad E' = \sum l \frac{\partial^2 x}{\partial u^2}, \qquad F' = \sum l \frac{\partial^2 x}{\partial u\,\partial v}, \qquad G' = \sum l \frac{\partial^2 x}{\partial v^2},$$

and

$$\text{(5)} \qquad \Psi\,(du, dv) = E'du^2 + 2F'du\,dv + G'dv^2 = 0.$$

With those notations, the relation that was found can be written:

$$E' \cdot du\,\delta u + 2F' \cdot (du\,\delta v + dv\,\delta u) + G' \cdot dv\,\delta v = 0,$$

or:

$$\text{(6)} \qquad \frac{\partial \Psi(du \cdot dv)}{\partial du}\delta u + \frac{\partial \Psi(du \cdot dv)}{\partial dv}\delta v = 0.$$

That relation, whose left-hand side is the polar form of the form Ψ, is symmetric with respect to d, δ. *There is reciprocity then between the direction of the tangent to the contact curve of the developable and the direction of the characteristic of the tangent plane to that developable.* Those two directions are called *conjugate directions.*

In particular, we seek the condition for the curves u = const., v = const. to form a *conjugate net*; i.e., for their tangents to have conjugate directions at each point of the surface. One would then have $dv = 0$, $du = 0$, so the condition is that one must have the identity $F' = 0$.

Remark 1. – From the relation:

$$dx = \frac{\partial x}{\partial u}du + \frac{\partial x}{\partial v}dv,$$

one will infer that:

$$d^2x = \frac{\partial x}{\partial u} d^2u + \frac{\partial x}{\partial v} d^2v + \frac{\partial^2 x}{\partial u^2} du^2 + 2\frac{\partial^2 x}{\partial u\,\partial v} du\,dv + \frac{\partial^2 x}{\partial v^2} dv^2 .$$

On the other hand:

$$\sum l \frac{\partial x}{\partial u} = 0, \qquad \sum l \frac{\partial x}{\partial v} = 0.$$

One concludes from this that:

$$\sum l\, d^2x = \sum l \frac{\partial^2 x}{\partial u^2} du^2 + 2\sum l \frac{\partial^2 x}{\partial u\,\partial v} du\,dv + \sum l \frac{\partial^2 x}{\partial v^2} dv^2 ;$$

i.e.:

$$\Psi\,(du, dv) \equiv \sum l\, d^2x .$$

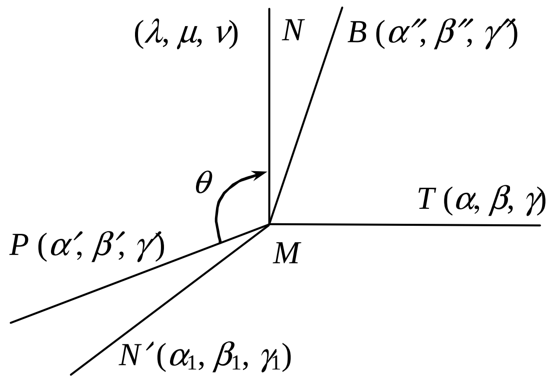
Remark 2. – In particular, if one takes $l = A$, $m = B$, $n = C$ then the form Ψ will be identical to $\sum A\, d^2x$, and its coefficients can be written in the form of determinants:

$$E' = \sum A \frac{\partial^2 x}{\partial u^2} = \begin{vmatrix} \frac{\partial^2 x}{\partial u^2} & \frac{\partial^2 y}{\partial u^2} & \frac{\partial^2 z}{\partial u^2} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}, \qquad F' = \sum A \frac{\partial^2 x}{\partial u\,\partial v} = \begin{vmatrix} \frac{\partial^2 x}{\partial u\,\partial v} & \frac{\partial^2 y}{\partial u\,\partial v} & \frac{\partial^2 z}{\partial u\,\partial v} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix},$$

$$G' = \sum A \frac{\partial^2 x}{\partial v^2} = \begin{vmatrix} \frac{\partial^2 x}{\partial v^2} & \frac{\partial^2 y}{\partial v^2} & \frac{\partial^2 z}{\partial v^2} \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix}.$$

Fundamental formulas that relate to a curve traced on a surface

4. – *Fundamental elements of a curve on a surface.* – We consider the Serret trihedron at a point on the curve and a trihedron that is composed of the tangent to the curve, the normal *MN* to the surface, and the tangent *MN′* to the surface that is normal to the curve. We choose the positive direction in such a fashion that the trihedron $M \cdot TN'N$ thus-constituted will have the same disposition as the coordinate trihedron, in such a way that if *l*, *m*, *n* are the direction cosines of the normal to the surface, and α_1, β_1, γ_1 are those of the tangent to the normal surface to the curve then one will have:



$$\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha_1 & \beta_1 & \gamma_1 \\ \lambda & \mu & \nu \end{vmatrix} = 1.$$

The two trihedra considered have a common axis and the same direction, which is that of the tangent. In order to define one of them in terms of the other, it will suffice to give the angle between one of the edges of one trihedron and one of the edges of the other. We give the angle $\theta = (MP, MN)$ through which one must turn the principal semi-normal MP on order to make it coincide with the semi-normal MN to the surface, and the positive sense of rotation is defined by the positive direction MT of the axis of rotation.

We seek the relations that exist between the direction cosines of the edges of those trihedra. When one passes from one to the other, in reality, one performs a coordinate transformation around the origin in the normal plane. Consider the unity point at a distance M along MN; its coordinates are λ, μ, ν. When referred to the system PMB, it will have coordinates cos θ and sin θ, so:

(1)
$$\begin{cases} \lambda = \alpha' \cos\theta + \alpha'' \sin\theta, \\ \mu = \beta' \cos\theta + \beta'' \sin\theta, \\ \nu = \gamma' \cos\theta + \gamma'' \sin\theta. \end{cases}$$

Similarly, the unity point at a distance on MN whose coordinates are α_1, β_1, γ_1 when referred to the system PMB will have coordinates cos $\left(\theta - \frac{\pi}{2}\right)$ = sin θ and sin $\left(\theta - \frac{\pi}{2}\right)$ = – cos θ, so:

(1)[*sic*]
$$\begin{cases} \alpha_1 = \alpha' \sin\theta - \alpha'' \cos\theta, \\ \beta_1 = \beta' \sin\theta - \beta'' \cos\theta, \\ \gamma_1 = \gamma' \sin\theta - \gamma'' \cos\theta. \end{cases}$$

Therefore, once more, upon performing the inverse coordinate transformation:

(2)
$$\begin{cases} \alpha' = \lambda\cos\theta + \alpha_1\sin\theta, \\ \beta' = \mu\cos\theta + \beta_1\sin\theta, \\ \gamma' = \nu\cos\theta + \gamma_1\sin\theta, \\ \alpha'' = \lambda\sin\theta - \alpha_1\cos\theta, \\ \beta'' = \mu\sin\theta - \beta_1\cos\theta, \\ \gamma'' = \nu\sin\theta - \gamma_1\cos\theta. \end{cases}$$

Differentiate formula (1) with respect to *s*; we will get:

$$\frac{d\lambda}{ds} = (-\alpha'\sin\theta + \alpha''\cos\theta)\frac{d\theta}{ds} + \cos\theta\,\frac{d\alpha'}{ds} + \sin\theta\,\frac{d\alpha''}{ds}$$

and its analogues;

$$\frac{d\alpha_1}{ds} = (\alpha'\cos\theta + \alpha''\sin\theta)\frac{d\theta}{ds} + \sin\theta\,\frac{d\alpha'}{ds} - \cos\theta\,\frac{d\alpha''}{ds}$$

and its analogues. Hence, upon taking the Frenet formulas and relations (1), (2) into account:

(3)
$$\frac{d\lambda}{ds} = \alpha_1\left(\frac{1}{T} - \frac{d\theta}{ds}\right) - \alpha\frac{\cos\theta}{R}$$

and its analogues; similarly:

(4)
$$\frac{d\alpha_1}{ds} = -\lambda\left(\frac{1}{T} - \frac{d\theta}{ds}\right) - \alpha\frac{\sin\theta}{R}$$

and its analogues. Finally:

(5)
$$\frac{d\alpha}{ds} = \frac{\alpha'}{R} = \lambda\frac{\cos\theta}{R} + \alpha_1\frac{\sin\theta}{R}$$

and its analogues.

The fundamental formulas (3), (4), (5) *permit one to calculate θ, R, T; i.e., to determine the osculating plane, curvature, and the torsion of the curve considered.*

Calculation of $\frac{\cos\theta}{R}$

The formulas (5) first give us:

$$\frac{\cos\theta}{R} = \sum\lambda\frac{d\alpha}{ds} = \sum\lambda\frac{d}{ds}\frac{d\alpha}{ds} = \sum\lambda\frac{ds\,d^2x - dx\,d^2s}{ds^2} = \frac{\sum\lambda\,d^2x}{ds^2} = \frac{\sum A\,d^2x}{H\,ds^2}.$$

Hence, from the preceding calculation, and upon setting, as we did at the end of that paragraph:

$$E' = \sum A \frac{\partial^2 x}{\partial u^2}, \qquad F' = \sum A \frac{\partial^2 x}{\partial u\, \partial v}, \qquad G' = \sum A \frac{\partial^2 x}{\partial v^2},$$

we will get:

$$\frac{\cos\theta}{R} = \frac{1}{H} \frac{E' \cdot du^2 + 2F' \cdot du\, dv + G' dv^2}{ds^2},$$

or finally:

$$\frac{\cos\theta}{R} = \frac{1}{H} \cdot \frac{\Psi(du, dv)}{\Phi(du, dv)}. \tag{6}$$

Calculation of $\frac{\sin\theta}{R}$

Formulas (5) again give:

$$\frac{\sin\theta}{R} = \sum \alpha_1 \frac{d\alpha}{ds} = \sum \alpha_1 \frac{d}{ds} \frac{dx}{ds} = \sum \alpha_1 \frac{ds\, d^2 x - dx\, d^2 s}{ds^2} = \frac{\sum \alpha_1 d^2 x}{ds^2}.$$

Note that:

$$\frac{\sum \alpha_1 d^2 x}{ds^2} = \frac{1}{ds^2} \begin{vmatrix} \alpha & \beta & \gamma \\ d^2 x & d^2 y & d^2 z \\ \lambda & \mu & \nu \end{vmatrix} = \frac{1}{ds^2} \begin{vmatrix} dx & dy & dz \\ d^2 x & d^2 y & d^2 z \\ \lambda & \mu & \nu \end{vmatrix}.$$

In order to calculate the last determinant, multiply it by:

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ \lambda & \mu & \nu \end{vmatrix} = A\,\lambda + B\,\mu + C\,\nu = \frac{A^2 + B^2 + C^2}{H} = H.$$

The product is:

$$\begin{vmatrix} \sum \frac{\partial x}{\partial u} dx & \sum \frac{\partial x}{\partial v} dx & \sum \lambda\, dx \\ \sum \frac{\partial x}{\partial u} d^2 x & \sum \frac{\partial x}{\partial v} d^2 x & \sum \lambda\, d^2 x \\ \sum \lambda \frac{\partial x}{\partial u} & \sum \lambda \frac{\partial x}{\partial v} & \sum \lambda^2 \end{vmatrix} = \begin{vmatrix} \sum \frac{\partial x}{\partial u} dx & \sum \frac{\partial x}{\partial v} dx & 0 \\ \sum \frac{\partial x}{\partial u} d^2 x & \sum \frac{\partial x}{\partial v} d^2 x & \sum \lambda\, d^2 x \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} \sum \dfrac{\partial x}{\partial u} dx & \sum \dfrac{\partial x}{\partial v} dx \\ \sum \dfrac{\partial x}{\partial u} d^2 x & \sum \dfrac{\partial x}{\partial v} d^2 x \end{vmatrix}.$$

Now:

$$\sum \frac{\partial x}{\partial u} \cdot dx = \sum \frac{\partial x}{\partial u} \cdot \left(\frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv \right) = E\, du + F\, dv,$$

$$\sum \frac{\partial x}{\partial v} \cdot dx = \sum \frac{\partial x}{\partial v} \cdot \left(\frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv \right) = F\, du + G\, dv,$$

and

$$\sum \frac{\partial x}{\partial u} d^2 x = \sum \frac{\partial x}{\partial u} \cdot \left(\frac{\partial x}{\partial u} d^2 u + \frac{\partial x}{\partial v} d^2 v + \frac{\partial^2 x}{\partial u^2} du^2 + 2 \frac{\partial^2 x}{\partial u\, \partial v} du\, dv + \frac{\partial^2 x}{\partial v^2} dv^2 \right)$$

$$= E\, d^2 x + F\, d^2 x + \frac{1}{2} \frac{\partial E}{\partial u} du^2 + \frac{\partial E}{\partial v} du\, dv + \left(\frac{\partial F}{\partial v} - \frac{1}{2} \frac{\partial G}{\partial u} \right) dv^2,$$

$$\sum \frac{\partial x}{\partial v} d^2 x = \sum \frac{\partial x}{\partial v} \cdot \left(\frac{\partial x}{\partial u} d^2 u + \frac{\partial x}{\partial v} d^2 v + \frac{\partial^2 x}{\partial u^2} du^2 + 2 \frac{\partial^2 x}{\partial u\, \partial v} du\, dv + \frac{\partial^2 x}{\partial v^2} dv^2 \right)$$

$$= F\, d^2 x + G\, d^2 x + \left(\frac{\partial F}{\partial u} - \frac{1}{2} \frac{\partial E}{\partial v} \right) du^2 + \frac{\partial G}{\partial u} du\, dv + \frac{1}{2} \frac{\partial G}{\partial v} dv^2 .$$

The preceding product will then be written:

$$- \begin{vmatrix} E\, d^2 u + F\, d^2 v + \dfrac{1}{2} \dfrac{\partial E}{\partial u} du^2 + \dfrac{\partial E}{\partial v} du\, dv + \left(\dfrac{\partial F}{\partial v} - \dfrac{1}{2} \dfrac{\partial G}{\partial u} \right) dv^2 & E\, du + F\, dv \\ F\, d^2 u + G\, d^2 v + \left(\dfrac{\partial F}{\partial u} - \dfrac{1}{2} \dfrac{\partial E}{\partial v} \right) du^2 + \dfrac{\partial G}{\partial u} du\, dv + \dfrac{1}{2} \dfrac{\partial G}{\partial v} dv^2 & F\, du + G\, dv \end{vmatrix}.$$

The determinant is the sum of two determinants, the first of which is:

$$- \begin{vmatrix} E\, d^2 u + F\, d^2 v & E\, du + F\, dv \\ F\, d^2 u + G\, d^2 v & F\, du + G\, dv \end{vmatrix} = H^2\, (du \cdot d^2 v - dv \cdot d^2 u),$$

and finally:

(7)
$$\frac{\sin \theta}{R} =$$

$$\frac{1}{H\, ds^2} \left[H^2 (du\, d^2 v - dv\, d^2 u) - \begin{vmatrix} \dfrac{1}{2} \dfrac{\partial E}{\partial u} du^2 + \dfrac{\partial E}{\partial v} du\, dv + \left(\dfrac{\partial F}{\partial v} - \dfrac{1}{2} \dfrac{\partial G}{\partial u} \right) dv^2 & E\, du + F\, dv \\ \left(\dfrac{\partial F}{\partial u} - \dfrac{1}{2} \dfrac{\partial E}{\partial v} \right) du^2 + \dfrac{\partial G}{\partial u} du\, dv + \dfrac{1}{2} \dfrac{\partial G}{\partial v} dv^2 & F\, du + G\, dv \end{vmatrix} \right].$$

Calculation of $\frac{1}{T} - \frac{d\theta}{ds}$

Finally, formula (4) gives us:

$$\frac{1}{T} - \frac{d\theta}{ds} = \sum \alpha_1 \frac{d\lambda}{ds} = \frac{1}{ds}\begin{vmatrix} \alpha & \beta & \gamma \\ d\lambda & d\mu & d\nu \\ \lambda & \mu & \nu \end{vmatrix} = \frac{1}{ds^2}\begin{vmatrix} dx & dy & dz \\ d\lambda & d\mu & d\nu \\ \lambda & \mu & \nu \end{vmatrix}.$$

In order to calculate the determinant, we again multiply it by the same determinant H. The product will be:

$$\begin{vmatrix} \sum \frac{\partial x}{\partial u} dx & \sum \frac{\partial x}{\partial v} dx & \sum \lambda\, dx \\ \sum \frac{\partial x}{\partial u} d\lambda & \sum \frac{\partial x}{\partial v} d\lambda & \sum \lambda\, d\lambda \\ \sum \lambda \frac{\partial x}{\partial u} & \sum \lambda \frac{\partial x}{\partial v} & \sum \lambda^2 \end{vmatrix} = \begin{vmatrix} E\,du + F\,dv & F\,du + G\,dv & 0 \\ \sum \frac{\partial x}{\partial u} d\lambda & \sum \frac{\partial x}{\partial v} d\lambda & 0 \\ 0 & 0 & 1 \end{vmatrix}.$$

Moreover, one will infer that:

$$\sum \lambda \frac{\partial x}{\partial u} = 0.$$

Upon differentiating:

$$\sum d\lambda \frac{\partial x}{\partial u} = -\sum \lambda \left(\frac{\partial^2 x}{\partial u^2} du + \frac{\partial^2 x}{\partial u\, \partial v} dv \right) = -\frac{1}{H}(E'du + F'dv)\ ;$$

similarly:

$$\sum d\lambda \frac{\partial x}{\partial v} = -\frac{1}{H}(F'du + G'dv)\ .$$

The product is then:

$$-\frac{1}{H}\begin{vmatrix} E\,du + F\,dv & F\,du + G\,dv \\ E'\,du + F'\,dv & F'\,du + G'\,dv \end{vmatrix}$$

and

$$\text{(8)} \qquad \frac{1}{T} - \frac{d\theta}{ds} = \frac{1}{H^2 ds^2}\begin{vmatrix} E'\,du + F'\,dv & E\,du + F\,dv \\ F'\,du + G'\,dv & F\,du + G\,dv \end{vmatrix}.$$

The three formulas (6), (7), (8) permit one calculate the three fundamental elements θ, R, T.

Kinematic interpretation

The auxiliary elements:

$$-\sum \alpha_1 \frac{d\lambda}{ds} = \frac{d\theta}{ds} - \frac{1}{T}, \qquad -\sum \lambda \frac{d\alpha}{ds} = \frac{\cos\theta}{R}, \qquad \sum \alpha_1 \frac{d\alpha}{ds} = \frac{\sin\theta}{R}$$

offer themselves up as the components of the instantaneous rotation of the trihedron $M \cdot TN'N$ around MT, MN', MN, resp., when the point M describes the curve (C) with the velocity + 1.

Along with that trihedron, consider the tri-rectangular trihedron that was introduced by Darboux:

Let MO be a direction of the tangent plane that is chosen independently of any curve (C) at each point M (u, v) of the surface according to a rule that is arbitrary, but continuous, and let MO' be the direction of the tangent plane that, along with MO and the normal MN, defines a tri-rectangular trihedron $M \cdot OO'N$ that has the same disposition as the coordinate trihedron. That is the trihedron that we shall consider.

Since the direction cosines λ_0, μ_0, ν_0 of MO and λ_0', μ_0', ν_0' of MO' are functions of (u, v), the projection of the instantaneous rotation of that trihedron along MN when M describes the curve (C) with the velocity + 1 will have the form:

$$\sum \lambda_0' \frac{d\lambda_0}{ds} = \frac{r\,du + r_1\,dv}{ds},$$

in which r and r_1 are functions of u, v.

Now, if one lets φ_0 denote the angle (MO, MT) whose magnitude and sign are evaluated in the oriented tangent plane through MN then the instantaneous relative motion of the trihedron $M \cdot TN'N$ with respect to $M \cdot OO'N$ will be a rotation that is represented by a vector whose algebraic value is $d\varphi_0 / ds$ and is carried along MN. That vector is the geometric difference of the ones that represent instantaneous rotations of the two trihedra. Upon projecting that equipollence onto MN, one will then have:

$$\frac{d\varphi_0}{ds} = \frac{\sin\theta}{R} - \frac{r\,du + r_1 dv}{ds},$$

which one can write:

(9) $$\frac{\sin\theta}{R} ds - d\varphi_0 = r\,du + r_1\,dv.$$

The geometric element $\left(\frac{\sin\theta}{R} ds - d\varphi_0\right)$ *is a linear form in du, dv then.*

It will be simple to calculate that linear form upon specifying the choice of the auxiliary direction from the origin MO (cf., Chap. IV, § 5).

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CHAPTER III

STUDY OF THE FUNDAMENTAL ELEMENTS OF CURVES ON A SURFACE

Normal curvature

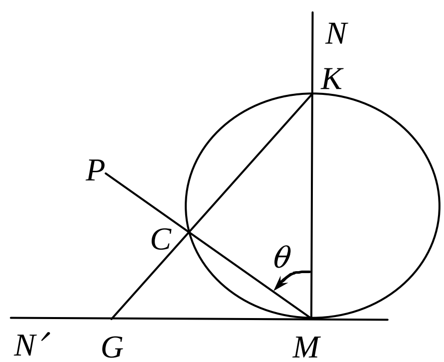
1. – Recall the first fundamental formula:

$$\frac{\cos\theta}{R} = \frac{1}{H}\frac{E' du^2 + 2F' du\,dv + G' dv^2}{E\,du^2 + 2F\,du\,dv + G\,dv^2},$$

in which the second differentials $d^2 u$, $d^2 v$ do not occur. $\cos\theta / R$ depends upon only the ratio dv / du; i.e., the direction of the tangent. Hence, $\cos\theta / R$ *is the same for all curves on the surface that are tangent to the same line.* Consider the center of curvature C along the principle normal MP then. If one takes the point M to be the pole, the normal MN to the surface to be the polar axis, and the rotation of MN to MN' to be the positive sense of the polar angles then R, θ will be the polar coordinates of the point C. The equation:

$$\frac{\cos\theta}{R} = \text{const.}$$

will represent a circle. Hence, *the locus of the point C is a circle*, which one can also see as follows:



Consider the polar plane. It is in the plane that is normal to the curve, and therefore it will meet the normal MN to the surface at a point K such that:

$$R = MK \cos\theta,$$

so:

$$MK = \frac{R}{\cos\theta}.$$

MK is constant, so:

The polar lines to all of the curves on a surface will pass through the same point M of that surface, and the tangents to the same line at that point will meet at the same point K on the normal to the surface at M. The locus of centers of curvature of all those curves will be the circle of diameter MK (Meusnier's circle).

In particular, suppose that $\theta = 0$. The principal normal will coincide with the normal to the surface, and the osculating plane will pass through the normal, so it will be normal to the surface. Cut the surface with that plane, so K will be the center of curvature at M of the section, and let R_n be the radius of curvature. We will have:

$$\frac{\cos\theta}{R} = \frac{1}{R_n},$$

which will lead us to give the name of *normal curvature* to the geometric element $\cos\theta / R$. We will then conclude that:

$$R = R_n \cos\theta.$$

Hence, we will have:

Meusnier's theorem: *The center of curvature at M of a curve that is traced on a surface is the projection of the center of curvature of the normal section that is tangent to the curve at M onto the osculating plane of that curve at M.*

The theorem breaks down when:

$$\Psi(du, dv) = E'du^2 + 2F'du\,dv + G'dv^2 = 0.$$

Hence, $\cos\theta / R = 0$, so R will be infinite, in general. The formula will become completely indeterminate when $\cos\theta = 0$. The principal normal will then be perpendicular to the normal to the surface, so the osculating plane to the curve will be tangent to the surface. The two tangents that correspond to that exceptional case are called the *two asymptotic directions* of the *asymptotic tangents* that correspond to the point M that is being considered.

The theorem will likewise break down when:

$$\Phi(du, dv) = E\,du^2 + 2F\,du\,dv + G\,dv^2 = 0.$$

$\cos\theta / R$ will then be infinite, so R will be zero, in general (cf., § 4).The direction of the tangent is such that:

$$dx^2 + dy^2 + dz^2 = 0,$$

so it will be one of the two isotropic lines that pass through in the tangent plane at M.

Remark. – Since du, dv are, in fact, homogeneous coordinates for the corresponding direction dx, dy, dz of the tangent plane, one will verify that the *orthogonality condition for the two tangents* [pp. 20, eq. (4)] expresses the idea that they are harmonic conjugates with respect to the isotropic directions of the tangent plane. Similarly, the *condition for the two tangents to be conjugate* [pp. 26, eq. (6)] expresses the idea that they are harmonic conjugates with respect to the asymptotic lines.

Variations of the normal curvature

2. – Meusnier's theorem shows us that in order to study the curvature of the various curves on a surface that pass through a point of that surface, it is sufficient to consider the normal sections that pass through the various tangents to the surface of the point considered.

We saw above that:

$$\frac{1}{R_n} = \frac{1}{H}\frac{E'du^2 + 2F'du\,dv + G'dv^2}{E\,du^2 + 2F\,du\,dv + G\,dv^2}.$$

In the tangent plane at *M*, trace out the tangents *MU*, *MV* to the coordinate curves v = const. and u = const., resp., that pass through *M*, and consider the trihedron that is composed of *MU*, *MV*, and the normal *MN* to the surface. If one chooses the senses of increasing u and increasing v along *MU* and *MV*, resp. to be the positive directions then the direction cosines of its axes will be:

MU: $$\frac{dx}{ds} = \frac{\partial x}{\partial u}\frac{du}{ds} = \frac{1}{\sqrt{E}}\cdot\frac{\partial x}{\partial u} = \lambda', \qquad \frac{1}{\sqrt{E}}\cdot\frac{\partial y}{\partial u} = \mu', \qquad \frac{1}{\sqrt{E}}\cdot\frac{\partial z}{\partial u} = \nu',$$

MV: $$\frac{dx}{ds} = \frac{\partial x}{\partial u}\frac{dv}{ds} = \frac{1}{\sqrt{G}}\cdot\frac{\partial x}{\partial v} = \lambda', \qquad \frac{1}{\sqrt{G}}\cdot\frac{\partial y}{\partial v} = \mu', \qquad \frac{1}{\sqrt{G}}\cdot\frac{\partial z}{\partial v} = \nu',$$

MN: λ , μ , ν .

Consider an arbitrary tangent *MT* then that is defined by the values du, dv of the differentials of the coordinates u, v. The direction cosines are:

$$\begin{cases} \dfrac{dx}{ds} = \dfrac{\partial x}{\partial u}\cdot\dfrac{du}{ds} + \dfrac{\partial x}{\partial v}\cdot\dfrac{dv}{ds} = \sqrt{E}\,\dfrac{du}{ds}\cdot\lambda' + \sqrt{G}\,\dfrac{dv}{ds}\cdot\lambda'', \\[2mm] \dfrac{dy}{ds} = \dfrac{\partial y}{\partial u}\cdot\dfrac{du}{ds} + \dfrac{\partial y}{\partial v}\cdot\dfrac{dv}{ds} = \sqrt{E}\,\dfrac{du}{ds}\cdot\mu' + \sqrt{G}\,\dfrac{dv}{ds}\cdot\mu'', \\[2mm] \dfrac{dz}{ds} = \dfrac{\partial z}{\partial u}\cdot\dfrac{du}{ds} + \dfrac{\partial z}{\partial v}\cdot\dfrac{dv}{ds} = \sqrt{E}\,\dfrac{du}{ds}\cdot\nu' + \sqrt{G}\,\dfrac{dv}{ds}\cdot\nu''. \end{cases}$$

Those formulas show that the director segment of *MT* is the geometric sum of two segments whose algebraic values are:

$$P = \sqrt{E}\frac{du}{ds}, \quad Q = \sqrt{G}\frac{dv}{ds},$$

which are measured along *MU* and *MV*, resp. In other words, *P*, *Q* are the direction parameters of *MT* in the coordinate system *UMV*.

Upon introducing those direction parameters, the formula that gives R_n will become:

$$\frac{1}{R_n} = \frac{1}{H}\left[E'\left(\frac{du}{ds}\right)^2 + 2F'\frac{du}{ds}\cdot\frac{dv}{ds} + G'\left(\frac{dv}{ds}\right)^2\right] = \frac{1}{H}\left[E'P^2 + \frac{2F'}{\sqrt{EG}}PQ + \frac{G'}{G}Q^2\right].$$

If one considers the point that is obtained by measuring out a segment that is equal to $\pm\sqrt{|R_n|}$ along *MT*, starting at *M*, then the locus of that point, whose coordinates in the *MUV* system are:

$$U = \pm P\sqrt{|R_n|}, \qquad V = \pm Q\sqrt{|R_n|},$$

will have the equation:

$$\frac{E'}{E}U^2 + \frac{2F'}{\sqrt{EG}}UV + \frac{G'}{G}V^2 = H.$$

It is a conic whose center is situated in the tangent plane, and one calls that conic the *indicatrix* of the surface at the point *M*. Once the conic has been traced, one will immediately find that the square of the measure of the radius vector will be the radius of curvature of an arbitrary normal section, and one will painlessly conclude the variation of the radius of curvature when *MT* varies.

The nature of the indicatrix depends upon the sign of $\frac{E'G' - F'^2}{E\cdot G}$, or, since *E*, *G* are positive, the sign of $E'G' - F'^2$:

1. $E'G' - F'^2 > 0$. The indicatrix is an ellipse, so all of the radii of curvature have the same sign, and one says that the surface is *convex* at the point *M*. It is completely on one side of the tangent plane at *M* in the neighborhood of the point *M*.

2. $E'G' - F'^2 < 0$. The indicatrix is a hyperbola. The surface crosses its tangent plane at the point *M*. It is said *to have opposite curvatures* at the point *M*.

3. $E'G' - F'^2 = 0$. The indicatrix has parabolic type, and since it has a center, it will reduce to a system of two parallel lines. The point *M* is then called a *parabolic point.*

Consider the particular case in which $1 / R_n$ is the same, no matter what section one considers. For that to be true, it is necessary and sufficient that $1 / R_n$ should be independent of du / dv; hence:

$$\frac{E'}{E} = \frac{F'}{F} = \frac{G'}{G}.$$

Now, the angle ω that *MU* makes with *MV* is given by the formula:

$$\cos\omega = \sum \lambda'\lambda'' = \frac{F}{\sqrt{EG}}.$$

The preceding conditions are then written:

$$\frac{E'}{E} = \frac{F/\sqrt{EG}}{\cos\omega} = \frac{G'}{G},$$

and express the idea that the indicatrix is a circle, which should be obvious *a priori*.

The point *M* is then an *umbilic*.

Remark. – In the case where the equation of the surface is:

$$z = f(x, y),$$

when we take the usual notations, the element of arc length will be expressed by:

$$ds^2 = (1 + p^2) \cdot dx^2 + 2pq \cdot dx \cdot dy + (1 + q^2)\, dy^2,$$

so:

$$E = 1 + p^2, \qquad F = p \cdot q, \qquad G = 1 + q^2,$$

and

$$H = \sqrt{E \cdot G - F^2} = \sqrt{1 + p^2 + q^2}.$$

Now, the coefficients of the tangent plane to the surface are:

$$A = -p, \qquad B = -q, \qquad C = 1,$$

and:

$$\sum \Lambda\, d^2 x = -\sum d\Lambda \cdot d\,x = dp \cdot dx + dq \cdot dy.$$

But:

$$dp = r\,dx + s\,dy, \qquad dq = s\,dx + t\,dy,$$

so:

$$E' = r, \qquad F' = s, \qquad G' = t,$$

and:

$$E'G' - F'^2 = rt - s^2.$$

Principal sections

3. – We seek the directions of the axes of the indicatrix. They are conjugate directions with respect to the asymptotic directions of the indicatrix that are defined by:

$$\Psi(du, dv) = 0,$$

and with respect to the isotropic directions of the tangent plane that are defined by:

$$\Phi\,(du,\,dv) = 0.$$

They are then defined by the condition:

$$\frac{\partial\Psi/\partial du}{\partial\Phi/\partial du} = \frac{\partial\Psi/\partial dv}{\partial\Phi/\partial dv} = \frac{\Psi(du,dv)}{\Phi(du,dv)} = \frac{H}{R} = S,$$

since *du*, *dv* are homogeneous coordinates for the directions *MT* of the tangent plane.

They are the *principal directions.* The corresponding radii of curvature are called *principal radii of curvature.*

The equation that defines the principal directions is then:

$$\begin{vmatrix} E\cdot du + F\cdot dv & F\cdot du + G\cdot dv \\ E'\cdot du + F'\cdot dv & F'\cdot du + G'\cdot dv \end{vmatrix} = 0.$$

The left-hand side $\dfrac{D(\Phi,\Psi)}{D(du,dv)}$ is a simultaneous covariant for the forms Φ, Ψ.

The equation of the principal radii of curvature is obtained by eliminating *du*, *dv* from the equations:

$$\frac{\partial\Psi}{\partial du} = S\,\frac{\partial\Phi}{\partial du}, \qquad \frac{\partial\Psi}{\partial dv} = S\,\frac{\partial\Phi}{\partial dv},$$

which gives:

$$\begin{vmatrix} E' - SE & F' - SF \\ F' - SF & G' - SG \end{vmatrix} = 0,$$

or:

$$S^2\,(E\cdot G - F^2) - S\,(E\cdot G' + G\cdot E' - 2FF') + E'G' - F'^2 = 0,$$

with:

$$S = \frac{H}{R}.$$

Euler's formula. – Now suppose that the coordinate curves are tangent to the principal directions; those directions are rectangular. Hence, the coordinate curves constitute an orthogonal net. Moreover, the indicatrix is referred to its axes, so:

$$F' = 0, \qquad H = \sqrt{EG},$$

and

$$\frac{1}{R_n} = P^2\,\frac{E'}{E\sqrt{EG}} + Q^2\,\frac{G'}{G\sqrt{EG}}.$$

If we suppose that $P = 1$, $Q = 0$ then we will have one of the principal radii of curvature R_1 :

$$\frac{1}{R_1} = \frac{E'}{E\sqrt{EG}},$$

and for $P = 0$, $Q = 1$, we will have the other principal radius of curvature R_2 :

$$\frac{1}{R_2} = \frac{G'}{G\sqrt{EG}},$$

and the formula will become:

$$\frac{1}{R_n} = \frac{P^2}{R_1} + \frac{Q^2}{R_2}.$$

However, since the coordinates are rectangular here, if φ is the angle (MU, MT) between the tangent MT and the principal direction MU then $P = \cos\varphi$, $Q = \sin\varphi$, and we will get *Euler's formula:*

$$\frac{1}{R_n} = \frac{\cos^2\varphi}{R_1} + \frac{\sin^2\varphi}{R_2}.$$

Consider the tangent MT', which is perpendicular to MT. One must then replace φ with $\varphi + \pi / 2$, and we will get:

$$\frac{1}{R'_n} = \frac{\sin^2\varphi}{R_1} + \frac{\cos^2\varphi}{R_2},$$

so:

$$\frac{1}{R_n} + \frac{1}{R'_n} = \frac{1}{R_1} + \frac{1}{R_2}.$$

Therefore, *the arithmetic mean of the curvatures of two arbitrary rectangular normal sections is equal to the arithmetic mean of the curvatures of the principal normal sections.* That constant quantity $\frac{1}{2}\left(\frac{1}{R_1} + \frac{1}{R_2}\right)$ is called the *mean curvature* of the surface at the point considered.

Minimal lines

4. – There are three remarkable pairs of directions in the tangent plane at each point of a surface: The isotropic lines of the tangent plane, which are defined by $\Phi(du, dv) = 0$, the asymptotic directions of the indicatrix, which are defined by $\Psi(du, dv) = 0$, and the principal directions, which are harmonically conjugate with respect to the preceding two pairs and are defined by $\frac{D(\Phi, \Psi)}{D(du, dv)} = 0$.

Consider the isotropic directions, and look for the existence of curves on the surface that are tangent to an isotropic direction at each of their points. That amounts to integrating the differential equation:

$$\Phi\,(du, dv) = 0.$$

One will then obtain the *minimal curves* of the surface. The preceding equation will decompose into two first-order equations of first-degree in dv / du. Hence, *there are two families of minimal curves on a surface, and one and only one curve of each family will pass through each point of the surface, in general.* Those curves are imaginaries. Along each of them, one has:

$$ds^2 = dx^2 + dy^2 + dz^2 = 0.$$

That is why one also calls them *lines of null length.* If one takes the lines to be coordinates lines then the equation $\Phi\,(du, dv) = 0$ will be verified for $du = 0$ and $dv = 0$, and one will have:

$$E = 0, \quad G = 0$$

identically, and the element of arc length will reduce to the characteristic form:

$$ds^2 = 2F\,du \cdot dv.$$

Remark. – The calculation that is necessary if one is to effectively refer the surface to its minimal lines was indicated incidentally in Chap. II (pp. 22). In general, two distinct families of curves on the surface are defined by two equations:

$$\varphi\,(u, v) = \text{const.}, \qquad \psi\,(u, v) = \text{const.},$$

in which φ and ψ are independent functions, so it will suffice to take those curves to be coordinate curves, and make the change of parameters u, v in the equations (S) of the surface (pp. 20) that is defined by the formulas:

$$u_1 = \varphi\,(u, v), \quad v_1 = \psi\,(u, v).$$

Isotropic developables. – *Equations of minimal curves.* – In general, the two systems of minimal lines are distinct. In order for them to coincide, it is necessary and sufficient that one must have:

$$EG - F^2 = H^2 = 0$$

identically. In that case, $A^2 + B^2 + C^2 = 0$, and the fundamental formulas will no longer apply. In order to study the nature of such a surface, consider the tangent plane:

$$A\,(X - x) + B\,(Y - y) + C\,(Z - z) = 0.$$

That plane will then be tangent to an isotropic cone; it is then an *isotropic plane. All tangent planes to the surface will be isotropic then.* We seek the general equation of the isotropic planes. Let:

$$ax + by + cz + d = 0$$

be the equation of such a plane. a, b, c are coupled by the condition:

$$a^2 + b^2 + c^2 = 0,$$

or

$$(a + ib)\,(a - ib) = -\,c^2.$$

Set:

$$a + ib = tc, \qquad a - ib = -\,\frac{1}{t}\,c,$$

or:

$$a + ib - tc = 0, \qquad\qquad ta - ibt + c = 0.$$

We infer from these two homogeneous relations in a, b, c that:

$$\frac{a}{1-t^2} = \frac{b}{i(1+t^2)} = \frac{c}{-2t}.$$

Hence, we have the general equation of the isotropic planes:

(1) $$(1 - t^2)\,x + i\,(1 + t^2)\,y - 2tz + 2w = 0.$$

An isotropic plane depends upon two parameters. The surface considered is the envelope of its isotropic planes. If those planes depend upon two parameters then it will reduce to the imaginary circle at infinity. Therefore, suppose that w is a function of t, for example. The tangent plane depends upon only one parameter, so the surface will be developable, namely, an *isotropic developable.*

We seek its edge of regression. Differentiate equation (1) twice with respect to t. Upon denoting derivatives with respect to t by primes, we will have:

(2) $$-\,tx + ity - z + w' = 0,$$
(3) $$-\,x + iy + w'' = 0.$$

Equations (1), (2), (3) will then define the edge of regression. (3) gives:

$$x - iy = w''.$$

(2) is written:

$$z = -\,t\,(x - iy) + w' = w' - tw'',$$

and (1) becomes:

$$x + iy = t^2\,(x - iy) + 2tz - 2w = t^2\,w'' + 2t\,(w' - tw'') - 2w.$$

Hence, the equations of the edge of regression will be:

(4) $$x - iy = w''', \qquad d\,(x + iy) = -\,t^2\,w'''^2\,dt, \qquad dz = -\,tw'''\,dt.$$

Hence:

$$d\,(x - iy) \cdot d\,(x + iy) = -\,t^2\, w'''^2\, dt^2 = -\,dz^2$$

or:

$$d\,(x - iy) \cdot d\,(x + iy) + dz^2 = 0,$$

so

$$dx^2 + dy^2 + dz^2 = 0.$$

The curve that was found will then be a minimal curve. *The edge of regression of an isotropic developable is a minimal curve.*

Conversely, consider a minimal curve. The coordinates x, y, z of one of its points are such that:

$$dx^2 + dy^2 + dz^2 = 0.$$

Differentiate that and get:

$$dx \cdot d^2x + dy \cdot d^2y + dz \cdot d^2z = 0.$$

However, the Lagrange identity will then give us:

$$\sum dx^2 \sum (d^2x)^2 - \sum dx \cdot d^2x = \sum (dy \cdot d^2z - dz \cdot d^2y)^2 = 0;$$

i.e., if A, B, C denote the coefficients of the osculating plane:

$$A^2 + B^2 + C^2 = 0.$$

The osculating plane at a point of a minimal curve is isotropic. Any minimal curve can be considered to be the edge of regression of an isotropic developable.

It then results that the edge of regression is the most general minimal curve, and that the coordinates of a point on an arbitrary minimal curve are given by formulas (4), in which w is an arbitrary function of t, and w', w'' are its first and second derivatives, resp.

Remark. – Those formulas can serve for the study of the minimal curves, since the classical theory of curvature and torsion does not apply to those curves. On that occasion, observe that the *plane curves that are situated in isotropic planes* will likewise be singular curves from that same viewpoint.

Asymptotic lines

5. – If we now seek the curves on a surface that are tangent to an asymptote of the indicatrix at each of their points then we will be led to integrate the equation:

$$\Psi\,(du, dv) = 0, \tag{1}$$

and we will obtain the *asymptotic lines*. As before, we see that *there are two families of asymptotic lines, and one and only one asymptote of each family will pass through any point of the surface, in general.*

From the remarks in § 3 of Chap. II (pp. 28), the preceding differential equation is written:

$$\sum A\, d^2x = 0.$$

Moreover:

$$\sum A\, dx = 0;$$

however, A, B, C are the coefficients of the tangent plane to the surface. Equation (1) then expresses the idea that the tangent plane contains the direction d^2x, d^2y, d^2z, in addition to the direction dx, dy, dz; i.e., that it coincides with the osculating plane of the curve. Therefore: *The asymptotic lines are defined by the condition that the osculating plane at each of their points should be tangent to the surface.* In particular, *any rectilinear generator of a surface is an asymptotic line*, because since the osculating plane at a point of a line is indeterminate, it can be considered to coincide with the tangent plane at that point of the surface. *Therefore, if a surface is ruled then one of the systems of asymptotic lines will be composed of rectilinear generators.*

If we take the asymptotic lines to be coordinate curves then we will have:

$$E' = G' = 0,$$

and the form Ψ will reduce to the characteristic form:

$$\Psi(du, dv) = 2\, F' du \cdot dv.$$

The asymptotic lines are real at the points where the surface has opposite curvatures, which will be imaginary at the points where it is convex. They are distinct, in general, as well as distinct from the minimal lines. *We shall examine the exceptional cases:*

1. *The asymptotic lines coincide.* – Take the equation of the surface in the form:

$$z = f(x, y).$$

The condition for the two families of asymptotic lines to coincide, namely:

$$E'G' - F'^2 = 0,$$

will then reduce to:

$$rt - s^2 = 0$$

here. *All of the points of the surface must be parabolic.* That expresses the idea that the total differentials:

$$dp = r\, dx + s\, dy, \qquad dq = s\, dx + t\, dy$$

are two linear forms in dx and dy that are not independent; i.e., that the functions p and q of x and y are functions of each other. (For example, q is a function of p). On the other hand, the tangent plane at a point has the equation:

$$p(X - x) + q(Y - y) - (Z - z) = 0,$$

or:

$$pX + qY - Z = px + qy - z.$$

However:

$$d\,(px + qy - z) = x \cdot dp + y \cdot dq,$$

and we see that if $dp = 0$, since that condition is already implied by $dq = 0$, then we will have, at the same time, $d\,(px + qy - z) = 0$. Hence, $px + qy - z$ is a function p, as well as q, and the tangent plane will then depend upon just one parameter, and the surface will be developable. The converse is immediate, because if the equation $pX + qY - Z = px + qy - z$ depends only upon one parameter θ then dp and dq will be proportional to $d\theta$, and the two linear forms $dp = r \cdot dx + s \cdot dy$, $dq = s \cdot dx + r \cdot dy$ will not be independent. One will then have:

$$\begin{vmatrix} r & s \\ s & t \end{vmatrix} = rt - s^2 = 0.$$

Hence, *the surfaces with double asymptotic lines are the developable surfaces, and the double asymptotic lines are the rectilinear generators. For the isotropic developables, the double asymptotic lines coincide with the double minimal lines, which are the isotropic rectilinear generators.*

Remark. – For the developable surfaces, since the edge of regression has its osculating plane tangent to the surface, it must be considered to be an asymptotic line. Indeed, it is a singular integral of the differential equation of the asymptotic lines.

2. *A family of asymptotic lines coincides with a family of minimal lines.* – Omit the case of isotropic developables, which was just examined. Take the minimal lines to be coordinates curves. We will then have $E = 0$, $G = 0$, and if we suppose that the family v = const. constitutes a family of asymptotes then $dv = 0$ must be a solution of $\Psi\,(du, dv) = 0$, so $E' = 0$; i.e.:

$$E' = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial v} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial^2 x}{\partial u^2} & \dfrac{\partial^2 y}{\partial u^2} & \dfrac{\partial^2 z}{\partial u^2} \end{vmatrix} = 0.$$

There will then exist the same homogeneous, linear relations between the elements of the rows of that determinant, namely:

$$\begin{cases} \dfrac{\partial^2 x}{\partial u^2} = M\,\dfrac{\partial x}{\partial u} + N\,\dfrac{\partial x}{\partial v}, \\ \dfrac{\partial^2 y}{\partial u^2} = M\,\dfrac{\partial y}{\partial u} + N\,\dfrac{\partial y}{\partial v}, \\ \dfrac{\partial^2 z}{\partial u^2} = M\,\dfrac{\partial z}{\partial u} + N\,\dfrac{\partial z}{\partial v}. \end{cases}$$

Multiply them by $\frac{\partial x}{\partial u}$, $\frac{\partial y}{\partial u}$, $\frac{\partial z}{\partial u}$, and add them. The coefficient of M is $E = 0$, and that of N is F, so the left-hand side will be $\frac{1}{2}\frac{\partial E}{\partial u} = 0$. Hence, $NF = 0$, and since $F \neq 0$ (since the minimal lines are distinct), $N = 0$, in such a way:

$$\frac{\frac{\partial^2 x}{\partial u^2}}{\frac{\partial x}{\partial u}} = \frac{\frac{\partial^2 y}{\partial u^2}}{\frac{\partial y}{\partial u}} = \frac{\frac{\partial^2 z}{\partial u^2}}{\frac{\partial z}{\partial u}} = M.$$

The curves v = const. will then be lines, and since they are minimal lines, they will be isotropic lines. Conversely, if the curves v = const. are lines then there will exist a function M of u, v such that:

$$\frac{\partial^2 x}{\partial u^2} = M\frac{\partial x}{\partial u}, \qquad \frac{\partial^2 y}{\partial u^2} = M\frac{\partial y}{\partial u}, \qquad \frac{\partial^2 z}{\partial u^2} = M\frac{\partial z}{\partial u};$$

hence:

$$\sum A\frac{\partial^2 x}{\partial u^2} = M \cdot \sum A\frac{\partial x}{\partial u} = 0,$$

in such a way that the curves v = const., which are minimal lines, will be asymptotic lines. Hence, *the surfaces that have a family of asymptotes that coincide with a family of minimal lines are the ruled surfaces with isotropic generators, and those generators will be the asymptotes that coincide with the minimal curves.*

3. *Both systems of asymptotes are minimal curves.* – The quadratic forms Φ and Ψ are proportional then, and:

$$\frac{E'}{E} = \frac{F'}{F} = \frac{G'}{G}.$$

The indicatrix at an arbitrary point is a circle, so *all of the points of the surface are umbilics.* Upon once more taking the minimal lines to be coordinate curves, the preceding conditions will reduce to $E' = G' = 0$. Upon repeating the calculations as before, one will see that the surface admits two systems of isotropic rectilinear generators, and conversely. *It is a sphere.*

Minimal surfaces

6. – The latter case leads us to study the surfaces for which the indicatrix is always a circle. We now examine the case in which *that indicatrix is always an equilateral hyperbola.* That amounts to looking for the surfaces for which the asymptotic lines are orthogonal. For that to be true, it is necessary and sufficient that:

$$EG' + GE' - 2FF' = 0,$$

or

$$\frac{1}{R_1} + \frac{1}{R_2} = 0.$$

The mean curvature is zero, so the radii of curvature of each point are opposite. The surface is called a *minimal surface.*

Take the minimal lines to be coordinates. One will then have $E = 0$, $G = 0$, and:

$$ds^2 = 2F \cdot du \cdot dv.$$

The preceding conditions then give $F' = 0$, and:

$$\Psi\,(du, dv) = E'\,du^2 + G'\,dv^2.$$

However,:

$$F' = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \\ \dfrac{\partial^2 x}{\partial u\,\partial v} & \dfrac{\partial^2 y}{\partial u\,\partial v} & \dfrac{\partial^2 z}{\partial u\,\partial v} \end{vmatrix} = 0.$$

There then exists the same homogeneous, linear relationship between the line elements, namely:

$$\begin{cases} \dfrac{\partial^2 x}{\partial u\,\partial v} = M\,\dfrac{\partial x}{\partial u} + N\,\dfrac{\partial x}{\partial v}, \\ \dfrac{\partial^2 y}{\partial u\,\partial v} = M\,\dfrac{\partial y}{\partial u} + N\,\dfrac{\partial y}{\partial v}, \\ \dfrac{\partial^2 z}{\partial u\,\partial v} = M\,\dfrac{\partial z}{\partial u} + N\,\dfrac{\partial z}{\partial v}. \end{cases}$$

Multiply these by $\frac{\partial x}{\partial u}$, $\frac{\partial y}{\partial u}$, $\frac{\partial z}{\partial u}$, resp., and add them. The left-hand side will be $\frac{1}{2}\frac{\partial E}{\partial u} = 0$, the coefficient of M will be $E = 0$, and that of N will be F. Hence, $NF = 0$, and since $F \neq 0$, $N = 0$. Similarly, upon multiplying them by $\frac{\partial x}{\partial v}$, $\frac{\partial y}{\partial v}$, $\frac{\partial z}{\partial v}$, resp., and adding them, one will find that $M = 0$; thus:

$$\frac{\partial^2 x}{\partial u\,\partial v} = 0, \qquad \frac{\partial^2 y}{\partial u\,\partial v} = 0, \qquad \frac{\partial^2 z}{\partial u\,\partial v} = 0,$$

which will give:

$$x = f(u) + \varphi(v), \qquad y = g(u) + \psi(v), \qquad z = h(u) + \chi(v).$$

The surfaces that are represented by equations of that form are called *surfaces of translation. They can be generated in two different ways by translating a curve of invariable form so that each point describes another curve.* Indeed, consider the four points M_0 (u_0, v_0), M_1 (u, v_0), M_2 (u_0, v), M (u, v) on the surface. From the preceding formulas, those points will be the vertices of a parallelogram. If one varies u, while fixing v_0 then the point M_1 will describe a curve (Γ) on the surface. Similarly, if one varies v, while leaving u_0 fixed then the point M_2 will describe another curve (Γ') on the surface. The point M_0 will belong to both curves. One can then consider the surface as being generated by the curve (Γ), when it is has been animated with a translational motion under which the point M_0 describes the curve (Γ'), or by the curve (Γ'), when it has been animated with a translational motion in which the point M_0 describes the curve (Γ).

The six functions f, g, h, φ, ψ, χ are not arbitrary for the minimal surfaces. It must then satisfy the relations:

$$E = f'^2 + g'^2 + h'^2 = 0, \qquad G = \varphi'^2 + \psi'^2 + \chi'^2 = 0.$$

It will then result that the curve:

$$x = f(u), \qquad y = g(u), \qquad z = h(u)$$

is a minimal curve, and if we refer to the general equations of a minimal curve then, if F is an arbitrary function of u and F', F'', F''' are its successive derivatives, we will see that we can write:

$$\begin{aligned} &f(u) - i\,g(u) = F''(u), \\ &f(u) + i\,g(u) = -2F(u) + 2u\,F'(u) - u^2 F''(u), \\ &h(u) = F'(u) - u\,F''(u). \end{aligned}$$

Likewise, if the curve:

$$x = \varphi(v), \qquad y = \psi(v), \qquad z = \chi(v)$$

is a minimal curve then, if G is an arbitrary function of v, and G', G'', G''' are its successive derivatives, one will have:

$$\begin{aligned} &\varphi(v) - i\,\psi(v) = G''(v), \\ &\varphi(v) + i\,\psi(v) = -2G(v) + 2v\,G'(v) - v^2 G''(v), \\ &h(v) = G'(v) - v\,G''(v), \end{aligned}$$

so the coordinates of a point on the most general minimal surface will be:

$$\begin{aligned} &x + iy = -2F(u) + 2u \cdot F'(u) - u^2 F''(u) - 2\,G(v) + 2v\,G'(v) - v^2 G''(v), \\ &x - iy = F''(u) + G''(v), \\ &z = F'(u) - u\,F''(u) + G'(v) - v\,G''(v). \end{aligned}$$

Remark. – In the case in which the equation of the surface has been put into the form:

$$z = f(x, y),$$

when the partial differential equation for minimal surfaces has been integrated, from the formulas on page 40, that will give:

$$(1 + p^2) \cdot t + (1 + q^2) \cdot r - 2pqs = 0.$$

Lines of curvature

7. – The *lines of curvature* are the lines that are tangent to the principal direction or the axes of the indicatrix at each of their points. They will then be integrals of the equation:

$$\frac{\partial \Phi}{\partial(du)} \cdot \frac{\partial \Psi}{\partial(dv)} - \frac{\partial \Phi}{\partial(dv)} \cdot \frac{\partial \Psi}{\partial(du)} = 0,$$

so the principal directions will be conjugate and orthogonal; i.e., they will be harmonic conjugates with respect to the isotropic directions and the asymptotic directions. If those two pairs constitute four distinct directions then the principal directions will also be distinct from each other and the preceding ones. It will then result that there are no other singular cases for the lines of curvature than the ones that have been encountered already for the minimal lines and asymptotic lines.

1. *Non-developable ruled surfaces with isotropic generators (except for the sphere).* A family of minimal lines is composed of asymptotic lines. If we take the minimal lines to be the coordinate lines then we will have:

$$\Phi = 2F \cdot du \cdot dv.$$

If we suppose the lines u = const. coincide with the asymptotes then $du = 0$ must annul Ψ; hence:

$$\Psi = E' du^2 + 2F \cdot du \cdot dv.$$

The differential equation of the lines of curvature is then:

$$F \cdot dv \cdot F' du - F \cdot du \, (E' \cdot du + F' \cdot dv) = 0$$

or

$$E' \cdot F \cdot du^2 = 0.$$

The lines of curvature are double, which are the isotropic rectilinear generators that already define minimal lines and asymptotes.

2. *The sphere.* Φ, Ψ are proportional, so the differential equation is verified identically. *All of the lines on the sphere are lines of curvature.*

3. *Non-isotropic developables.* Take the rectilinear generators to be the curves u = const., which are double asymptotic lines, and get:

$$\Phi = E \cdot du^2 + 2F \cdot du \cdot dv + G \cdot dv^2,$$
$$\Psi = E' \cdot du^2.$$

The differential equation of the lines of curvature will then be:

$$(F \cdot du + G \cdot dv)\, E' \cdot du = 0.$$

The lines of curvature are the rectilinear generators, which are already asymptotic lines, and their orthogonal trajectories.

4. *Isotropic developable surfaces.* If we take the curves v = const. to be the double minimal lines that coincide with the double asymptotic lines then we will have:

$$\Phi = E \cdot du^2, \quad \Psi = E' \cdot du^2.$$

The equation for the lines of curvature is verified identically. *All lines on isotropic developables are lines of curvature.*

5. *The plane.* – For a plane, the minimal curves are lines, and any line in the plane is an asymptotic line, as well as a line of curvature.

Remark. – In order for the coordinate curves to be lines of curvature, it is first necessary that they should be orthogonal, so $F = 0$. The differential equation of the lines of curvature will then reduce to:

$$EF' du^2 + (EG' - GE')\, du\, dv - GF' dv^2 = 0.$$

Hence, upon omitting the singular cases, the fact that the lines of curvature are coordinate curves is characterized by the identities $F = 0$, $F' = 0$.

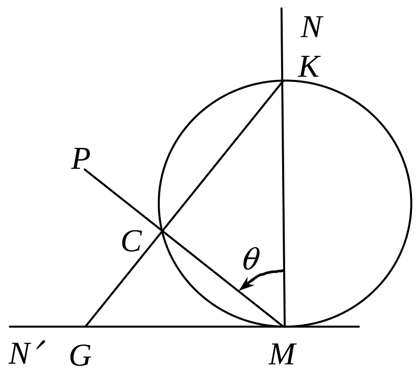
In Chap. II, § 3, it was shown that only the identity $F' = 0$ expresses the idea that the tangents to the coordinate curves have conjugate directions at each point of the surface, which one can express by saying that those curves form a *conjugate net.*

From that, one can characterize the lines of curvature by saying that they form an *orthogonal conjugate net.*

Geodesic curvature

8. – Let us now examine the second fundamental formula:

$$\frac{\sin\theta}{R} = \frac{1}{H\,ds^3}\left[H^2(du\,d^2v - dv\,d^2u) - \begin{vmatrix} \frac{1}{2}\frac{\partial E}{\partial u}du^2 + \frac{\partial E}{\partial v}du\,dv + \left(\frac{\partial F}{\partial v} - \frac{1}{2}\frac{\partial G}{\partial u}\right)dv^2 & E\,du + F\,dv \\ \left(\frac{\partial F}{\partial u} - \frac{1}{2}\frac{\partial E}{\partial v}\right)du^2 + \frac{\partial G}{\partial u}du\,dv + \frac{1}{2}\frac{\partial G}{\partial v}dv^2 & F\,du + G\,dv \end{vmatrix}\right].$$



θ is the angle (*MN*, *MP*) between the principal normal and the normal to the surface (§ 1). Let *C* be the center of curvature. Consider the polar line that meets the tangent plane along *MN′* at *G*.

$$MC = MG \cos\left(\theta - \frac{\pi}{2}\right) = MG \sin\theta.$$

MG is what one calls the *radius of geodesic curvature* R_g . One will then have:

$$R = R_g \sin\theta.$$

The point *G* is the *center of geodesic curvature*. *The projection of the center of geodesic curvature onto the principal normal is the center of curvature.* The inverse of the radius of geodesic curvature is called the *geodesic curvature.* Its expression depends upon only *E*, *F*, *G*, and their derivatives. *The geodesic curvature is preserved when one deforms the surface.*

We seek whether there exist curves on the surface whose radius of geodesic curvature is constantly infinite; such curves are called *geodesic lines.* $\sin\theta / R$ is constantly zero then, and if those curves are not lines, so *R* is not constantly infinite, then $\sin\theta = 0$. *The osculating plane is normal to the surface at each point of the curve, and conversely.* Any line that is traced on the surface is, moreover, obviously a geodesic line, and can be considered to satisfy the preceding condition.

The geodesic lines are defined by a differential equation of the form:

$$v'' = \Phi(u, v, v').$$

It results from the study of equations of that form that:

There is, in general, one and only one geodesic line that passes through each point of the surface and is tangent to a given direction in the tangent plane at that point. There is, *in general, one and only one of them that joins two give points in a sufficiently-small domain.*

Take the coordinate lines to the minimal lines. Hence:

$$E = G = 0 \qquad \text{and} \qquad H^2 = -F^2.$$

The differential equation of the geodesic lines becomes:

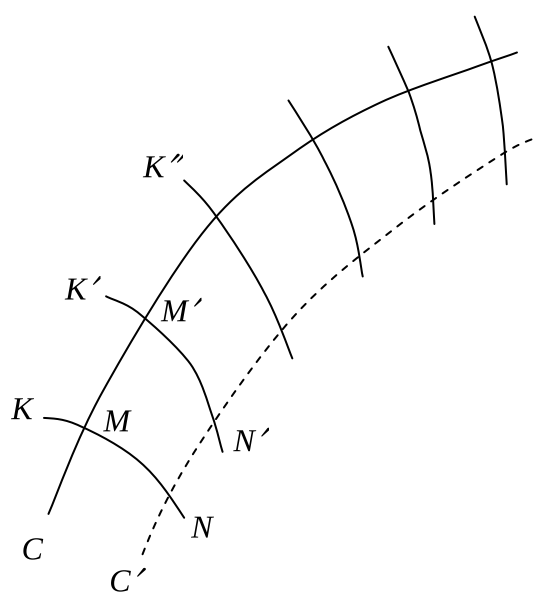
$$-F^2(du \cdot d^2v - dv\, d^2u) - \begin{vmatrix} \dfrac{\partial F}{\partial v} dv^2 & F\,dv \\ \dfrac{\partial F}{\partial u} du^2 & F\,du \end{vmatrix} = 0$$

or:

$$du \cdot d^2v - dv\, d^2u + \frac{\partial \cdot \log F}{\partial v} du \cdot dv^2 - \frac{\partial \cdot \log F}{\partial u} du^2\, dv = 0.$$

One sees that it is verified for $du = 0$, $dv = 0$. Hence, the *minimal lines are geodesic lines.*

Remark. – If the osculating plane coincides with the tangent plane then the center of curvature will coincide with the center of geodesic curvature. In particular, if one considers a plane then *there will be no other curvature in the plane besides geodesic curvature*, which one can easily verify by calculation.



Direct definition of geodesic curvature. Consider a curve (*C*) on a surface and a family of curves (*K*) that are orthogonal to (*C*). Measure out a constant arc length *MN* on each curve (*K*), starting from the point where it meets the curve (*C*). For each value of that constant, we will get a curve (*C*′) that is the locus of the point *N*. Take the curves (*C*), (*C*′), … to be coordinates curves (v = const.), where the curve (*C*) is $v = 0$, and take the curves (*K*) to be the coordinate curves (u = const.). We can take the coordinate v to be the arc length *MN*. Now consider the square of arc length:

$$ds^2 = E\,du^2 + 2F \cdot du \cdot dv + G \cdot dv^2.$$

The curve $v = 0$ is orthogonal to all the curves (*K*), so for any u, one will have:

$$F(u, 0) = 0.$$

Since v represents the arc length *MN*, one will have $ds^2 = dv^2$ for $du = 0$, hence $G = 1$, and then:

$$ds^2 = E \cdot du^2 + 2F \cdot du \cdot dv + dv^2.$$

We suppose that u represents the arc length of the curve (*C*). For $v = 0$, one will then have $ds = du$, hence:

$$E(u, 0) = 1,$$

and on that curve (*C*):

$$H^2 = E \cdot G - F^2 = 1,$$

so, for example, $H = 1$. One then has:

$$\frac{\sin\theta}{R} = -\frac{1}{ds^3}\begin{vmatrix} \dfrac{1}{2}\dfrac{\partial E}{\partial u}du^2 & E\,du \\ \left(\dfrac{\partial F}{\partial u} - \dfrac{1}{2}\dfrac{\partial E}{\partial v}\right)du^2 & F\,du \end{vmatrix} = -\frac{1}{2}\frac{\partial E}{\partial v}$$

for that curve.

For the curve (*C*′), we will have:

$$ds'^2 = E \cdot du^2,$$

if we denote the arc length of that curve by s', hence:

$$ds' = \sqrt{E}\,du, \qquad \frac{ds'}{du} = \sqrt{E},$$

and if we take the logarithmic derivative with respect to v then:

$$\frac{\partial \log \dfrac{ds'}{du}}{\partial v} = \frac{\partial \log \sqrt{E}}{\partial v} = \frac{1}{2E}\frac{\partial E}{\partial v}.$$

If one makes v tend to zero then (*C*′) will tend to (*C*), E will tend to 1, and in the limit:

$$\frac{\partial \log \dfrac{ds'}{du}}{\partial v} = \frac{1}{2}\frac{\partial E}{\partial v}.$$

If one uses the letter *s* to denote the arc length of (*C*), instead of *u*, then one can conclude that:

$$\frac{1}{R_g} = \frac{\sin\theta}{R} = -\left(\frac{\partial \log \dfrac{ds'}{ds}}{\partial v}\right)_{v=0},$$

which gives a definition for the geodesic curvature that is not borrowed from any element that is external to the surface. *s* and *s′* denote the homologous arc lengths on (*C*) and (*C′*), and *v* is the constant arc length *MN* that is found between (*C*) and (*C′*) on the curves (*K*). That definition makes the invariance of the geodesic curvature under the *deformation* of surfaces more intuitive.

Remark. – The consideration that concluded the preceding chapter lead one to introduce the geometric element:

$$\frac{\sin\theta}{R} - \frac{d\varphi_0}{ds} = \frac{r_1\,du + r\,dv}{\sqrt{\Phi(du, dv)}},$$

at the same time as the geodesic curvature, which, like the normal curvature, depends upon only the ratio *du* / *dv*; i.e., the direction of the tangent. However, that will have a precise sense to it only if one has specialized the choice of directions *MO* that are tangent to the origin. It is the geodesic torsion of a curve that is tangent to the proposed one and makes a constant angle with the directions at the origin that correspond to its various points.

Properties of geodesic lines

9. – In particular, suppose that all of the curves (*K*) are geodesic. With the same conventions as before, *du* = 0 must be a solution to the differential equation of the geodesic lines, which will give the identity:

$$\begin{vmatrix} \dfrac{\partial F}{\partial v} & F \\ 0 & 1 \end{vmatrix} = \frac{\partial F}{\partial v} = 0.$$

Hence, *F* is a function of only *u*, and since *F* = 0 for *v* = 0, *F* will be identically zero, and:

$$ds^2 = E\,du^2 + dv^2,$$

so all of the curves (C) will cut the geodesics (K) orthogonally. Hence:

If we consider a curve (C), *draw the geodesic through each point of* (C) *that is orthogonal to it, and measure out a constant arc along each of those geodesics then the locus of the extremities of those arcs will be a curve* (C') *that is normal to the geodesics.*

We will then get the *parallel curves* on an arbitrary surface.

Conversely, if we consider a family of geodesics and their orthogonal trajectories then those trajectories will determine equal arc lengths along the geodesics.

Always under the same hypotheses, since the curves u = const. and v = const. are orthogonal, $F = 0$. Since the u = const. are geodesics, it is necessary that:

$$\begin{vmatrix} -\frac{1}{2}\frac{\partial G}{\partial u}dv^2 & 0 \\ \frac{1}{2}\frac{\partial G}{\partial v}dv^2 & G \end{vmatrix} = -\frac{1}{2}G\frac{\partial G}{\partial u}dv^2 = 0.$$

$G \neq 0$, since otherwise the curves u = const. would be minimal curves, hence $\partial G / \partial u = 0$ and $G = \varphi(v)$. Then calculate the arc length of a curve (K) that is found between the curve $v = v_0$ and the curve $v = v_1$:

$$ds^2 = G\, dv^2 = \varphi(v)\, dv^2,$$

and:

$$s = \int_{v_0}^{v_1} \sqrt{\varphi(v)} \cdot dv\,.$$

s is independent of u, so the arc length will indeed be the same on all geodesics.

If one once more takes v to be the arc length along the curves u = const. then:

$$ds^2 = E\, du^2 + dv^2,$$

and *that form will be characteristic of the coordinate system employed, which is composed of a family of geodesics and their orthogonal trajectories.*

Take two points A, B on the surface then. There will then exist one and only one geodesic line in the domain of those two points that will join them. Consider it as belonging to a family of neighboring geodesics that do not intersect in the domain, and take those geodesics and their orthogonal trajectories to be coordinate curves. Let there be an arbitrary line of the surface that goes from A to B, and define it by the equation:

$$u = f(v).$$

If A has the coordinates u_0, v_0, and if u_1, v_1 are those of B then the arc length AB of the line is:

$$\int_{v_0}^{v_1} \sqrt{E\,du^2 + dv^2} = \int_{v_0}^{v_1} \sqrt{E(f(v), v)\, f'^2(v) + 1} \cdot dv.$$

That integral will obviously be minimal if $f'(v) = 0$; i.e., if the line AB that joins them is a geodesic. Hence:

In a sufficiently small domain that surrounds two points of a surface, the geodesic will be the shortest path between those two points.

Geodesic torsion

10. – Finally, we study the third fundamental formula:

$$\frac{1}{T} - \frac{d\theta}{ds} = \frac{1}{H^2 ds^2} \begin{vmatrix} E'du + F'dv & F'du + G'dv \\ E\,du + F dv & F\,du + G\,dv \end{vmatrix}.$$

If θ is constant, and in particular, if it is constantly zero, then the preceding formula will give the torsion; in particular, it will then give the torsion of a geodesic. The preceding expression will depend upon only du / dv; i.e., the direction of the tangent. Consider a curve (C) on the surface then and a point M. There exists a geodesic that is tangent to (C) at the point M, and $\frac{1}{T} - \frac{d\theta}{ds}$ will be the torsion of that geodesic. That is why $\frac{1}{T} - \frac{d\theta}{ds}$ is called the *geodesic torsion.* One then sees that *the geodesic torsion at a point of a curve is the torsion of the geodesic that is tangent to the given curve at that point.* Set:

$$\frac{1}{T_g} = \frac{1}{T} - \frac{d\theta}{ds}.$$

T_g is the *radius of geodesic torsion.* As opposed to the radius of geodesic curvature, it will change under the deformation of surfaces.

The preceding formula shows that the geodesic torsion is zero if the direction du, dv is a principal direction. *The geodesic torsion is zero for any curve that is tangent to a line of curvature.* It will then result that *the lines of curvature have a geodesic torsion that is constantly zero (Lancret's theorem).*

$1 / T_g$ is the quotient of the two trinomials of second degree in du, dv, so one can study its variation. Take the lines of curvature to be the coordinate curves, in such a way that (§ 7) $F = F' = 0$, and:

$$\frac{1}{T_g} = \frac{1}{H^2 ds^2} (E'G - G'E)\, du\, dv = \left(\frac{E'}{E} - \frac{G'}{G}\right) \frac{du}{ds} \frac{dv}{ds}.$$

If we return to the notation that was employed in § 2 for the study of the normal curvature then the direction parameters of the tangent in the tangent plane will be:

$$P = \sqrt{E}\,\frac{du}{ds}, \quad Q = \sqrt{G}\,\frac{dv}{ds},$$

and then:

$$\frac{1}{T_g} = \frac{1}{\sqrt{EG}}\left(\frac{E'}{E} - \frac{G'}{G}\right) PQ.$$

The principal radii of curvature are:

$$\frac{1}{R_1} = \frac{1}{\sqrt{EG}}\frac{E'}{E}, \qquad \frac{1}{R_2} = \frac{1}{\sqrt{EG}}\frac{G'}{G}.$$

Hence:

$$\frac{1}{T_g} = \left(\frac{1}{R_1} - \frac{1}{R_2}\right) PQ,$$

and one will then get *Ossian Bonnet's formula*, which is analogous to Euler's formula:

$$\frac{1}{T_g} = \left(\frac{1}{R_1} - \frac{1}{R_2}\right) \sin\varphi \cdot \cos\varphi.$$

Joachimsthal's theorems

11. – Consider a curve (C) that is the intersection of two surfaces. The normal plane to (C) at one of its points M contains the principal normal MP to the curve and the normals MN, MN_1 to the two surfaces. Let V be the angle between the normals MN, MN_1, and let θ, θ' be the angles that they make with MP.

$$V = \theta' - \theta.$$

However:

$$\frac{1}{T} - \frac{d\theta}{ds} = \frac{1}{T_g}, \qquad \frac{1}{T} - \frac{d\theta'}{ds} = \frac{1}{T'_g},$$

so, upon subtracting these, one will get:

$$\frac{dV}{ds} = \frac{1}{T_g} - \frac{1}{T'_g}.$$

Suppose that (C) is a line of curvature of the two surfaces then. $1 / T_g$ and $1/T'_g$ will then be zero, so $dV / ds = 0$, and V will be constant. Hence, one has:

Joachimsthals' theorems:

If two surfaces cut along a line of curvature then their angle will be constant along that line,

and the same formula will show immediately that, conversely:

If two surfaces cut at a constant angle, and if the intersection is a line of curvature for one of the surfaces then it will also be a line of curvature for the other one.

On a plane or a sphere, all of the lines will be lines of curvature. Hence:

If a line of curvature of a surface is planar or spherical then the plane or sphere that contains it will cut the surface at a constant angle, and conversely, if a plane or sphere cuts a surface at a constant angle then the intersection will be a line of curvature of the surface.

Finally, if a circle is a line of curvature of a surface then there will be a sphere that passes through that circle that is tangent to the surface at one point of the circle, and as a result, at all points of the circle. Therefore:
Hence:

Any circular line of curvature is the contact curve of a sphere that is inscribed or circumscribed on the surface.

Similarly:

Any rectilinear line of curvature is the contact curve of a tangent plane to the surface at all points of that line.

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CHAPTER IV

THE SIX INVARIANTS – TOTAL CURVATURE

The six invariants $E, F, G; E', F', G'$

1. – The only things that intervene in the study of the curves that are traced on a surface (S) are the coefficients of the two fundamental quadratic forms:

$$\Phi\,(du, dv) = ds^2 \qquad = E\;du^2 + 2F\;du\,dv + G\;dv^2,$$
$$\Psi\,(du, dv) = \sum A\,d^2x = E'du^2 + 2F'du\,dv + G'dv^2,$$

and the differentials of u, v, which are considered to be functions of one independent variable t that corresponds to each particular curve that one considers.

If one displaces the surface (S) in space without deforming it and does not change the surface coordinates u, v that one employs then those quadratic forms will remain the same in such a way that *their six coefficients E, F, G, E', F', G' will be six differential invariants for the group of motions in space.*

For the form $ds^2 = \Phi\,(du, dv)$, that will result from the fact that it represents the square of the differential of an arc that will remain the same under the stated conditions.

Furthermore, $H = \sqrt{EG - F^2}$ is an invariant, and the formula:

$$\Psi\,(du, dv) = H\,\Phi\,(du, dv) \cdot \frac{\cos\theta}{R},$$

in which all of the factors on the right-hand side are invariants, shows that Ψ is again an invariant.

Moreover, there is no difficulty associated with verifying the invariance of the coefficients by a direct calculation that is based upon the formulas that define them:

$$(1) \qquad \sum\left(\frac{\partial x}{\partial u}\right)^2 = E, \qquad \sum\frac{\partial x}{\partial u}\frac{\partial x}{\partial v} = F, \qquad \sum\left(\frac{\partial x}{\partial v}\right)^2 = G,$$

$$(2) \qquad \sum A\frac{\partial^2 x}{\partial u^2} = E', \qquad \sum B\frac{\partial^2 x}{\partial u\,\partial v} = F', \qquad \sum C\frac{\partial^2 x}{\partial v^2} = G',$$

in which A, B, C are the three functional determinants:

$$A = \frac{D(y,z)}{D(u,v)}, \quad B = \frac{D(z,x)}{D(u,v)}, \quad C = \frac{D(x,y)}{D(u,v)}.$$

Finally, recall that:

$$H = \pm\sqrt{A^2 + B^2 + C^2} = \pm\sqrt{EG - F^2}.$$

The form of the surface defined by the six invariants

2. – Now suppose that *E, F, G, E′, F′, G′* have been calculated as functions of *u*, *v* for a particular surface (*S*):

(3) $$x = f(u, v), \qquad y = g(u, v), \qquad z = h(u, v),$$

and consider equations (1), (2) to be a system of partial differential equations, in which *x*, *y*, *z* are unknown functions, *u*, *v* are independent variables, and *E*, *F*, *G*, *E′*, *F′*, *G′* are given functions. By virtue of the invariance that we just established, that differential system will admit not just the functions (3) that define (*S*) as integrals, but also all functions:

(4) $$\begin{cases} x = x_0 + \alpha f + \alpha' g + \alpha'' h, \\ y = y_0 + \beta f + \beta' g + \beta' h, \\ z = z_0 + \gamma f + \gamma' g + \gamma'' h, \end{cases}$$

which define the surfaces that are obtained by displacing (*S*) in all possible ways when one gives all possible constant values to x_0, y_0, z_0, and all constant values to α, β, γ, α', β', γ', α'', β'', γ'' that are compatible with the six well-known orthogonality conditions.

We then obtain integrals that depend upon six arbitrary constants. We show that the system (1), (2) has no other ones. We express that by saying that *the form of the surface is defined entirely by the six invariants E, F, G, E′, F′, G′.*

In the theory of partial differential equations, one shows that *in any system whose general integral depends upon only arbitrary constants, all of the partial derivatives of a certain order can be expressed as functions of the independent and dependent variables and their lower-order derivatives.* We shall first verify that the same thing is true for the system (1), (2).

Differentiate equations (1). We obtain the formulas that were used before:

(5) $$\begin{cases} \sum \dfrac{\partial x}{\partial u}\cdot\dfrac{\partial^2 x}{\partial u^2} = \dfrac{1}{2}\dfrac{\partial E}{\partial u}, & \sum \dfrac{\partial x}{\partial u}\cdot\dfrac{\partial^2 x}{\partial u\,\partial v} = \dfrac{1}{2}\dfrac{\partial E}{\partial v}, \quad \sum \dfrac{\partial x}{\partial u}\cdot\dfrac{\partial^2 x}{\partial v^2} = -\dfrac{1}{2}\dfrac{\partial G}{\partial u}, \\ \sum \dfrac{\partial x}{\partial v}\cdot\dfrac{\partial^2 x}{\partial u^2} = \dfrac{\partial F}{\partial u} - \dfrac{1}{2}\dfrac{\partial E}{\partial v}, & \sum \dfrac{\partial x}{\partial v}\cdot\dfrac{\partial^2 x}{\partial u\,\partial v} = \dfrac{1}{2}\dfrac{\partial G}{\partial u}, \quad \sum \dfrac{\partial x}{\partial v}\cdot\dfrac{\partial^2 x}{\partial v^2} = \dfrac{1}{2}\dfrac{\partial G}{\partial v}. \end{cases}$$

We also predict that upon associating those equations with equations (2), one will effectively obtain expressions for all of the second-order derivatives as functions of *u*, *v*, $\dfrac{\partial x}{\partial u}, \dfrac{\partial x}{\partial v}, \dfrac{\partial y}{\partial u}, \dfrac{\partial y}{\partial v}, \dfrac{\partial z}{\partial u}, \dfrac{\partial z}{\partial v}$.

In order to facilitate that calculation, we introduce the direction cosines of the normal:

(6) $$\lambda = \frac{A}{H}, \qquad \mu = \frac{B}{H}, \qquad \nu = \frac{C}{H}.$$

We then replace the form $\sum A\, d^2x$ with the form:

$$\sum \lambda \cdot d^2x = \frac{1}{H}\sum \lambda \cdot d^2x = L \cdot du^2 + 2M \cdot du\, dv + N \cdot dv^2, \tag{7}$$

in which:

$$L = \frac{E'}{H}, \qquad M = \frac{F'}{H}, \qquad N = \frac{G'}{H}. \tag{8}$$

Equations (2) will then be replaced with the equations:

$$\sum \lambda \frac{\partial^2 x}{\partial u^2} = L, \qquad \sum \lambda \frac{\partial^2 x}{\partial u\, \partial v} = M, \qquad \sum \lambda \frac{\partial^2 x}{\partial v^2} = N. \tag{9}$$

We then set:

$$\frac{\partial^2 x}{\partial u^2} = L' \frac{\partial x}{\partial u} + L'' \frac{\partial x}{\partial v} + L''' \lambda,$$
$$\frac{\partial^2 y}{\partial u^2} = L' \frac{\partial y}{\partial u} + L'' \frac{\partial y}{\partial v} + L''' \mu,$$
$$\frac{\partial^2 z}{\partial u^2} = L' \frac{\partial z}{\partial u} + L'' \frac{\partial z}{\partial v} + L''' \nu,$$

in which, L', L'', L''' are coefficients to be determined; we deduce from this that:

$$\sum \frac{\partial x}{\partial u}\frac{\partial^2 x}{\partial u^2} = E\,L' + F\,L'', \qquad \sum \frac{\partial x}{\partial v}\frac{\partial^2 x}{\partial u^2} = F\,L' + G\,L'', \qquad \sum \lambda \frac{\partial^2 x}{\partial u^2} = L'''.$$

The third of these conditions shows that $L''' = L$, and the first two are two linear equations that will provide L' and L'' when one takes formulas (5) into account.

Upon doing the same thing with the other derivatives, one will get the following results:

$$\left\{\begin{aligned} \frac{\partial^2 x}{\partial u^2} &= L' \frac{\partial x}{\partial u} + L'' \frac{\partial x}{\partial v} + L \cdot \lambda, \\ \frac{\partial^2 x}{\partial u\, \partial v} &= M' \frac{\partial x}{\partial u} + M'' \frac{\partial x}{\partial v} + M \cdot \lambda, \\ \frac{\partial^2 x}{\partial v^2} &= N' \frac{\partial x}{\partial u} + N'' \frac{\partial x}{\partial v} + N \cdot \lambda, \end{aligned}\right. \tag{10}$$

with the auxiliary equations:

(11)
$$\begin{cases} EL' + FL'' = \dfrac{1}{2}\dfrac{\partial E}{\partial u}, & FL' + GL'' = \dfrac{\partial F}{\partial u} - \dfrac{1}{2}\dfrac{\partial E}{\partial v}, \\ EM' + FM'' = \dfrac{1}{2}\dfrac{\partial E}{\partial v}, & FM' + GM'' = \dfrac{1}{2}\dfrac{\partial G}{\partial u}, \\ EN' + FM'' = \dfrac{\partial F}{\partial v} - \dfrac{1}{2}\dfrac{\partial G}{\partial u}, & FN' + GN'' = \dfrac{1}{2}\dfrac{\partial G}{\partial v}, \end{cases}$$

from which, one can deduce the values of the coefficients L', L'', M', M'', N', N''. One notes that they depend upon only the coefficients E, F, G of the linear element $ds^2 = \Phi\,(du, dv)$ and the first derivatives of those coefficients.

Finally, the same equations (10) will persist for the other coordinates y, z. One will only need to keep the same coefficients and replace the letter x with the letter y or z at the same time as one changes λ into μ or ν, resp.

We conclude from this that if one knows the values of x, y, z, and their first derivatives for a system of values of u, v then one can calculate the values of their second derivatives, and by new differentiations, those of all their higher-order derivatives. As a result, the Taylor series developments of an arbitrary solution cannot contain any other arbitrary contributions than the initial values of:

$$x,\ y,\ z,\ \frac{\partial x}{\partial u},\ \frac{\partial x}{\partial v},\ \frac{\partial y}{\partial u},\ \frac{\partial y}{\partial v},\ \frac{\partial z}{\partial u},\ \frac{\partial z}{\partial v},$$

which are linked by equations (1), moreover, and the integral will be determined entirely when those initial values are given.

Hence, in order to prove that equations (4) give the general integral, it will suffice to show that the functions x, y, z that are defined by equations (4) can satisfy the stated initial conditions. Now, if we introduce the direction cosines λ', μ', ν'; λ'', μ'', ν'' of the tangents MU, MV, resp., to the two stated coordinate curves that pass through an arbitrary point M of the surface then we will know that:

$$\begin{cases} \dfrac{\partial x}{\partial u} = \lambda'\sqrt{E}, & \dfrac{\partial y}{\partial u} = \mu'\sqrt{E}, & \dfrac{\partial z}{\partial u} = \nu'\sqrt{E}, \\ \dfrac{\partial x}{\partial v} = \lambda''\sqrt{G}, & \dfrac{\partial y}{\partial v} = \mu'\sqrt{G}, & \dfrac{\partial z}{\partial v} = \nu''\sqrt{G}, \end{cases}$$

and the conditions (1) will reduce to:

$$\sum \lambda'^2 = 1, \qquad \sum \lambda''^2 = 1, \qquad \sum \lambda'\lambda'' = \cos\omega,$$

in which ω is the angle $\widehat{UMV}$.

The initial conditions then signify that one arbitrarily gives the position of the point M that corresponds to the initial values of u, v, and the directions of the tangents MU, MV, with the single reservation that those directions must form the same angle between them that they make with the corresponding point of (*S*). There is, in fact, a position for (*S*) that satisfies those conditions, and our result is found to be established definitively.

Remark. – The preceding argument will break down when the coordinate curves are minimal lines (because $E = G = 0$ then). However, it suffices to remark that if Φ and Ψ are known for a coordinate system *u*, *v* then one can define their expressions in another coordinate system *u*, *v* by performing the corresponding change of variables directly. Our theorem will then be true for any system of surface coordinates as long as it is true for one of them.

The integrability conditions

3. – The coefficients in formulas (10) satisfy certain conditions that are called *integrability conditions*, which one will obtain from the theory of partial differential equations, by writing down that each of the third-order derivatives $\frac{\partial^2 x}{\partial u^2 \partial v}$, $\frac{\partial^2 x}{\partial u \partial v^2}$ has the same value that one obtains by differentiating one or the other of formulas (10).

In order to obtain those conditions, it is convenient to have some formulas that give the derivatives of the direction cosines λ, μ, ν of the normal. Those cosines are defined by the equations:

$$\sum \lambda \frac{\partial x}{\partial u} = 0, \quad \sum \lambda \frac{\partial x}{\partial v} = 0, \quad \sum \lambda^2 = 1,$$

which will give, by differentiation:

$$(12) \qquad \left\{ \begin{array}{ll} \sum \frac{\partial \lambda}{\partial u} \frac{\partial x}{\partial u} = -\sum \lambda \frac{\partial^2 x}{\partial u^2} = -L, & \sum \frac{\partial \lambda}{\partial v} \frac{\partial x}{\partial u} = -\sum \lambda \frac{\partial^2 x}{\partial u \partial v} = -M, \\ \sum \frac{\partial \lambda}{\partial u} \frac{\partial x}{\partial v} = -\sum \lambda \frac{\partial^2 x}{\partial u \partial v} = -M, & \sum \frac{\partial \lambda}{\partial v} \frac{\partial x}{\partial v} = -\sum \lambda \frac{\partial^2 x}{\partial v^2} = -N, \\ \sum \lambda \frac{\partial \lambda}{\partial u} = 0, & \sum \lambda \frac{\partial \lambda}{\partial v} = 0. \end{array} \right.$$

While following the same method as in the preceding paragraph, if one sets:

$$\left\{ \begin{array}{l|l} \frac{\partial \lambda}{\partial u} = P' \frac{\partial x}{\partial u} + P'' \frac{\partial x}{\partial v} + P\lambda, & \frac{\partial \lambda}{\partial v} = Q' \frac{\partial x}{\partial u} + Q'' \frac{\partial x}{\partial v} + Q\lambda, \\ \frac{\partial \mu}{\partial u} = P' \frac{\partial y}{\partial u} + P'' \frac{\partial y}{\partial v} + P\mu, & \frac{\partial \mu}{\partial v} = Q' \frac{\partial y}{\partial u} + Q'' \frac{\partial y}{\partial v} + Q\mu, \\ \frac{\partial \nu}{\partial u} = P' \frac{\partial z}{\partial u} + P'' \frac{\partial z}{\partial v} + P\nu, & \frac{\partial \nu}{\partial v} = Q' \frac{\partial z}{\partial u} + Q'' \frac{\partial z}{\partial v} + Q\nu \end{array} \right.$$

then one will find that:

$$\sum \frac{\partial x}{\partial u}\frac{\partial \lambda}{\partial u} = EP' + FP'', \qquad \sum \frac{\partial x}{\partial v}\frac{\partial \lambda}{\partial u} = FP' + GP'', \qquad \sum \lambda \frac{\partial \lambda}{\partial u} = P = 0,$$
$$\sum \frac{\partial x}{\partial u}\frac{\partial \lambda}{\partial v} = EQ' + FQ'', \qquad \sum \frac{\partial x}{\partial v}\frac{\partial \lambda}{\partial v} = FQ' + GQ'', \qquad \sum \lambda \frac{\partial \lambda}{\partial v} = Q = 0.$$

Hence:

(13)
$$\begin{cases} \dfrac{\partial \lambda}{\partial u} = P'\dfrac{\partial x}{\partial u} + P''\dfrac{\partial x}{\partial v}, \\[2ex] \dfrac{\partial \lambda}{\partial v} = Q'\dfrac{\partial x}{\partial u} + Q''\dfrac{\partial x}{\partial v}, \end{cases}$$

in which the coefficients P', P'', Q', Q'' are defined by the equations:

(14)
$$\begin{cases} EP' + FP'' = -L, \quad FP' + GP'' = -M, \\ EQ' + FQ'' = -M, \quad FQ' + GQ'' = -N. \end{cases}$$

For μ, v, it will suffice to change x into y and z, respectively.

We can carry out the calculations by assuming that the surface is referred to its minimal lines. The preceding calculations then simplify considerably. If we apply the formulas that we have found directly, upon taking into account the fact that E and G are zero, then we will get:

$$L'' = 0, \quad L' = \frac{\partial \log F}{\partial u}, \quad M'' = 0, \quad M' = 0, \quad N'' = \frac{\partial \log F}{\partial v}, \quad N' = 0$$

for formulas (11), and:

$$P'' = -\frac{L}{F}, \qquad P' = -\frac{M}{F}, \qquad Q'' = -\frac{M}{F}, \qquad Q' = -\frac{N}{F}$$

for formulas (14); i.e.:

(15)
$$\begin{cases} \dfrac{\partial^2 x}{\partial u^2} = \dfrac{\partial \log F}{\partial u}\cdot\dfrac{\partial x}{\partial u} + L\cdot\lambda, \quad \cdots, \quad \cdots, \\[2ex] \dfrac{\partial^2 x}{\partial u\,\partial v} = M\cdot\lambda, \qquad \cdots, \quad \cdots, \\[2ex] \dfrac{\partial^2 x}{\partial v^2} = \dfrac{\partial \log F}{\partial v}\cdot\dfrac{\partial x}{\partial v} + N\cdot\lambda, \quad \cdots, \quad \cdots, \end{cases}$$

(16)
$$\begin{cases} \dfrac{\partial\lambda}{\partial u} = -\dfrac{1}{F}\left(M\dfrac{\partial x}{\partial u} + L\dfrac{\partial x}{\partial v}\right), & \cdots, \quad \cdots, \\ \dfrac{\partial\lambda}{\partial u} = -\dfrac{1}{F}\left(N\dfrac{\partial x}{\partial u} + M\dfrac{\partial x}{\partial v}\right), & \cdots, \quad \cdots \end{cases}$$

Differentiate the first of equations (15) with respect v, upon taking equations (15) and (16) into account:

$$\frac{\partial^3 x}{\partial u^2\,\partial v} = \left(\frac{\partial^2 \log F}{\partial u\,\partial v} - \frac{NL}{F}\right)\frac{\partial x}{\partial u} - \frac{LM}{F}\frac{\partial x}{\partial v} + \left(M\frac{\partial \log F}{\partial u} + \frac{\partial L}{\partial v}\right)\lambda.$$

Likewise, differentiate the second of equations (15) with respect to u:

$$\frac{\partial^3 x}{\partial u^2\,\partial v} = -\frac{M^2}{F}\cdot\frac{\partial x}{\partial u} - \frac{LM}{F}\frac{\partial x}{\partial v} + \frac{\partial M}{\partial u}\lambda.$$

Upon equating them, we will get:

(17)
$$\left(\frac{\partial^2 \log F}{\partial u\,\partial v} - \frac{LN - M^2}{F}\right)\frac{\partial x}{\partial u} + \left(M\frac{\partial \log F}{\partial u} + \frac{\partial L}{\partial v} - \frac{\partial M}{\partial u}\right)\lambda = 0.$$

This is a condition of the form:

$$S'\frac{\partial x}{\partial u} + S''\frac{\partial x}{\partial v} + S\lambda = 0,$$

and upon repeating the same calculation for y and z, one will obtain the analogous conditions:

$$S'\frac{\partial y}{\partial u} + S''\frac{\partial y}{\partial v} + S\mu = 0,$$
$$S'\frac{\partial z}{\partial u} + S''\frac{\partial z}{\partial v} + S\nu = 0.$$

One then concludes that one necessarily has $S = S' = S'' = 0$; i.e.:

(18)
$$\frac{\partial^2 \log F}{\partial u\,\partial v} - \frac{LN - M^2}{F} = 0, \qquad M\frac{\partial \log F}{\partial u} + \frac{\partial L}{\partial v} - \frac{\partial M}{\partial u} = 0,$$

and those conditions will imply the condition (17).

Upon similarly equating the two values of $\frac{\partial^3 x}{\partial u\,\partial v^2}$, one will get some conditions that are deduced from (18) by changing the roles of the variables *u*, *v*; that will modify only the second of those conditions.

The desired integrability conditions are then:

$$(19)\qquad \begin{cases} M\,\dfrac{\partial \log F}{\partial u} = \dfrac{\partial M}{\partial u} - \dfrac{\partial L}{\partial v}, \\[2ex] \dfrac{\partial \log F}{\partial u\,\partial v} = \dfrac{LN - M^2}{F}, \\[2ex] M\,\dfrac{\partial \log F}{\partial v} = \dfrac{\partial M}{\partial v} - \dfrac{\partial N}{\partial u}, \end{cases}$$

and from the theory of differential equations, they will be the only integrability conditions for the system considered.

Total curvature

The second of the preceding formulas:

$$(20)\qquad \frac{\partial^2 \log F}{\partial u\,\partial v} = \frac{LN - M^2}{F},$$

which is due to Gauss, leads to an important consequence. Indeed, recall the equation for the radii of principle curvature, which is:

$$H^2 (LN - M^2) + 2\,SFHM - S^2 F^2 = 0$$

here, with:

$$S = \frac{H}{R}.$$

It is written:

$$LN - M^2 + 2FM \cdot \frac{1}{R} - \frac{F^2}{R^2} = 0,$$

so:

$$\frac{1}{R_1 R_2} = -\frac{LN - M^2}{F^2};$$

i.e., from formula (20):

$$(21)\qquad \frac{1}{R_1 R_2} = -\frac{1}{F}\frac{\partial^2 \log F}{\partial u\,\partial v}.$$

The product of the radii of principal curvature depends upon only the linear element; it is then preserved under the deformation of surfaces. One gives the name of *total curvature* to $\frac{1}{R_1 R_2}$.

Remark. – From the preceding, *surfaces with zero total curvature* are characterized by the condition $LN - M^2 = 0$ or $EG - F^2 = 0$, which expresses the idea that the surface considered is the envelope of ∞^1 planes (page 47); namely, by the condition $\frac{\partial^2 \log F}{\partial u \, \partial v} = 0$, which expresses the idea that since the linear element is $ds^2 = 2F\, du\, dv$, the surface can be mapped to a plane (page 26). One then concludes that *the surfaces that can be mapped to a plane are the developable surfaces.* (Cf., Chap. V, § 4.)

Spherical representation. – Just as one can make a curve correspond to its spherical indicatrix, one can imagine a correspondence between an arbitrary surface and the sphere of radius 1, in which the homologue of a point (u, v) of the surface will be the point (λ, μ, ν). An area on the surface will correspond to an area on the sphere. The consideration of the limit with respect to those areas when they become infinitely small in all of their dimensions leads us to a *direct definition of the total curvature.*

The area on the surface has the expression:

$$\mathcal{A} = \iint \sqrt{A^2 + B^2 + C^2}\, du\, dv = \iint H\, du\, dv.$$

In order to get the homologous area on the sphere, one must first calculate the linear element $d\lambda^2 + d\mu^2 + d\nu^2$. From formulas (16):

$$d\lambda = \frac{\partial \lambda}{\partial u} du + \frac{\partial \lambda}{\partial v} dv = -\frac{du}{F}\left(M \frac{\partial x}{\partial u} + L \frac{\partial x}{\partial v} \right) - \frac{dv}{F}\left(N \frac{\partial x}{\partial u} + M \frac{\partial x}{\partial v} \right)$$
$$= -\frac{1}{F}\left[L \frac{\partial x}{\partial u} du + M\, dx + N \frac{\partial x}{\partial u} dv \right];$$

hence:

$$\sum d\lambda^2 = \frac{1}{F^2}[M^2 \cdot 2F\, du\, dv + 2\, LMF \cdot du^2 + 2MNF \cdot dv^2 + 2LNF \cdot du\, dv],$$

so

$$\sum d\lambda^2 = \frac{2LM}{F} du^2 + 2 \frac{LN + M^2}{F} du\, dv + \frac{2MN}{F} dv^2 .$$

The function that is analogous to H for the sphere is then:

$$\sqrt{4 \frac{LM^2 N}{F^2} - \frac{(LN + M^2)^2}{F^2}} = \frac{LN - M^2}{iF} = \frac{LN - M^2}{H},$$

and the spherical area will have the expression:

$$\mathcal{A}' = \iint \frac{LN - M^2}{H} du\, dv,$$

and upon noting that:

$$d\mathcal{A} = H \cdot du\, dv,$$

it can be written:

$$\mathcal{A}' = \iint \frac{LN - M^2}{H^2} d\mathcal{A}, \qquad d\mathcal{A} = \iint \frac{1}{R_1 R_2} d\mathcal{A},$$

so:

$$d\mathcal{A}' = \frac{1}{R_1 R_2} \mathcal{A}.$$

The ratio of the homologous areas on the sphere and the surface will have the total curvature for its limit when those areas become infinitely small in all of their dimensions.

Orthogonal, isothermal coordinates

4. – In order to avoid the use of imaginaries in the preceding considerations, we shall introduce a new curvilinear coordinate system. Since the surface is assumed to be real, we first choose the minimal coordinates in such a fashion that u, v are conjugate imaginaries. We then set:

$$u = u' + i\, v', \quad v = u' - i\, v',$$

in which u', v' are real quantities. We then infer that:

$$du = du' + i\, dv', \qquad dv = du' - i\, dv',$$

so:

$$du\, dv = du'^2 + dv'^2.$$

The linear element will then take the form:

$$ds^2 = 2F \cdot du\, dv = 2F\, (du'^2 + dv'^2).$$

The coordinates u', v' are orthogonal; one gives them the name of *orthogonal, isothermal coordinates.* One can say that *those coordinates divide the surface into a net of infinitely-small squares.* Indeed, consider the coordinate curves u', $u' + h$, $u' + 2h$, … and v', $v' + h$, $v' + 2h$, … If one takes one of the curvilinear quadrilaterals thus-obtained then its angles will be right angles. Its edges are $\sqrt{2F} \cdot du'$ and $\sqrt{2F} \cdot dv'$; i.e., $\sqrt{2F} \cdot h$, up to higher-order infinitesimals. Those arcs are equal.

With this particular coordinate system, upon denoting the values of the functions that are analogous to E, F, G, H by $\bar{E}$, $\bar{F}$, $\bar{G}$, $\bar{H}$, we will have:

$$\bar{E} = 2F, \quad \bar{G} = 2F, \quad \bar{F} = 0, \quad \bar{H}^2 = \bar{E}\bar{G} - \bar{F}^2 = 4F^2, \quad \bar{H} = 2F,$$

hence:

$$ds^2 = \bar{H}\,(du'^2 + dv'^2).$$

However, for an arbitrary function Φ, we have:

$$\frac{\partial \Phi}{\partial u'} = \frac{\partial \Phi}{\partial u} + \frac{\partial \Phi}{\partial v}, \qquad \frac{\partial \Phi}{\partial v'} = i\left(\frac{\partial \Phi}{\partial u} - \frac{\partial \Phi}{\partial v}\right);$$

hence:

$$\frac{\partial^2 \Phi}{\partial u'^2} = \frac{\partial^2 \Phi}{\partial u^2} + 2\frac{\partial^2 \Phi}{\partial u\,\partial v} + \frac{\partial^2 \Phi}{\partial v^2}, \qquad \frac{\partial^2 \Phi}{\partial v'^2} = -\frac{\partial^2 \Phi}{\partial u^2} + 2\frac{\partial^2 \Phi}{\partial u\,\partial v} - \frac{\partial^2 \Phi}{\partial v^2},$$

and:

$$\frac{\partial^2 \Phi}{\partial u'^2} + \frac{\partial^2 \Phi}{\partial v'^2} = 4\,\frac{\partial^2 \Phi}{\partial u\,\partial v}.$$

Hence, as a consequence:

$$4\,\frac{\partial^2 \log F}{\partial u\,\partial v} = \frac{\partial^2 \log H}{\partial u\,\partial v} = \frac{\partial^2 \log H}{\partial u'^2} + \frac{\partial^2 \log H}{\partial v'^2}.$$

Upon suppressing the primes and the overbars, we will get the following formulas in orthogonal, isothermal coordinates:

$$ds^2 = H\,(du^2 + dv^2),$$

$$\frac{1}{R_1 R_2} = -\frac{1}{2H}\left(\frac{\partial^2 \log H}{\partial u^2} + \frac{\partial^2 \log H}{\partial v^2}\right).$$

We again set:

$$\sum \lambda\, d^2x = L\,du^2 + 2M\,du\,dv + N\,dv^2.$$

The equation of the principal radii of curvature will be:

$$(LN - M^2) - \frac{H}{R}(L + N) + \frac{H^2}{R^2} = 0,$$

and one will have:

$$\frac{1}{R_1 R_2} = \frac{LN - M^2}{H^2}.$$

Calculate the spherical representation. As in § 2, set:

$$\lambda' = \frac{1}{\sqrt{H}}\frac{\partial x}{\partial u}, \qquad \mu' = \frac{1}{\sqrt{H}}\cdot\frac{\partial y}{\partial u}, \qquad \nu' = \frac{1}{\sqrt{H}}\cdot\frac{\partial z}{\partial u},$$

$$\lambda'' = \frac{1}{\sqrt{H}}\frac{\partial x}{\partial v}, \qquad \mu'' = \frac{1}{\sqrt{H}}\cdot\frac{\partial y}{\partial v}, \qquad \nu'' = \frac{1}{\sqrt{H}}\cdot\frac{\partial z}{\partial v}.$$

From the relation:

$$\sum \lambda^2 = 1,$$

we infer that:

$$\sum \lambda \frac{\partial \lambda}{\partial u} = 0.$$

On the other hand:

$$L = \sum \lambda \frac{\partial^2 x}{\partial u^2} = -\sum \frac{\partial \lambda}{\partial u}\cdot\frac{\partial x}{\partial u} = -\sqrt{H}\cdot\sum \lambda' \frac{\partial \lambda}{\partial u};$$

hence:

$$\sum \lambda' \frac{\partial \lambda}{\partial u} = -\frac{L}{\sqrt{H}}.$$

Similarly:

$$M = \sum \lambda \frac{\partial^2 x}{\partial u\,\partial v} = -\sum \frac{\partial \lambda}{\partial u}\cdot\frac{\partial x}{\partial v} = -\sqrt{H}\cdot\sum \lambda'' \frac{\partial \lambda}{\partial u},$$

so

$$\sum \lambda'' \frac{\partial \lambda}{\partial u} = -\frac{M}{\sqrt{H}}.$$

One then gets three equations in $\frac{\partial \lambda}{\partial u}, \frac{\partial \mu}{\partial u}, \frac{\partial \nu}{\partial u}$. If one multiplies them by $\lambda, \lambda', \lambda''$, resp., and adds them then one will get (†):

$$\frac{\partial \lambda}{\partial u} = -\frac{L}{H}\cdot\frac{\partial x}{\partial u} - \frac{M}{H}\cdot\frac{\partial x}{\partial \upsilon},$$

and similarly:

$$\frac{\partial \mu}{\partial u} = -\frac{L}{H}\cdot\frac{\partial y}{\partial u} - \frac{M}{H}\cdot\frac{\partial y}{\partial \upsilon},$$

$$\frac{\partial \nu}{\partial u} = -\frac{L}{H}\cdot\frac{\partial z}{\partial u} - \frac{M}{H}\cdot\frac{\partial z}{\partial \upsilon}.$$

One will get:

$$\frac{\partial \lambda}{\partial \upsilon} = -\frac{1}{H}\left(M\frac{\partial x}{\partial u} + N\frac{\partial x}{\partial \upsilon}\right),$$

$$\frac{\partial \mu}{\partial \upsilon} = -\frac{1}{H}\left(M\frac{\partial y}{\partial u} + N\frac{\partial y}{\partial \upsilon}\right),$$

(†) Translator: We have temporarily replaced (italic v) with υ, since the fonts that we are using make (italic v) identical to (Greek nu).

$$\frac{\partial \nu}{\partial v} = -\frac{1}{H}\left(M \frac{\partial z}{\partial u} + N \frac{\partial z}{\partial v} \right)$$

by an analogous calculation.

Hence, the functions on the sphere that are analogous to E, F, G¸ H will be:

$$\mathcal{E} = \sum \left(\frac{\partial \lambda}{\partial u} \right)^2 = \frac{1}{H^2} \sum \left(L \frac{\partial x}{\partial u} + M \frac{\partial x}{\partial v} \right)^2 = \frac{L^2 + M^2}{H},$$

$$\mathcal{F} = \sum \frac{\partial \lambda}{\partial u} \cdot \frac{\partial \lambda}{\partial v} = \frac{1}{H^2} \sum \left(L \frac{\partial x}{\partial u} + M \frac{\partial x}{\partial v} \right)\left(M \frac{\partial x}{\partial u} + N \frac{\partial x}{\partial v} \right) = \frac{M(L+N)}{H},$$

$$\mathcal{G} = \sum \left(\frac{\partial \lambda}{\partial v} \right)^2 = \frac{1}{H^2} \sum \left(M \frac{\partial x}{\partial u} + N \frac{\partial x}{\partial v} \right)^2 = \frac{M^2 + N^2}{H};$$

hence:

$$\mathcal{H}^2 = \mathcal{E} \cdot \mathcal{G} - \mathcal{F}^2 = \frac{(L^2 + M^2)(M^2 + N^2) - M^2(L^2 + N^2)}{H^2} = \left(\frac{LN - M^2}{H} \right)^2,$$

and the area on the sphere will have the expression:

$$\mathcal{A}' = \iint \frac{LN - M^2}{H} \, du \, dv.$$

One recovers the same expression as before, and one will likewise arrive at the direct definition of total curvature.

Remark. – $\mathcal{A}'$ has a sign in the preceding expression, which is that of $LN - M^2$, because $du\,dv$ is considered to be positive.

The interpretation of that sign results from the identity:

$$\begin{vmatrix} \lambda & \mu & \nu \\ \frac{\partial \lambda}{\partial u} & \frac{\partial \mu}{\partial u} & \frac{\partial \nu}{\partial u} \\ \frac{\partial \lambda}{\partial v} & \frac{\partial \mu}{\partial v} & \frac{\partial \nu}{\partial v} \end{vmatrix} = \frac{LN - M^2}{H^2} \begin{vmatrix} \lambda & \mu & \nu \\ \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix},$$

which indicates whether the two trihedra that are defined by the common direction to the normal to the surface and the normal to the sphere and the positive directions to the curves v = const. and u = const. (when considered on the surface and the sphere, respectively), have the same disposition.

One then concludes that if $\mathcal{A}' > 0$ then the moving point x, y, z will describe the contour that bounds the area on the surface in the direct sense, and the point λ, μ, ν will describe the contour that bounds the homologous area on the sphere, also in the direct sense. If $\mathcal{A}' < 0$ then the conclusions will be the opposite ones.

Relations between total curvature and geodesic curvature

5. – The total curvature is an element that remains invariant under the deformation of surfaces. We shall seek to find what relations exist between it and the other elements that are invariant under deformation. Consider the geodesic curvature. Its expression in orthogonal, isothermal coordinates will be:

$$\frac{1}{R_g} = \frac{1}{H\,ds^2}\left[H^2(du\,d^2v - dv\,d^2u) - \begin{vmatrix} \frac{1}{2}\frac{\partial H}{\partial u}du^2 + \frac{\partial H}{\partial v}du\,dv - \frac{1}{2}\frac{\partial H}{\partial v}dv^2 & H\,du \\ -\frac{1}{2}\frac{\partial H}{\partial v}du^2 + \frac{\partial H}{\partial u}du\,dv + \frac{1}{2}\frac{\partial H}{\partial v}dv^2 & H\,dv \end{vmatrix}\right],$$

or:

$$\frac{1}{R_g} = \frac{1}{ds^2}\left[H^2(du\,d^2v - dv\,d^2u) + \frac{1}{2}\left(\frac{\partial H}{\partial u}dv - \frac{\partial H}{\partial v}du\right)(du^2 + dv^2)\right];$$

however:

$$ds^2 = H\,(du^2 + dv^2),$$

and the preceding formula can be written:

$$\frac{ds}{R_g} = \frac{du\,d^2v - dv\,d^2u}{du^2 + dv^2} + \frac{1}{2}\frac{\partial \log H}{\partial u}dv - \frac{1}{2}\frac{\partial \log H}{\partial v}du\,,$$

or rather:

$$\frac{ds}{R_g} = d\left(\arctan\frac{dv}{du}\right) + \frac{1}{2}\frac{\partial \log H}{\partial u}dv - \frac{1}{2}\frac{\partial \log H}{\partial v}du\,.$$

Now, imagine the semi-tangents MU, MV in the tangent plane to the coordinate curves in the sense of increasing u, v, respectively. Consider the tangent to an arbitrary curve MT to the surface, and let $(MU, MT) = \varphi$:

$$\cos\varphi = \sqrt{H}\,\frac{du}{ds},$$

$$\sin\varphi = \sqrt{H}\,\frac{dv}{ds};$$

hence:

$$\tan\varphi = \frac{dv}{du},$$

so:

$$\varphi = \arctan \frac{dv}{du},$$

and the preceding formula will become:

$$\frac{ds}{R_g} = d\varphi + \frac{1}{2}\frac{\partial \log H}{\partial u} dv - \frac{1}{2}\frac{\partial \log H}{\partial v} du .$$

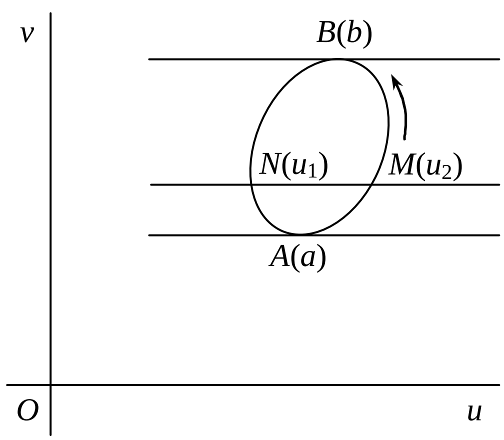
Now take a closed contour on the surface (*S*) and integrate along it in the direct sense:

$$\int \frac{ds}{R_g} = \int d\varphi + \tfrac{1}{2}\int \frac{\partial \log H}{\partial u} dv - \tfrac{1}{2}\int \frac{\partial \log H}{\partial v} du .$$

Recall *Green's theorem*, which will allow us to transform that result. The point (u, v) describes a closed contour in the *uv*-plane, also in the direct sense. Suppose that it is composed of two tangents that are parallel to the *u*-axis; let *A*, *B* be their contact points. We will then have two arcs *AMB* and *ANB*, and if we denote the contour by *C* then we will have:

$$\int_C \frac{\partial f}{\partial u} dv = \int_{AMB} \frac{\partial f}{\partial u} dv + \int_{BNA} \frac{\partial f}{\partial u} dv$$

for any function $f(u, v)$.



Suppose that a parallel to *Ou* that is found between the two tangents considered cuts the contour at two points $M(u_2)$ and $N(u_1)$.

Finally, let *a*, *b* be the values of *u* that correspond to the two points *A*, *B*. We will have:

$$\int_C \frac{\partial f}{\partial u} dv = \int_a^b \left(\frac{\partial f}{\partial u}\right)_{u=u_2} dv - \int_a^b \left(\frac{\partial f}{\partial u}\right)_{u=u_1} dv = \int_a^b \left[\left(\frac{\partial f}{\partial u}\right)_2 - \left(\frac{\partial f}{\partial u}\right)_1\right] dv .$$

However:

$$\left(\frac{\partial f}{\partial u}\right)_2 - \left(\frac{\partial f}{\partial u}\right)_1 = \int_{u_1}^{u_2} \frac{\partial^2 f}{\partial u^2} du ,$$

and then:

$$\int_C \frac{\partial f}{\partial u} dv = \int_a^b dv \int_{u_1}^{u_2} \frac{\partial^2 f}{\partial u^2} du = \iint \frac{\partial^2 f}{\partial u^2} du\, dv,$$

in which the double integral is taken over the entire area that is bounded by the contour.

That formula will persist for an arbitrary simple contour.

Similarly:

$$\int_C \frac{\partial f}{\partial v} du = -\iint \frac{\partial^2 f}{\partial v^2} du\, dv.$$

Hence:

$$\int \frac{ds}{R_g} = -\int d\varphi - \iint \frac{H}{R_1 R_2} \cdot du\, dv = \int d\varphi - \iint \frac{d\mathcal{A}}{R_1 R_2} ,$$

and one gets the *formula of Ossian Bonnet:*

$$\mathcal{A}' = \iint \frac{d\mathcal{A}}{R_1 R_2} = \int d\varphi - \int \frac{ds}{R_g} .$$

Remark. – The angle φ is the angle that MU makes with the tangent MT to the curve. Suppose that at each point of the surface, one has determined a direction MO whose direction cosines are well-defined functions of u, v as in Chap. II, § 4 (page 35). Let $\psi = (MO, MU)$ and $\varphi_0 = (MO, MT)$. One will then have:

$$\varphi_0 = \psi + \varphi,$$

so:

$$d\varphi_0 = d\psi + d\varphi.$$

Integrate along an arbitrary closed contour:

$$\int d\varphi_0 = \int d\psi + \int d\varphi .$$

Now, ψ is a function of u, v, so along any closed contour one will have:

$$\int d\psi(u,v) = 0;$$

hence:

$$\int d\varphi_0 = \int d\varphi ,$$

and one can replace the angle φ with the angle φ_0 that was defined before.

One then sees the geometric element $\left(\frac{ds}{R_g} - d\varphi_0\right)$ that was introduced in Chap. II, pp. 34 appear in the study of total curvature. (Cf., Chap. III, pp. 55)

Geodesic triangles

We call the figure that is formed by three geodesic lines a *geodesic triangle.* Along each of its edges:

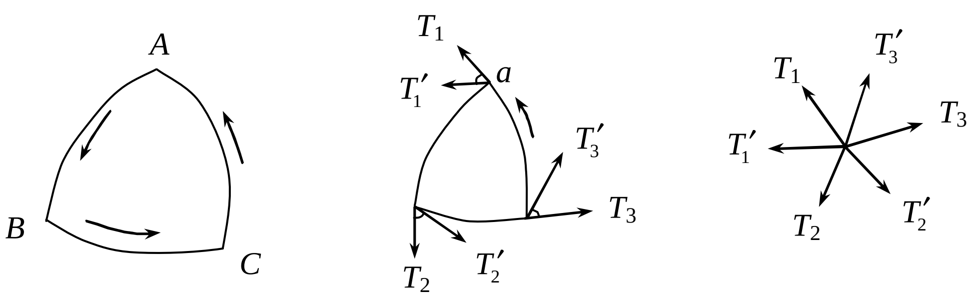
$$\int \frac{ds}{R_g} = \int \frac{\sin\theta}{R} ds = 0,$$

and the formula of O. Bonnet shows us that:

$$\mathcal{A}_0' = \int d\varphi;$$

i.e.:

$$\mathcal{A}_0' = \int_{AB} d\varphi + \int_{BC} d\varphi + \int_{CA} d\varphi .$$



The orthogonal, isothermal coordinates provide a conformal representation of the surface on the *uv*-plane. Hence, consider the representation *abc* of the triangle *ABC* on that plane. Draw tangents to the edges at the extremities *a*, *b*, *c* in the direct sense; let T_1, T_2, T_3, T_1', T_2', T_3' be those tangents. If the = sign indicates the equalities that are true up to a multiple of 2π then we will have:

$$\int_{AB} d\varphi = (T_1', T_2), \qquad \int_{BC} d\varphi = (T_2', T_3), \qquad \int_{CA} d\varphi = (T_3', T_1).$$

Hence, if we call the angles of the geodesic triangle *a*, *b*, *c* then we will get the following value for $\mathcal{A}_0'$:

$$(T_1', T_2) + (T_2', T_3) + (T_3', T_1) = -[(T_1, T_1') + (T_2, T_2') + (T_3, T_3')] + [(T_1, T_2) + (T_2, T_3) + (T_3, T_1)]$$

$$\equiv 2\pi - [(\pi - a) + (\pi - b) + (\pi - c)] \equiv a + b + c - \pi,$$

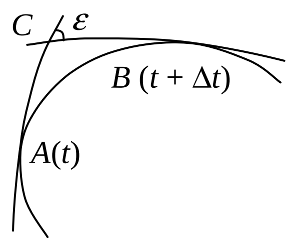
and therefore *Gauss's formula:*

$$a + b + c - \pi = \mathcal{A}'_0,$$

in which we have used the = sign because the two sides tend to zero when the three summits of the triangle *ABC* tend to the same point.

In particular, if the surface is a sphere of radius *R* then one will get the formula that gives the area of a spherical triangle:

$$\mathcal{A} = R^2 \mathcal{A}' = R^2 (a + b + c - \pi).$$



New definition of the geodesic curvature

Consider an arc *AB*. Draw the geodesics that are tangent to that curve at *A* and *B*, and which intersect at *C´* with an angle that we call the *geodesic contingency angle.* Along the contour of that triangle:

$$\int d\varphi = -\varepsilon,$$

and the formula of O. Bonnet gives us:

$$-\varepsilon - \int_{AB} \frac{ds}{R_g} = \iint d\mathcal{A}'.$$

Suppose that *A* corresponds to the parameter *t*, and *B*, to $t + \Delta t$, and that Δt tends to 0; let Δs be the arc *AB*. We will have:

$$-\frac{\varepsilon}{\Delta s} - \frac{1}{\Delta s}\int_{AB} \frac{ds}{R_g} = \frac{1}{\Delta s}\iint d\mathcal{A}'_0 .$$

Let $\left(\frac{1}{R_g}\right)_m$ be the mean value of the geodesic curvature on the arc *AB*, so;

$$\frac{1}{\Delta s}\int_{AB} \frac{ds}{R_g} = \left(\frac{1}{R_g}\right)_m ,$$

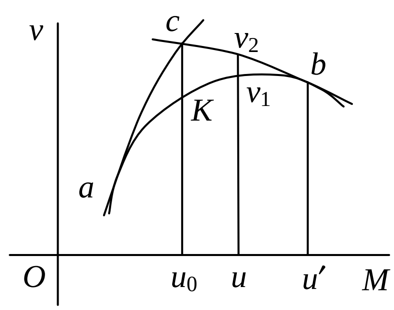
and as a result:

$$-\frac{1}{\Delta s}-\left(\frac{1}{R_g}\right)_m = \frac{1}{\Delta s}\iint d\mathcal{A}'_0 .$$

If Δs tends to 0 then $\left(\frac{1}{R_g}\right)_m$ will have the geodesic curvature at the point A as its limit. I say that the right-hand side has the limit 0; it will suffice to show that $\iint d\mathcal{A}'$ is infinitely small of at least second order. Consider the representation *abc* of the triangle *ABC* on the *uv*-plane.

$$\iint d\mathcal{A}' = \iint \psi(u,v)\, du\, dv = [\psi(u, v)]_m \iint du\, dv .$$

From Green's theorem, $\iint du\, dv$ is equal to the curvilinear integral $\int v\, du$, up to sign. Let v_2, v_1 be the expressions for v as functions of u in the arcs *bc* and *bk*. The part of the integral $\int v\, du$ that is given by those arcs is $\int_{u_0}^{u'}(v_2 - v_1)\, du$. Now, since the curves *ab* and *bc* are tangent at *b*, $v_2 - v_1$ is infinitely small of at least second order with respect to $u' - u$ and *a fortiori* with respect to $(u' - u_0)$. The integral $\int_{u_0}^{u'}(v_2 - v_1)\, du$, which is equal to the product of $(u' - u_0)$ with the mean value of $v_2 - v_1$, will then be of at least third order with respect to $(u' - u_0)$, and as a result with respect to Δs. The same argument applies to the other arcs *ac* and *ak*, so one sees that $\iint d\mathcal{A}'_0$ has order at least three, and the property is established.



The geodesic curvature can then be defined to be the curvature in plane geometry: i.e., *the limit of the ratio of the (geodesic) contingency angle to the arc of the curve* when the latter tends to zero.

Surfaces of constant total curvature

6. – We have seen that the surfaces of constant zero total curvature are the plane and the developable surfaces (§ 3). Now consider the surfaces of constant non-zero total curvature. Among them, one finds the spheres, and a sphere of radius R will have a total curvature of $1 / R^2$. We shall seek the linear element of the surfaces of constant total curvature $\frac{1}{R_1 R_2} = k$ in the form:

$$ds^2 = 2F\, du \cdot dv.$$

Now:

$$\frac{1}{R_1 R_2} = -\frac{1}{F}\frac{\partial^2 \log F}{\partial u\, \partial v},$$

so the problem will amount to the integration of the partial differential equation (*Liouville's equation*):

$$(1) \qquad \frac{\partial^2 \log F}{\partial u\, \partial v} = -k\, F.$$

The solution that is provided by the spheres of radius R for $k = 1 / R^2$ permits one to predict what the general integral will be.

Indeed, refer the sphere of radius R that has the origin for its center to its minimal lines; i.e., its rectilinear generators. One immediately deduces the parametric equations of the sphere from the equations of those generators:

$$x + iy = u\,(R - z), \qquad x - iy = \frac{1}{u}(R + z),$$

$$x + iy = v\,(R - z), \qquad x - iy = \frac{1}{v}(R + z),$$

namely:

$$(2) \qquad x + iy = \frac{2Ruv}{u+v}, \qquad x - iy = \frac{2R}{u+v}, \qquad z = R\frac{u-v}{u+v};$$

hence, for:

$$ds^2 = d\,(x + iy) \cdot d\,(x - iy) + dz^2,$$

one will infer the value:

$$(3) \qquad ds^2 = -\frac{4R^2 du\, dv}{(u+v)^2}.$$

The most general change of curvilinear coordinates that preserves the minimal lines as coordinate lines is:

$$u = V\,(u_1), \qquad v = V\,(v_1),$$

in which U, V are arbitrary functions of their arguments. Upon making that change in formula (3) and putting the letters u, v back in place of u_1, v_1, one will get the expression:

$$(4) \qquad ds^2 = -\frac{4R^2 U' V'}{(U+V)^2} du\, dv$$

for the ds^2 of any sphere of radius R when it is referred to its minimal lines.

Equation (1) is then verified by:

$$F = -\frac{2U'V'}{k(U+V)^2}, \tag{5}$$

and since U and V are two arbitrary functions, one predicts that this will be the general integral of (1).

We prove that by integrating (1) directly. Set:

$$-k\,F = w, \tag{6}$$

which will reduce equation (1) to the equation:

$$\frac{\partial^2 \log w}{\partial u\,\partial v} = w. \tag{7}$$

Upon introducing an auxiliary unknown φ, this will be equivalent to the system:

$$\frac{\partial \varphi}{\partial u} = w, \qquad \frac{\partial w}{\partial v} = \varphi\, w; \tag{8}$$

hence, one concludes that:

$$\varphi \frac{\partial \varphi}{\partial u} = \frac{\partial w}{\partial v}, \quad \text{or} \qquad \frac{\partial \varphi^2}{\partial u} = \frac{\partial (2w)}{\partial v},$$

in such a way that upon denoting a new auxiliary unknown by ψ, equation (7) will be equivalent to the system:

$$2w = \frac{\partial \psi}{\partial u}, \qquad \varphi^2 = \frac{\partial \psi}{\partial v}, \qquad \frac{\partial \varphi}{\partial u} = w = \frac{1}{2}\cdot\frac{\partial \psi}{\partial u}. \tag{9}$$

It results from these equations when one integrates the last one that:

$$\varphi = \tfrac{1}{2}\,\psi + V_0\,,$$

in which V_0 is a function of only v, and in turn:

$$\frac{\partial \psi}{\partial v} = \left(\tfrac{1}{2}\psi + V_0\right)^2.$$

That equation is a Ricatti equation (cf., Chap. V, § 10), so ψ will have the form:

$$\psi = \frac{U\,V_1 + V_2}{U+V},$$

in which U, which is a function of only u, plays the role of integration constant with respect to v, and V, V_1, V_2 are functions of only v.

The first of equations (9) then gives:

$$w = \frac{U' V_3}{2(U+V)^2} \qquad (V_3 = V_1 V - V_2).$$

If one substitutes that value in equation (7) then one will find immediately that $V_3 = 4V'$, while U and V remain arbitrary. One has, in fact, formula (5) as the general integral of (1) then.

Hence: *The ds^2 of any surface of constant total curvature* $\frac{1}{R_1 R_2} = k$ *will be:*

$$(10) \qquad ds^2 = -\frac{2U'V'}{k(U+V)^2} du\, dv$$

when referred to its minimal lines, and can be reduced to the typical form:

$$(1) \qquad ds^2 = -\frac{2du\, dv}{k(u+v)^2}$$

by a convenient choice of coordinates.

It then results from this that *in order for two surfaces of constant total curvature to be mappable to each other, it is necessary and sufficient that they have the same curvature.* The question of the reality of the correspondence that realizes the map from one surface to the other one is contained in that statement, moreover.

Pseudo-sphere. – The spheres of radius R serve as examples of surfaces of positive constant total curvature $k = 1 / R^2$. We seek a surface of revolution of negative constant curvature $k = -1 / R^2$. Let Oz be the axis of revolution, let M be a point on the principal meridian that is situated in the plane zOx, let x, z be its coordinates, and let $\theta = (Ox, MT)$ be the angle between the positive semi-tangent MT and Ox, when measured positively from Ox to Oz. Since the positive semi-normal MN is defined by $(Ox, MN) = \theta + \pi / 2$, the center of curvature C_1 of the principal meridian, which is one of the principal sections of the surface, will be given by the formula:

$$MC_1 = \frac{ds}{d\theta} = \frac{dx}{\cos\theta\, d\theta},$$

which is true in magnitude and sign.

The second principal section is tangent to the parallel to the point M, so Meusnier's theorem shows that its center of curvature C_2 is at the intersection of Oz and the normal to the meridian, and one will have:

$$MC_2 = \frac{x}{\sin\theta}$$

in magnitude and sign.

The equation of the problem $MC_1 \cdot MC_2 = R_1 R_2 = -R^2$ is then written:

$$x\,dx = -R^2 \sin\theta \cos\theta\, d\theta.$$

We confine ourselves to the solution:

$$x = -R\cos\theta. \tag{12}$$

If one denotes the point where the tangent meets Oz by S then one will have:

$$MS = -\frac{x}{\cos\theta}$$

in magnitude and sign. Equation (12) then expresses the idea that *the desired meridian is the curve of equal tangents, or* ***tractrix***. One succeeds in determining it by integrating:

$$dz = \tan\theta \cdot dx = R\frac{\sin^2\theta}{\cos\theta}d\theta = R\left(\frac{1}{\cos\theta} - \cos\theta\right)d\theta.$$

One can suppress the constant of integration, on the condition that one must choose the origin O conveniently on the axis of revolution, and one will get:

$$x = -R\cos\theta, \qquad z = R\left[\log\tan\left(\frac{\pi}{4} + \frac{\theta}{2}\right) - \sin\theta\right] \tag{13}$$

for the equations of the desired meridian (i.e., the tractrix).

The surface of revolution that it generates by turning around its base Oz is called a ***pseudo-sphere***.

Remark. – The importance of the surfaces of constant total curvature amounts to the fact that, like the plane, they can be mapped to themselves in an infinitude of ways. Such a surface can then slide over itself by way of ∞^3 continuous motions, under which the surface can deform, but in such a manner that any arc of a curve is traced on the surface will keep the same length. It will then result from this that the *geometries* of those surfaces – which are called *non-Euclidian geometries* – are analogous to plane geometry, but from the preceding (geodesic lines play the role of lines in the plane), the sum of the angles of a triangle will be greater or less than π according to whether the total curvature is positive or negative (spherical or pseudo-spherical geometry, resp.), respectively.

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CHAPTER V

RULED SURFACES

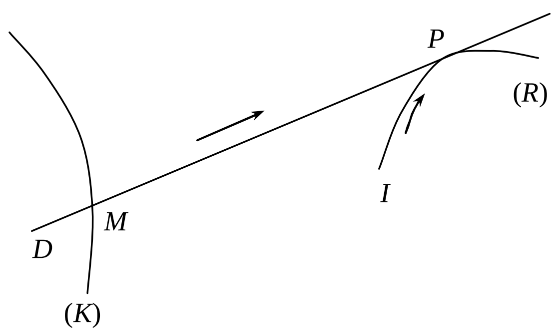
Developable surfaces

1. – In order to define the variation of the line that generates a ruled surface, we give the trajectory of a point *M* on that line and the direction of that line for each position of the point *M*. The coordinates of a point on the surface are then expressed as functions of two parameters, one of which defines the position of the point *M* along its trajectory (*K*), while the other one defines the position of the point *P* considered along the line (*D*). Let:

$$x = f(v), \qquad y = g(v), \qquad z = h(v)$$

be the expressions for the coordinates of a point on the curve *K*. Let $l_0(v)$, $m_0(v)$, $n_0(v)$ be the direction coefficients of the generator (*D*), and let *u* be the ratio of the vector *MP* to the vector whose components are l_0, m_0, n_0 . The coordinates of *P* are:

(1) $$x = f(v) + u \cdot l_0(v), \quad y = g(v) + u \cdot m_0(v), \quad z = h(v) + u \cdot n_0(v).$$



Let us look for the condition for the surface that is defined by the preceding equations to be developable. If we exclude the cases of the cylinder and the cone then the necessary and sufficient condition will be that the generators must be tangent to the same skew curve. One must then be able to find a point *P* on the generator (*D*) such that its trajectory is constantly tangent to (*D*); the coordinates *x*, *y*, *z* of such a point must be such that:

$$\frac{dx}{l_0} = \frac{dy}{m_0} = \frac{dz}{n_0} = d\rho\,;$$

hence:

(2) $$dx = l_0\, d\rho, \qquad dy = m_0\, d\rho, \qquad dz = n_0\, d\rho.$$

However equations (1) give:

$$dx = df + u\, dl_0 + l_0\, du, \qquad dy = dg + u\, dm_0 + m_0\, du, \qquad dz = dh + u\, dn_0 + n_0\, du,$$

and equations (2) will be written:

$$df + u\,dl_0 + l_0\,(du - d\rho) = 0,$$
$$dg + u\,dm_0 + m_0\,(du - d\rho) = 0,$$
$$dh + u\,dn_0 + n_0\,(du - d\rho) = 0,$$

or, upon setting:

(3) $$d\sigma = du - d\rho,$$

(4) $$df + u\,dl_0 + l_0\,d\sigma = 0, \qquad dg + u\,dm_0 + m_0\,d\sigma = 0, \qquad dh + u\,dn_0 + n_0\,d\sigma = 0.$$

$d\sigma$ and u must satisfy these three linear equations. Hence, the determinant of these equations must be zero:

(5) $$\begin{vmatrix} df & dl_0 & l_0 \\ dg & dm_0 & m_0 \\ dh & dn_0 & n_0 \end{vmatrix} = 0.$$

If the three determinants that are deduced from the matrix:

$$\begin{Vmatrix} dl_0 & dm_0 & dn_0 \\ l_0 & m_0 & n_0 \end{Vmatrix}$$

are not all zero then there will exist values of u and $d\sigma$ that satisfy equations (4), and the condition (5) will be sufficient. If those three determinants are identically zero then one will have:

$$\frac{dl_0}{l_0} = \frac{dm_0}{m_0} = \frac{dn_0}{n_0},$$

and the integration of those equations shows that l_0, m_0, n_0 are proportional to fixed quantities; the surface will then be a cylinder. If we discard that case then the condition (5) will be necessary and sufficient.

Remark 1. – In order for the point P to effectively describe a curve, it is necessary that dx, dy, dz, and in turn $d\rho$, must not be identically zero. If $d\rho$ is identically zero then all of the generators will pass through a fixed point, and the surface will be a cone. The condition (5) will then be applied to the case of the cone.

Remark 2. – One often employs the equations of the generator in the form:

$$x = Mz + P, \qquad y = Nz + Q,$$

in which M, N, P, Q are functions of an arbitrary parameter. This is a particular case of the general representation (1) in which one sets $h(v) = 0$ and $n_0(v) = 1$; one will then have $z = u$, and:

(6) $$x = f(v) + z \cdot l_0(v), \qquad y = g(v) + z \cdot m_0(v).$$

The direction coefficients are l_0, m_0, 1. The curve (K) is then the section by the plane z = 0. In this case, the condition (5) takes the simple form:

(7) $$\begin{vmatrix} df & dl_0 \\ dg & dm_0 \end{vmatrix} = 0, \qquad \text{i.e.,} \qquad \begin{vmatrix} dM & dP \\ dN & dQ \end{vmatrix} = 0.$$

Properties of developables

Let us return to the general case. Suppose that l_0, m_0, n_0 are the direction cosines of the generator; hence:

$$l_0^2 + m_0^2 + n_0^2 = 1,$$

so:

$$l_0\, dl_0 + m_0\, dm_0 + n_0\, dn_0 = 0.$$

Multiply equations (4) by dl_0, dm_0, dn_0, respectively, and add them, which will give:

$$n = -\frac{\sum dl_0\, df}{\sum dl_0^2}.$$

Suppose, in addition, that the generator (D) is normal to the curve (K). Indeed, it is possible to find orthogonal trajectories to the generators on a ruled surface. It will suffice that x, y, z are such that:

$$\sum l_0\, dx = 0,$$

or

$$\sum l_0\, df + u \sum l_0\, dl_0 + \sum l_0^2\, du = 0.$$

Since one has:

$$\sum l_0^2 = 1, \qquad \sum l_0\, dl_0 = 0$$

here, that condition will reduce to:

$$\sum l_0\, df + du = 0,$$

and the determination of the orthogonal trajectory will be accomplished by means of one quadrature.

Therefore, if we suppose that (K) is normal to the generator then we will have:

$$\sum l_0\, df = 0.$$

If we multiply equations (4) by l_0, m_0, n_0 , respectively, and add them then we will get $d\sigma = 0$, so $d\rho = du$, and equations (2) will become:

$$dx = l_0\, du, \qquad dy = m_0\, du, \qquad dz = n_0\, du.$$

However, since l_0, m_0, n_0 are the direction cosines of the tangent to the edge of regression (R), u will represent the arc length of that curve, as measured in the positive sense that is chosen on the generator by starting with an arbitrary origin I, and since u also represents the segment MP, one will see that:

$$d \cdot MP = d \cdot (\text{arc } IP);$$

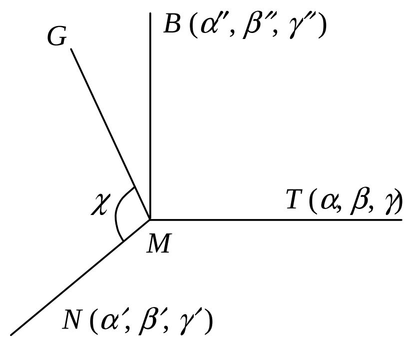
hence:

$$MP = \text{arc } IP + \text{const.}$$

One can always choose the origin I of the arc in such a fashion that the constant is zero. One will then have MP = arc IP. The curve (K) is a development of the curve (R). *The orthogonal trajectories of the generators on a developable surface are involutes of the edge of regression.*

Formulas (4) then give:

$$(4') \qquad df + u\, dl_0 = 0, \qquad dg + u\, dm_0 = 0, \qquad dh + u\, dn_0 = 0.$$



Developments of skew curves

2. – Suppose that one is given the curve (K), and one seeks to draw a normal to that curve at each of its points in such a fashion as to obtain a developable surface. We take that variable to be the arc length s of the curve (K). Consider the Serret trihedron at the point M of the curve. Let MG be the desired normal; it is in the normal plane to the curve. In order to define it, it will then suffice to give the angle (MN, MG) = χ. The point at a unit distance along MG has coordinates 0, cos χ, sin χ with respect to the Serret trihedron. Hence, if l_0, m_0, n_0 are the direction cosines of MG then:

$$\begin{cases} l_0 = \alpha' \cos\chi + \alpha'' \sin\chi, \\ m_0 = \beta' \cos\chi + \beta'' \sin\chi, \\ n_0 = \gamma' \cos\chi + \gamma'' \sin\chi. \end{cases}$$

Now, since v is the arc length on the curve (K):

$$df = \alpha\, dv, \qquad dg = \beta\, dv, \qquad dh = \gamma\, dv.$$

If one takes the Frenet formulas into account then formulas (4′) will give:

$$\alpha\,dv + u\left[(-\alpha'\sin\chi+\alpha''\cos\chi)\,d\chi-\left(\frac{\alpha}{R}+\frac{\alpha''}{T}\right)\cos\chi\cdot dv+\frac{\alpha''}{T}\sin\chi\cdot dv\right]=0,$$

or

$$\alpha\left[1-\frac{u}{R}\cos\chi\right]+\alpha' u\left[\frac{1}{T}-\frac{d\chi}{dv}\right]\sin\chi+\alpha'' u\left[\frac{d\chi}{dv}-\frac{1}{T}\right]\cos\chi = 0,$$

and two analogous equations in β, β', β'', and $\gamma, \gamma', \gamma''$. We will then have three equations that are linear and homogeneous in the coefficients $\alpha, \alpha', \alpha'', \beta, \beta', \beta'', \gamma, \gamma', \gamma''$. The determinant of those equations is 1, so the unknowns are all zero, and since u is not constantly zero:

$$1-\frac{u\cos\chi}{R}=0,\qquad \sin\chi\left[\frac{d\chi}{dv}-\frac{1}{T}\right]=0,\qquad \cos\chi\left[\frac{d\chi}{dv}-\frac{1}{T}\right]=0.$$

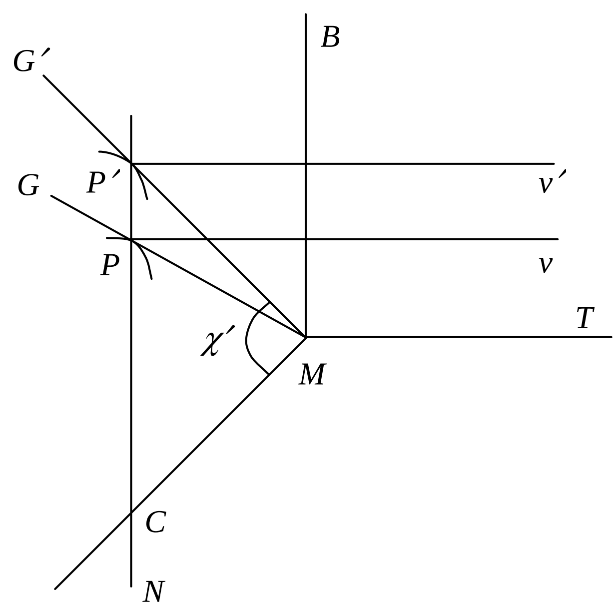
If one replaces v with the arc length s then the last two will give:

(1) $$\frac{d\chi}{ds}=\frac{1}{T},$$

and the first one will give:

(2) $$u=\frac{R}{\cos\chi}.$$

There is then an infinity of solutions: χ is determined by a quadrature.



Formula (2) shows us that:

$$R = u\cos\chi.$$

Hence, the projection of the point P where the normal MG touches its envelope on the principal normal is the center of curvature C. *The contact point of the normal with its envelope is on the polar line. The developments of a curve are on the polar surface.*

Consider two solutions χ, χ' of equation (1), so their difference is a constant. The two normals MG, MG' cut at a constant angle. Therefore, *when a normal to a curve describes a developable surface, if one rotates it in each of its positions through a constant angle around the tangent then the line that one obtains will again describe a developable.*

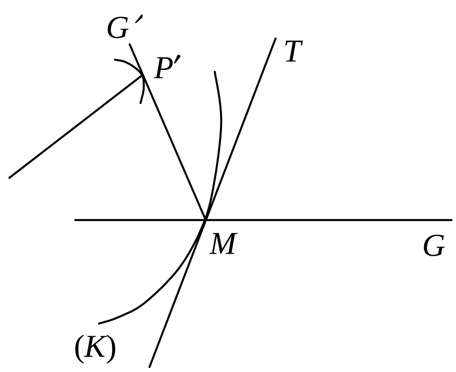
The osculating plane to a developable is the tangent plane to the corresponding developable: It is the plane GMT. That plane is normal to the plane BMC, which is the tangent plane to the polar surface. *Hence, the developments are the geodesics of the polar surface.*

Consider the principal normal Pv to the development at P. It is in the osculating plane GMT and perpendicular to the tangent MP, and therefore parallel to MT. *The principal normals to the developments of a curve are parallel to the tangents to the curve. The plane normal to the curve is the rectifying plane to all of its developables.*

Upon starting from a curve (R) and remarking that the given curve (K) is the involute, one can state the preceding properties in such a fashion as to obtain properties of the involutes of a curve.

Lines of curvature

3. – Consider a line of curvature (K) on a surface (S) and the circumscribed developable to (S) along (K). The direction of a generator MG of that developable is conjugate to the tangent MT to the line of curvature, and consequently, it will be perpendicular to MT; i.e., normal to (K). That generator MG will then be constantly tangent to a development of the line of curvature, and we see that *the normals to a line of curvature that are tangent to a surface will generate a developable.* The converse is established by an analogous argument.



If we rotate MG through a right angle around the tangent then we will get a line MG' that will be normal to the surface, since it is perpendicular to the two tangents to the surface MT, MG. Therefore, *the normals to the surface at all points of a line of curvature will generate a developable and conversely.*

Consider the point P' where the line MG' touches its envelope. It is the point where the polar line to the line of curvature meets the normal to the surface. Now, from Meusnier's theorem, the polar lines to all curves on the surface that are tangent at M will meet the normal at M at the same point, which is the center of the curvature of the corresponding normal section. P' will then be the center of curvature of the principal section $G'MT$, so it will be one of the principal centers of curvature of the surface at the point M.

Therefore, recall formulas (4) of § 1 for the normal MG', which we write:

$$dx + u\,d\lambda = 0, \qquad dy + u\,d\mu = 0, \qquad dz + u\,dv = 0.$$

Replace f, g, h in them with the coordinates x, y, z of the point M and l_0, m_0, n_0 with the direction cosines λ, μ, v of the normal to the surface: u is the radius of principal curvature R. We then obtain the *formulas of Olinde Rodrigues:*

$$dx + R\,d\lambda = 0, \qquad dy + R\,d\mu = 0, \qquad dz + R\,dv = 0$$

for a displacement along a line of curvature.

Joachimsthal's theorems are easily deduced from the preceding. Suppose that the intersection (K) of the two surfaces (S), (S_1) is a line of curvature for each of them. Let MG', MG_1' be the normals to the two surfaces at a point M of (K). They generate two developables, and thus envelop two developments of (K), and in turn the angle between them will be constant. *Conversely*, if the intersection (K) of (S), (S_1) is a line of curvature of (S_1), and if the angle between the two surfaces is constant along (K) then the normal MG_1' to (S_1) will generate a developable, and since MG' makes a constant angle with MG_1', it will also generate a developable, so (K) will be a line of curvature on (S).

Differential equation of the lines of curvature. – When the condition (5) for a line to generate a developable surface is applied to the normal MG', it will be written:

$$\begin{vmatrix} dx & d\lambda & \lambda \\ dy & d\mu & \mu \\ dz & d\nu & \nu \end{vmatrix} = 0$$

here, or:

$$\begin{vmatrix} \dfrac{\partial x}{\partial u}\cdot du + \dfrac{\partial x}{\partial v}\cdot dv & \dfrac{\partial \lambda}{\partial u}\cdot du + \dfrac{\partial \lambda}{\partial v}\cdot dv & \lambda \\ \cdots & \cdots & \mu \\ \cdots & \cdots & \nu \end{vmatrix} = 0.$$

Multiply this by the determinant:

$$\begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} & \lambda \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} & \mu \\ \dfrac{\partial z}{\partial u} & \dfrac{\partial z}{\partial v} & \nu \end{vmatrix},$$

which is not zero.

We will get:

$$\begin{vmatrix} E\,du + F\,dv & -L\,du - M\,dv & 0 \\ F\,du + G\,dv & -M\,du - N\,dv & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0,$$

and we will then recover the *differential equation for the lines of curvature:*

$$\begin{vmatrix} E\,du + F\,dv & L\,du + M\,dv \\ F\,du + G\,dv & M\,du + N\,dv \end{vmatrix} = 0.$$

Remark. – If the equation of the surface is taken in the form $z = f(x, y)$ then the equations of the normal will be:

$$X = (x + pz) - p\,Z, \qquad Y = (y + qz) - q\,Z,$$

and the same method, when applied to them, and appealing to the condition (7) [§ 1]:

$$\begin{vmatrix} dM & dP \\ dN & dQ \end{vmatrix} = 0$$

will easily give the differential equation:

$$\begin{vmatrix} dx + p\,dz & dp \\ dy + q\,dz & dq \end{vmatrix} = 0.$$

Development of a developable surface onto a plane

4. – *Any developable surface can be mapped to a plane.*

Incidentally, that theorem and its converse were obtained in Chap. IV, § 3.

We shall establish it directly and study the actual development of a developable surface onto a plane.

Indeed, one must observe that we have not discussed the reality of the pairs of homologous points in the correspondences that were considered in Chap. II (§ 2).

First consider the case of the cylinder whose equations are:

$$x = f(v) + u \cdot l_0, \qquad y = g(v) + u \cdot m_0, \qquad z = h(v) + u \cdot n_0,$$

in which l_0, m_0, n_0 are constants. We deduce from them that:

$$dx = f'(v)\,dv + l_0 \cdot du, \qquad dy = g'(v)\,dv + m_0 \cdot du, \qquad dz = h'(v)\,dv + n_0 \cdot du,$$

so:

$$ds^2 = \sum f'^2(v)\,dv^2 + 2\sum l_0\, f'(v) \cdot du\,dv + \sum l_0^2 \cdot du^2 .$$

We suppose that the directrix:

$$x = f(v), \qquad y = g(v), \qquad z = h(v)$$

is a cross section, in such a way that $\sum l_0 f' = 0$. Suppose that l_0, m_0, n_0 are direction cosines, so $\sum l_0^2 = 1$. Finally, since v is the arc length along the cross section, $\sum f'^2 = 1$. Hence:

(1) $$ds^2 = du^2 + dv^2,$$

which is the linear element of a plane in rectangular coordinates. *A cylinder can be mapped to a plane*, and (1) gives the well-known law of that development.

Now, look at the case of the cone:

$$x = u \cdot l_0(v), \quad y = u \cdot m_0(v), \quad z = u \cdot n_0(v),$$

where u is the length along the generator when one starts from its summit. Suppose that l_0, m_0, n_0 are direction cosines of the generator, and v is the arc length of the spherical curve $u = 1$ that is the intersection of the cone with the sphere of radius one. Hence:

$$dx = u\, l_0'(v)\,dv + l_0(v)\,du, \qquad dy = u\, m_0'(v)\,dv + m_0(v)\,du, \qquad dz = u\, n_0'(v)\,dv + n_0(v)\,du,$$

and

(2) $$ds^2 = u^2\,dv^2 + du^2.$$

This is the linear element of a plane in polar coordinates. A *cone can be mapped to a plane*. (2) gives the well-known law of the development.

Finally, we pass to the general case.

$$x = f(v) + u \cdot l_0(v), \qquad y = g(v) + u \cdot m_0(v), \qquad z = h(v) + u \cdot n_0(v).$$

We suppose that the curve $x = f(v)$, $y = g(v)$, $z = h(v)$ is the edge of regression, where v is the arc length along that curve, l_0, m_0, n_0 are the direction cosines of the tangent at a point, and u is the distance when reckoned along that tangent when starting from the point of contact. Hence:

$$l_0 = f' = \alpha, \qquad m_0 = g' = \beta, \qquad n_0 = h' = \gamma,$$

and:

$$l_0' = \frac{d\alpha}{dv} = \frac{\alpha'}{R}, \qquad m_0' = \frac{d\beta}{dv} = \frac{\beta'}{R}, \qquad n_0' = \frac{d\gamma}{dv} = \frac{\gamma'}{R}.$$

Hence:

$$dx = \alpha\, dv + u\frac{\alpha'}{R}dv + \alpha\, du, \quad dy = \beta\, dv + u\frac{\beta'}{R}dv + \beta\, du, \quad dz = \gamma\, dv + u\frac{\gamma'}{R}dv + \gamma\, du,$$

and:

$$ds^2 = [d\,(u + v)]^2 + \frac{u^2}{R^2}dv^2.$$

That element will remain the same if R keeps the same expression as a function of v. Hence, *the linear element is the same for all developable surfaces whose edges of regression are curves whose radius of curvature has the same expression as a function of the arc length:*

$$R = \Phi\,(v).$$

We can determine a planar curve whose radius of curvature is expressed as a function of the arc length by means of the preceding equation. We take the coordinates in the plane of that curve to be the arc length s of the curve and the distance when reckoned along the tangent by starting from the contact point. The developable will then be mappable onto that plane. When the developable is given, one can determine its edge of regression by algebraic operations and then determine the arc length along that edge of regression by a quadrature. Its radius of curvature will then determined by an equation of the form:

$$R = \Phi\,(s).$$

One must construct a planar curve that satisfies that condition. If θ is the angle between the tangent and Ox then one knows that:

$$R = \frac{ds}{d\theta};$$

hence:

$$\frac{ds}{d\theta} = \Phi\,(s),$$

so

$$\theta = \int \frac{ds}{\Phi(s)},$$

and therefore:

$$dx = \cos\,\theta\, ds, \qquad dy = \sin\,\theta\, ds.$$

x, y are determined by means of three quadratures. The curve that one obtains is homologous to the edge of regression in the development.

Converse

Conversely, any surface that can be mapped to a plane is a developable surface.

Let the surface be:

$$x = f(u, v), \qquad y = g(u, v), \qquad z = h(u, v),$$

which we assume to be mappable to a plane. Upon choosing the coordinates u, v suitably, we will have:

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2 = du^2 + dv^2,$$

so:

$$\sum\left(\frac{\partial x}{\partial u}\right)^2 = 1, \qquad \sum\frac{\partial x}{\partial u}\cdot\frac{\partial x}{\partial v} = 0, \qquad \sum\left(\frac{\partial x}{\partial v}\right)^2 = 1.$$

If we differentiate these relations with respect to u, v in succession then we will get:

$$\sum\frac{\partial x}{\partial u}\cdot\frac{\partial^2 x}{\partial u^2} = 0, \qquad \sum\frac{\partial^2 x}{\partial u^2}\cdot\frac{\partial x}{\partial v} + \sum\frac{\partial x}{\partial u}\cdot\frac{\partial^2 x}{\partial u\,\partial v} = 0, \qquad \sum\frac{\partial x}{\partial v}\cdot\frac{\partial^2 x}{\partial u\,\partial v} = 0,$$

$$\sum\frac{\partial x}{\partial u}\cdot\frac{\partial^2 x}{\partial u\,\partial v} = 0, \qquad \sum\frac{\partial^2 x}{\partial u\,\partial v}\cdot\frac{\partial x}{\partial v} + \sum\frac{\partial x}{\partial u}\cdot\frac{\partial^2 x}{\partial v^2} = 0, \qquad \sum\frac{\partial x}{\partial v}\cdot\frac{\partial^2 x}{\partial v^2} = 0.$$

We then infer that:

$$\sum\frac{\partial x}{\partial v}\cdot\frac{\partial^2 x}{\partial u^2} = 0, \qquad \sum\frac{\partial x}{\partial u}\cdot\frac{\partial^2 x}{\partial v^2} = 0.$$

Now, consider the equations:

$$X\frac{\partial^2 x}{\partial u^2} + Y\frac{\partial^2 y}{\partial u^2} + Z\frac{\partial^2 z}{\partial u^2} = 0,$$

$$X\frac{\partial^2 x}{\partial u\,\partial v} + Y\frac{\partial^2 y}{\partial u\,\partial v} + Z\frac{\partial^2 z}{\partial u\,\partial v} = 0.$$

From the relations that were written down previously, that system will admit the two solutions:

$$X = \frac{\partial x}{\partial u}, \qquad Y = \frac{\partial y}{\partial u}, \qquad Z = \frac{\partial z}{\partial u},$$

$$X = \frac{\partial x}{\partial v}, \qquad Y = \frac{\partial y}{\partial v}, \qquad Z = \frac{\partial z}{\partial v}.$$

These solutions are not proportional to each other, since otherwise the curves u = const. and v = const. would always be tangent. Hence, the three determinants that are deduced from the matrix:

$$\left\| \begin{matrix} \dfrac{\partial^2 x}{\partial u^2} & \dfrac{\partial^2 y}{\partial u^2} & \dfrac{\partial^2 z}{\partial u^2} \\ \dfrac{\partial^2 x}{\partial u\,\partial v} & \dfrac{\partial^2 y}{\partial u\,\partial v} & \dfrac{\partial^2 z}{\partial u\,\partial v} \end{matrix} \right\|$$

will be zero. Now, they are the functional determinants of the three quantities $\partial x / \partial u$, $\partial y / \partial u$, $\partial z / \partial u$, taken two at a time, and thus those three quantities are functions of just one of them; i.e., of just one variable t. Similarly, $\partial x / \partial v$, $\partial y / \partial v$, $\partial z / \partial v$ are functions of just one variable θ. Moreover, the relation:

$$\sum \frac{\partial x}{\partial u} \cdot \frac{\partial x}{\partial v} = 0$$

shows that θ, for example, can be expressed as a function of t.

The six partial derivatives are then functions of the same variable. The same thing will then be true for the derivatives $p = \dfrac{D(y,z)}{D(u,v)} : \dfrac{D(x,y)}{D(u,v)}$, $q = -\dfrac{D(z,x)}{D(u,v)} : \dfrac{D(x,y)}{D(u,v)}$ of z, when considered to be functions of x and y. The surface is then developable [Chap. III, pp. 46].

Remark I. – The geodesics are preserved under the development. Now, the geodesics in the plane are lines. *The geodesic lines on the developable surface are then lines that correspond to the lines in that plane under the development of that surface onto a plane.*

In particular, consider the rectifying surface of a curve that is the envelope of the rectifying plane. That curve is a geodesic of its rectifying surface, since its osculating plane is perpendicular to the tangent plane. It is then developed along a line when one performs the development of the rectifying surface onto a plane. *Hence, the name "rectifying plane."*

Remark II. – It results from this that the search for geodesics on a developable surface reduces to its development, and consequently to four quadratures.

Remark III. – The determination of the lines of curvature, which are involutes of the edge of regression, reduces to one quadrature.

Geodesic lines on a developable surface

5. – We have reduced the search for geodesic lines on a developable plane to the development of that surface onto a plane. One can look for them directly. Let the edge of regression be:

(1) $$x = f(s), \qquad y = g(s), \qquad z = h(s),$$

in which s denotes the arc length. If α, β, γ are the direction cosines of the tangent, and u is a length that is measured along that tangent when one starts from the point of contact then the surface will be represented by the equations:

$$x = f + u\,\alpha, \qquad y = g + u\,\beta, \qquad z = h + u\,\gamma.$$

Upon denoting the first and second derivatives of u with respect to s by u' and u'', resp., one will deduce from this that:

$$\frac{dx}{ds} = \alpha + u\frac{\alpha'}{R} + \alpha\,u', \qquad \frac{dy}{ds} = \ldots, \qquad \frac{dz}{ds} = \ldots,$$

or

$$\frac{dx}{ds} = \alpha\,(1 + u') + \alpha'\,\frac{u}{R}, \qquad \frac{dy}{ds} = \ldots, \qquad \frac{dz}{ds} = \ldots,$$

and

$$\frac{d^2x}{ds^2} = \alpha\left(u'' - \frac{u}{R^2}\right) + \alpha' \cdot \frac{1}{R}\left(1 + 2u' - u\frac{R'}{R}\right) - \alpha''\frac{u}{RT},$$

along with their analogues.

Upon remarking that the normal to the surface is nothing but the binormal to the edge of regression, the equation for the geodesic lines will be:

$$\begin{vmatrix} \dfrac{d^2x}{ds^2} & \dfrac{d^2y}{ds^2} & \dfrac{d^2z}{ds^2} \\ \dfrac{dx}{ds} & \dfrac{dy}{ds} & \dfrac{dz}{ds} \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix} = 0$$

or:

$$\begin{vmatrix} \alpha\left(u'' - \dfrac{u}{R^2}\right) + \dfrac{\alpha'}{R}\left(1 + 2u' - u\dfrac{R'}{R}\right) - \alpha''\dfrac{u}{RT} & \cdots & \cdots \\ \alpha(1 + u') + \alpha'\dfrac{u}{R} & \cdots & \cdots \\ \alpha'' & \cdots & \cdots \end{vmatrix} = 0.$$

The left-hand side is the product of two determinants, and the equation can be written:

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix} \times \begin{vmatrix} u'' - \dfrac{u}{R^2} & \dfrac{1}{R}\left(1 + 2u' - u\dfrac{R}{R'}\right) & -\dfrac{u}{RT} \\ 1 + u' & \dfrac{u}{R} & 0 \\ 0 & 0 & 1 \end{vmatrix} = 0$$

or

$$\frac{u}{R}\left(u'' - \frac{u}{R^2}\right) - \frac{1}{R}(1+u')\left(1 + 2u' - u\frac{R'}{R}\right) = 0;$$

i.e.:

$$u \cdot u'' - 2u'^2 - u'\left(3 - u\frac{R'}{R}\right) - \frac{u^2}{R^2} + u \cdot \frac{R'}{R} - 1 = 0. \tag{2}$$

That is the differential equation that determines u.

Let us seek to understand the nature of the general integral. If we develop the surface onto a plane then the curve (1) will be represented by a curve:

$$X = F(s), \qquad Y = G(s)$$

whose radius of curvature will again be R. The homologous point to the point (u, s) on the surface will be:

$$X = F + u\,F', \quad Y = G + u\,G'.$$

The lines in the plane are defined by the general equation:

$$A(F + u\,F') + B(G + u\,G') + C = 0,$$

so

$$u = -\frac{AF + BG + C}{AF' + BG'}.$$

Upon remarking that the denominator is the derivative of the numerator, we will then be led to set:

$$u = -\frac{w}{w'}$$

and to predict that the equation in w will be linear and homogeneous of third order. Effectively:

$$u' = -1 + \frac{w\,w''}{w'^2}$$

and

$$u'' = \frac{w\,w'''}{w'^2} + \frac{w''}{w'} - \frac{2ww''^2}{w'^3}.$$

(2) will then become:

$$-\frac{w}{w'}\left(\frac{w\,w'''}{w'^2} + \frac{w''}{w'} - \frac{2ww''^2}{w'^3}\right) - 2\left(-1 + \frac{w\,w''}{w'^2}\right)^2 - 3\left(-1 + \frac{w\,w''}{w'^2}\right) - \frac{R'}{R}\frac{w}{w'}\left(-1 + \frac{w\,w''}{w'^2}\right)$$
$$-\frac{1}{R^2}\cdot\frac{w^2}{w'^2} - \frac{R'}{R}\cdot\frac{w}{w'} - 1 = 0.$$

If we set:

$$w' = \theta$$

then we will obtain:

(3) $$\theta'' + \frac{R'}{R}\theta' + \frac{1}{R^2}\theta = 0,$$

which is a linear equation of second order in θ. We can make the second term disappear by a change of variable:

$$\sigma = \theta(s),$$

so

$$\theta' = \frac{d\theta}{ds} = \frac{d\theta}{d\sigma}\cdot\varphi',$$

and

$$\theta'' = \frac{d^2\theta}{ds^2} = \frac{d^2\theta}{d\sigma^2}\cdot\varphi'^2 + \frac{d\theta}{d\sigma}\cdot\varphi''.$$

Equation (3) will then become:

$$\frac{d^2\theta}{d\sigma^2}\cdot\varphi'^2 + \frac{d\theta}{d\sigma}\left(\varphi'' + \frac{R'}{R}\varphi'\right) + \frac{1}{R^2}\theta = 0.$$

Choose the function φ in such a fashion that:

$$\varphi'' + \frac{R'}{R}\varphi' = 0,$$

or

$$\frac{\varphi''}{\varphi'} = -\frac{R'}{R}.$$

It suffices to take:

$$\varphi' = \frac{1}{R} = \frac{d\sigma}{ds},$$

so:

$$ds = R\,d\sigma.$$

We will then get the equation:

$$\frac{d^2\theta}{d\sigma^2} + \theta = 0,$$

whose general integral is:

$$\theta = A \cos \sigma + B \sin \sigma = \frac{dw}{ds};$$

hence:

$$w = A \int \cos \sigma \cdot ds + B \int \cos \sigma \cdot ds + C,$$

and finally:

$$u = -\frac{A\int \cos\sigma \cdot ds + B\int \sin\sigma \cdot ds + C}{A\cos\sigma + B\sin\sigma},$$

with

$$\sigma = \int \frac{ds}{R}.$$

One can dispense with the explicit introduction of the arc length *s*, because it will only enter into these formulas by way of its differential. Hence, the geodesic lines on a developable surface are obtained by at most three quadratures. One confirms, moreover, that the two methods will lead to the same calculations.

Skew ruled surfaces. Orthogonal trajectories of generators

6. – Let a ruled surface be:

$$x = f(v) + u \cdot l_0(v), \qquad y = g(v) + u \cdot m_0(v), \qquad z = h(v) + u \cdot m_0(v).$$

Since the generators are geodesics, it will then result that *the orthogonal trajectories of the generators will determine equal segments along those generators.* We have already seen how one obtains those orthogonal trajectories: One must determine u as a function of v in such a fashion that:

$$\sum l_0\, dx = 0.$$

To simplify, we suppose that l_0, m_0, n_0 are direction cosines; then:

$$\sum l_0^2 = 1, \qquad \sum l_0\, dl_0 = 0,$$

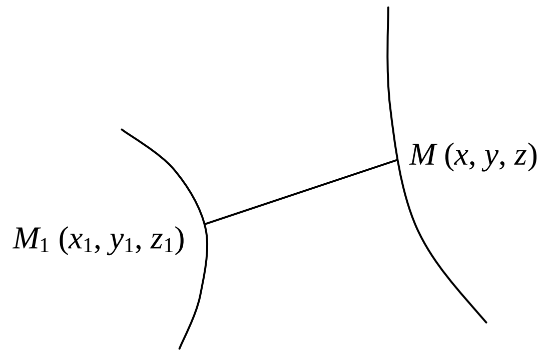
and the differential equation will become:

$$\sum l_0 \cdot df + du = 0;$$

hence:

$$u = -\int \sum l_0 \cdot df.$$

The determination of the orthogonal trajectories of the generators of a ruled surface comes down to one quadrature.



Remark. – One can attach that fact to the formula that gives the variation of a line segment. Take a positive direction on the line MM_1 . Let r be the absolute value of the distance MM_1 . Let x, y, z, and x_1, y_1, z_1 be the coordinates of the two extremities, which describe two given curves. The distance MM_1 is given by the formula:

$$r^2 = (x_1 - x)^2 + (y_1 - y)^2 + (z_1 - z)^2,$$

so

$$r\, dr = (x_1 - x)\,(dx_1 - dx) + (y_1 - y)\,(dy_1 - dy) + (z_1 - z)\,(dz_1 - dz),$$

or

$$dr = \left(\frac{x_1 - x}{r}dx_1 + \frac{y_1 - y}{r}dy_1 + \frac{z_1 - z}{r}dz_1\right) - \left(\frac{x_1 - x}{r}dx + \frac{y_1 - y}{r}dy + \frac{z_1 - z}{r}dz\right).$$

Let α, β, γ, α_1, β_1, γ_1 be the direction cosines of the tangents to the curves at M, M_1 , directed in the sense of increasing arc length. Let λ, μ, ν be the direction cosines of the positive direction of the line MM_1 . The preceding formula can then be written:

$$dr = (\lambda\alpha_1 + \mu\beta_1 + \nu\gamma_1)\, ds_1 - (\lambda\,\alpha + \mu\,\beta + \nu\,\gamma)\, ds,$$

and if one introduces the angles θ, θ_1 between MM_1 and the two tangents then one will get the important formula:

$$dr = \cos\theta_1\, ds_1 - \cos\theta\, ds.$$

Suppose that the line MM_1 is tangent to the first curve and normal to the second one:

$$\theta = 0, \quad \theta_1 = \pm\frac{\pi}{2},$$

and the formula will reduce to:

$$dr = -\, ds.$$

We then recover the properties of involutes and developments.

Suppose that we have the normal line to the two curves $\theta = \pm\frac{\pi}{2}$, $\theta_1 = \pm\frac{\pi}{2}$, so $dr = 0$, r = const., and we will recover the properties of the orthogonal trajectories of the generators.

Director cone. Central point. Line of striction

7. – One calls the surface of the cone:

$$x = u \cdot l_0 (v), \qquad y = u \cdot m_0 (v), \qquad z = u \cdot m_0 (v)$$

the *director cone.*

If that cone reduces to a plane then that plane will be called the *director plane*, and the generators will all be parallel to that plane.

The tangent plane at any point of the surface will have the determinants that are deduced from the matrix:

$$\text{(1)} \qquad \begin{Vmatrix} l_0 & m_0 & n_0 \\ df + u\,dl_0 & dg + u\,dm_0 & dh + u\,dn_0 \end{Vmatrix}$$

for their coefficients. The tangent plane to the director cone along the generator that corresponds to the one that passes through the point considered will have the determinants that are deduced from the matrix:

$$\begin{Vmatrix} l_0 & m_0 & n_0 \\ dl_0 & dm_0 & dn_0 \end{Vmatrix}$$

for their coefficients. Those planes will be parallel if u is infinite. One will then have the tangent plane to the point at infinity on the generator of the surface, which one calls the *asymptote plane. The asymptote planes are parallel to the tangent planes to the director cone along the corresponding generators.*

All of the asymptote planes to a surface with a director plane will be parallel to the director plane.

In order for the two tangent planes to the surface and the director cone to be rectangular, it is necessary that the sum of the products of the preceding determinants should be zero, which will give:

$$\begin{vmatrix} \sum l_0^2 & \sum l_0 df + u \sum l_0 dl_0 \\ \sum l_0 dl_0 & \sum dl_0 \cdot df + u \sum dl_0^2 \end{vmatrix} = 0,$$

which is an equation of first degree in u. *There will then exist a point on any generator, in general, where the tangent plane is perpendicular to the tangent plane to the director cone, i.e., to the asymptote plane. That is the central point,* and the tangent to that point is called the *central plane.*

The locus of central points is called the *line of striction.*

We suppose, to simplify, that $\sum l_0^2 = 1$, which eliminates the case of the ruled surface with isotropic generators. Hence, $\sum l_0 \, dl_0 = 0$, and the equation in u which gives the general point reduces to:

$$u \sum dl_0^2 + \sum l_0 \, df = 0;$$

the central point always exists then, unless:

$$\sum dl_0^2 = 0.$$

In that case, the spherical curve that is at the base of the director cone will be a minimal curve of the sphere; i.e., an isotropic generator. The cone will then be a tangent plane to the asymptote cone of the sphere, which is an isotropic cone, so it will be an isotropic plane. The surfaces considered are *ruled surfaces with isotropic director planes.* All of them are imaginary, except for the paraboloid of revolution.

Remark. – The tangent plane will be indeterminate when all of the determinants in the matrix (1) are zero. There will then exist a factor K such that:

$$df + u \, dl_0 + K \, l_0 = 0, \qquad dy + u \, dm_0 + K \, m_0 = 0, \qquad dh + u \, dn_0 + K \, n_0 = 0,$$

which demands that:

$$\begin{vmatrix} df & dl_0 & l_0 \\ dg & dm_0 & m_0 \\ dh & dn_0 & n_0 \end{vmatrix} = 0.$$

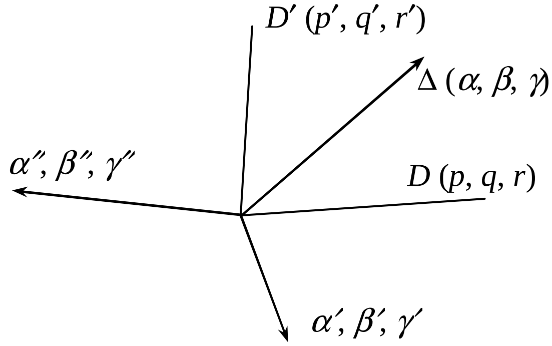
That condition, which expresses the idea that the generator considered meets the infinitely-close generator, can be true for the exceptional generators. If it is an identity then the surface will be developable. In order to find the point where the tangent plane is indeterminate in this case, multiply the condition by dl_0, dm_0, dn_0, resp., and add; we will get:

$$u \sum dl_0^2 + \sum dl_0 \cdot df = 0.$$

This equation determines the contact point of the generator and the edge of regression on page 97. The indeterminacy in the tangent plane at that point explains why the preceding formula, which gives the line of striction for an arbitrary ruled surface, gives the edge of regression for a developable surface. Indeed, it is the only point of the generator of a developable surface where the tangent plane does not coincide with the asymptote plane, and where one can, due to the indeterminacy of the tangent plane, consider the plane that is perpendicular to the asymptote plane to be tangent to the surface.

Variation of the tangent plane along a generator

8. – We propose to seek the angle between the tangent planes to a ruled surface at two points along the same generator. To that effect, we first treat the following problem: One is given a line Δ whose direction cosines are α, β, γ, and the direction coefficients of the two lines D (p, q, r) and D' (p', q', r') that meet it. Calculate the angle V between the two plane $D\Delta$ and $D'\Delta$.



Consider a direct auxiliary tri-rectangular trihedron, one of whose axes is Δ. Let α', β', γ', α'', β'', γ'' be the direction cosines of the other axes, and let u, v, w, and u', v', w' be the direction coefficients of D and D' in that system. One will then have:

$$\tan V = \frac{vw' - wv'}{vv' + ww'}.$$

On the other hand:

$$\begin{aligned} u &= \alpha p + \beta q + \gamma r, & v &= \alpha' p + \beta' q + \gamma' r, & w &= \alpha'' p + \beta'' q + \gamma'' r, \\ u' &= \alpha p' + \beta q' + \gamma r', & v' &= \alpha' p' + \beta' q' + \gamma' r', & w' &= \alpha'' p' + \beta'' q' + \gamma'' r', \end{aligned}$$

so

$$vw' - vw' = \begin{vmatrix} \alpha' p + \beta' q + \gamma' r & \alpha' p' + \beta' q' + \gamma' r' \\ \alpha'' p + \beta'' q + \gamma'' r & \alpha'' p' + \beta'' q' + \gamma'' r' \end{vmatrix} = \begin{Vmatrix} \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{Vmatrix} \begin{Vmatrix} p & p' \\ q & q' \\ r & r' \end{Vmatrix}$$

$$= \begin{vmatrix} \alpha & \beta & \gamma \\ p & q & r \\ p' & q' & r' \end{vmatrix}.$$

Furthermore:

$$uu' + vv' + ww' = pp' + qq' + rr',$$

so

$$vv' + ww' = \sum pp' - \sum \alpha p \cdot \sum \alpha p'.$$

Hence:

$$\tan V = \frac{\begin{vmatrix} \alpha & \beta & \gamma \\ p & q & r \\ p' & q' & r' \end{vmatrix}}{\sum pp' - \sum \alpha p \cdot \sum \alpha p'} = \frac{\sqrt{\sum \alpha^2}\begin{vmatrix} \alpha & \beta & \gamma \\ p & q & r \\ p' & q' & r' \end{vmatrix}}{\sum \alpha^2 \cdot \sum pp' - \sum \alpha p \cdot \sum \alpha p'}, \quad \text{since } \sum \alpha^2 = 1.$$

In that form, one can then introduce the direction coefficients l, m, n of the direction Δ.

$$\tan V = \frac{\sqrt{l^2 + m^2 + n^2}\begin{vmatrix} l & m & n \\ p & q & r \\ p' & q' & r' \end{vmatrix}}{\sum l^2 \cdot \sum pp' - \sum lp \cdot \sum lp'}. \tag{1}$$

Apply that formula to the angle between the tangent planes at two points M, M' along the same generator. We take the directions D, D' to be the directions tangent to the curves n = const.:

$$p = df + u\,dl_0, \qquad q = dg + u\,dm_0, \qquad r = dh + u\,dn_0,$$
$$p' = df + u'\,dl_0, \qquad q' = dg + u'\,dm_0, \qquad r' = dh + u'\,dn_0;$$

the determinant of the formula (1) becomes:

$$\begin{vmatrix} l_0 & df + u\,dl_0 & df + u'\,dl_0 \\ m_0 & dg + u\,dm_0 & dg + u'\,dm_0 \\ n_0 & dh + u\,dn_0 & dh + u'\,dn_0 \end{vmatrix} = \begin{vmatrix} l_0 & dl_0 & df \\ m_0 & dm_0 & dg \\ n_0 & dn_0 & dh \end{vmatrix} (u - u')$$

and

$$\tan V = \frac{(u' - u)\sqrt{l_0^2 + m_0^2 + n_0^2}\begin{vmatrix} df & dg & dh \\ dl_0 & dm_0 & dn_0 \\ l_0 & m_0 & n_0 \end{vmatrix}}{\begin{vmatrix} \sum l_0^2 & \sum l_0(df + u\,dl_0) \\ \sum l_0(df + u'\,dl_0) & \sum (df + u\,dl_0)(df + u'\,dl_0) \end{vmatrix}}.$$

We set:

$$D = \begin{vmatrix} df & dg & dh \\ dl_0 & dm_0 & dn_0 \\ l_0 & m_0 & n_0 \end{vmatrix},$$

and in order to simplify the result, we take l_0, m_0, n_0 to be the direction cosines of the generator; hence, $\sum l_0^2 = 1$, $\sum l_0\,dl_0 = 0$. We suppose, moreover, that the curve $x = f(v)$, $y = y(v)$, $z = h(v)$ is an orthogonal trajectory of the generator, so $\sum l_0\,df = 0$. Finally, we determine u by the relation:

$$u\sum dl_0^2 + \sum dl_0 \cdot df = 0,$$

which amounts to taking the central point to be one of its points.

The denominator then becomes:

$$\begin{vmatrix} 1 & 0 \\ 0 & \sum df^2 + u\sum dl_0\, df + u'[u\sum dl_0^2 + \sum dl_0\, df] \end{vmatrix},$$

which reduces to:

$$\sum df^2 + u\sum dl_0\, df = \frac{\sum df^2 \cdot \sum dl_0^2 - \left(\sum dl_0 \cdot df\right)^2}{\sum dl_0^2},$$

and then:

$$\tan V = \frac{(u'-u)D\cdot\sum dl_0^2}{\sum df^2 \cdot \sum dl_0^2 - \left(\sum dl_0 \cdot df\right)^2}.$$

Upon setting:

$$K = \frac{\sum df^2 \cdot \sum dl_0^2 - \left(\sum dl_0 \cdot df\right)^2}{D\cdot\sum dl_0^2}$$

and remarking that $u' - u = CM$, one will then obtain the *Chasles formula:*

(2) $$\tan V = \frac{CM}{K},$$

which has the following well-known consequences, which break down for singular generators:

1. *When M describes the generator from one end to the other, the tangent plane* (P) *at M will always turn around the generator in the same sense, and the total rotation that it experiences will be* 180°. The tangent planes at two different points will be different.

2. *The distribution of points M and the sheaf of planes* (P) *are in homographic correspondence.*

3. Since three pairs define a homography, *two ruled surfaces that have a common generator and are tangent to that generator at three points will be tangent to that generator at all other points,* i.e., they will agree all along that generator.

We seek to simplify the expression for K. In order to do that, we remark that:

$$D^2 = \begin{vmatrix} \sum df^2 & \sum dl_0 \cdot df_0 & \sum l_0 \cdot df \\ \sum dl_0 \cdot df & \sum dl_0^2 & \sum l_0 \cdot dl_0 \\ \sum l_0 \cdot df & \sum l_0 \cdot dl_0 & \sum l_0^2 \end{vmatrix} = \begin{vmatrix} \sum df^2 & \sum dl_0 \cdot df_0 & 0 \\ \sum dl_0 \cdot df & \sum dl_0^2 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$= \sum dl_0^2 \sum df^2 - (\sum dl_0 \cdot df)^2 ,$$

so

(3) $$K = \frac{D}{\sum dl_0^2} .$$

In the general case, one will likewise find that:

(4) $$K = \frac{D \cdot \sum l_0^2}{\sum l_0^2 \cdot \sum dl_0^2 - (\sum l_0 \, dl_0)^2} .$$

K is the *distribution parameter;* it is rational. Formula (2) shows that if M is displaced in an arbitrary direction along the generator then the tangent plane will turn in the positive sense of rotation with respect to that direction if K is positive and in the negative sense if K is negative.

The sign of K then corresponds to a geometric property of the surface. From (3) or (4), *the distribution parameter is zero for a developable surface.*

If one abstracts from the sign then formula (3) will exhibit the fact that *the distribution parameters is the quotient of the shortest distance from the generator considered to the infinitely-close generator with the angle between the two generators,* since that distance is $D : \sqrt{\sum dl_0^2}$ and that angle is $\sqrt{\sum dl_0^2}$, up to higher-order infinitesimals.

Remark. – Let M, M' be two points along the same generator where the tangent planes are rectangular. The angles V, V' are such that:

$$\tan V \cdot \tan V' = -1,$$

so, by virtue of (2):

$$CM \cdot CM' = -K^2.$$

The points on a generator where the tangent planes are rectangular define an involution whose central point is C.

Example 1. – Surface generated by the binormals to a skew curve.

Let the curve be:

$$x = f(s), \qquad y = g(s), \qquad z = h(s).$$

With the usual notations, we will have:

$$\begin{cases} df = \alpha\, ds & dg = \beta\, ds & dh = \gamma\, ds \\ l_0 = \alpha'' & m_0 = \beta'' & n_0 = \gamma'' \\ dl_0 = \dfrac{\alpha'}{T} ds & dm_0 = \dfrac{\beta'}{T} ds & dn_0 = \dfrac{\gamma'}{T} ds. \end{cases}$$

The central point is defined by $u = 0$ here, so *the curve is the line of striction of the surface that is generated by its binormals.* The distribution parameter is:

$$K = \begin{vmatrix} \alpha\, ds & \beta\, ds & \gamma\, ds \\ \dfrac{\alpha'}{T} ds & \dfrac{\beta'}{T} ds & \dfrac{\gamma'}{T} ds \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix} \dfrac{T^2}{ds^2} = T.$$

The distribution parameter is equal to the radius of torsion of the curve at the corresponding point. The curve is a line of striction that is an orthogonal trajectory to the generators and a geodesic.

Example 2. – Surface generated by the principal normals to a curve.

Here, one has:

$$df = \alpha\, ds, \qquad dg = \beta\, ds, \qquad dh = \gamma ds,$$

$$l_0 = \alpha', \qquad m_0 = \beta', \qquad m_0 = \gamma',$$

$$dl_0 = \left(-\frac{\alpha}{R} - \frac{\alpha''}{T}\right) ds, \qquad dm_0 = \left(-\frac{\beta}{R} - \frac{\beta''}{T}\right) ds, \qquad dn_0 = \left(-\frac{\gamma}{R} - \frac{\gamma''}{T}\right) ds\,.$$

The central point C is defined by the equation:

$$u = \frac{\sum \alpha\left(\dfrac{\alpha}{R} + \dfrac{\alpha''}{T}\right)}{\sum\left(\dfrac{\alpha}{R} + \dfrac{\alpha''}{T}\right)^2} = \frac{\dfrac{1}{R}}{\dfrac{1}{R^2} + \dfrac{1}{T^2}} = \frac{RT^2}{R^2 + T^2} = MC.$$

The distribution parameter is:

$$K = -\frac{RT^2}{R^2 + T^2} \begin{vmatrix} \alpha & \beta & \gamma \\ \dfrac{\alpha}{R} + \dfrac{\alpha''}{T} & \dfrac{\beta}{R} + \dfrac{\beta''}{T} & \dfrac{\gamma}{R} + \dfrac{\gamma''}{T} \\ \alpha' & \beta' & \gamma' \end{vmatrix} = \frac{R^2 T}{R^2 + T^2}\,.$$

We now seek the tangent plane to the center of curvature O. The Chasles formula gives:

$$\tan V = \frac{CO}{K} = \frac{MO - MC}{K} = \frac{1}{K}\left(R - \frac{RT^2}{R^2 + T^2}\right) = \frac{1}{K} \cdot \frac{R^3}{R^2 + T^2} = \frac{R}{T}.$$

For the point M, which is on the curve, one will likewise obtain:

$$\tan V = \frac{CM}{K} = -\frac{T}{R},$$

so:

$$\tan V \cdot \tan V = -1.$$

The tangent planes at M and O are rectangular, which is a particular case of a proposition that will verify later on (§ 12).

Canonical form of the linear element

9. – We now seek the linear element of a ruled surface that is defined by the equations:

$$x = f(v) + u\, l_0(v), \qquad y = g(v) + u\, m_0(v), \qquad z = h(v) + u\, n_0(v).$$

Upon denoting derivatives with respect to v by primes, we infer from those equations that:

$$dx = (f' + u\, l_0')\, dv + l_0\, du, \qquad dy = (g' + u\, m_0')\, dv + l_0\, du, \qquad dz = (h' + u\, n_0')\, dv + m_0\, du,$$

and

$$ds^2 = E\, du^2 + 2\, F\, du\, dv + G\, dv^2,$$

with

$$E = \sum l_0^2, \qquad F = u \sum l_0\, l_0' + \sum l_0\, f', \qquad G = u^2 \sum l_0'^2 + 2u \sum l_0'\, f' + \sum f'^2.$$

Suppose that l_0, m_0, n_0 are direction cosines, so:

$$\sum l_0^2 = 1, \qquad \sum l_0\, l_0' = 0,$$

so

$$E = 1, \qquad F = \sum l_0\, f', \qquad G = u^2 \sum l_0'^2 + 2u \sum l_0'\, f' + \sum f'^2.$$

These results are obtained directly by making the change of parameter:

$$\sqrt{E} \cdot n = u_1,$$

so

$$du_1 = \sqrt{E}\, du + u \frac{dE / dv}{2\sqrt{E}} dv \, .$$

Upon suppressing the indices, we will indeed obtain an expression of the form:

$$ds^2 = du^2 + 2F\, du\, dv + G\, dv^2.$$

Suppose, moreover, that the curve:

$$x = f(v), \qquad y = g(v), \qquad z = h(v)$$

is an orthogonal trajectory of the generators, so:

$$\sum l_0\, f' = 0, \qquad F = 0,$$

and the linear element reduces to:

$$ds^2 = du^2 + G\, dv^2.$$

One should expect that this would be its form, since the coordinate curves are orthogonal. One will also arrive at that expression by setting:

$$du + F\, dv = du_1 \, ,$$

so

$$u_1 = u + \int F\, dv,$$

which demands a quadrature.

The variable u is defined up to a constant, so it is a length that is carried by each generator when one starts with the same orthogonal trajectory. In order to specify the variable v, consider the direction of the generator:

$$x = l_0(v), \qquad y = m_0(v), \qquad z = n_0(v).$$

These equations are the ones for the trace of the director cone on the sphere of radius 1. We take v to be the arc length along that curve, so:

$$\sum l_0'^2 = 1$$

and

$$G = u^2 + 2u \sum l_0' f' + \sum f'^2 \, .$$

Set:

$$\sum l_0' f' = G_0 \, , \quad \sum f'^2 = G_1 \, ,$$

in such a way that:

$$G = u^2 + 2u\, G_0 + G_1 \, .$$

The quantities G_0, G_1 thus-introduced are linked with the central point and the distribution parameter in a simple way. Indeed, consider the involution of the points M,

M' where the tangent planes are rectangular. Its central point is the central point of the generator, and upon denoting the distribution parameter by K:

$$CM \cdot CM' = -K^2.$$

The coefficients of the tangent plane at a point u of the generator will be the determinants that are deduced from the matrix:

$$\left\| \begin{matrix} l_0 & m_0 & n_0 \\ f' + u\,l_0' & g' + u\,m_0' & h' + u\,n_0' \end{matrix} \right\|.$$

Similarly, the coefficients of the tangent plane at the point u' will be deduced from the matrix:

$$\left\| \begin{matrix} l_0 & m_0 & n_0 \\ f' + u'\,l_0' & g' + u'\,m_0' & h' + u'\,n_0' \end{matrix} \right\|.$$

We express the idea that these tangent planes are rectangular. The sum of the products of the preceding determinants, and in turn, the product of the matrices, must be zero, which gives:

$$\begin{vmatrix} 1 & 0 \\ 0 & G_1 + (u + u')G_0 + uu' \end{vmatrix} = 0.$$

The involution relation is then:

$$uu' + (u + u')\, G_0 + G_1 = 0,$$

or

$$(u + G_0)\,(u + G_0) = G_0^2 - G_1\,.$$

Since the central point is the homologue of a point at infinity, it is given by:

$$u + G_0 = 0.$$

Hence $-G_0$ is the u of the central point. We denote it by:

$$P = -G_0 = -\sum l_0'\, f'.$$

On the other hand:

$$G_0^2 - G_1 = -K^2,$$

so

$$G_1 = G_0^2 + K^2 = P^2 + K^2 = \sum f'^2\,.$$

Hence:

$$G = u^2 - 2u\,P + P^2 + K^2 = (u - P)^2 + K^2.$$

In summary, *if v is the arc length of the trace of the director cone on the sphere of radius 1, and u is the length that is carried by the generator when one starts from an orthogonal trajectory then the linear element will be given by the formula:*

(1) $$ds^2 = du^2 + [(u - P)^2 + K^2]\, dv^2,$$

in which P is the value of u for the central point, and K is the distribution parameter.

Remark. – That can serve to calculate the distribution parameter. Indeed:

$$\begin{vmatrix} f' & g' & h' \\ l_0' & m_0' & n_0' \\ l_0 & m_0 & n_0 \end{vmatrix}^2 = \begin{vmatrix} G_1 & G_0 & 0 \\ G_0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = G_1 - G_0^2 = K^2,$$

so

(2) $$K = \begin{vmatrix} f' & g' & h' \\ l_0' & m_0' & n_0' \\ l_0 & m_0 & n_0 \end{vmatrix}, \qquad P = -\sum l_0' f', \qquad P^2 + K^2 = \sum f'^2 .$$

Conversely, let a surface have a linear element of the form:

$$ds^2 = du^2 + [(u - P)^2 + K^2]\, dv^2.$$

We look for ruled surfaces that might be mappable to that surface. The elements of such a ruled surface will be determined by the relations:

$$\sum l_0^2 = 1, \qquad \sum l_0 f' = 0, \qquad \sum l_0'^2 = 1, \qquad \sum l_0' f' = -P, \qquad \sum f'^2 = K^2 + P^2.$$

From the expression (2) for K, the last of these relations can be further written:

$$\sum f'(m_0 n_0' - n_0 m_0') = -K.$$

We can initially give the director cone arbitrarily in such a fashion that the two equations $\sum l_0^2 = 1$, $\sum l_0'^2 = 1$ are satisfied. It will then remain for us to satisfy three linear equations in f', g', h' whose determinant is non-zero. f', g', h' will be determined perfectly, but f, g, h will be determined up to an additive constant, which amounts to adding constant quantities to x, y, z; i.e., to subjecting the surface to a translation. *There is then an infinitude of ruled surfaces that can be mapped to a given ruled surface in such a manner that the generators will correspond to generators*, since one can take the director cone arbitrarily. We remark that it is not K that figures in the linear element, but K^2, in such a way that, in particular, *there exist two ruled surfaces that have the same director cone and distribution parameters that are equal, but opposite in sign, and can be mapped to each other.*

In order to find f, g, h explicitly, solve the system of linear equations:

$$\sum l_0\, f' = 0, \qquad \sum l'_0\, f' = -P, \qquad \sum (m_0\, n'_0 - n_0\, m'_0)\, f' = -K.$$

l_0, m_0, n_0 ; l'_0, m'_0, n'_0 are direction cosines of two rectangular directions here. Introduce a new direction with cosines l_0, m_0, n_0 that defines a direct tri-rectangular trihedron with the preceding ones:

$$l_2 = m_0\, n'_0 - n_0\, m'_0, \qquad m_2 = n_0\, l'_0 - l_0\, n'_0, \qquad n_2 = l_0\, m'_0 - m_0\, l'_0 .$$

The system becomes:

$$\sum l_0\, f' = 0, \qquad \sum l'_0\, f' = -P, \qquad \sum l_2\, f' = -K;$$

hence:

$$(3) \qquad \begin{cases} f' = -P\, l'_0 - K(m_0\, n'_0 - n_0\, m'_0), \\ g' = -P\, m'_0 - K(n_0\, l'_0 - l_0\, n'_0), \\ h' = -P\, n'_0 - K(l_0\, m'_0 - m_0\, l'_0). \end{cases}$$

One deduces f, g, h by quadratures.

The form Ψ and asymptotic lines

10. – We can take the second fundamental form (page 29) to be:

$$\Psi\,(du, dv) = \sum A\, d^2x = \begin{vmatrix} d^2x & d^2y & d^2z \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \end{vmatrix} = \begin{vmatrix} (f' + u\, l'_0)\, dv^2 + 2 l'_0\, du\, dv & \cdots & \cdots \\ l_0 & \cdots & \cdots \\ f' + u\, l'_0 & \cdots & \cdots \end{vmatrix},$$

so **Ψ** will have an expression of the form:

$$\Psi\,(du, dv) = 2\, F'\, du\, dv + G'\, dv^2,$$

in which F' is a function of v, and G' is a trinomial of degree two in u. We naturally find that the asymptotic lines are the curves $dv = 0$ or v = const., which are the generators. The other asymptotic lines are determined by the differential equation:

$$\frac{du}{dv} = -\frac{G'}{2F'},$$

which has the form:

$$\frac{du}{dv} = R\,u^2 + 2Su + T, \tag{1}$$

in which R, S, T are functions of v. It is a *Riccati equation.* Let us recall the properties of that equation.

Properties of the Ricatti equation

1. *Suppose that one knows an integral* u_1 of that equation. Set:

$$u = u_1 + \frac{1}{w}, \tag{2}$$

so

$$du = du_1 - \frac{dw}{w^2}.$$

Equation (1) becomes:

$$\frac{du_1}{dv} - \frac{1}{w^2}\frac{dw}{dv} = R u_1^2 + 2R\frac{u_1}{w} + R\frac{1}{w^2} + 2Su_1 + 2S\frac{1}{w} + T .$$

However, since u_1 is an integral of (1):

$$\frac{du_1}{dv} = R u_1^2 + 2S\,u_1 + T,$$

in such a way that the equation will become:

$$-\frac{dw}{dv} = 2\,(R\,u_1 + S)\,w + R,$$

which will have the form:

$$\frac{dw}{dv} = Qw - R. \tag{3}$$

This is a linear equation *whose integration will involve two quadratures.*

2. *Suppose that one knows two integrals* u_1, u_2 of the equation. Set:

$$u_2 = u_1 + \frac{1}{w_0},$$

so

$$w_0 = \frac{1}{u_2 - u_1}.$$

w_0 will be an integral of equation (3). We then set:

(4) $$w = w_0 + \theta,$$

so

$$dw = dw_0 + d\theta.$$

(3) will become:

$$\frac{dw_0}{dv} + \frac{d\theta}{dv} = Q\,w_0 + Q\,\theta - R,$$

or, since w_0 is an integral of (1):

(5) $$\frac{d\theta}{dv} = Q\,\theta,$$

which is a linear equation with no right-hand side that integrates immediately *by just one quadrature:*

$$\frac{d\theta}{\theta} = Q\,dv,$$

so

$$\log|\,\theta\,| = \int Q\,dv\,,$$

and

$$|\,\theta\,| = e^{\int Q\,dv}.$$

3. *Suppose that one knows three integrals* u_1, u_2, u_3 of equation (1). One then knows two integrals of equation (3). Let:

$$w_1 = \frac{1}{u_3 - u_1}.$$

w_1 is an integral of (3), and in turn, one will know an integral θ_0 of (5):

$$\theta_0 = w_1 - w_0 = \frac{1}{u_3 - u_1} - \frac{1}{u_2 - u_1} = \frac{u_2 - u_3}{(u_3 - u_1)(u_2 - u_1)}.$$

Set:

$$\theta = \theta_0\,\psi,$$

so

$$d\theta = \theta_0\,d\psi + \psi \cdot d\theta_0\,.$$

(5) becomes:

$$\theta_0\,\frac{d\psi}{dv} + \psi \cdot \frac{d\theta_0}{dv} = Q\,\psi\,\theta_0\,,$$

or, since θ_0 is an integral of (5):

$$\theta_0\,\frac{d\psi}{dv} = 0,$$

so

$$\frac{d\psi}{dv} = 0.$$

ψ is a constant C, and the general integral of (5) is:

(6) $$\theta = C\,\theta_0 .$$

The equation is integrated completely by means of algebraic operations. If we seek the expression for the general integral u as a function of the particular integrals u_1, u_2, u_3 then, by virtue of (2), (4), (6), we will have:

$$u = u_1 + \frac{1}{w} = u_1 + \frac{1}{\dfrac{1}{u_2 - u_1} + \theta} = u_1 + \frac{1}{\dfrac{1}{u_2 - u_1} + C\dfrac{u_2 - u_1}{(u_3 - u_1)(u_2 - u_1)}},$$

so

$$\frac{1}{u - u_1} = \frac{1}{u_2 - u_1} + C\frac{u_2 - u_1}{(u_3 - u_1)(u_2 - u_1)} = \frac{u_2 - u_1 + C(u_2 - u_3)}{(u_3 - u_1)(u_2 - u_1)},$$

so

$$C\,(u_2 - u_3) = \frac{(u_3 - u_1)(u_2 - u_1)}{u - u_1} - (u_3 - u_1) = \frac{(u_3 - u_1)(u_2 - u_1)}{u - u_1},$$

and

$$C = \frac{u - u_2}{u - u_1} : \frac{u_2 - u_3}{u_3 - u_1},$$

or

(7) $$(u, u_1, u_2, u_3) = C.$$

Hence, the anharmonic ratio of the four arbitrary integrals of a Ricatti equation is constant. Upon remarking that in the present case, those integrals are precisely the u of the points of intersection of an arbitrary generator with the asymptotes, one will see that *four asymptotic lines of a ruled surface will cut the generators with a constant anharmonic ratio.*

Remark. – When equation (7) is solved for u, that will give:

(8) $$u = \frac{VC + V_0}{V_1 C + V_2},$$

in which V, V_0, V_1, V_2 are functions of v. The general solution is then a fraction of degree one in the arbitrary constant. Conversely, any function of the form (8) will satisfy a Ricatti equation, because if one eliminates the constant C by means of a differentiation then one will recover a differential equation of the form (1).

Application to the asymptotes of particular ruled surfaces

If the ruled surface has a rectilinear directrix then that directrix will be an asymptote, and one knows a particular integral of the Ricatti equation (1). The determination of the asymptotic lines is accomplished by means of two quadratures. That is the case for *ruled surfaces with a director plane* (one directrix at infinity).

If the surface admits two rectilinear directions then those two lines will be asymptotes, and one will know two particular integrals of equation (1). That is the case for *conoidal surfaces with a director plane.* From the preceding, more than one quadrature will be necessary in order to determine the asymptotic lines. However, in reality, one can obtain them without a quadrature.

Indeed, consider a ruled surface that admits two rectilinear directrices. One can perform a homographic transformation in such a fashion that one of the directrices goes to infinity, and the surface will be transformed into a conoid with a director plane.

Let:

$$z = \varphi\left(\frac{y}{x}\right)$$

be the equation of such a conoid. It is equivalent to the equations:

$$x = u, \qquad y = uv, \qquad z = \varphi(v).$$

The coefficients l, m, n of the tangent plane must satisfy the relations:

$$l\frac{\partial x}{\partial u} + m\frac{\partial y}{\partial u} + n\frac{\partial z}{\partial u} = 0, \qquad l\frac{\partial x}{\partial v} + m\frac{\partial y}{\partial v} + n\frac{\partial z}{\partial v} = 0,$$

or

$$l + m\,v = 0, \qquad m\,u + n\,\varphi'(v) = 0,$$

which are equations that will be satisfied if one takes:

$$n = -u, \qquad m = \varphi'(v), \qquad l = -v\,\varphi'(v).$$

The differential equation of the asymptotic lines:

$$\Psi\,(du, dv) = \sum l\,d^2x = -\sum dl\,dx = 0$$

will then be:

$$[\varphi'(v)\,dv + v\,\varphi''(v)\,dv]\,du - \varphi''\,(v)\,dv \cdot (v\,du + u\,dv) + du \cdot \varphi'(v) \cdot dv = 0$$

here, or:

$$u\,\varphi''(v) \cdot dv^2 - 2\,\varphi'(v)\,du\,dv = 0.$$

We find the solution v = const., which gives us the generators, and what will remain is:

$$\frac{\varphi''(v)\,dv}{\varphi'(v)} = \frac{2\,du}{u},$$

so

$$\ln|\varphi'(v)| = \ln u^2 - \ln|C|,$$

or

$$u^2 = C\,\varphi'(v).$$

We will then get the asymptotic lines of a conoid with no quadrature.

Remark. – If there are three rectilinear directrices then the surface will be a second-degree surface, and it will be doubly ruled. The two systems of asymptotic lines will be the two systems of rectilinear generators, and one will see that *four generators of the same system of a quadric will meet the generators of the other system with a constant anharmonic ratio.*

Calculating the form Ψ

We now seek the general expression for the form Ψ. In order to do that, we introduce the canonical variables u, v, which permitted us to arrive at the form that has the type of the linear element. Consider the Serret trihedron of the curve (Σ) that is the trace of the director cone on the sphere of radius 1 that has its center at the summit of that cone. The generator (l_0, m_0, n_0) is in the normal plane to that curve: Let θ be the angle that it makes with the principal normal; with the usual notations, we will have:

$$\begin{cases} l_0 = \alpha'\cos\theta + \alpha''\sin\theta, \\ m_0 = \beta'\cos\theta + \beta''\sin\theta, \\ n_0 = \gamma'\cos\theta + \gamma''\sin\theta; \end{cases}$$

hence, we will get:

$$\alpha = l_0' = \theta'(-\alpha\sin\theta + \alpha''\cos\theta) - \left(\frac{\alpha}{R} + \frac{\alpha''}{T}\right)\cos\theta + \frac{\alpha'}{T}\sin\theta,$$

and some analogous ones. Thus:

$$\frac{\cos\theta}{R} = -1, \qquad \theta' = \frac{1}{T}.$$

Therefore:

$$\begin{cases} m_0\,n_0' - n_0\,m_0' = m_0\gamma - n_0\beta = \alpha'\sin\theta - \alpha''\cos\theta, \\ n_0 l_0' - l_0 n_0' = \beta'\sin\theta - \beta''\cos\theta, \\ l_0 m_0' - m_0 l_0' = \gamma'\sin\theta - \gamma''\cos\theta, \end{cases}$$

and by using formulas (3) of § 9, page 113, we will get:

$$\begin{cases} f' + u\, l_0' = (u-P)l_0' - K(m_0 n_0' - n_0 m_0') = (u-P)\alpha - K\alpha'\sin\theta + K\alpha''\cos\theta, \\ g' + u\, m_0' = \cdots, \\ h' + u\, n_0' = \cdots \end{cases}$$

Then, upon taking derivatives with respect to v:

$$f'' + u\, l_0' = -P'\,\alpha + (u-P)\frac{\alpha'}{R} - K'\,\alpha'\sin\,\theta - K\,\frac{\alpha'}{T}\,\cos\,\theta$$

$$+ K\left(\frac{\alpha}{R} + \frac{\alpha''}{T}\right)\sin\,\theta + K'\,\alpha''\cos\,\theta - K\,\frac{\alpha''}{T}\sin\,\theta + K\,\frac{\alpha'}{T}\cos\,\theta,$$

or:

$$\begin{cases} f'' + u\, l_0'' = \left(\dfrac{K\sin\theta}{R} - P'\right) + \alpha'\left(\dfrac{u-P}{R} - K'\sin\theta\right) + \alpha''\cdot K'\cos\theta, \\ g'' + u\, m_0'' = \cdots, \\ h'' + u\, n_0'' = \cdots \end{cases}$$

The formula of § 10 then gives:

$$\Psi = \begin{vmatrix} 2\alpha\cdot du\,dv + \left[\alpha\left(\dfrac{K\sin\theta}{R} - P'\right) + \alpha'\left(\dfrac{u-P}{R} - K'\sin\theta\right) + \alpha''\cdot K'\cos\theta\right]dv^2 & \cdots & \cdots \\ \alpha'\cos\theta + \alpha''\sin\theta & \cdots & \cdots \\ (u-P)\,\alpha - K\alpha'\sin\theta + K\alpha''\cos\theta & \cdots & \cdots \end{vmatrix}$$

That determinant is the product of the determinant of the nine cosines with the determinant:

$$\begin{vmatrix} 2\,du\,dv + \left(\dfrac{K\sin\theta}{R} - P'\right)dv^2 & \left(\dfrac{u-P}{R} - K'\sin\theta\right)dv^2 & K'\cos\theta\cdot dv^2 \\ 0 & \cos\theta & \sin\theta \\ u-P & -K\sin\theta & K\cos\theta \end{vmatrix}.$$

One then obtains:

$$\Psi = K\left[2\,du\,dv + \left(\frac{K\sin\theta}{R} - P'\right)dv^2\right] + (u-P)\left[(u-P)\frac{\sin\theta}{R} - K'\right]dv^2,$$

or finally:

$$\Psi = 2K\,du\,dv - \left\{(u-P)K' + KP - \frac{\sin\theta}{R}\left[(u-P)^2 + K^2\right]\right\} dv^2.$$

The only new element that intervenes is the geodesic curvature (sin θ) / R of the curve (Σ) on the sphere; that element will suffice to determine (Σ). Indeed, suppose that one is given:

$$\frac{\sin\theta}{R} = \varphi(v).$$

We saw above that:

$$\frac{\cos\theta}{R} = -1, \qquad \frac{1}{T} = \theta'.$$

We deduce the following formulas from it:

$$\text{(1)} \qquad \tan\theta = -\varphi(v), \qquad R = -\cos\theta, \qquad T = \frac{dv}{d\theta},$$

which give the radius of curvature and the radius of torsion of the curve (Σ) as functions of its arc length v. One knows that the form of a skew curve is then defined entirely.

Remark. – Formulas (1) permit us to find the condition for a curve to be traced on a sphere of radius 1. Indeed, one infers that:

$$\frac{dR}{dv} = \sin\theta \cdot \frac{d\theta}{dv} = \frac{\sin\theta}{T},$$

so, upon replacing s with the letter v, which denotes the arc length on (Σ):

$$\text{(2)} \qquad R^2 + T^2\left(\frac{dR}{ds}\right)^2 = 1.$$

That gives the condition (which is obvious *a priori*) for the radius of the osculating sphere to be equal to 1.

Conversely, suppose that this condition is realized. We can set:

$$R = -\cos\theta, \qquad T\frac{dR}{ds} = \sin\theta,$$

from which we infer that:

$$T = \frac{ds}{d\theta}.$$

A comparison of these equations with formulas (1) and (2) of § 2 shows that one of the developments of the curve is (upon setting $c = \theta$, $u = -1$ in the formulas of § 2):

$$x = f - \alpha' \cos\theta - \alpha'' \sin\theta, \qquad y = g - \beta' \cos\theta - \beta'' \sin\theta, \qquad z = h - \gamma' \cos\theta - \gamma'' \sin\theta.$$

We then infer that:

$$\frac{dx}{ds} = \alpha + \left(\frac{\alpha}{R} + \frac{\alpha''}{T}\right) \cos\theta - \frac{\alpha'}{T} \sin\theta + (\alpha' \sin\theta - \alpha'' \cos\theta) \cdot \frac{1}{T} = 0,$$

and similarly, $dy = dz = 0$, in such a way that this development reduces to a point, which we can assume to be the coordinate origin.

Since the normal to the curve constantly passes through the origin, one will have the identity:

$$f \cdot df + g \cdot dg + h \cdot dg = 0,$$

moreover. Hence:

$$f^2 + g^2 + h^2 = \text{const.}$$

The curve is then indeed a spherical curve, and the radius of the sphere on which it is traced is equal to unity, since it is the radius of the osculating sphere.

Differential equation of the lines of curvature

11. – The differential equation of the lines of curvature is [Chap. III, § 7]:

$$\begin{vmatrix} \dfrac{\partial(ds^2)}{\partial(du)} & \dfrac{\partial(ds^2)}{\partial(dv)} \\ \dfrac{\partial\Psi}{\partial(du)} & \dfrac{\partial\Psi}{\partial(dv)} \end{vmatrix} = 0,$$

or:

$$\begin{vmatrix} du & [(u-P)^2 + K^2]\,dv \\ K\,dv & K\,du - \left\{(u-P)K' + KP' - \dfrac{\sin\theta}{R}\left[(u-P)^2 + K^2\right]\right\} dv \end{vmatrix} = 0;$$

i.e.:

$$K\,du^2 - \{(u-P)\,K' + KP' - \varphi(v)\,[(u-P)^2 + K^2]\}\,du\,dv - K\,[(u-P)^2 + K^2]\,dv^2 = 0.$$

That is the differential equation for the lines of curvature, in which $\varphi(v)$ represents the geodesic curvature of the curve (Σ).

Center of geodesic curvature

12. – Consider an orthogonal trajectory of the generators – for example, $u = 0$:

$$x = f(v), \qquad y = g(v), \qquad z = h(v).$$

We seek to find its center of geodesic curvature. It is the point where the polar line meets the tangent plane. Now, since the generator is normal to its orthogonal trajectory, it is the intersection of the normal plane and the tangent plane: *The center of geodesic curvature is then at the intersection of the polar line with the generator.* The normal plane has the equation:

$$\sum (x - f) f' = 0.$$

The characteristic is defined by the preceding equation, and by:

$$\sum (x - f) f'' - \sum f'^2 = 0.$$

In order to determine the center of geodesic curvature, it will suffice to determine the u for the point of intersection of the preceding line with the generator:

$$x = f(v) + u\, l_0(v), \qquad y = g(v) + u\, m_0(v), \qquad z = h(v) + u\, n_0(v).$$

The first equation reduces to an identity, while the second one will give:

$$u \sum l_0 f'' - \sum f'^2 = 0.$$

However:

$$\sum l_0 f' = 0,$$

so

$$\sum l_0' f' + \sum l_0 f'' = 0,$$

and the equation that gives the u of the desired point will become:

$$u \sum l_0' f' + \sum f'^2 = 0,$$

or [eq. (2), § 9]:

$$-u P + P^2 + K^2 = 0,$$

which can be written:

$$P(u - P) = K^2.$$

If C is the central point, M is the point considered on the orthogonal trajectory, and M' is the center of geodesic curvature then the preceding equation will give:

$$CM \cdot CM' = -K^2.$$

Hence, the tangent planes at M and M' are rectangular (cf., pp. 105). Therefore, *the center of geodesic curvature at a point M of an orthogonal trajectory of the generators of a ruled surface is the point of the generator where the tangent plane is perpendicular to the tangent plane at M.*

Application. – If we now consider (see figures on pages 30, 36, 53) a curve (*C*) that is traced on an arbitrary surface (*S*) then the normals *MN′* to (*C*) that are tangent to (*S*) will generate a ruled surface (Σ_t). Since the surfaces (*S*), (Σ_t) are tangent all along (*C*), the curve (*C*) will have the same center of geodesic curvature *G* at *M* on (*S*) and on (Σ_t). Therefore, *G* is the homologue of *M* under the involution of the rectangular tangent planes that relates to the generator *MN′* of (Σ_t). *The center of geodesic curvature G is the point of MN′ where the normal plane to* (*C*) *is tangent to* (Σ_t).

Likewise, since the center of normal curvature *K* is on the polar line of (*C*), it is the center of geodesic curvature at *M* on the ruled surface (Σ_n) that is generated by the normals *MN* that are drawn from (*C*) to the various points of (*C*). It is then homologous to *M* under the involution of the rectangular tangent planes that relates to the generator *MN* of (Σ_n): *The center of normal curvature K is the point of MN where the normal plane to* (*C*) *is tangent to* (Σ_n).

For the same reason, the center of curvature *C* will possess the same property in relation to the ruled surface that is generated by the principal normals of (*C*) [cf., page 108].

Remark. – The results of this paragraph will become obvious if one notes that any normal to a curve (*C*) at a point *M* of that curve will touch the polar surface at the point where it meets the polar line that corresponds to *M*, in such a way that any ruled surface that is generated by the normals to (*C*) will be circumscribed by the polar surface; i.e., tangent to each normal plane, such that the contact point with any of those normal planes will be on the corresponding polar line.

CHAPTER VI

CONGRUENCES OF LINES

Focal points and focal planes

1. – One calls a set of lines that depend upon two parameters a *congruence* or *ray system*. All lines that meet two fixed lines constitute a congruence. Similarly, the lines that pass through a fixed point and the normals to a surface will also constitute congruences. If one considers a one-parameter family of curves on a surface then the set of all their tangents will again constitute a congruence.

The fundamental properties of congruences that are defined by the normals to the same surface (which play an essential role in geometrical optics) are due to Monge. The principal notions of the general theory of congruences were introduced by Hamilton.

An arbitrary line (*D*) of a given congruence will be represented by the equations:

$$(1) \qquad x = f(v, w) + u \cdot a(v, w), \quad y = f(v, w) + u \cdot b(v, w), \quad z = f(v, w) + u \cdot c(v, w).$$

The equations:

$$(2) \qquad x = f(v, w), \quad y = g(v, w), \quad z = h(v, w)$$

define what we call the *support* of the congruence, to simplify the language. *a*, *b*, *c* define the directions of the *lines of the congruence* or *rays of the congruence* that pass through each point of the support. That support will be a surface, in general, and the congruence will be composed of lines with given directions that pass through all points of a surface. It can happen that *f*, *g*, *h* depend upon only one parameter, so the support will be a curve, and an infinitude of lines will pass through each point of the curve, which will define a cone. Finally, *f*, *g*, *h* can reduce to constants, and the congruence will be composed of all lines that pass through the fixed point whose coordinates are *f*, *g*, *h*.

Suppose that one establishes a relation between *v* and *w*; that amounts to choosing ∞^1 lines of the congruence, which will constitute a *ruled surface of the congruence.* Equations (1) will then become the equations of a ruled surface. Consider all of the ruled surfaces of the congruence that pass through a line (*D*) of the congruence. Two of those surfaces will agree at two points of the line (*D*). We shall show that those two points are independent of the ruled surfaces that one considers. In other words, *there exist two points F, F′ on each line* (*D*) *of the congruence that correspond to two planes* (*P*), (*P′*) *that pass through the line D, and are such that all of the ruled surfaces of the congruence that pass through the line D will have the planes* (*P*), (*P′*) *for their tangent planes at F, F′, respectively.* Those points *F*, *F* are called *foci* or *focal points* of the line (*D*), while the planes (*P*), (*P′*) are the *focal planes* that are associated with *F*, *F′*. In order to prove the proposition, we seek the tangent plane to any point of the generator (1). The parameters *l*, *m*, *n* of that tangent plane satisfy the equations:

(3) $$l\,a + m\,b + n\,c = 0,$$

(3′) $$l\,(df + u\,da) + m\,(dg + u\,db) + n\,(dh + u\,dc) = 0.$$

We shall show that one can choose u in such a fashion that the tangent plane is independent of the differentials dv, dw, and in turn, independent of the relation that exists between v and w; i.e., independent of the ruled surface. Develop the second equation in (3):

$$0 = \left[l\left(\frac{\partial f}{\partial v}+u\frac{\partial a}{\partial v}\right)+m\left(\frac{\partial g}{\partial v}+u\frac{\partial b}{\partial v}\right)+n\left(\frac{\partial h}{\partial v}+u\frac{\partial c}{\partial v}\right)\right]dv$$
$$+\left[l\left(\frac{\partial f}{\partial w}+u\frac{\partial a}{\partial w}\right)+m\left(\frac{\partial g}{\partial w}+u\frac{\partial b}{\partial w}\right)+n\left(\frac{\partial h}{\partial w}+u\frac{\partial c}{\partial w}\right)\right]dw.$$

In order for the tangent plane to be independent of dv, dw, it is necessary and sufficient that one must have both:

(4) $$\begin{cases} l\left(\dfrac{\partial f}{\partial v}+u\dfrac{\partial a}{\partial v}\right)+m\left(\dfrac{\partial g}{\partial v}+u\dfrac{\partial b}{\partial v}\right)+n\left(\dfrac{\partial h}{\partial v}+u\dfrac{\partial c}{\partial v}\right)=0, \\ l\left(\dfrac{\partial f}{\partial w}+u\dfrac{\partial a}{\partial w}\right)+m\left(\dfrac{\partial g}{\partial w}+u\dfrac{\partial b}{\partial w}\right)+n\left(\dfrac{\partial h}{\partial w}+u\dfrac{\partial c}{\partial w}\right)=0. \end{cases}$$

Relations (4) and relation (3) must be satisfied for all non-zero values of l, m, n, so their determinant must be zero:

(5) $$\begin{vmatrix} a & b & c \\ \dfrac{\partial f}{\partial v}+u\dfrac{\partial a}{\partial v} & \dfrac{\partial g}{\partial v}+u\dfrac{\partial b}{\partial v} & \dfrac{\partial h}{\partial v}+u\dfrac{\partial c}{\partial v} \\ \dfrac{\partial f}{\partial w}+u\dfrac{\partial a}{\partial w} & \dfrac{\partial g}{\partial w}+u\dfrac{\partial b}{\partial w} & \dfrac{\partial h}{\partial w}+u\dfrac{\partial c}{\partial w} \end{vmatrix} = 0.$$

That is the equation that gives the u of the focal points. It has degree two, so there will be two focal points. The coefficients of the focal plane that corresponds to each of them will have the values of l, m, n that satisfy equations (3) and (4).

Remark. – Equation (5) cannot be an identity in u for any v and w, because the constant term will be annulled only if the ray of the congruence is tangent to the support. One can then suppose that the support has been chosen in such a manner that this term is not zero for the ray in question, as long as it is not singular.

As for equations (3) and (4) in l, m, n, the relations between the focal planes and the locus of foci that we shall study will show that the indeterminate case can present itself for singular rays, as well.

In order for that to be true, it is necessary that the minors of the left-hand side of equation (5) must be zero, and consequently, that u must be a double root; i.e., the foci of the ray must coincide. However, the latter condition is not sufficient.

The two indeterminate cases will be excluded from consideration in what follows. The properties of the lines of the congruence that we obtain will apply to only non-singular rays, in general.

The congruences that are composed of either the lines of a plane or the lines that pass through a point are the only ones for which all lines are singular, from one of the two preceding viewpoints. They have been implicitly excluded from the foregoing.

Focal surfaces. Focal curves

The locus of the foci is obtained with no difficulty. It will suffice to infer u from (5) and substitute its value into (1). Equation (5) has degree two, so it will give two values for u, in such a way that the locus is composed of two distinct components in the neighborhood of the line (D). Consider one of those components. It can be a surface, which one calls the *focal surface*, or a curve, which one calls the *focal curve*, or it can even reduce to a point, and the congruence will then be composed of all the lines that pass through the point. If one discards that case then one will see that the locus of the foci will be composed of two surfaces, a curve and a surface, or two curves.

1. Suppose that the locus of the foci is a surface (Φ). Take that surface to be the support of the congruence. Equation (5) has the root $u = 0$, so:

$$\begin{vmatrix} a & b & c \\ \frac{\partial f}{\partial v} & \frac{\partial g}{\partial v} & \frac{\partial h}{\partial v} \\ \frac{\partial f}{\partial w} & \frac{\partial g}{\partial w} & \frac{\partial h}{\partial w} \end{vmatrix} = 0.$$

This expresses the idea that the line (D) is in the tangent plane to the surface (Φ) at the point M (u = 0), which is one of the foci, namely, F. *Hence, the lines of the congruence are tangent to the focal surface at the corresponding focus.* We seek the focal plane that corresponds to F. Its coefficients l, m, n are determined by the equations:

$$\begin{cases} la + mb + nc = 0, \\ l\frac{\partial f}{\partial v} + m\frac{\partial g}{\partial v} + n\frac{\partial h}{\partial v} = 0, \\ l\frac{\partial f}{\partial w} + m\frac{\partial g}{\partial w} + n\frac{\partial h}{\partial w} = 0. \end{cases}$$

From the condition that was written down before, those equations will reduce to two, and they express the idea that *the focal plane that corresponds to the focus F is the tangent plane to the surface* (Φ) *at F. All of the skew ruled surfaces of the congruence are circumscribed by the focal surface.* The case of developable surfaces will be discussed later on [§ 2]. The preceding argument will break down if *F* is a point of the edge of regression.

2. It results from the foregoing that *if the locus of foci F, F′ consists of two focal surfaces* (Φ), (Φ′) *then the lines of the congruence will be tangent to the two focal surfaces, the foci F, F′ will be the contact points, and the focal planes will be the tangent planes to the focal surfaces at the corresponding foci. The locus of the foci will coincide with the envelope of the focal planes.*

Conversely, if one is give two arbitrary surfaces (Φ), (Φ′) then their common tangents will depend upon two parameters. Indeed, let *F* be a point of (Φ). Consider the tangent plane to (Φ) at *F*. It cuts (Φ′) along a certain curve. If we draw tangents to that curve through *F* then those lines, which will be tangent to the two surfaces (Φ), (Φ′), will be determined when the point *F* is determined. They depend upon just as many parameters as the point *F*, and therefore, two parameters. They constitute a congruence whose ruled surfaces will be circumscribed by the surfaces (Φ), (Φ′), which are the focal surfaces.

If the surfaces (Φ), (Φ′) constitute two sheets of the same surface (*S*) (which will be true, in general) then the congruence will be composed of the double tangents to the surface (*S*).

3. Suppose that one portion of the locus of the foci is a curve (φ), which we take to be the support of the congruence. *f*, *g*, *h* depend upon only one parameter then – *v*, for example. $\partial f / \partial w$, $\partial g / \partial w$, $\partial h / \partial w$ are zero, and $u = 0$ is a root of equation (5). *If the lines of a congruence meet a fixed curve then the points of that curve will be foci for the lines of the congruence that pass through it.* We seek the corresponding focal plane. Its coefficients will be determined by the equations:

$$\begin{cases} la + mb + nc = 0, \\ l\dfrac{\partial f}{\partial v} + m\dfrac{\partial g}{\partial v} + n\dfrac{\partial h}{\partial v} = 0. \end{cases}$$

Therefore, the focal plane passes through the line (*D*) *and is tangent to the focal curve. All of the skew ruled surfaces of the congruences pass through the focal curve, and at a point M of that curve, they will be tangent to the tangent plane to that curve that passes through the line* (*D*). The case of developable surfaces will be studied in § 2.

4. Suppose that one has a focal surface (Φ) and a focal curve (φ'). *The congruence is composed of the lines that meet* (φ') *and are tangent to* (Φ). One immediately gets the foci and the focal planes from the foregoing. *Conversely, the lines that meet a curve* (φ') *and are tangent to a surface* (Φ) *constitute a congruence that admits* (φ') *and* (Φ) *for its locus of foci.*

5. Suppose that one has two focal curves (φ), (φ'). *The congruence is composed of the lines that meet* (φ), (φ'), *and its skew ruled surfaces contain the two focal curves. Conversely, the lines that meet two given curves constitute a congruence that admits those two curves for its focal curves.* If (φ), (φ') constitute two components of the same curve (c) then the congruence will be composed of lines that meet (c) at two points; i.e., the chords of (c).

Singular cases

Let us see which cases are the ones in which the two foci coincide on all lines of a congruence.

From the definition itself of the foci and the focal planes, the latter will also coincide, and conversely, since the focal planes are tangent to the same ruled surface at the corresponding foci, as one will see in § 2, one can therefore suppose that the ruled surface is not developable.

1. First of all, examine the case of two coincident focal surfaces. To that effect, first consider a focal surface (Φ) of an arbitrary congruence. A line (D) of the congruence is tangent to each point F of that surface. If one associates those focal points with the corresponding lines then there will exist a family of curves on the surface that are tangent to the corresponding line of the congruence at all of their points. In order to show that, take the focal surface (Φ) to be the support of the congruence: The line (D) is tangent to that support, so if P and Q are functions of u, w then its direction coefficients will be:

$$a = P\frac{\partial f}{\partial v}+Q\frac{\partial f}{\partial w}, \qquad b = P\frac{\partial g}{\partial v}+Q\frac{\partial g}{\partial w}, \qquad c = P\frac{\partial h}{\partial v}+Q\frac{\partial h}{\partial w}.$$

Let a curve on the surface (Φ) be defined by expressing v, w as functions of one parameter. The direction coefficients of the tangent are:

$$dx = \frac{\partial f}{\partial v}\cdot dv+\frac{\partial f}{\partial w}\cdot dw, \qquad dy = \frac{\partial g}{\partial v}\cdot dv+\frac{\partial g}{\partial w}\cdot dw, \qquad dz = \frac{\partial h}{\partial v}\cdot dv+\frac{\partial h}{\partial w}\cdot dw,$$

and in order for that tangent to be the line (D), it is necessary and sufficient that:

$$\frac{dv}{P} = \frac{dw}{Q}.$$

In order to determine one of the parameters v, w as a function of the other one, one must then integrate a first-order differential equation. The family of curves thus-determined will depend upon one parameter: Let us take it to be the family w = const. The direction coefficients of the rays of the congruence will be:

$$a = \frac{\partial f}{\partial v}, \qquad b = \frac{\partial g}{\partial v}, \qquad c = \frac{\partial h}{\partial v},$$

and the general equations of those rays will be written:

$$x = f(v, w) + u\,\frac{\partial f}{\partial v}, \quad y = g(v, w) + u\,\frac{\partial g}{\partial v}, \quad z = h(v, w) + u\,\frac{\partial h}{\partial v}. \tag{6}$$

The equation of the focal points (5) will become:

$$\begin{vmatrix} \frac{\partial f}{\partial v} & \frac{\partial g}{\partial v} & \frac{\partial h}{\partial v} \\ \frac{\partial f}{\partial v} + u\frac{\partial^2 f}{\partial v^2} & \frac{\partial g}{\partial v} + u\frac{\partial^2 g}{\partial v^2} & \frac{\partial h}{\partial v} + u\frac{\partial^2 h}{\partial v^2} \\ \frac{\partial f}{\partial w} + u\frac{\partial^2 f}{\partial v\,\partial w} & \frac{\partial g}{\partial w} + u\frac{\partial^2 g}{\partial v\,\partial w} & \frac{\partial h}{\partial w} + u\frac{\partial^2 h}{\partial v\,\partial w} \end{vmatrix} = 0,$$

and upon subtracting the first row from the second one, u will become a factor.

Having said that, suppose that the focal points coincide pair-wise. In order for that to be true, it is necessary and sufficient that the determinant should once more vanish for $u = 0$, which will give:

$$\begin{vmatrix} \frac{\partial f}{\partial v} & \frac{\partial g}{\partial v} & \frac{\partial h}{\partial v} \\ \frac{\partial^2 f}{\partial v^2} & \frac{\partial^2 g}{\partial v^2} & \frac{\partial^2 h}{\partial v^2} \\ \frac{\partial f}{\partial w} & \frac{\partial g}{\partial w} & \frac{\partial h}{\partial w} \end{vmatrix} = 0,$$

or $E' = 0$. That expresses the idea that the equation of the asymptotic lines of the surface (Φ), which is:

$$E' dv^2 + 2\, F' dv \cdot dw + G' dw^2 = 0,$$

must be satisfied for $dw = 0$; i.e., that the curves w = const. must be asymptotic lines of the surface (Φ). Hence: *The congruences with double focal surfaces are composed of the tangents to the asymptotic lines of an arbitrary, non-developable surface.*

The hypothesis of a developable double focal surface is found to be excluded by our conclusion, since the asymptotic lines are generators, so their tangents will no longer depend upon one parameter.

We shall return to that hypothesis in § 3, and we shall see that it is inadmissible.

2. Now consider the case of two coincident focal curves. Take the double focal curve (φ) to be the support; f, g, h are functions of only v. If we then express the idea that equation (5) admits $u = 0$ for a double root then we will get the condition:

$$\begin{vmatrix} a & b & c \\ \dfrac{\partial f}{\partial v} & \dfrac{\partial g}{\partial v} & \dfrac{\partial h}{\partial v} \\ \dfrac{\partial a}{\partial w} & \dfrac{\partial b}{\partial w} & \dfrac{\partial c}{\partial w} \end{vmatrix} = 0.$$

The lines (*D*) of the congruence that pass through a point *F* of the curve (φ) will generate a cone. The coefficients of the tangent plane to that cone will be the determinants that are deduced from the matrix:

$$\begin{Vmatrix} a & b & c \\ \dfrac{\partial a}{\partial w} & \dfrac{\partial b}{\partial w} & \dfrac{\partial c}{\partial w} \end{Vmatrix},$$

and the preceding condition expresses the idea that the tangent *FT* to the focal curve is on the tangent plane to the cone. That must be true for any generator of the cone that one considers, so all of the tangent planes to the cone will pass through *FT*, and the cone will reduce to a plane. *A congruence with a double focal curve is generated by the lines that radiate around each point F of a curve* (φ) *in a plane that passes through the tangent to* (φ), *and conversely*. The envelope of the focal planes no longer coincides with the locus of focal points here.

Developables of the congruence

2. – Let us see if one can associate the lines of a congruence in such a fashion as to obtain a developable surface. To that effect, recall the equations of the line (*D*):

(1) $\quad x = f(v, w) + u \cdot a(v, w), \quad y = g(v, w) + u \cdot b(v, w), \quad z = h(v, w) + u \cdot c(v, w).$

The condition for that line to generate a developable surface is [Chap. V, § 1, eq. (5)]:

$$\begin{vmatrix} a & b & c \\ da & db & dc \\ df & dg & dh \end{vmatrix} = 0,$$

or

(2) $$\begin{vmatrix} a & b & c \\ \dfrac{\partial a}{\partial v}dv + \dfrac{\partial a}{\partial w}dw & \dfrac{\partial b}{\partial v}dv + \dfrac{\partial b}{\partial w}dw & \dfrac{\partial c}{\partial v}dv + \dfrac{\partial c}{\partial w}dw \\ \dfrac{\partial f}{\partial w}dv + \dfrac{\partial f}{\partial w}dw & \dfrac{\partial g}{\partial w}dv + \dfrac{\partial g}{\partial w}dw & \dfrac{\partial h}{\partial w}dv + \dfrac{\partial h}{\partial w}dw \end{vmatrix} = 0.$$

That is the differential equation that expresses the idea that the line of the congruence generates a developable surface. It has the form:

$$A\,dv^2 + 2B\,dv \cdot dw + C\,dw^2 = 0.$$

It gives two values for dv / dw, so there will be two families of ∞^1 developables that are generated by the rays of the congruence, which one calls *developables of the congruence. Two developables of the congruence pass through each line of the congruence.*

Let us seek the contact points of that line with the edge of regression. The value of u that provides the coordinates (1) of one of those points must verify the equations [Chap. V, § 1, eq. (4)]:

$$\begin{cases} df + u \cdot da + a \cdot d\rho = 0, \\ dg + u \cdot db + b \cdot d\rho = 0, \\ dh + u \cdot dc + c \cdot d\rho = 0, \end{cases}$$

or

$$\begin{cases} \left(\dfrac{\partial f}{\partial v} + u\dfrac{\partial a}{\partial v}\right) dv + \left(\dfrac{\partial f}{\partial w} + u\dfrac{\partial a}{\partial w}\right) dw + a \cdot d\rho = 0, \\ \left(\dfrac{\partial g}{\partial v} + u\dfrac{\partial b}{\partial v}\right) dv + \left(\dfrac{\partial g}{\partial w} + u\dfrac{\partial b}{\partial w}\right) dw + b \cdot d\rho = 0, \\ \left(\dfrac{\partial h}{\partial v} + u\dfrac{\partial c}{\partial v}\right) dv + \left(\dfrac{\partial h}{\partial w} + u\dfrac{\partial c}{\partial w}\right) dw + c \cdot d\rho = 0. \end{cases}$$

If we eliminate dv, dw, $d\rho$ from these equations then their determinant will give the u of the contact point of the line with the edge of regression, equation (5) [§ 1], which gives the focal points. Therefore, *the points where one line* (D) *of the congruence touches the edges of regression of two developables of the congruence that pass through that line will be foci of the line* (D).

These results can be obtained without calculation. Indeed, let (Δ) be one of the two developables that pass through (D). At least one of the foci is not on the edge of regression; let F be that focus. The tangent plane to (Δ) at that point is the focal plane (P) that is associated with F. At the focus F', the second focal plane (P'), which is different from (P), must be tangent to (Δ). That demands that F' must be on the edge of regression, since (Δ) is developable, so the tangent plane will be the plane (P) all along the generator, except at the point where (D) is tangent to the edge of regression, for which the tangent plane will be indeterminate.

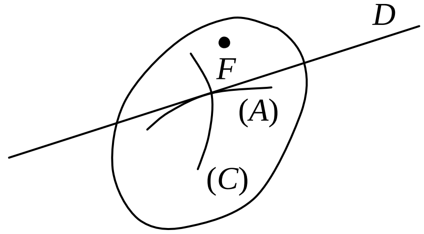
One also sees that *the tangent plane along* (D) *to one of the developables of the congruence that pass through* (D) *is the focal plane that is associated with the focus that is not on the edge of regression of that developable.*

If the developable is a cone or a cylinder then one must interpret the preceding results by considering the summit of the surface (which is situated at a finite or infinite point) to constitute the edge of regression.

One can say, in a general manner, that *each ray* (D) *is met by two infinitely close rays. The points of intersection are the foci, their planes pass through* (D), *and the focal planes* *pass through each of the two infinitely-close rays, and the focal plane that is furnished by one of those rays is associated with the focus that is provided by the other one.*

Developables and the focal surface

Suppose that the locus of focal points consists of a surface (Φ). It results from the foregoing that any developable of the congruence is either circumscribed by that surface or it has its edge of regression on it. Let us examine that situation more closely.



A line (*D*) of the congruence passes through each point *F* of the surface (Φ) that is tangent to (Φ) at *F* and admits *F* for its focus. We showed incidentally on page 127 that there exists a family of curves (*A*) on the surface (Φ) that are tangent to the lines (*D*). The developable that has one of the curves (*A*) for its edge of regression will be a developable of the congruence. We then obtain one of the families of developables. Consider the curves (*C*) that define a conjugate net on (Φ), along with (*A*), and the developable that is the envelope of the tangent planes to (Φ) all along one of those curves (*C*). The generator of that developable at a point *F* of (*C*) is the characteristic of the tangent plane, so it is the tangent that is conjugate to the tangent to (*C*), and thus, the line (*D*). We then get the second family of developables by taking the envelope of the tangent planes to (Φ) at all points of each of the curves (*C*) that are conjugate to the curves (*A*).

One can recover those results analytically by taking the equations of the congruence in the form (6), [§ 1], which will exhibit the curves (*A*). They are then the curves w = const.

Equation (2), which defines the developables, will then become:

$$\begin{vmatrix} \dfrac{\partial f}{\partial v} & \cdots & \cdots \\ \dfrac{\partial^2 f}{\partial v^2}\cdot dv + \dfrac{\partial^2 f}{\partial v\,\partial w}dw & \cdots & \cdots \\ \dfrac{\partial f}{\partial v}\cdot dv + \dfrac{\partial f}{\partial w}dw & \cdots & \cdots \end{vmatrix} = 0.$$

Subtract the elements of the third row from those of the first one, multiplied by dv; the equation will take the form:

$$(E'dv + F'dw)\,dw = 0.$$

We first find that $dw = 0$ (viz., the curves *A*), and the relation:

$$E'dv + F'dw = 0$$

defines precisely the curves (*C*) that are conjugate to the curves w = const.

Developables and the focal curve

Now let us examine the case of a focal curve (φ), which we take to be the support:

$$x = f(v), \qquad y = g(v), \qquad z = h(v).$$

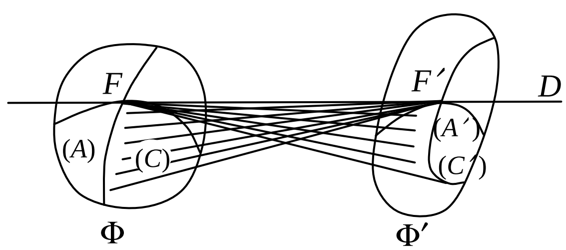
$\partial f / \partial w$, $\partial g / \partial w$, $\partial h / \partial w$ are zero then, and equation (2) will become:

$$\begin{vmatrix} a & \cdots & \cdots \\ \dfrac{\partial a}{\partial v}dv + \dfrac{\partial a}{\partial w}dw & \cdots & \cdots \\ \dfrac{\partial f}{\partial v}dv & \cdots & \cdots \end{vmatrix} = 0;$$

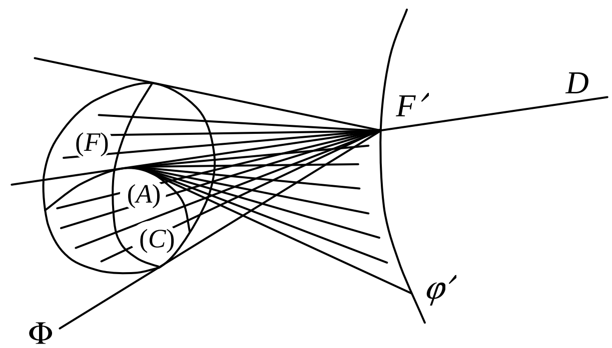
dv is a factor. One of the families of developables is composed of the lines v = const.; i.e., all of the lines of the congruence that pass through the same point *F* of (φ). They are cones.

Examination of various possible cases

Let us examine the various possible cases that relate to the nature of the locus of the foci.



1. Suppose that one has two focal surfaces (Φ), (Φ′). Any line (*D*) of the congruence is tangent to (Φ), (Φ′) at two points *F*, *F′*, resp., that are foci of (*D*). Consider one of the developables that have one of the curves (*A*) for their edge of regression. All of its generators are tangent to (Φ′), so that developable will be circumscribed by (Φ′) along a curve (*C′*) that we call the *contact curve*. The focal plane that corresponds to *F* is the tangent plane to the surface (Φ) at *F*. The second focal plane is the tangent plane to (Φ′) at *F′*, and since the developable is circumscribed by (Φ′), that tangent plane will be the tangent plane to the developable at the point *F′*; i.e., along the generator (*D*). It is the osculating plane to the edge of regression (*A*) at the point *F*.



There is obviously reciprocity between (Φ), (Φ′). The other sequence of developables will have the envelopes of the lines (*D*) on the surface (Φ′) for their edges of regression. Let (*A*′) be those edges of regression. Those developables will be circumscribed by (Φ) along curves of contact (*C*). *We have thus determined two conjugate nets on* (Φ) *and* (Φ′) *that correspond* in such a manner that the curves (*A*) correspond to the curves (*C*), and the curves (*C*) correspond to the curves (*A*′). One of the families of corresponding curves is composed of the edges of regression, and the other is composed of the contact curves.

The second focus *F*′ is the contact point of the line (*D*) with its envelope when *F* is displaced along the curve (*C*) [cf., Chap. VIII, § 3].

2. Suppose that one has a focal surface (Φ) and a focal curve (φ'). One sequence of developables is comprised of the cones that have their summits on (φ'). The curves (*C*) on (Φ) are the contact curves of the cones that are circumscribed by (Φ) and have the various points of (φ') for their summits. The focal planes are: The osculating plane to (*A*) at the point *F* and the tangent plane to (Φ) at the point *F*; i.e., the tangent plane to (φ') that passes through *D* and the tangent plane to the cone of the congruence with its summit at *F*′ along *D*. The curves (*C*), (*A*) define a conjugate net on (Φ).

3. Finally, suppose that (φ), (φ') are two focal curves. The two families of developables are the cones that pass through one of the curves and have their summits on the other one.

Singular cases

Now let us look at the case of coincident foci.

1. There is a non-developable *double focal surface.* In this case, the congruence is composed of the tangents to one family of asymptotes of that surface [§ 1, page 129]. There is no longer a family of developables that have those asymptotes for their edges of regression. Indeed, take that surface to be the support and take those asymptotes to be the curves w = const. As we have seen (page 132), the differential equation that determines the developables is:

$$(E' dv + F' dw)\, dw = 0.$$

The equation of the asymptotic lines is:

$$E'dv^2 + 2F'dv \cdot dw + G'dw^2 = 0.$$

It must be verified for $dw = 0$, so $E' = 0$, and the equation that determines the developables will become $dw^2 = 0$, which proves the stated result.

2. There is a *double focal curve* (φ). The lines of the congruence are in the tangent planes to the various points of (φ) then. Those will plane then constitute a family of developables. One immediately perceives two other particular developables, namely, the envelope of the preceding tangent planes and the developable that has the curve (φ) for its edge of regression. It is easy to see that there are no other ones.

Indeed, let the curve (φ) be:

$$x = f(v), \qquad y = g(v), \qquad z = h(v).$$

The direction coefficients of the tangent are the derivatives f', g', h'. Give the direction coefficients of a particular line of the congruence $a_0(v)$, $b_0(v)$, $c_0(v)$ at each point. An arbitrary line of the congruence will have the direction coefficients:

$$a = f'(v) + w\,a_0(v), \quad b = g'(v) + w\,b_0(v), \quad c = h'(v) + w\,c_0(v).$$

The differential equation of the developables is then:

$$\begin{vmatrix} f' + w a_0 & \cdots & \cdots \\ (f'' + w a_0')\,dv + a_0 \cdot dw & \cdots & \cdots \\ f'\,dv & \cdots & \cdots \end{vmatrix} = 0;$$

dv is a factor. Upon subtracting the third line, divided by dv, from the first, w will be a factor, and the equation will reduce to:

$$w \cdot dv^2 \begin{vmatrix} a_0 & \cdots & \cdots \\ f'' + w a_0' & \cdots & \cdots \\ f' & \cdots & \cdots \end{vmatrix} = 0.$$

We find $dv = 0$, which corresponds to the tangent planes, $w = 0$, which corresponds to the developable whose edge of regression is (φ), and finally:

$$\begin{vmatrix} a_0 & b_0 & c_0 \\ f'' & g'' & h'' \\ f' & g' & h' \end{vmatrix} + w \cdot \begin{vmatrix} a_0 & b_0 & c_0 \\ a_0' & b_0' & c_0' \\ f' & g' & h' \end{vmatrix} = 0, \tag{3}$$

which remains to be interpreted.

Now, the tangent plane considered at a point of the curve (φ) will have the equation:

$$\begin{vmatrix} x-f & y-g & z-h \\ f' & g' & h' \\ a_0 & b_0 & c_0 \end{vmatrix} = 0.$$

We seek its envelope: The characteristic is the intersection of that plane with the plane:

$$\begin{vmatrix} x-f & y-g & z-h \\ f'' & g'' & h'' \\ a_0 & b_0 & c_0 \end{vmatrix} + \begin{vmatrix} x-f & y-g & z-h \\ f' & g' & h' \\ a'_0 & b'_0 & c'_0 \end{vmatrix} = 0.$$

The line (D):

$$x = f + u\,[f' + w\,a_0\,(v)], \qquad y = \ldots, \qquad z = \ldots$$

is in the first plane for all w.

We express the idea that it is in the second plane: In order to determine w, it will give the equation:

$$\begin{vmatrix} f' + w a_0 & \cdots & \cdots \\ f'' & \cdots & \cdots \\ a_0 & \cdots & \cdots \end{vmatrix} + \begin{vmatrix} f' + w a_0 & \cdots & \cdots \\ f' & \cdots & \cdots \\ a'_0 & \cdots & \cdots \end{vmatrix} = 0,$$

which is nothing but equation (1).

That will indeed define the envelope of the planes that contain the lines of the congruence then.

Case of developable focal surfaces

3. – We have found a curve as a particular case of the locus of foci. Upon examining the question from the *correlative viewpoint* of the *duality principle*, we will be led to examine *the case in which the envelope of the focal planes is a developable surface*, namely, (Φ). Let (Φ') be the other sheet of the focal surface. The lines of the congruence are tangents to (Φ), (Φ'). Now, a tangent to the developable (Φ) must be in one of the tangent planes that envelop that developable. The lines of the congruence are then the tangents to (Φ') that are in the tangent planes to (Φ), which are the tangent to the sections of (Φ') by the planes that envelop (Φ). In that case, the edges of regression (A') on the surface (Φ') are plane curves, so the corresponding developables will be the planes of those curves. The foci of a line (D) are: The contact point with (Φ') and the point of intersection with the characteristic of the tangent plane to the developable (Φ). The other family of developables will have its edges of regression on the surface (Φ) and will correspond to the curves (C') that are conjugate to the curves (A').

Conversely, if the edges of regression of the developables that are situated on one of the sheets of the focal surface are planar curves then the corresponding developables will be planes, and their envelope will be the second sheet of the focal surface.

In order to have a congruence of that type, one can take the developable (Φ) arbitrarily, and an arbitrary family of curves on that developable. The tangents to those curves will generate congruences of the type considered, because one of the families of developables is obviously composed of the tangent planes to the developable (Φ). The contact curves on the developable will be the generators, which can be considered to be conjugate to the entire family of curves.

The case in which the congruence possesses *a focal curve and a developable focal surface*, which it correlative to itself, will be studied in § 5.

Suppose that the two sheets of the focal surface are developables. It suffices to start with a developable (Φ), and to cut them with a family of planes that depend upon one parameter. The sections will be the curves (*A*), and the planes of those sections will envelop the other focal developable. One can say in that case that one has two one-parameter families of planes, so the lines of the congruence will be the intersections of each plane of one family with each plane of the other.

One can verify that *the hypothesis of a developable double focal surface must be rejected.* Indeed, if one is given a developable surface:

$$x = f(v) + w f'(v), \qquad y = g(v) + w g'(v), \qquad z = h(v) + w h'(v) \tag{1}$$

then any line (*D*) of a congruence that admits that surface for a focal surface will be tangent to that surface, and it will have direction coefficients of the form:

$$a = f'(v) + \theta f''(v), \qquad b = g'(v) + \theta g''(v), \qquad c = h'(v) + \theta h''(v), \tag{2}$$

in which θ is a certain function of v and w. One will then recognize that if one takes the focal surface (1) to be the support of the congruence then the equation of the focal points [§ 1, eq. (5)] can be written:

$$\begin{vmatrix} f' & f'' & f''' \\ g' & g'' & g''' \\ h' & h'' & h''' \end{vmatrix} \times \begin{vmatrix} 1 & 0 & 0 \\ 1 & w+u+u\dfrac{\partial\theta}{\partial v} & u\,\theta \\ 1 & u\dfrac{\partial\theta}{\partial w} & 0 \end{vmatrix} = 0.$$

The first factor is non-zero, since the edge of regression is not planar. The second one reduces to:

$$u\,\theta\left(u\frac{\partial\theta}{\partial w} - \theta\right) = 0.$$

Now, θ is not zero, since otherwise only the lines (*D*) would be generators of the developable. It would then be impossible for the focal points to coincide.

The *two singular cases*, in which the focal surfaces are double, correspond to themselves from the correlative viewpoint. For the case of a double focal surface, that will result from the remark that the asymptotes of a surface correspond to themselves, because an asymptote is such that the osculating plane at one of its points is tangent to the surface, and from the correlative viewpoint, a point of one curve will transform into the osculating plane of an edge of regression, and conversely.

In the case where the locus of foci is a double focal curve, each point of which is associated with a unique focal plane that is tangent to the curves, a duality transformation will make ∞^1 focal planes correspond to ∞^1 foci, and each of them will be associated with a unique focus that is situated on the envelope of those focal planes. Once again, one will indeed have a unique curve for the locus of foci then that has ∞^1 focal planes that are tangent to that curve.

Introduction of contact elements. – Rectilinear foci. – Koenigs congruences.

4. – There is *another special case that is correlative to itself*, to which one will be led quite naturally when one will introduces the fundamental notions of the *geometry of contact elements* (Sophus Lie) into the theory of congruences. [Cf., Chap. XI, § 1]

One calls the system that is composed of a point *M* and a plane that passes through that point a *contact element*. Surfaces, curves, and points can be considered to be *multiplicities*, each of which is composed of ∞^2 contact elements. At each point of a surface, there is one and only one tangent plane, which gives ∞^2 contact elements. There are ∞^1 points on a curve and ∞^1 tangent planes at each point, which will again give ∞^2 contact elements. For the developables, we have ∞^1 planes and ∞^2 points, which will give ∞^2 contact elements. Similarly, a line is composed of ∞^2 contact elements that are obtained by associating the ∞^1 points of the line with the ∞^1 planes that pass through the line in all possible manners. The contact elements of a plane will be the ∞^2 elements that it defines with its various points. The contact elements of a point will be the ∞^2 elements that it defines with the various planes that contain it. The contact of two surfaces, or a surface and a line, the intersection of two lines, or the fact that a point belongs to a surface or a line, all of those geometric relations that appear to be diverse can then be interpreted in a single manner: The two multiplicities considered have a common contact element.

In the theory of congruences, the foci and the associated focal planes of a ray constitute the *focal contact element* of that ray, which are common to all of the ruled surfaces of the congruence that pass through that ray. The focal surfaces, focal curves, and developable surfaces are *focal multiplicities that are generated by the focal contact elements*, and each of them has a contact element in common with each ray.

A focal multiplicity is the locus of ∞^2 focal elements, but there are more than ∞^1 in the case of a double focal curve: They then constitute a *strip* or *strip of elements* that has that curve for its *support* [cf., Chap. VII, § 4].

We have considered all possible cases that relate to the special nature of focal multiplicities, except for the one in which one of the focal multiplicities is a line.

We discard the cases in which a focal multiplicity is a plane or a point: The congruence will then reduce to lines in a plane or rays that issue from a point.

The line can be considered to be a locus of ∞^1 points or the envelope of ∞^1 planes. Hence, it is at the same time a curve and a developable. It then results that for a congruence with a line for one focal surface, one of the families of developables of the congruence will be composed of cones that have their summits on the line, and the other one will be composed of planes that pass through the line. In particular, if the congruence has a line (δ) and a surface (Φ) for its focal multiplicities then the families of developables will be, on the one hand, the cones that are circumscribed by (Φ) at the various points of (δ), which will give the contact curves (C), and on the other hand, the planes that pass through (δ), which will cut (Φ) along the edges of regression (A). Moreover, the curves (A), (C) will define a system of conjugate curves (pp. 128). One will then get:

The Koenigs theorem:

The contact curves of the cones that are circumscribed by a surface with the various points of a line (δ) *and the sections of that surface with the planes that pass through* (δ) *constitute a conjugate net.*

Remark. – If the focal multiplicities are two lines (δ) and (δ') then the congruence will be composed of the lines that meet those two lines. It will be a *linear congruence* whose lines (δ) and (δ') will be its *directrices.*

In the case of a *double focal line* (δ) – i.e., a rectilinear double focal line – each point A of the line will correspond to a plane (P) that passes through that line, and the congruence will be composed of the lines (D) that are situated in the planes (P) and pass through the points A of (δ). If the correspondence between the points A and the planes (P) is homographic then one will obtain a *special linear congruence* with a double directrix (see Chap. X).

Application: Joachimsthal's surfaces.

We now seek the surfaces whose lines of curvature of one system are in planes that pass through a fixed line (δ).

Let (Φ) be a surface that meets that demand. Imagine the tangents to the lines of curvature considered. Those tangents (D) constitute a congruence, and since the lines of curvature are in planes that pass through (δ), those lines (D) will meet the line (δ); (Φ) is one of the sheets of the focal surface: The developables are, on the one hand, the planes of the lines of curvature, and on the other, the cones that are circumscribed by (Φ) that have their summits at the various points of (δ). Therefore, from the Koenigs theorem, the contact curves constitute a system that is conjugate to the first system of lines of curvature, and in turn, will define the second system of lines of curvature. If we consider the second system then the circumscribed cone will cut the surface (Φ) along an angle that is constantly zero. From the Koenigs theorem, the contact curve, which is a line of curvature of (Φ), will also be a line of curvature of the circumscribed cone then. It will then be an orthogonal trajectory of the generators; i.e., the intersection of the cone with a sphere that has its center at the summit. The second system of the lines of curvature will then be composed of the spherical curves, and the corresponding spheres will cut the circumscribed cones orthogonally, and in turn, the surface (Φ), along lines of curvature. *The surface* (Φ) *will then be an orthogonal trajectory of one family of spheres that have their centers on* (δ).

That property is characteristic of the surface (Φ). Indeed, suppose that a family of spheres that have their centers on (δ) and a surface (Φ) is orthogonal to each of those spheres all along the curve of intersection. The intersection is a line of curvature of the sphere, and since the angle between (Φ) and the sphere is constantly a right angle, it will be a line of curvature of (Φ). If one joins the center A of the sphere to a point M of the line of curvature then that line will be normal to the sphere and therefore tangent to the surface (Φ), in such a way that the line of curvature will be the contact curve of the circumscribed cone to (Φ) that has the point A for its summit. One of the families of the lines of curvature is composed of the contact curves of the cones that circumscribe (Φ) that have their summits on (δ), so, from the Koenigs theorem, the other family will be, in fact, composed of plane sections that of (Φ) that are made by the planes that pass through (δ).

We are then led to look for the surfaces that cut a given family of spheres at a right angle, and have their centers on (δ), all along the curves of intersection. Let (Φ) be one such surface, and let (Σ) be one of the spheres of the family. The plane that passes through (δ) and a point M of the intersection of (Φ) and (Σ) is also orthogonal to (Σ). Hence, the section of (Φ) with that plane is orthogonal to (Σ) at M, and consequently, to the great circle of (Σ) that is situated in that plane.

Hence, the section of (Φ) by an arbitrary plane that passes through (δ) [which, from the foregoing, is one of the planar lines of curvature of (Φ)] will be an orthogonal trajectory to the family of great circles that are determined by that plane in the given spheres. If one considers another plane that passes through (δ) then the line of curvature that is situated in that plane will be an orthogonal trajectory to the family of great circles that is obtained similarly. Upon folding the second plane over the first one, the two families of great circles will be superimposed, and one will have another orthogonal trajectory of the same family of great circles.

One then considers a family of circles in a plane that passes through (δ) *that have their centers on* (δ), *determines their orthogonal trajectories, and makes each of those orthogonal trajectories turn around* (δ) *through an angle that corresponds to it and varies in a continuous manner when one passes from one trajectory to the infinitely-close trajectory.* If the family of circles and the law of rotation are chosen conveniently then *the locus of curves thus-obtained will be the surface* (Φ).

No matter what that law of rotation is, moreover, and no matter what the family of circles is, one will always obtain a surface that meets the requirements above: Indeed, that surface will be generated by the curves that orthogonally cut the family of spheres that have the circles considered for their great circles, and consequently the surface will cut all of the spheres at a right angle all along the curves of intersection.

We shall then look for the orthogonal trajectories of a family of circles that are situated in a plane and have their centers on a line (d). *More generally, we shall look for the orthogonal trajectories to an arbitrary family of circles in a plane,* which we define by giving the coordinates (a, b) of their centers I and their radii R as functions of one parameter u. Consider an orthogonal trajectory that meets one of the circles at a point M. The coordinates of the point M will then be, as functions of the parameter u:

(1) $$x = a + R \cos \varphi, \qquad y = b + R \sin \varphi,$$

in which φ is a conveniently-chosen function of u. Everything comes down to determining that function of u in such a manner that the curve that is represented by equations (1) will be normal to all of the circles. The normal IM to the circle will have $\cos \varphi$, $\sin \varphi$ for its direction parameters. It must be tangent to the curve, which gives the condition:

(2) $$\begin{vmatrix} dx & dy \\ \cos\varphi & \sin\varphi \end{vmatrix} = 0;$$

i.e.:

$$\begin{vmatrix} da+\cos\varphi\cdot dR - R\sin\varphi\cdot d\varphi & da+\sin\varphi\cdot dR + R\cos\varphi\cdot d\varphi \\ \cos\varphi & \sin\varphi \end{vmatrix} = 0$$

or

$$\sin \varphi \cdot da - \cos \varphi \cdot db - R\, d\varphi = 0,$$

or rather:

(3) $$\frac{d\varphi}{du} = \frac{a'}{R}\sin\varphi - \frac{b'}{R}\cos\varphi \qquad \left(a' = \frac{da}{du}, b' = \frac{db}{du}\right).$$

If we set:

$$\tan\frac{\varphi}{2} = w$$

then:

$$d\varphi = \frac{2\,dw}{1+w^2},$$

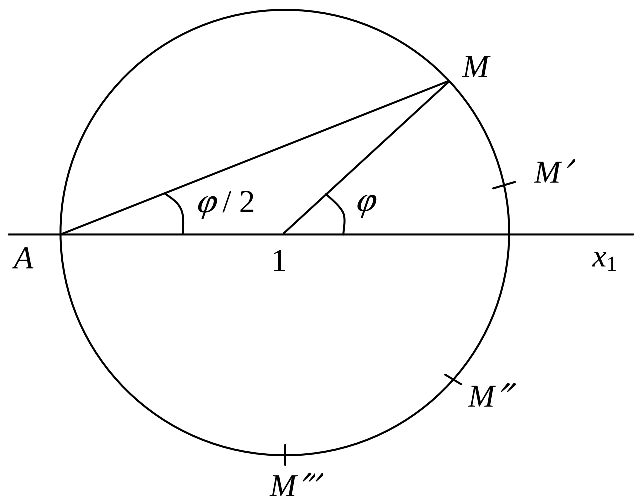
and the differential equation will become:

$$\frac{1}{du}\frac{2\,dw}{1+w^2} = 2A\,\frac{2\,w}{1+w^2} - 2B\frac{1-w^2}{1+w^2} \qquad \left(2A = \frac{a'}{R}, 2B = +\frac{b'}{R}\right)$$

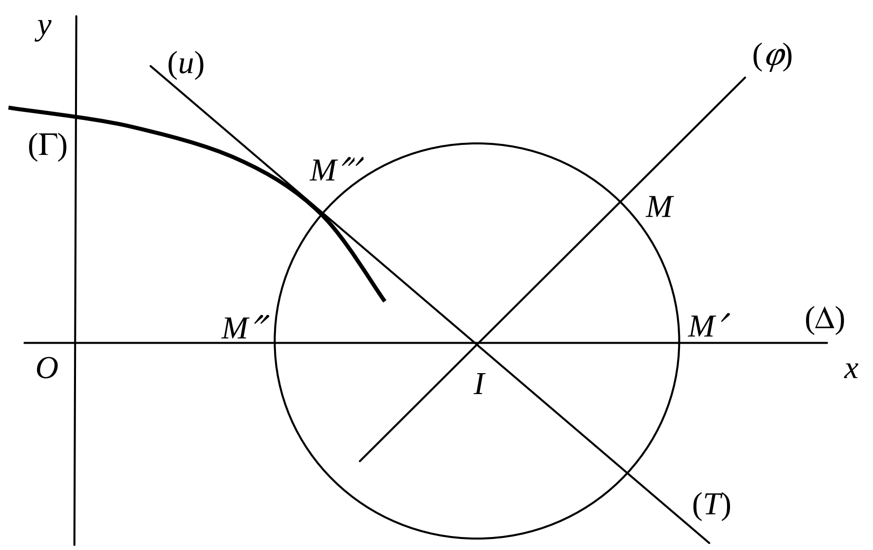
or:

$$\frac{dw}{du} = B\,w^2 + 2A\,w - B.$$

That is a Ricatti equation. The anharmonic ratio of four integrals w will be constant. In order to interpret that result, imagine one of the circles of the family. Let M be the point where it is cut by one of the orthogonal trajectories: $\tan \varphi / 2$ is the angular coefficient of the line AM (cf., the figure). If one considers four orthogonal trajectories that cut the circle at the points M, M', M'', M''' then the four corresponding values of w will be the angular coefficients of the four lines AM, AM', AM'', AM''', and the anharmonic ratio of the four integrals w will be the anharmonic ratio of the sheaf (A, M, M', M'', M'''); i.e., the anharmonic ratio (M, M', M'', M''') of the four points on the circle. It will then result that *four orthogonal trajectories of one family of circles cut all of the circles of the family with the same anharmonic ratio.*



In the special case in which the circles have their centers on a line (δ), the points M', M'' of intersection of the circle with (δ) will correspond to two orthogonal trajectories. One will then know two integrals of the Riccati equation, and the determination of the orthogonal trajectories will come down to one quadrature. In order to define the family, instead of giving a, b, R as functions of one parameter, one can give an orthogonal trajectory (Γ). One will then know three integrals of the Riccati equation, and the general integral will be obtained by writing down that its anharmonic ratio with the three known integrals is constant.



Suppose that (δ) is the axis Ox, and give (Γ) by its tangent (T). One of them is defined by the equations:

$$x = a + \rho \cos u, \qquad y = \rho \sin u,$$

in which a is a given function of u. In order to determine the ρ of the contact point M with (Γ), according to the principles of the theory of envelopes, it will suffice to differentiate the latter equations while considering x and y to be constants, which will give:

$$da - \rho \sin u \, du + \cos u \, d\rho = 0, \qquad \rho \cos u \, du + \sin u \, d\rho = 0,$$

so:

$$\rho = \frac{da}{du} \sin u = R.$$

That formula gives the radius $R = IM'''$ of the circles of a family whose centers have the coordinates $x = a$, $y = 0$. From the foregoing, an arbitrary orthogonal trajectory will then be represented by:

$$x = a + \frac{da}{du} \sin u \cdot \cos \varphi, \qquad y = \frac{da}{du} \sin u \cdot \sin \varphi, \tag{4}$$

in which the angle φ is linked with u by the constancy of the anharmonic ratio (M, M', M'', M'''), which is expressed by the formula:

$$\tan \frac{\varphi}{2} = m \cdot \tan \frac{u}{2} \qquad (m = \text{const.}). \tag{5}$$

Now return to the Joachimsthal surfaces.

If one turns the curve (4) through an angle v around Ox, and if one sets:

$$a = f(u), \qquad \frac{da}{du} = f(u)$$

then one will get the following equations for an arbitrary orthogonal trajectory of the family of spheres that has the circles considered for their great circles:

$$\left\{ \begin{aligned} x &= f(u) + f'(u) \sin u \cos \varphi, \\ y &= f'(u) \sin u \cos \varphi \cos v, \\ z &= f'(u) \sin u \sin \varphi \sin v, \end{aligned} \right. \tag{6}$$

in which φ is always linked with u by formula (5). From the mode of generation that is obtained, those formulas will represent any of the orthogonal surfaces to the spheres considered, on the condition that one must consider m to be a function $m = g(v)$, which can be chosen arbitrarily. One will then suppose that sin φ and cos φ are replaced in equations (6) with their expressions as functions of:

$$\tan\frac{\varphi}{2} = g(v)\cdot\tan\frac{u}{2}, \tag{7}$$

and upon considering u and v to be arbitrary parameters, they will represent the most general Joachimsthal surface.

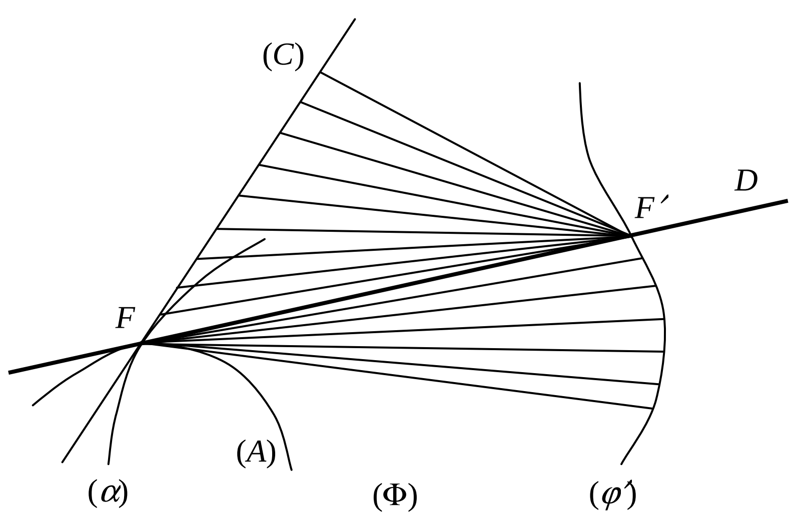
Determining the developables of a congruence

5. – We have seen that the determination of the developables of a congruence depends upon integrating a first-order differential equation of degree two. That integration can be simplified in some cases.

One will obtain the developables without any quadrature if the congruence admits two focal curves, or correlatively, two focal developables. In the former case, one will obtain cones, and in the latter case, tangent planes, as one saw before.

If the congruence admits a focal curve, or correlatively, a focal developable, then one will immediately have one of the families of developables of the congruence. In order to have the other one, one comes down to integrating a first-order differential equation of degree one.

That equation has some special properties in a case that is correlative to itself, which is *the case in which the congruence admits a focal curve and a focal developable.* Let (α) be the edge of regression of the focal developable (Φ). Consider an arbitrary generator (C) of that developable. The lines of the congruence meet the focal curve (φ') and are in the tangent planes to (Φ). Suppose that one has a tangent plane to (Φ) that meets (φ') at F'. All of the lines of that plane that pass through F' are lines of the congruence. Consider the developables of the congruence that pass through one of those lines (D). One will first have the planes that envelop the developable and admit the generator (C) for their contact curve. The foci of the line (D) are F' on (φ') and F on (C). The second developable has a curve (A) of (Φ) for its edge of regression whose tangent must meet (φ'). The problem then amounts to *finding the curves of a developable* (Φ) *whose tangents must meet a curve* (φ').



We shall look for the developables of the congruence directly, which we define by starting from the curve (φ') and associating each of its points with a certain plane in which one will find all of the lines of the congruence that pass through that point; the developable (Φ) will the envelope of that plane.

Let the curve (φ') be:

$$x = f(v), \qquad y = g(v), \qquad z = h(v).$$

In order to define a plane that passes through one of its points, it will suffice to give two directions $a_1(v)$, $b_1(v)$, $c_1(v)$ and $a_2(v)$, $b_2(v)$, $c_2(v)$. That plane will contain all lines of the congruence, so the direction coefficients of one such line will be:

$$a = a_1 + w\,a_2, \qquad b = b_1 + w\,b_2, \qquad c = c_1 + w\,c_2.$$

The differential equation of a developable:

$$\begin{vmatrix} a & b & c \\ df & dg & dh \\ da & db & dc \end{vmatrix} = 0$$

will become:

$$dv \begin{vmatrix} a_1 + w\,a_2 & \cdots & \cdots \\ f'(v) & \cdots & \cdots \\ (a_1' + w\,a_2')\,dv + a_2\,dw & \cdots & \cdots \end{vmatrix} = 0$$

here, when one denotes the derivatives with respect to v by primes. We find $dv = 0$, $v =$ const, which gives us the planes of the lines of the congruence. The other solution will be obtained by integrating the equation:

$$dw \begin{vmatrix} a_1 & \cdots & \cdots \\ f'(v) & \cdots & \cdots \\ a_2 & \cdots & \cdots \end{vmatrix} + dv \begin{vmatrix} a_1 + w\,a_2 & \cdots & \cdots \\ f' & \cdots & \cdots \\ a_1' + w\,a_2' & \cdots & \cdots \end{vmatrix} = 0,$$

which is an equation with the form:

$$\frac{dw}{dv} = P\,w^2 + Q\,w + R,$$

in which P, Q, R are functions of only v. That is a Riccati equation.

We shall point out some cases in which one can have particular integrals of that equation. If the curve (φ') is planar, and if one cuts (Φ) with that plane then the section will be a curve whose tangents meet, so it will be a curve (A). One knows a particular integral, so the problem will be solved by means of two quadratures. In particular, if (φ') is the imaginary circle at infinity then one must determine curves on (Φ) whose tangents meet the imaginary circle at infinity, and those will be the minimal curves. *The determination of the minimal curves of a developable comes down to two quadratures.*

Correlatively, if (Φ) is a cone then consider the cone with the same summit that has (φ') for its base; it is a developable of the second family. One knows a particular integral, and the problem will be solved by two quadratures.

If (Φ) is a cone and (φ') is a planar curve then one will know two particular integrals and come down to only one quadrature.

Suppose, moreover, that the planes that envelop the developable (Φ) are normal to the curve (φ'). We will have the *normal congruence* to the curve (φ'), and the search for developables will lead to the search for the *developments* of (φ'). The normal plane to (φ') at one of its points F' is perpendicular to the tangent $F'T$. If one considers the isotropic cone (J) with summit F' then the normal plane will be the polar plane to the tangent with respect to that isotropic cone. Among the normals, there will then be two of them that are contact generators of the tangent planes that are drawn through the tangent to the isotropic cone. Let (G) be one of them, which one obtains algebraically. Consider the ruled surface (R) that it generates when F' describes the curve (φ'). The asymptotic plane, which is the tangent plane at infinity on (G), is the tangent plane to the isotropic cone (J) along (G). The ruled surface contains the curve (φ'), and the tangent plane at the point F' is the plane that is defined by (G) and $F'T$, which is again the tangent plane to the isotropic cone along (G). The tangent plane to R is then the same at two points of (G), and in turn, is the same along (G): That line will then generate a developable surface. Hence, *the isotropic lines of the normal planes to a skew curve generate two developables and envelop two developables of the skew curve.* We have two particular integrals then, and the determination of the developments will be accomplished by just one quadrature.

Effectively, upon supposing that v is the arc length s of (φ') and that a_1, b_1, c_1; a_2, b_2, c_2 are the direction cosines α', β', γ' of the principal normal and those α'', β'', γ'' of the binormal, resp., the preceding equation will become:

$$dw \begin{vmatrix} \alpha' & \beta' & \gamma' \\ \alpha & \beta & \gamma \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix} + ds \begin{vmatrix} \alpha' + w\alpha'' & \cdots & \cdots \\ \alpha & \cdots & \cdots \\ -\dfrac{\alpha}{R} - \dfrac{\alpha''}{T} + w\dfrac{\alpha}{T} & \cdots & \cdots \end{vmatrix} = 0$$

when one denotes the direction cosines of the tangent by α, β, γ; i.e.:

$$-dw + \frac{ds}{T}(1 + w^2) = 0,$$

which will give the solution:

$$w = \tan \int \frac{ds}{T}.$$

Since w is nothing but the tangent of the angle χ between a normal and the principal normal here, the concordance with formula (1) of Chapter V, § 2 will be obvious.

One verifies that the differential equation in w admits the two solutions $w = \pm i$, which correspond to the isotropic developables.

If one remarks, moreover, that the focal surface of the congruence of normals is the polar surface to (φ') – i.e., that the contact points of the normals with the developments are on the polar line – then one will recover all of the essential results that were obtained in Chapter V on the subject of the developments of skew curves.

Infinitesimal metric properties of congruences

6. – We shall study an arbitrary congruence in the neighborhood of one of its lines; i.e., analyze the properties that result from simultaneous consideration of that line and some infinitely-close lines that also belong to the congruence. That amounts to considering the various ruled surfaces of the congruence (i.e., the ones that are generated by the lines of the congruence) for which the line in question is a generator and studying the tangent planes to those ruled surfaces at the various points of that generator. The notion of the foci and the focal planes is the starting point of that study.

Let (D) be the line considered. Take it to be the z-axis and place the origin of the coordinates at the midpoint between the two foci. Finally, take the xz and yz planes to be the bisecting planes of the focal planes. If the congruence is real then the foci, as well as the focal planes, can be real or conjugate imaginaries, in such a way that the midpoint of the foci and the bisecting planes of the focal planes will always be real.

Recall the notations of § 1, but with the following choice of givens: The support of the congruence will pass through O and will be normal to Oz there. The coordinate lines $w = 0$, $v = 0$ will be the ones that cross at O. The variables v and w will be the arc lengths of those curves, which will admit Ox and Oy for their tangents, moreover. On the other hand, a, b, c will be the direction cosines for (D).

That being the case, one will have:

$$\frac{\partial f}{\partial v} = \frac{\partial g}{\partial w} = c = 1, \qquad \frac{\partial f}{\partial w} = \frac{\partial g}{\partial v} = \frac{\partial h}{\partial v} = \frac{\partial h}{\partial w} = a = b = 0$$

for $v = w = 0$, and as a result:

$$df = dv, \qquad dy = dw, \qquad dh = 0. \tag{1}$$

Moreover, one will have:

$$a^2 + b^2 + c^2 = 1, \qquad a\frac{\partial a}{\partial v} + b\frac{\partial b}{\partial v} + c\frac{\partial c}{\partial v} = 0, \qquad a\frac{\partial a}{\partial w} + b\frac{\partial b}{\partial w} + c\frac{\partial c}{\partial w} = 0$$

for any v, w, and in turn, for $v = w = 0$. $\partial c / \partial v$ and $\partial c / \partial w$ are zero, and one will have:

$$du = a'\,dv + a''\,dw, \qquad db = b'\,dv + b''\,dw, \qquad dc = 0, \tag{2}$$

upon setting:

(3) $$a' = \frac{\partial a}{\partial v}, \qquad a'' = \frac{\partial a}{\partial w}, \qquad b' = \frac{\partial b}{\partial v}, \qquad b'' = \frac{\partial b}{\partial w} \qquad \text{(for } v = w = 0\text{)}.$$

Since we have confined ourselves to first-order infinitesimal properties, the arbitrary ruled surface (R) that we shall consider, which passes through (D) and whose generators belong to the congruence, will enter in only by the direction of the tangent OT to its trace on the plane xOy: We set:

$$(Ox, OT) = \varphi,$$

in such a way that an infinitely-small displacement along its trace will be:

$$dx = \cos \varphi \cdot ds, \qquad dx = \sin \varphi \cdot ds, \qquad dz = 0.$$

Hence, if one considers formulas (1) then one will see that the generator of (R) that is infinitely-close to (D) that one has to introduce is obtained by giving infinitely-small increments to v and w:

(4) $$dv = \cos \varphi \cdot ds, \qquad dw = \sin \varphi \cdot ds .$$

The tangent plane to (R) at the point M of (D) that has $z = u$ for its parameter value will be defined by the angle $\theta = (Ox, OP)$ that it makes with the plane zOx, where OP is the trace of that plane on the plane xOy; its equation will be:

(5) $$x \sin \theta - y \cos \theta = 0.$$

In order to calculate the angle θ, it will suffice to write down that this plane must contain the tangent to the curve u = constant that passes through M, which is a tangent whose direction coefficients are:

$$dx = df + u\, da, \qquad dy = dg + u\, db, \qquad dz = 0.$$

Upon taking formulas (1), (2), and (4) into account, one will then get:

$$[\cos \varphi + (a' \cos \varphi + a'' \sin \varphi)\, u] \sin \theta - [\sin \varphi + (b' \cos \varphi + b'' \sin \varphi)\, u] \cos \theta = 0$$

or

(6) $$\tan \theta = \frac{b'u + (1 + b''u) \tan \varphi}{(1 + \alpha' u) + a'' u \cdot \tan \varphi}.$$

Upon observing that the left-hand side does not depend upon $\tan \varphi$, one will get the equation of the parameter values of the *foci:*

(7) $$(1 + a'\, u)\,(1 + b''\, u) - b'\, a'' \cdot u^2 = 0.$$

Since the origin is at the midpoint of the foci, the sum of the squares will be zero:

$$a' + b'' = 0.$$

Moreover, the values of tan θ that correspond to the two roots u and $-u$ (i.e., the ones that give the *focal planes*) will be:

$$\tan\theta = \frac{\pm b'u}{1 \pm a'u},$$

and since they must be equal and opposite in sign, from the choice of coordinate planes, a' will be zero. One will then have:

(8) $$a' = b'' = 0.$$

We set:

(9) $$a'' = \frac{1}{p}, \quad b' = \frac{1}{q}.$$

The equation of the foci (7) will then reduce to:

(10) $$u^2 = pq,$$

and the focal planes will be defined at the same time by:

(11) $$\tan\theta = \frac{u}{q} = \frac{p}{u}, \qquad \tan^2\theta = \frac{p}{q}.$$

Equation (6), which gives the law of simultaneous variation for the associated geometric elements q, u, and φ, will finally become the *fundamental formula:*

(12) $$\tan\theta = \frac{p}{q} \cdot \frac{u + q\tan\varphi}{p + u\tan\varphi}.$$

The correspondence between any two of the three elements tan φ, u, tan θ is homographic, since the third one is assumed to be constant. In particular, one sees that when (R) varies, the tangent plane to (R) at a given point M of (D) will turn in the same sense as the tangent plane at the midpoint O or in the opposite sense according to whether pq ($pq - u^2$) is positive or negative, resp. Therefore, if the foci are imaginary (i.e., $pq < 0$) then the two rotations will always be in the same sense. If the foci are real (i.e., $pq > 0$) then they will be in the same sense when M is between the foci, and in the contrary sense when M is not between the foci.

Recall once more that equation (12) can be written:

(13) $$u = pq \cdot \frac{\tan\theta - \tan\varphi}{p - q\tan\theta\tan\varphi} \qquad \text{or} \qquad (13') \quad \tan\varphi = \frac{p}{q} \cdot \frac{-u + q\tan\theta}{p - u\tan\theta},$$

which exhibits a law of reciprocity between θ and φ, up to the sign of u.

Limit points and principal planes

We again look for the central point of the generator (*D*) of (*R*). If we suppose that *u* is infinite then formula (12) will give:

$$\tan\theta = \frac{p}{q\tan\varphi}$$

for the asymptotic plane; we will have:

$$\tan\theta = -\frac{q\tan\varphi}{p} \tag{14}$$

for the *central plane*, and that value, when substituted in (13), will give:

$$u = -pq\,(p+q)\frac{\tan\varphi}{p^2+q^2\tan^2\varphi} = \frac{p+q}{2}\cdot\sin 2\theta \tag{15}$$

for the central point. The parameter value of the central point will always be finite then, and its extreme values, which correspond to:

$$\theta = \pm\frac{\pi}{4}, \qquad \tan\varphi = \pm\frac{p}{q},$$

will be:

$$u = \pm\frac{p+q}{2}. \tag{16}$$

One calls those extreme positions of the central point *limit points.* They are always real, as well as the corresponding central planes, which one calls *principal planes* of the ray. They are rectangular and have the same bisecting planes as the focal planes (Hamilton).

If the foci are real then one-half their separation distance, which is the geometric mean of $|p|$ and $|q|$, will be less than that of the limit points, which is the arithmetic mean. The foci will then be between the limit points, and the pairs of points will have the same center, which is called the *center of the ray.*

If one denotes the distance between the foci and the limit points by $2d$ and 2δ, resp., then formulas (10) and (17) will give the geometric interpretation of the quantities *p* and *q*:

$$d^2 = pq, \qquad 2\delta = |p+q| \tag{17}$$

From formulas (11), the angle 2ϖ between the focal planes is given, at the same time, by the formulas:

$$\tan\varpi = \frac{d}{q} = \frac{p}{d}, \qquad \tan^2\varpi = \frac{p}{q}. \tag{17′}$$

Remark. – In the following chapter, we will see that *for any congruence that is composed of one surface, the foci are the centers of principal curvature, and the focal planes are the planes of the principal sections of the surface.* The focal planes will then be rectangular, and will coincide, moreover, with the principal planes of the ray that we just defined. Furthermore, from formula (11), the orthogonality of the focal planes is expressed by the condition:

$$\frac{d}{q}\cdot\frac{-d}{q} = \frac{p}{d}\cdot\frac{d}{-q} = 1.$$

Upon considering (17), we will then have:

$$d^2 = p^2 = q^2 = pq, \qquad p = q = \pm\, d, \quad d = \delta.$$

Hence:

$$p = q = \pm\, d, \quad d = \delta.$$

Hence, *the limit points of each ray in a normal congruence will coincide with its foci, and the focal planes will coincide with the principal planes of the ray.* The same situation will prevail for an arbitrary congruence for the rays that satisfy the condition $p = q$; i.e., the ones whose focal planes are rectangular.

Study of the deviation

Now consider two arbitrary points M and M' of the line (D), and look for the relation that exists between the relative parameter value $u' - u = \rho$ of those two points and the *deviation* that the tangent plane to (R) experiences when one passes from one to the other; i.e., the angle $\theta' - \theta = \psi$. We denote the parameter value of M' by u' and the angle that the tangent plane at M' makes with the plane zOx by θ', in such a way that, from (13′), we can write:

$$\tan\varphi = \frac{p}{q}\cdot\frac{-u' + q\tan\theta'}{p - u'\tan\theta'} = \frac{p}{q}\cdot\frac{-u + q\tan\theta}{p - u\tan\theta}.$$

We then conclude that:

$$(u' - u)\,(p - q\tan\theta\tan\theta') + (uu' - pq)\,(\tan\theta' - \tan\theta) = 0$$

or

$$\rho\,(p\cos\theta\cos\theta' - q\sin\theta\sin\theta' + u\sin\psi) + (u^2 - pq)\sin\psi = 0,$$

which can be further written:

$$\rho\left[\frac{p-q}{2}\cos\psi + \frac{p+q}{2}\cos(\psi + 2\theta) + u\sin\psi\right] = (u^2 - pq)\sin\psi. \tag{18}$$

If one is given the deviation ψ and varies (R) (i.e., θ) while leaving the point M fixed (i.e., u) then one will see that ρ has a maximum and minimum that are given by:

$$(19)\qquad \begin{cases} \rho_1\left[\dfrac{p-q}{2}\cos\psi+\dfrac{p+q}{2}+u\sin\psi\right]=(pq-u^2)\sin\psi, \\ \rho_2\left[\dfrac{p-q}{2}\cos\psi-\dfrac{p+q}{2}+u\sin\psi\right]=(pq-u^2)\sin\psi, \end{cases}$$

resp.

One infers from this that:

$$\frac{p-q}{2}\cos\psi+u\sin\psi=\tfrac{1}{2}(pq-u^2)\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)\sin\psi,$$

so

$$\frac{p+q}{2}=\tfrac{1}{2}(pq-u^2)\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)\sin\psi,$$

which will permit one write formula (18) in the form:

$$\frac{1}{2}\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)+\frac{1}{2}\left(\frac{1}{\rho_1}-\frac{1}{\rho_2}\right)\cos(\psi+2\theta)=\frac{1}{\rho},$$

so one will conclude the *Kummer formula:*

$$(20)\qquad \frac{1}{\rho}=\frac{\cos^2\left(\dfrac{\psi}{2}+\theta\right)}{\rho_1}+\frac{\sin^2\left(\dfrac{\psi}{2}+\theta\right)}{\rho_2}.$$

In the particular case where the deviation ψ is assumed to be equal to $\pi / 2$, it will reduce to:

$$\frac{1}{\rho}=\frac{\cos^2\left(\theta+\dfrac{\pi}{4}\right)}{\rho_1}+\frac{\sin^2\left(\theta+\dfrac{\pi}{4}\right)}{\rho_2},$$

which one can write more elegantly by introducing the angle $\theta + \pi / 4 = \theta_0$ that the tangent plane at M makes with one of the principal planes of the ray:

$$(21)\qquad \frac{1}{\rho}=\frac{\cos^2\theta_0}{\rho_1}+\frac{\sin^2\theta_0}{\rho_2}.$$

This *Hamilton formula* has the same form as the Euler formula (page 42) that relates to the variation of the normal curvature and will have analogous consequences. Euler's formula is, in fact, a special case of Hamilton's. Indeed, suppose that the congruence considered is the congruence of normals to a surface (S) and that M is a point of that surface. Let (G) be the trace of the ruled surface (R) on the surface (S). The angle θ_0 will be precisely the angle between the tangent to (C) at M and one of the principal planes of the ray – i.e., from the remark that was made in the preceding paragraph, one of the principal directions of the surface. On the other hand, the surface (R) is nothing but the surface (Σ_n) that was considered in the remark that concluded Chapter V, in such a way that the point M' for which the tangent plane to (R) is perpendicular to the tangent plane to (R) at M – i.e., is normal to (G) – is the center K of the normal curvature to (C). Formula (21) then expresses, in the special case in question, the normal curvature $1 / \rho$ of the curve (C) of (S) as a function of the principal curvatures $1 / \rho_1$ and $1 / \rho_2$ of (S) and the angle θ_0 between (C) and one of the principal directions of the surface.

Distribution parameter. – Suppose that in the general formula for the deviation (18), M is the central point of (D) on (R) – i.e., that u is given by (15). We then obtain:

$$(22) \qquad pq - u^2 = pq - (p+q)^2 \sin^2\theta \cos^2\theta = (p \cos^2\theta - q \sin^2\theta)\,(q \cos^2\theta - p \sin^2\theta),$$

and formula (18) will become:

$$\rho\left[\frac{p-q}{2}\cos\psi + \frac{p+q}{2}\cos\psi\cos 2\theta\right] = (p \cos^2\theta - q \sin^2\theta)\,(q \cos^2\theta - p \sin^2\theta)\sin\psi;$$

i.e., after dividing by the factor $(p \cos^2\theta - q \sin^2\theta)$:

$$(23) \qquad 0 = (q \cos^2\theta - p \sin^2\theta) \cdot \tan\psi.$$

We then get the Chasles formula (page 105), and the distribution parameter for (D) for each surface (R) will be given by the equation:

$$(24) \qquad K = q \cos^2\theta - p \sin^2\theta = \frac{p+q}{2}\cos 2\theta - \frac{p-q}{2}$$

as a function of the angle θ between the central plane and the plane zOx. At the same time, the central point is given by formula (15):

$$(15) \qquad u = \frac{p+q}{2} \cdot \sin 2\theta.$$

One sees that q and $-p$ are the extreme values of the distribution parameter: They correspond to two cases in which the central point is the center of the ray. The central planes will then be the coordinate planes – i.e., the bisector planes of the focal planes and the principal planes.

The distribution parameter is annulled when the central plane becomes perpendicular to the one the focal planes. The central point then tends towards the focus that corresponds to the other focal plane.

Properties of pencils of rays

7. – *Density at a point.* – Imagine a ruled surface (Σ) of the congruence that contains a ray (*D*) in its interior. The section of that ruled surface by the perpendicular plane to (*D*) at an arbitrary point of that line is a closed curve (σ) that contains *M* in its interior. Consider all of its points to be situated at an infinitely-small distance from *M*: The set of all rays of the congruence that are contained in the interior of (Σ) will then be called an *infinitely-thin pencil of rays* that has the ray (*D*) for its *axis*. The sections, such as (σ), will be called *cross-sections of the pencil.*

The fundamental property of these pencils results from the interpretation of the product $u_1\, u_2$ of the roots of equation (5) of § 1, which determine the foci of the ray (*D*). That product is:

$$u_1\, u_2 = \frac{P}{\Pi}, \qquad P = \begin{vmatrix} a & b & c \\ \dfrac{\partial f}{\partial v} & \dfrac{\partial g}{\partial v} & \dfrac{\partial h}{\partial v} \\ \dfrac{\partial f}{\partial w} & \dfrac{\partial g}{\partial w} & \dfrac{\partial h}{\partial w} \end{vmatrix}, \qquad \Pi = \begin{vmatrix} a & b & c \\ \dfrac{\partial a}{\partial v} & \dfrac{\partial b}{\partial v} & \dfrac{\partial c}{\partial v} \\ \dfrac{\partial a}{\partial w} & \dfrac{\partial b}{\partial w} & \dfrac{\partial c}{\partial w} \end{vmatrix}.$$

If *dv*, *dw* denote positive infinitesimals then we will write it as:

$$u_1\, u_2 = \frac{P\, dv \cdot dw}{\Pi\, dv \cdot dw}. \tag{1}$$

The numerator in that formula is developed in the form:

$$P\, dv\, dw = a \cdot \frac{D(g,h)}{D(v,w)} \cdot dv\, dw + b \cdot \frac{D(h,f)}{D(v,w)}\, dv\, dw + c\, \frac{D(f,g)}{D(v,w)}\, dv\, dw. \tag{2}$$

Now:

$$\frac{D(g,h)}{D(v,w)}\, dv\, dw, \qquad \frac{D(h,f)}{D(v,w)}\, dv\, dw, \qquad \frac{D(f,g)}{D(v,w)}\, dv\, dw \tag{3}$$

are the three components of a vector that is normal to the surface:

$$x = f(v, w), \qquad y = g(v, w), \qquad z = h(v, w) \tag{4}$$

at the point (*v*, *w*) of that surface, and whose length measures the area element of the surface at that point. Hence, if one supposes that *a*, *b*, *c* are the direction cosines of the ray that has its foot at that point then the quantity (2), which is the projection of the vector (3) onto the direction *a*, *b*, *c*, is the projection of that area element onto the plane perpendicular to the ray that is drawn through the point considered. Moreover, since the vector (3) and the positive directions:

$$\left(\frac{\partial f}{\partial v}, \frac{\partial g}{\partial v}, \frac{\partial h}{\partial v}\right), \qquad \left(\frac{\partial f}{\partial w}, \frac{\partial g}{\partial w}, \frac{\partial h}{\partial w}\right)$$

of the coordinate curves at the point considered define a direct trihedron, that projection will be positive if the direction a, b, c and the preceding positive directions also define a direct trihedron.

If we suppose that the support (4) is normal to the ray, and that its foot is the point M of the ray (D) that was considered before then that projection of the area element will reduce to the area element itself; i.e., up to higher-order infinitesimals, the area of the cross-section (σ) of the pencil, with the same sign convention.

Apply the same considerations to the denominator of formula (1). The support (4) is replaced by a sphere of radius (1), which is also normal to (D) at the point (v, w). That denominator $\Pi \, dv \, dw$ (with an entirely similar sign convention) will then measure the elementary spherical area that is homologous to $|\sigma|$; i.e., the elementary solid angle that is filled with the directions of the rays that constitute the pencil, which are supposed to issue from the same point: That is what one can call the measure of the solid angle of the pencil.

Furthermore, one sees that the ratio (1) will be positive or negative according to whether the homologous points of the contours of the cross-section (σ) and the spherical area that they correspond to describe those two contours with respect to the positive direction of the axis of the pencil in the same or opposite sense when one makes a moving ray describe the ruled surface (Σ) that bounds the pencil.

Therefore, *the product of the algebraic measures of the distances from a point M of an arbitrary ray of one congruence to the two foci of that ray is equal to the quotient of the area of the cross-section that one makes at M in an infinitely-thin pencil that has that ray for its axis with the measure of the solid angle of that pencil*, and that quotient will have the sign that was just specified (Kummer). That is equivalent to saying that it is the limit to which the analogous ratio that relates to a pencil with a finite cross-section will tend to when that cross-section tends to zero in all of its dimensions without the pencil ceasing to contain the ray in question in its interior. One takes the inverse of that limit, which will not depend upon the manner by which the pencil reduces to its axis, to be the measure of the *density of the infinitely-thin pencil* at the point M.

Hence:

The measure of the density of a pencil of the infinitely-thin congruence at an arbitrary point of its axis will be the inverse of the product of the algebraic distances of that point to the foci of that axis.

That theorem will reduce to Gauss's theorem on the total curvature (cf., page 70) for the normal congruence to a surface (S). That results from the following remarks: If one takes M to be the foot of a normal on (S) then the algebraic distance from M to the foci of that normal, which are the centers of principal curvature of the surface (§ 5), will become the principal radii of curvature of (S) at M. Moreover, one can then consider (S) to be the support of the congruence, and since that surface is normal to the ray considered (D) at M, its elementary area at M will be equal to the cross-section of an infinitely-thin pencil with axis (D). Finally, the correspondence that is established between the support (S) and a sphere of radius 1 by the directions of the rays is the spherical representation of (*S*) here. The solid angle of a pencil will then be the elementary area of the sphere that is homologous to the elementary area of the surface (*S*) in its spherical representation.

Study of the cross-section. – If one imagines two cross-sections of the same infinitely-thin pencil with axis (*D*) then the ratio of their areas σ and σ' will be equal to the inverse ratio of the densities at the point *M*, *M′* of (*D*) where those sections are made. Hence, if r_1, r_2 ; r_1', r_2' are the distances from *M* and *M′* to the two foci, respectively, then one will have:

$$\frac{\sigma'}{\sigma} = \frac{r_1' r_2'}{r_1 r_2} \tag{5}$$

for the ratio of the area. That ratio will tend to zero if *M* stays fixed and *M′* tends to a focus. The pencil will then flatten into its two foci, in such a manner that the areas of the corresponding cross-sections will be infinitesimals of higher order than the other cross-sections.

Meanwhile, we can specify it by means of the formulas of § 6. Suppose that the pencil is given by the cross-section that is made at the *center* of the ray. From the choice of the coordinate axes, it is the section that is made by the *xy*-plane. Upon neglecting the infinitesimals of order higher than one, the coordinates of a point of the contour of that section will be:

$$x = dv, \qquad y = dw, \qquad z = 0. \tag{6}$$

The coordinates of an arbitrary point of the ray of the congruence that passes through that point are:

$$x = f(v + dv, w + dw) + u \cdot a(v + dv, w + dw), \qquad y = \ldots, \qquad z = \ldots,$$

or, upon neglecting the higher-order infinitesimals and taking formulas (1), (2), (8), (9) of § 6 into account:

$$x = dv + u\frac{dw}{p}, \qquad y = u\frac{dv}{q} + dw, \qquad z = u. \tag{7}$$

If one considers *u* to be constant then formulas (6) and (7) will express the correspondence that is established by the rays of the congruence between the points of the plane $z = 0$ and those of the plane $z = u$. If we keep the letters *x*, *y* for the former and denote the latter by *X*, *Y* then that correspondence will be defined by the formulas:

$$X = x + \frac{u}{p}y, \qquad Y = \frac{u}{q}x + y. \tag{8}$$

This is a linear correspondence that will become singular when the determinant of the coefficients of *x* and *y* is zero; i.e., for:

$$u^2 - pq = 0.$$

That condition expresses the idea [eq. (10), § 6] that the section $z = u$ is drawn through one if the foci F; it can be realized only if the foci are real.

In the case where it is realized, one will have identically:

$$Y = \frac{a}{q} X$$

for any x and y, or [eq. (11), § 6], if θ denotes the angle between the focal plane (P) that is associated with the focus F and the plane zOx then:

$$Y = X \tan \theta.$$

Therefore, no matter what form the central cross-section (i.e., the plane that passes through the center of the ray) has, *the pencil is cut by the plane of the cross-section that passes through one focus along a rectilinear segment that is situated in the focal plane that is associated with that focus.* If one neglects the infinitesimals of order higher than the diameter of the central cross-section then the external surface of the pencil will have the appearance of a ruled surface that has two rectilinear directrices that pass through the foci of the axis of the pencil, are perpendicular to that axis, and are situated in the associated focal planes to those foci, respectively.

For example, suppose that the central cross-section is a circle of radius r. The section by the plane of the parameter value $z = u$ will be the ellipse:

(9) $$\left(X - \frac{u}{p} Y\right)^2 + \left(Y - \frac{u}{q} X\right)^2 = r^2 \left(1 - \frac{u^2}{pq}\right)^2,$$

which effectively reduces to a double line for $u^2 = pq$ in the case where the foci are real.

The angle ω between one axis of that ellipse and the plane zOx will be given by the formula:

(10) $$\tan 2\omega = \frac{2pq}{q-p} \cdot \frac{1}{u}.$$

In the case $p = q$ (i.e., in the case of normal congruences, if that situation is true for all the rays, and more generally, whenever the focal planes are rectangular), the axes will then always be in the focal planes, and will then coincide with the principal planes of the ray.

If one discards that case then if one projects the section onto the plane $z = 0$ then one will see that when the parameter value u of the second plane varies from $- \infty$ to $+ \infty$, the right angle that is defined by the two axes of the section will always turn in the same sense. The total rotation will be $\pi / 2$, and when the parameter u tends to zero, those axes will tend to be located in the principal planes of the ray. The two ellipses that are provided by two planes that are equidistant from the center of the ray will be symmetric to each other with respect to Ox and Oy upon projection, moreover.

In the case of real foci, if one regards the formulas (17) of § 6 then one can put formula (10) into the form:

$$\tan 2\omega = \frac{d}{u} \cdot \tan 2\varpi,$$

into which the half-distance d between the foci and the angle 2ϖ between the focal planes enter.

The lengths l of the axes of the ellipse (9) are given by the equation:

$$l^4 - \left[2 + u^2\left(\frac{1}{p^2} + \frac{1}{q^2}\right)\right] r^2 l^2 + \left(1 - \frac{u^2}{pq}\right)^2 r^4 = 0, \tag{11}$$

and the law of their variation will result from the study of the hyperbola that is represented by the equation that one will deduce by setting:

$$l^2 = r^2 \cdot y, \qquad u^2 = pq \cdot x.$$

One will then see that if u varies from 0 to $\pm\infty$ then one of these axes will constantly increase, while the other one will first decrease, pass through a minimum, and then also increase constantly: The two axes will become infinite along with the parameter u.

If the foci are real ($pq > 0$) then the minimum of the second axis will be zero, and in conformity to what we have seen, that will be true when the plane of the section goes through a focus. The first axis, which is then situated in the corresponding focal plane, will have the length:

$$2l = \frac{4R}{\sin 2\varpi}.$$

One can then say that the pencil is smeared along its rectilinear directrices over a length that is, in general, greater than twice its central diameter and equal to twice that diameter in the case where the focal planes are rectangular.

Remark. – The case in which the foci coincide on the ray considered is treated by letting the origin O be arbitrary on that ray. One supposes only that the plane zOx is the double focal plane. Hence, if h is the parameter value of the double focus and the other hypotheses about the choice of axes that were made in § 6 are maintained then the correspondence between the plane $z = 0$ and the plane of the parameter value u will be expressed by the formulas:

$$X = \left(1 - \frac{u}{h}\right)x + \frac{u}{k}y, \quad Y = \left(1 - \frac{u}{h}\right)y,$$

in which we have set:

$$h = -\frac{1}{a'}, \qquad k = \frac{1}{a''},$$

here.

Indeed, upon writing down that the roots of equation (7) are equal to h, and that formula (6) gives the value 0 for $u = h$, one will get:

$$a' = b'' = -\frac{1}{h}, \qquad b' = 0.$$

One sees that if one is given the section (6) of the pencil arbitrarily in the plane $z = 0$, which an arbitrary cross-sectional plane here, then the section by the plane $z = h$, which passes through the focus, is only rectilinear: It will then be in the focal plane, and its length will be proportional to the dimension of the section (6) that is perpendicular to that focal plane.

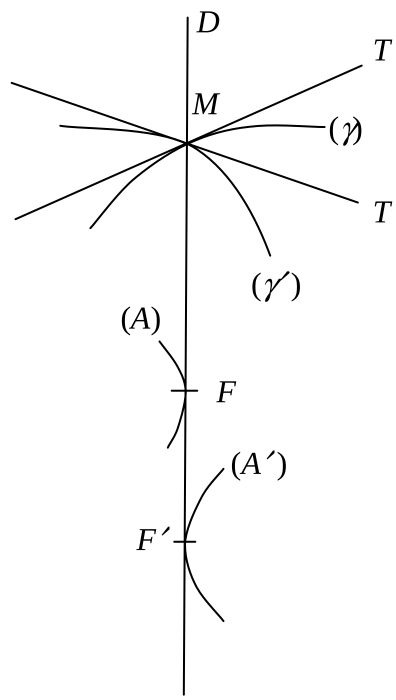
The hypothesis $a'' = 0$, which was discarded implicitly, corresponds to the case in which the focal plane is indeterminate. The cross-section of the pencil by a plane that passes through the focus will then reduce to a point.

CHAPTER VII

NORMAL CONGRUENCES

Characteristic property of normal congruences

1. – Consider a surface, so the coordinates of one of its points will depend upon two parameters. The set of all normals to that surface will depend upon two parameters and will constitute a congruence. In order to obtain the developables of that congruence, it will suffice to consider the two families of curvature lines on the surface, since the normals to a surface at all points of a curvature line generate a developable surface. The tangent plane to that developable passes through the normal (*D*) and the tangent to the corresponding line of curvature. It is one of the focal planes of the line (*D*). Therefore, the *focal planes are the planes of the principal sections of the surface. The focal planes of a normal congruence are rectangular.* It then results that an arbitrary congruence is not generally composed of the normals to a surface.



Consider the two lines of curvature (γ), (γ') that pass through a point *M* of the surface. The developable of (γ) corresponds to an edge of regression (*A*) whose osculating plane is the focal plane, so the contact point *F* of (*A*) and the line *D* will be one of the focal points. The edge of regression (*A*) is the envelope of the line (*D*) when the point *M* displaces along the curve (γ). The point *F* is then one of the centers of principal curvature of the surface at the point *M*. The associated focal plane is the second plane of the principal section *FMT′*. One will likewise get a second edge of regression (*A′*) upon considering the curve (γ').

One will easily see that these properties of the centers of principal curvature and the planes of the principal sections will persist regardless of the nature of the focal multiplicities of the congruence considered.

Conversely, let a congruence be composed of the lines (D):

$$x = f(v, w) + u \cdot a(v, w), \qquad y = g(v, w) + u \cdot b(v, w), \qquad z = h(v, w) + u \cdot c(v, w).$$

We seek the conditions under which one can choose a point M on each line (D) whose locus is a surface that is constantly normal to (D). In order for that to be true, it is necessary and sufficient that one can determine u as a function of v, w in such a fashion that:

$$\sum a\, dx = 0$$

or:

$$\sum a\,(df + u\, da + a\, du) = 0.$$

Suppose that a, b, c are the direction cosines of (D). u will then represent the distance from the point P where the line meets the support to the point M, and one will have:

$$\sum a^2 = 1, \qquad \sum a\, da = 0.$$

The preceding condition will, in turn, become:

$$du + \sum a\, da = 0$$

or

(1) $$-du = \sum a\, da\,.$$

That equation expresses the idea that $\sum a\, da$ *is an exact total differential.* Now:

$$\sum a\, da = \sum a\,\frac{\partial f}{\partial v}\,dv + \sum a\,\frac{\partial f}{\partial w}\,dw\,,$$

so the condition will be:

$$\frac{\partial}{\partial w}\sum a\,\frac{\partial f}{\partial v} = \frac{\partial}{\partial v}\sum a\,\frac{\partial f}{\partial w}$$

or

$$\sum \frac{\partial a}{\partial w}\frac{\partial f}{\partial v} = \sum \frac{\partial a}{\partial v}\frac{\partial f}{\partial w}$$

or finally:

(2) $$\sum\left(\frac{\partial a}{\partial w}\cdot\frac{\partial f}{\partial v} - \frac{\partial a}{\partial v}\cdot\frac{\partial f}{\partial w}\right) = 0.$$

We have found a unique condition. Now, we have previously found that a necessary condition is the orthogonality of the focal planes. We will then be led to compare the two conditions. The direction coefficients A, B, C of a focal plane verify the relations:

(3) $$Aa + Bb + Cc = 0,$$

$$\begin{cases} A\left(\dfrac{\partial f}{\partial v}+u\dfrac{\partial a}{\partial v}\right)+B\left(\dfrac{\partial g}{\partial v}+u\dfrac{\partial b}{\partial v}\right)+C\left(\dfrac{\partial h}{\partial v}+u\dfrac{\partial c}{\partial v}\right)=0, \\ A\left(\dfrac{\partial f}{\partial w}+u\dfrac{\partial a}{\partial w}\right)+B\left(\dfrac{\partial g}{\partial w}+u\dfrac{\partial b}{\partial w}\right)+C\left(\dfrac{\partial h}{\partial w}+u\dfrac{\partial c}{\partial w}\right)=0. \end{cases}$$

Eliminating u between the last two equations, we will obtain:

(4)
$$\begin{vmatrix} \sum A\dfrac{\partial f}{\partial v} & \sum A\dfrac{\partial a}{\partial v} \\ \sum A\dfrac{\partial f}{\partial w} & \sum A\dfrac{\partial a}{\partial w} \end{vmatrix} = 0.$$

The direction coefficients of the normals to the focal planes are defined by (3) and (4). If we consider A, B, C to be current coordinates then (3) will represent a plane that passes through the origin, and (3) will be a cone that has the origin for its summit, and the generators of the intersection will be precisely the desired normals. We express the idea that those two lines are rectangular. The plane (3) is perpendicular to the line (a, b, c), which is on the cone (4), because from the conditions $\sum a^2 = 1$ and $\sum a\,da = 0$, one will deduce that:

$$\sum a\frac{\partial a}{\partial v} = 0, \qquad \sum a\frac{\partial a}{\partial w} = 0.$$

Hence, the two normals will be perpendicular to the line (a, b, c). If they are rectangular then the cone (4) will admit an inscribed tri-rectangular trihedron, which will give the condition:

$$\sum\left(\frac{\partial f}{\partial v}\cdot\frac{\partial a}{\partial w}-\frac{\partial f}{\partial w}\cdot\frac{\partial a}{\partial v}\right) = 0.$$

That is precisely the condition (2). Hence, *the necessary and sufficient condition for the congruence to be a normal congruence is that the focal planes of each ray must be rectangular.*

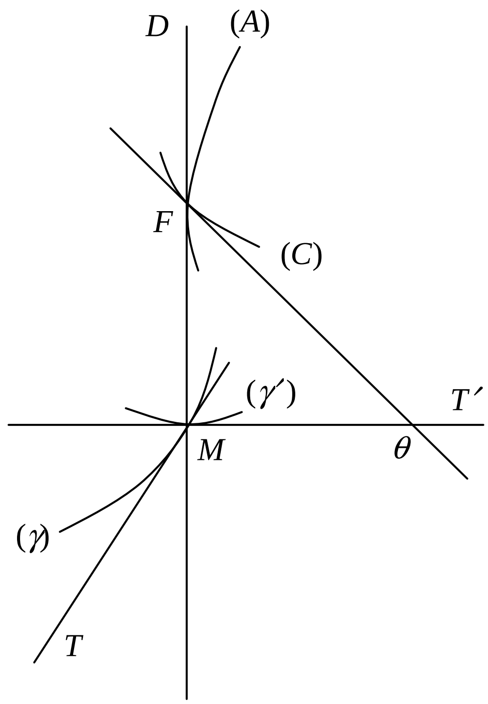
Suppose that condition (2) is satisfied. In order to obtain a normal surface to all lines of the congruence, it will suffice to calculate u as a function of v, w, which one does with equation (1). By hypothesis, it has the form:

$$du = d\Phi\,(v, w),$$

so

(5)
$$u = \Phi\,(v, w) + \text{const.}$$

There is then an infinitude of surfaces that meet the requirement. If two points M and M' of (D) describe two of those surfaces (S) and (S'), respectively, which correspond to two functions $u = PM$ and $u' = PM'$, resp., and which are given by formula (5), then the distance $MM' = u' - u$ will be a constant quantity. The surfaces (S), (S') are called *parallel surfaces,* and *a family of parallel surfaces will admit the same centers of principal curvature and the same focal multiplicities for each normal.* Those focal multiplicities constitute the *development* of any of those surfaces.

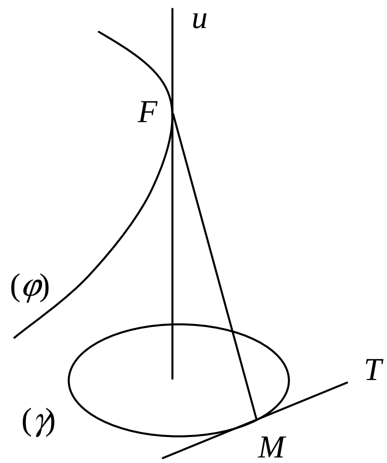


Relations between a surface and its development

2. – Consider one sheet of the development of a surface (*S*). First, suppose that it is a surface (Φ). Consider a line (*D*) of the congruence of normals to (*S*). That line is tangent at *F* to the edge of regression (*A*) that belongs to (Φ). The focal planes that are associated with (*D*) are the osculating plane to (*A*) and the tangent plane to (Φ). In order for the congruence to be a normal congruence, it is necessary and sufficient that the osculating plane to (*A*) must be normal to (Φ), and therefore that (*A*) must be a geodesic of (Φ). *The congruence of normals to a surface* (*S*) *is composed of the tangents to a family of geodesics of its development* (Φ), *and conversely, the tangents to a family of* ∞^1 *geodesics of an arbitrary surface* (Φ) *constitute a normal congruence.*

Let *M* be the point where the line (*D*) cuts the surface (*S*). When the line (*D*) envelops the edge of regression (*A*), the point *M* will describe a line of curvature (*γ*) of (*S*). Each point *M* of (*S*) corresponds to a point *F* of (Φ), so there is a point-by-point correspondence between the two surfaces. The family of lines of curvature (*γ*) of (*S*) corresponds to a family of geodesics of (Φ).

Now, look at the contact curves (*C*) with (Φ). Consider the tangent *Fθ* to (*C*). It is the characteristic of the tangent plane to (Φ) when the point *M* describes (*γ*). Now, that tangent plane to (Φ) is the second focal plane, so it is the plane perpendicular to the plane *FMT* that passes through *FM*, and thus the normal plane to (*γ*) to the point *M*. Hence, *Fθ* is the characteristic to the normal plane to (*γ*), so it is the polar line to (*γ*). Since *Fθ* is in the plane normal to (γ), it will meet the tangent to the second principal section; it will pass through the center of geodesic curvature of (γ) on (S).



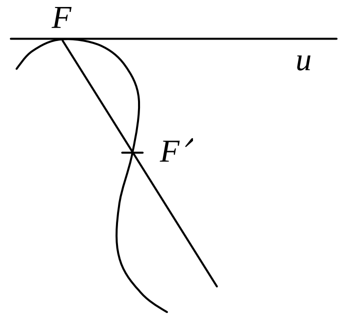
Canal surfaces

Suppose that one of the sheets of the development reduces to a curve (φ). The line (D) meets (φ) at one of the focal points F. One of the developables that passes through (D) is a cone with summit F. One of the lines of curvature (γ) of (S) that passes through M is situated on that cone with its summit at F. Now, (γ) is constantly normal to D, so it will be an orthogonal trajectory to the generators of the cone; i.e., the intersection of that cone with a sphere of center F. At each point M of that sphere, it is normal to the line (D). It will then be tangent to the surface (S) all along the curve (γ). Each point F of (φ) corresponds to a sphere that has that point for its center and is tangent to (S) all along the corresponding line of curvature. Hence, *a surface* (S) *that has a curve for one sheet of its development is the envelope of a family of spheres that depend upon one parameter*. We call such a surface a *canal surface.* Meanwhile, one sometime reserves that name for the envelopes of ∞^1 equal spheres. The converse of the preceding proposition is true, as we will see later on.

The curve (γ) is then the intersection of a sphere with an infinitely-close sphere; it is a circle. The cone F is one of revolution whose axis is the limiting position of the line of centers, so it will be the tangent Fu to (φ). Consider the tangent MT to (γ). MT, which is tangent to a point of the circle, is orthogonal to Fu. Fu is then in the second plane of the principal section. *The congruences considered are then composed of the generators of the* ∞^1 *cones of revolution whose axes are tangents to the curve that is the locus of summits of those cones. Conversely, any congruence, thus-constituted, is a normal congruence*, because the focal planes will be the tangent planes and the meridian planes of those cones and will consequently be rectangular.

Dupin cyclide

Let us see if the two sheets of the development can reduce to two curves (φ) and (φ'). The developables of the congruence are the cones that have their summits on one of the curves and pass through the other one. All of the cones (F) of revolution must pass through the curve (φ'). That curve (φ') is such that an infinitude of cones of revolution will pass through that curve, and similarly for (φ). Hence, (φ), (φ') can only be skew biquadratics or their elements of decomposition. However, neither of those curves can be a skew biquadratic, since otherwise four cones of only second degree would pass through one of them.



Now, let us see if one of them can be a twisted cubic. The cones of degree two that pass through a twisted cubic (φ') have their summits on (φ). The two curves (φ) and (φ') will then coincide. We then examine whether there can exist twisted cubics such that the cones of second degree that contain them will be cones of revolution. Such a cone will have the tangent Fu for its axis. Now, it contains that tangent, so it will decompose. Therefore, neither (φ) nor (φ') can be twisted cubics.

Suppose then that (φ') is a cubic. The locus of summits of the cones of revolution that pass through that conic is, as one knows, another conic, which is the focal surface of the first one. There is reciprocity between those conics, and the cones of revolution have the tangents to the focal surfaces for their axes. Therefore, *the lines that meet both of two focal conics constitute a normal congruence.* The normal surfaces to those lines are called *Dupin cyclides. Their two systems of lines of curvature are circles.*

Special cases. – Suppose, in particular, that (φ') is a circle. The locus of the summits of the cones of revolution that pass through (φ') is the axis (φ) of that circle, and we see that *all of the lines that are supported by a circle* (φ') *and its axis* (φ) *are normal to a family of surfaces.* Those surfaces are *torii* of revolution around the axis (φ), and the locus of the center of the meridian circle is the circle (φ').

Suppose that (φ') is a line: The surface is the envelope of a family of spheres that have their centers on that line. It is a surface of revolution around (φ'). The first sheet of the development is the line (φ'), while the second one generated by the rotation of the development of the principal meridian. In order for it to be a curve, it is necessary that the development must be a point, and thus that the meridian must be a circle, and we come back to the case of the torus.

Singular case

Finally, let us see whether the two sheets of the development can coincide. If that were true then the two families of lines of curvature of the surface (S) would coincide: That is the case for *ruled surfaces with isotropic generators.* For those surfaces, the two sheets of the development will reduce to just one curve, as we will see in the following paragraph.

Study of the enveloping surfaces of spheres

3. – While discussing the nature of the development of a surface, we were led to consider surfaces that were the envelopes of spheres. The study of those surfaces will now lead us to the converses of those preceding properties.

Consider a surface (S) that is the envelope of ∞^1 spheres (Σ). Each sphere cuts the sphere infinitely close to a circle, and the normals to (S) at all points of the circle will pass through the center of the sphere. The locus of the centers of the spheres is a curve that is met by all of the normals to (S), so it will be one of the sheets of the development. On the other hand, the sphere (S) is tangent to the surface (S) all along the characteristic circle, so that circle will be a line of curvature of the surface (S), from Joachimsthal's theorem. *The surfaces that are envelopes of spheres have a family of circular lines of curvature. Conversely, any surface that has a family of circular lines of curvature is an envelope of spheres.* Indeed, consider a circular line of curvature (K). Any sphere that passes through (K) cuts the surface (S) at a constant angle, from Joachimsthal's theorem. Now, there exists a sphere that passes through (K) and is tangent to (S) at one of the points of that circle. That sphere will then be tangent to (S) at all points of the circle (K), and any circular line of curvature will be a contact curve of a sphere with the surface. The surface is the development of the spheres thus-determined.

Let (a, b, c) be the center, and let r be the radius of one of the ∞^1 spheres considered; a, b, c, r are functions of the same parameter.

The sphere has the equation:

$$(x - a)^2 + (y - b)^2 + (z - c)^2 - r^2 = 0.$$

The characteristic is defined by that equation and the equation:

$$(x - a)\, da + (y - b)\, db + (z - c)\, dc + r\, dr = 0.$$

One indeed verifies that it is a circle whose plane is perpendicular to the direction da, db, dc of the tangent to the locus of centers of spheres.

We just considered surfaces with one family of lines of curvature that is composed of circles. Let us see if the two families of lines of curvature can be circular. The corresponding surface can be considered to be the envelope of ∞^1 spheres in two different ways. The two sheets of the development will be curves. The surfaces will then be a *Dupin cyclide*, and that will provide us with a new viewpoint for the study of that cyclide.

Correspondence between lines and spheres

Lines and spheres are geometric elements that depend upon four parameters. That fact permits one to predict that there will be a correspondence between the study of systems of lines and that of systems of spheres. That correspondence finds its analytical expression in a transformation that is due to Sophus Lie that we will present later on. However, we shall see that it manifests itself in various questions before that. Therefore, in the geometry of spheres, one can consider envelopes of ∞^1 spheres to correspond to ruled surfaces, which are loci of ∞^1 lines. The Dupin cyclides then correspond to doubly-ruled surfaces, and thus to ruled surfaces of degree two. We shall see the development of that analogy in what follows.

Let (Σ) be a sphere of the first family, and let (Σ') be a sphere of the second family, such that (Σ) touches (S) along a circle (K) and (Σ') touches (S) along a circle (K'). The surface (S) is generated by the circle (K) or by the circle (K'), so it will result that those two circles will have at least one common point M. Let O, O' be the centers of the spheres (Σ), (Σ'), resp., so OM and OM' will be normals to the spheres (Σ), (Σ'), and in turn, normals to M at the surface. They will then coincide, so O, M, O' will be on the same line. The spheres (Σ), (Σ') are tangent at M. *A sphere of one of the families is tangent to any sphere of the other family.* (More precisely: Two generators of different systems of a quadric will meet.)

Consider three fixed spheres (Σ), (Σ_1), (Σ_2) of one of the families. They are tangent to all of the spheres of the other family, and in turn, *the surface will be the envelope of spheres that are tangent to three fixed spheres.* (A quadric is the locus of all lines that meet three fixed lines.) The three spheres (Σ), (Σ_1), (Σ_2) will cut at two points that can be considered to be spheres of radius zero that are tangent to (Σ), (Σ_1), (Σ_2). Hence, there will be two spheres of radius zero in each family of spheres that is enveloped by the cyclide. The spheres of the other family must be tangent to those two spheres of radius zero that pass through their centers. Those two points are on the locus of centers of the spheres, and therefore on the focal conics. *Hence, if we consider the two focal conics then the spheres of one of the families will have their centers on one of the conics and pass through two fixed points of the other one that are symmetric with respect to the plane of the first one.* With that manner of generation, it will then be easy to find the equation of the cyclide.

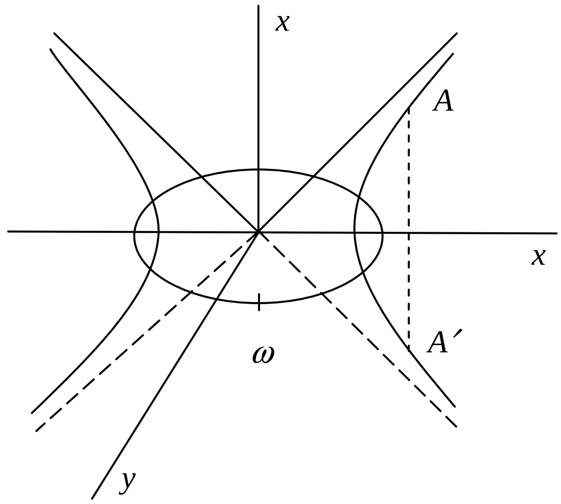
Equation of the Dupin cyclide

1. First suppose that one of the conics is an ellipse, for example, while the other one is a hyperbola. Take the Ox, Oy axes to be the axes of the ellipse, whose equation in its plane is:

$$(E) \qquad \frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0.$$

The focal hyperbola is in the plane $y = 0$. It has the equation in that plane:

$$(H) \qquad \frac{x^2}{a^2-b^2}+\frac{z^2}{b^2}-1=0.$$



A point ω of the ellipse (E) has the coordinates:

$$x = a \cos \varphi, \quad y = b \sin \varphi, \quad z = 0.$$

Let the fixed points A and A' on the hyperbola (H) be defined by the formulas:

$$x_0, \quad y_0 = 0, \quad z_0^2 = b^2\left(\frac{x_0^2}{a^2 - b^2} - 1\right).$$

The equation of a sphere (Σ) that has ω for its center and passes through the points A and A' will be:

$$(x - a \cos \varphi)^2 + (y - b \sin \varphi)^2 + z^2 = (x_0 - a \cos \varphi)^2 + b^2 \sin^2 \varphi + b^2\left(\frac{x_0^2}{a^2 - b^2} - 1\right),$$

or

$$x^2 + y^2 + z^2 - 2ax \cos \varphi - 2by \sin \varphi = x_0^2 + b^2\frac{x_0^2}{a^2 - b^2} - b^2 - 2ax_0 \cos \varphi,$$

which is written:

$$2a (x - x_0) \cos \varphi + 2by \sin \varphi = x^2 + y^2 + z^2 + b^2 - \frac{a^2 x_0^2}{c^2},$$

upon setting:

$$c^2 = a^2 - b^2,$$

according to habit.

The equation of the sphere (Σ) will then have the form:

$$A \cos \varphi + B \sin \varphi = C,$$

and the equation of the envelope, which expresses the idea that the preceding equation has a double root will be, in turn:

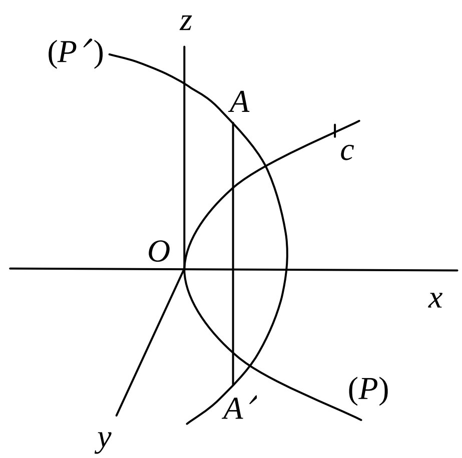
$$A^2 + B^2 = C^2.$$

Therefore, the cyclide will have the equation:

$$4a^2 (x - x_0)^2 + 4b^2 y^2 = \left(x^2 + y^2 + z^2 + b^2 - \frac{a^2 x_0^2}{c^2} \right)^2 .$$

2. Now suppose that one of the conics is a parabola. The other one is also a parabola. Take *Ox* and *Oy* to be the axis and the tangent at the summit of one of those parabolas, resp. The equations of those two conics will be:

(P)	$z = 0,$	$y^2 = 2px,$
(P')	$y = 0,$	$x^2 + z^2 = (x - p)^2.$



The center *C* of the sphere on the parabola *P* has the coordinates:

$$x = 2p\lambda^2, \qquad y = 2p\lambda, \qquad z = 0.$$

The fixed points *A* and *A*′on the parabola (*P*′) are defined by the formulas:

$$x_0, \qquad y_0 = 0, \qquad z_0^2 = (x_0 - p)^2 - x_0^2 .$$

The equation of the sphere is:

$$(x - 2p\lambda^2)^2 + (y - 2p\lambda^2)^2 + z^2 = (x_0 - 2p\lambda^2)^2 + 4p^2\lambda^2 + (x_0 - p)^2 - x_0^2$$

or

$$x^2 + y^2 + z^2 - (x_0 - p)^2 - 4p\lambda y - 4p\,(x - x_0)\,\lambda^2 = 0,$$

and the equation of the envelope – i.e., the cyclide – is:

$$[x^2 + y^2 + z^2 - (x_0 - p)^2]\,(x - x_0) + p\,y^2 = 0.$$

The surface, which has order four, in general, is only three here.

Isotropic canal surfaces

Among the ruled surfaces, we have considered the developable surfaces, for which each generator meets the infinitely-close generator. The corresponding case for the envelopes of the spheres will be the one in which *each sphere is tangent to the infinitely-close sphere.* In order for that to be true, it is necessary and sufficient that the "radical plane" of the two spheres must be tangent to both of them.

Let:

(1) $$(x-a)^2 + (y-b)^2 + (z-c)^2 - r^2 = 0$$

be a sphere.

The *radical plane* of that sphere and the infinitely-close sphere is:

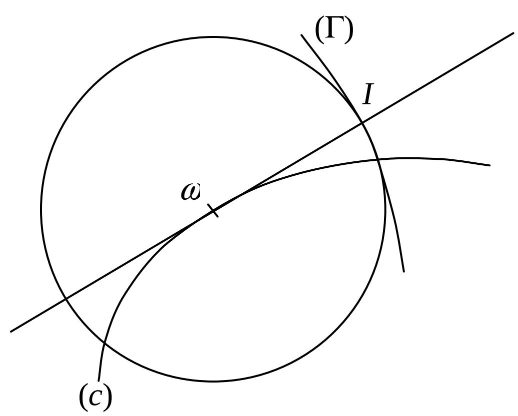
(2) $$(x-a)\,da + (y-b)\,db + (z-c)\,dc + r\,dr = 0.$$

In order for it to be tangent to the sphere (1), it is necessary and sufficient that the square of its distance to the center (a, b, c) must be equal to r^2, so:

$$\frac{r^2\,dr^2}{da^2+db^2+dc^2} = r^2,$$

or

(3) $$da^2 + db^2 + dc^2 = dr^2.$$



That condition expresses the idea that the radius r is equal (up to sign) to the arc length s of the curve (C) that is the locus of centers of the spheres, when measured from an arbitrary origin. Since r enters into equation (1) only by its square, one can adopt the solution $r = s$.

We seek the contact point of the sphere with the infinitely-close sphere. It is the foot of the perpendicular that is based at the center of the tangent plane (2). Its coordinates will then satisfy the equations:

$$\frac{x-a}{da} = a - s\,\frac{da}{ds} = a - s\alpha, \quad y = b - s\,\beta, \qquad z = c - s\,\gamma,$$

in which α, β, γ are direction cosines of the tangent. One then obtains the point I, which describes an involution (Γ) of the curve (C).

The intersection of a sphere with the infinitely-close sphere is nothing but the intersection of that sphere with one of the tangent planes to the infinitely-close one: It is a pair of isotropic lines that cut at the point I. *The envelope is composed of two ruled surfaces with isotropic generators.* We call it an *isotropic canal surface.* Conversely, *a ruled surface with isotropic generators is one sheet of the envelope of a family of spheres, each of which is tangent to the infinitely-close sphere.* Indeed, consider an isotropic generator (D) of one such surface (S). An infinitude of spheres pass through that isotropic generator (D). Those spheres contain the line (D) and the imaginary circle at infinity, which gives seven conditions; they depend upon two arbitrary parameters. If we impose the condition upon such a sphere that it must be tangent to the surface considered (S) at two points at a finite distance from the line (D) then it will be determined completely. However, it is tangent to the surface (S) at the point at infinity on (D), moreover. Therefore, that sphere (Σ) will coincide with (S) all along the generator (D). The surface (S) will be a component of the envelope of those spheres. Moreover, the sphere (Σ) has a generator (D) in common with the infinitely-close sphere, so it will be tangent to that generator at two points: One of them I will be at a finite distance.

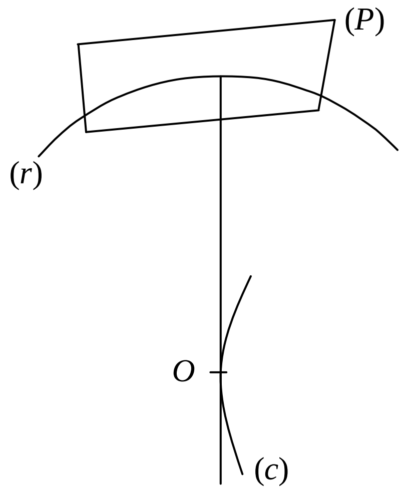
The two systems of line of curvature on such a surface (S) will coincide with the isotropic generators [Chap. III, § 7, pp. 51]. The two sheets of the development will coincide with the curve (C), since the normals to (S) at the various points of the same isotropic generator (D) must pass through the center ω of the corresponding sphere (Σ). Here, the curve (Γ) plays a role that is analogous to the edge of regression of a developable surface. Indeed, for a developable, there is a contact element (viz., a point of the edge of regression and the osculating plane at that point) that is common to a generator and the infinitely-close generator. Here, it is the contact element that is composed of the point I and the tangent plane to the sphere at that point, which is the normal plane to $I\omega$, that is common to the sphere (Σ) and the infinitely-close sphere.

The point I is an *umbilic* of the surface (S), because from what was just said, the locus (Γ) of the point I is normal to $I\omega$, and will have the locus (C) of the centers ω of the spheres (Σ) for its development. Hence, ω, which is the center of double principal curvature at any point of (D), is again the center of normal curvature of (Γ) at I, since it is on the normal to the surface and the polar surface of (Γ). Moreover, all of the normal curves are equal at I, and I is indeed an umbilic.

For the envelope of spheres (Σ), the line (Γ) is a double line, so it is a locus of umbilics for each of the two surfaces (S) that comprise it, and which are tangent to any point of that line. We call it the *umbilical line* of the isotropic canal surface.

Curvature bands and asymptotic bands

4. – Consider a surface (S_0) and an asymptotic line. The tangents to that line at each of its points will generate a developable surface, and the contact element that is common to the generator and the infinitely-close generator, which consists of a point of the line and the osculating plane, which is tangent to (S_0), is a contact element of (S_0).



Similarly, consider a line of curvature (Γ) of a surface (S_0): The normals to that surface at the various points *I* of (Γ) will generate a developable surface. Let (*C*) be the edge of regression, and let *O* be the contact point with the normal; *OI* will then be equal to the arc length along (*C*). Hence, if we consider the spheres with centers *O* and *OI* then each of those spheres will touch the infinitely-close sphere, and the contact element [*I*, (*P*)] that is common to those two spheres will be a contact element of the surface (S_0).

We call any sphere that has its center at a principal center of curvature and the corresponding radius of principal curvature for its radius a *curvature sphere* of (S_0). We see that:

The spheres of curvature of (S_0) *that correspond to the same line of curvature* (Γ) *envelop an isotropic canal surface that has* (Γ) *for an umbilical line.*

Conversely, if an isotropic canal surface (*S*) *is circumscribed by the surface* (S_0) *along its umbilical line then the latter will be a line of curvature for* (S_0), because the normals that are common to (S_0) and (*S*) at the various points *I* of (Γ) envelop the locus of centers *O* of the spheres (Σ) that have (*S*) for their envelope. Moreover, *the spheres* (Σ) *that envelope the surface* (*S*) *will be the curvature spheres of* (S_0) *that correspond to the line of curvature* (Γ), because the center *O* of each of them is the contact point of the normal *IO* with the locus those centers.

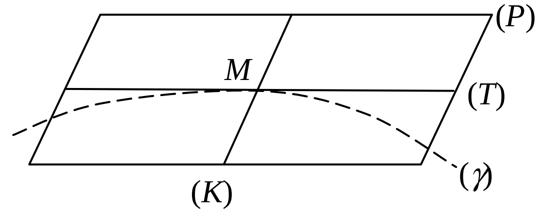
Things can be phrased in a more concise manner when one substitutes the notion of *band* or *bandeau* of contact elements for the notion of curve. By definition, a band is composed of ∞^1 contact elements that belong to the same multiplicity [Chap. VI, § 3]. The locus of points of those contact elements is a curve, and the planes of those contact elements will be tangent to the curve at the corresponding points. *A band that belongs to a surface* is composed of the points of a curve that is traced on the surface and the planes tangent to the surface at those points that that they are associated with. In other words, it is composed of the contact elements that are common to the curve and the surface.

One calls the locus of contact elements to a developable surface that are common to each generator and to the infinitely-close generator its *band of regression,* and one calls the locus of contact elements that are common to each of the spheres that are inscribed on the surface and the infinitely-close sphere the *umbilical band.*

Similarly, we call the locus of contact elements of a surface that belong to an asymptotic line or a line of curvature of that surface an *asymptotic band* or a *curvature band*, respectively, and we can state the preceding results:

An asymptotic band of a surface is the band of regression of a developable surface. A curvature band of a surface is the umbilical band of an isotropic canal surface. Conversely: Any band of regression of a developable that belongs to a surface (S_0) *is an asymptotic band of* (S_0). *Any umbilical band of an isotropic canal surface that belongs to a surface* (S_0) *is a curvature band for* (S_0).

In particular, one then sees that from the standpoint of the correspondence between lines and spheres, the asymptotic lines correspond to the lines of curvature.



Remarks. – There are *two linear elements* to consider on each contact element [*M*, (*P*)] of a band, since a linear element is composed of a point and a line that passes through that point. They are: *The tangent linear element* that is composed of the point *M* of the element and the tangent (*T*) to the curve that serves as the *support* of the band, which is a curve that one can simply call the *curve of the band*, and the *characteristic linear element* that is composed of the point *M* and the characteristic (*K*) of the plane (*P*); i.e., the rectilinear generator of the developable that is enveloped by the planes (*P*), or the *developable of the band.* Those two linear elements are correlative from the standpoint of duality; a band is correlative to a band.

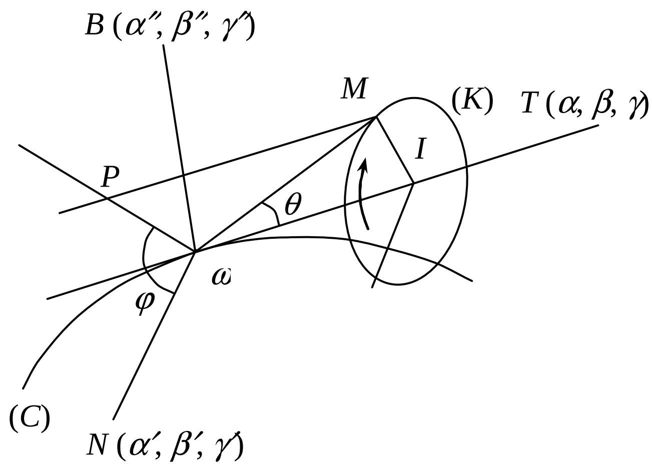
In an asymptotic band, the tangent linear elements and characteristic (*T*) *and* (*K*) *coincide for any contact element of the band, and conversely. In a curvature band, they are rectangular, and conversely.* The terms of the asymptotic band and the curvature band then have meaning in their right, without having to suppose that there is a surface (S_0) to which the band considered will belong.

If the band of regression is given then the corresponding developable will be the developable of the band. If the curvature band is given then its curve (γ) will be a line of curvature of the developable of the band, and the isotropic canal surface whose umbilical band coincides with that curvature band will be the envelope of the spheres of curvature of the developable that is constructed from the various points of *M* of the support of the band. *The terms: "umbilical band" and "curvature band" are equivalent then. Similarly, the terms "asymptotic band" and "band of regression" will also be equivalent.*

We further remark that if one is given a curvature band then the curvature sphere that corresponds to a contact element [*M*, (*P*)] of the band will be defined by the conditions that it must admit [*M*, (*P*)] for one of its contact elements, and that it must have its center on the polar line of the curve (γ) that is the locus of point *M*. (See § 2 and § 3.) That second condition expresses the idea that the sphere has second-order contact with (γ). Similarly, each plane (*P*) in an asymptotic band osculates (γ). That is therefore a new analogy between the curvature bands and the asymptotic bands.

Lines of curvature of the envelopes of spheres

5. – We already know one of the families of lines of curvature, which is composed of the characteristics of the spheres. Let us determine the second family.



Let (*C*) be the locus of centers of the spheres (Σ) considered. Express the coordinates *x*, *y*, *z* of one of its points as functions of the arc length (*s*). One of the spheres with center ω will meet the infinitely-close sphere along a circle (*K*) whose plane is normal to the tangent ωT. Introduce the Serret trihedron that is constructed from the point ω of the curve (*C*) and define the coordinates of a point *M* of the surface – i.e., of the circle (*K*) – with respect to that trihedron. Let θ denote the angle $\widehat{T\omega M}$; that angle is the same for all points of the circle (*K*). Project *M* at *P* onto the normal plane, and let φ be the angle (ωN, ωP) between ωP and ωN, when measured positively from ωN to ωB. The coordinates of *M* with respect to the Serret trihedron are, if one denotes the radius of the sphere (Σ) by ρ:

$$\xi = \rho \cos \theta, \qquad \eta = \rho \sin \theta \cos \varphi, \qquad \zeta = \rho \sin \theta \sin \varphi. \tag{1}$$

Those coordinates, with respect to an arbitrary system of axes, are:

$$X = x + \alpha\xi + \alpha'\eta + \alpha''\zeta, \qquad Y = y + b\xi + b'\eta + b''\zeta, \qquad Z = z + c\xi + c'\eta + c''\zeta. \tag{2}$$

Write down that (*K*) is the characteristic circle of the sphere (Σ). That circle has the equations:

$$\sum (X - x)^2 - \rho^2 = 0,$$
$$\sum \alpha (X - x)^2 + \rho \frac{d\rho}{ds} = 0.$$

Upon supposing that the coordinate trihedron coincides with the Serret trihedron, the second equation will become:

$$\xi + \rho \frac{d\rho}{ds} = 0;$$

i.e.:

$$\rho\cos\theta + \rho\,\frac{d\rho}{ds} = 0,$$

so

(3) $$\cos\theta = -\frac{d\rho}{ds}.$$

The angle θ is thus defined as a function of s, and the enveloping surface of the spheres (Σ) is represented by equations (2) by means of the parameters s and φ, moreover.

We seek the lines of curvature. They are the orthogonal trajectories of the circles (K) that are defined by s = const. The tangent to any curve that passes through M will have the direction coefficients:

$$dX = \alpha\,ds + \xi\frac{\alpha'}{R}ds - \eta\left(\frac{\alpha}{R}+\frac{\alpha''}{R}\right)ds + \zeta\frac{\alpha'}{T}ds + \alpha\,d\xi + \alpha'\,d\eta + \alpha''\,d\zeta,$$

$$dY = \ldots, \qquad dZ = \ldots$$

Upon once more taking the Serret trihedron to be the coordinate trihedron, those direction coefficients will become:

$$\left(1-\frac{\eta}{R}\right)ds + d\xi, \qquad \left(\frac{\xi}{R}+\frac{\zeta}{T}\right)ds + d\eta, \qquad -\frac{\eta}{T}ds + d\zeta.$$

The condition that defines the orthogonal trajectories of the circles (K) will then be:

$$-\left[\left(\frac{\xi}{R}+\frac{\zeta}{T}\right)ds + d\eta\right]\sin\varphi + \left[-\frac{\eta}{T}ds + d\zeta\right]\cos\varphi = 0.$$

Upon replacing ξ, η, ζ with their values (1), that will become:

$$\frac{\rho\cos\theta}{R}\sin\varphi\,ds + \frac{\rho\sin\theta}{T}ds - \rho\sin\theta\cdot d\varphi = 0,$$

or:

(4) $$\frac{d\varphi}{ds} = \frac{1}{T} + \frac{\cot\theta\cdot\sin\varphi}{R}.$$

This is an equation of the form $d\varphi / ds = A\sin\varphi + B$.

If one takes the unknown function to be $\tan\varphi/2$ then *one will come down to a Ricatti equation.*

The angle φ is the angle between the radius IM and a ray through the origin that is determined for each circle (K). One then concludes, by an argument that is similar to the one in Chap. VI, § 4, pp. 140, that *four non-circular lines of curvature of an envelope of* *spheres cut the characteristic circles at four points whose anharmonic ratio is constant.* That gives a new analogy with the asymptotic lines of a ruled surface.

One will get the usual simplifications if one knows one or more integrals of the equation *a priori*. Hence, if one considers an envelope of spheres (Σ) that have their centers in a plane then all of the characteristic circles will be orthogonal to the section of the surface by that plane, which will then be a line of curvature. The determination of the lines of curvature will then come down to two quadratures in that case.

Remark 1. – The search for *orthogonal trajectories to* ∞^1 *circles* that generate an arbitrary circled surface also leads to a Ricatti equation, as we shall see. Let x_0, y_0, z_0 be the coordinates of the center I of any of the circles considered, and let ρ_0 be its radius. Let α, β, γ be the direction cosines of its axis IT, which will not be tangent to a fixed curve (C) here, in general. Finally, let α', β', γ', α'', β'', γ'' be the direction cosines of two directions IT', IT'' that are chosen in such a manner that the trihedron $I\cdot TT'T''$ is a direct tri-rectangular trihedron. If one lets φ denotes the angle (IT', IM) between IT' and any radius IM of the circle, when measured positively from IT' to IT'', then the direction cosines of that radius IM will be:

$$(5) \qquad \alpha_0 = \alpha'\cos\varphi + \alpha''\sin\varphi, \qquad \beta_0 = \beta'\cos\varphi + \beta''\sin\varphi, \qquad \gamma_0 = \gamma'\cos\varphi + \gamma''\sin\varphi,$$

and the equations of the circled surface can be written:

$$(6) \qquad X = x_0 + \rho_0\,\alpha_0\,, \qquad Y = y_0 + \rho_0\,\beta_0\,, \qquad Z = z_0 + \rho_0\,\gamma_0\,,$$

in which x_0, y_0, z_0 ; α, β, γ; α', β', γ'; α'', β'', γ'' are functions of the same parameter t. The trihedron $I\cdot TT'T''$ displaces when t varies, and it will be convenient to interpret that displacement from the kinematical viewpoint while considering t to be a measure of time.

Let us seek the components of an infinitesimal displacement dX, dY, dZ of the surface relative to the axes IT, IT', IT''. Introduce the components of the velocity of the center I along the same axes into the calculations, along with those of the instantaneous rotation of the trihedron, which we denote by the notations:

$$(7) \qquad \begin{cases} u_0 = \sum \alpha \dfrac{dx_0}{dt}, \quad v_0 = \sum \alpha' \dfrac{dx_0}{dt}, \quad w_0 = \sum \alpha'' \dfrac{dx_0}{dt}, \\ p = \sum \alpha'' \dfrac{d\alpha'}{dt}, \quad q = \sum \alpha \dfrac{d\alpha''}{dt}, \quad r = \sum \alpha' \dfrac{d\alpha}{dt}. \end{cases}$$

The summation $\sum$ extends to the letters α, β, γ, x, y, z. We get the formulas:

$$(8) \qquad \begin{cases} U = \sum \alpha\, dX = [u_0 + \rho_0(q\sin\varphi - r\cos\varphi)]\,dt, \\ V = \sum \alpha'\, dX = (v_0 - \rho_0\, p\sin\varphi)dt + \cos\varphi \cdot d\rho_0 - \rho_0 \sin\varphi\, d\varphi, \\ W = \sum \alpha''\, dX = (w_0 + \rho_0\, p\cos\varphi)dt + \sin\varphi \cdot d\rho_0 + \rho_0 \cos\varphi\, d\varphi, \end{cases}$$

which will be easy to deduce directly from the theory of relative motion.

If one expresses the idea that the displacement (8) is normal to the displacement along the generating circle, which has the components 0, – sin φ, cos φ along the same axes, then one will get the condition that defines the desired orthogonal trajectories:

$$\rho_0 \frac{d\varphi}{dt} - v_0 \sin \varphi + w_0 \cos \varphi + \rho_0 \, p = 0. \tag{9}$$

One effortlessly verifies that equation (3) that was found in Chap. VI, § 3 for the trajectories of a family of circles in a plane is a special case of this one.

Upon taking tan φ / 2 to be the unknown, as before, one will reduce equation (9) to the form of a Ricatti equation, and as before, one can conclude from this that, in particular, *the orthogonal trajectories of a family of circles establish a homographic correspondence between the points of any two of those circles.*

Remark 2. – From the calculation that was made above in order to arrive at the parametric equations of an envelope of spheres, one concludes that in order for the ∞^1 circles to be the characteristic circles of ∞^1 spheres, it is necessary and sufficient that:

1. Their axes must generate a developable surface.

2. If one then defines each of those circles by the intersection of a sphere whose center is the contact point ω of the axis of the circle and the curve (*C*) that the axis envelops and a semi-cone of revolution that has its summit at the same point ω then the arc length *s* of (*C*), the radius ρ of that sphere, and the angle θ that the positive direction of the tangent to (*C*) makes with the generators of the semi-cone will be coupled by formula (3); i.e., by the condition:

$$d\rho + \cos \theta \cdot ds = 0, \tag{10}$$

which one will recover, moreover, upon applying the general formula on the variation of a line segment [Chap. V, § 6] to ωM.

However, we can replace those conditions with another one. Indeed, we point out that the characteristic circle of a variable sphere:

$$\sum (X - x)^2 - \rho^2 = 0, \quad \sum (X - x) \, dx - \rho \, d\rho = 0 \tag{11}$$

will meet the infinitely-close circle at two points that are defined by those equations (11) and the equation that is obtained by differentiating the second one. We also seek to express the idea that any variable circle that is represented by equations (6) will effectively meet its infinitely-close circle at two points.

The points at which that circle meets the infinitely-close circle (if there are any) are defined by the equations $dX = dY = dZ = 0$; i.e., by the equivalent equations $U = V = W = 0$. If one eliminates the auxiliary unknown $d\varphi$ then one will get the two equations:

$$r \cos \varphi - q \sin \varphi - \frac{u_0}{\rho_0} = 0, \quad v_0 \cos \varphi + w_0 \sin \varphi + \frac{d\rho_0}{dt} = 0. \tag{12}$$

One easily deduces the condition that expresses the idea that these equations have a common solution in tan $\varphi / 2$ from this:

$$\left(q\rho_0 \frac{d\rho_0}{dt} - u_0 w_0\right)^2 + \left(r\rho_0 \frac{d\rho_0}{dt} + u_0 w_0\right)^2 = (q\, v_0 + r\, w_0)^2 \rho_0^2 .$$

That is the condition for each circle to meet the infinitely-close circle at a point – i.e., for the ∞^1 circles considered to have an enveloping curve.

In order for there to be two common points, it is necessary and sufficient that equations (12) should be identical. That will first give the condition:

$$q\, v_0 + r\, w_0 = 0.$$

Upon taking formulas (7) into account, that condition will be written:

$$0 = \begin{vmatrix} \sum \alpha' dx_0 & \sum \alpha'' dx_0 \\ \sum \alpha' d\alpha & \sum \alpha'' d\alpha \end{vmatrix} = \sum (\beta'\gamma'' - \gamma'\beta'')\,(dy_0 \cdot d\gamma - dz_0 \cdot d\beta) = \begin{vmatrix} \alpha & \beta & \gamma \\ dx_0 & dy_0 & dz_0 \\ d\alpha & d\beta & d\gamma \end{vmatrix}.$$

They then express the idea that the axis of the circle generates a developable, which is the first of the conditions that were stated above for the circle generators of a canal surface.

If that condition is assumed to be satisfied then we will re-introduce the notations at the beginning of the paragraph, and the coordinates x, y, z of the contact point ω of the axis of the circle with its envelope (C). Upon setting:

$$h = \omega I = \rho \cos \theta, \tag{13}$$

we will then have, in succession:

$$x_0 = x + h\alpha, \quad y_0 = y + h\beta, \quad z_0 = z + h\gamma,$$
$$dx_0 = \alpha\,(ds + dh) + h\, d\alpha, \quad dy_0 = \ldots, \quad dz_0 = \ldots,$$
$$u_0\, dt = ds + dh, \quad v_0 = hr, \quad w_0 = -hq,$$

in such a way that equations (12) will become:

$$r \cos \varphi - q \sin \varphi - \frac{ds + dh}{\rho_0\, dt} = 0, \qquad r \cos \varphi - q \sin \varphi - \frac{d\rho_0}{h\, dt} = 0.$$

The identity condition of those equations then reduces to:

$$h\,(ds + dh) + \rho_0\, d\rho_0 = 0,$$

and upon observing that:

$$h^2 + \rho_0^2 = \rho^2, \qquad h\, dh + \rho_0\, d\rho_0 = \rho\, d\rho,$$

it can be written as:

$$h\,ds + \rho\,d\rho = 0.$$

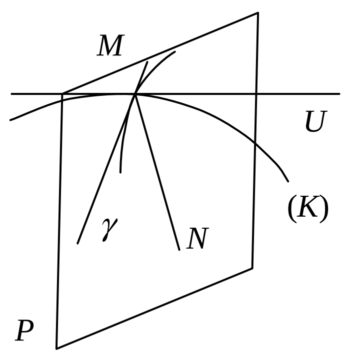
All that remains is to replace h with its value (13) in order to recover the condition (10), which succeeds in characterizing the characteristic circles for the ∞^1 spheres, from what we have seen.

We then conclude that *the necessary and sufficient condition for ∞^1 circles to generate a canal surface, or more precisely, for them to be the characteristic circles of ∞^1 spheres, is that each of them must meet the infinitely-close circle at two points.*

Case in which one sheet of the development is a developable

6. – We just considered the case in which one of the sheets of the development of a surface is a curve. Correlatively, we now consider the case in which one of the sheets of the development is a developable surface. The tangent planes to that developable will then constitute one of the families of developables of the congruence. Such a plane (*P*) will cut the surface along a curve that is normal to all of the lines of the congruences that are situated in that plane and which will be a line of curvature. At any point of that line, the normal to the surface is in the plane (*P*). Therefore, the plane (*P*) will cut the surface (*S*) orthogonally all along the line of curvature.

Conversely, from Joachimsthal's theorem, if a surface cuts a family of planes orthogonally then its sections by those planes will be lines of curvature, and those planes, which constitute one of the families of developables of the congruence of normals, will envelop a developable that is one of the sheets of the development of the surface.



Consider the second line of curvature that passes through a point *M* of the surface. Its tangent *MU* is perpendicular to the tangent *MT* to the first line of curvature and the normal *MN* to the surface. Those two lines are in the plane (*P*), so *MU* will be perpendicular to the plane (*P*). *The lines of curvature of the second family are orthogonal trajectories to the planes* (*P*).

Consider one of those orthogonal trajectories (*K*). The planes (*P*) are normal to the curve (*K*). One of the sheets of the developments, namely, the one that is a developable, will then be the envelope of the normal planes or the polar surface to the curve (*K*). *All of the non-planar lines of curvature* (*K*) *will then have the same polar surface, which is the envelope of the planes of the planar lines of curvature. The edge of regression of that surface will then be the locus of the centers of the osculating spheres to the various curves* (*K*) [Chap. I, § 12]. The curve (*K*) is a line of curvature, so the normals to the surface at all points of (*K*) will define a developable, and in turn, envelop a development of the curve (*K*), which is a geodesic of its polar surface. Therefore, if one starts from the planes (*P*), in order to get the curves (*K*), one will be reduced to the search for geodesics of a developable surface, which will reduce to quadratures, and since the desired surface can be considered to be generated by the curves (*K*), one will see that one will obtain that surface by quadratures.

Start with the planes (*P*) and look for their orthogonal trajectories directly. Consider the edge of regression (*A*) of the envelope of the planes (*P*) and introduce the Serret trihedron at each point ω of that curve, namely, (ω, $\xi\eta\zeta$). The plane (*P*) will be the osculating plane $\xi\omega\eta$, and we would like to look for a point M (ξ, η) in that plane whose locus is normal to (*P*). The coordinates of M are:

$$X = x + \alpha\xi + \alpha'\eta, \qquad Y = y + \beta\xi + \beta'\eta, \qquad Z = z + \gamma\xi + \gamma'\eta .$$

The direction of the tangent at the locus of the point M will have the direction coefficients:

$$(1) \qquad dX = \alpha\, ds + \xi \frac{\alpha'}{R} ds - \eta \left(\frac{\alpha}{R} + \frac{\alpha''}{T} \right) ds + \alpha\, dx + \alpha'\, d\eta, \qquad dY = \ldots, \qquad dZ = \ldots,$$

which are expressions of the form:

$$dX = A\alpha + B\alpha' + C\alpha'', \qquad dY = A\beta + B\beta' + C\beta'', \qquad dZ = A\gamma + B\gamma' + C\gamma''.$$

We write down that this direction is normal to the plane $\xi\omega\eta$ – i.e., parallel to the binormal α'', β'', γ''. That will give us $A = B = 0$, or:

$$ds - \frac{\eta}{R} \cdot ds + d\xi = 0, \quad \frac{\xi}{R} ds + d\eta = 0,$$

or:

$$(2) \qquad \frac{d\xi}{ds} = \frac{\eta}{R} - 1, \qquad \frac{d\eta}{ds} = -\frac{\xi}{R} .$$

ξ, η are then given by two first-order differential equations. It then results that one and only one orthogonal trajectory will pass through each point of the plane (*P*). A point-to-point correspondence will then exist between the various planes (*P*) such that the corresponding points are along the same orthogonal trajectory. Consider two points M, N in a plane (*P*), and let (*D*) be the line MN. When the plane (*P*) varies, the line (*D*) will generate a ruled surface on which the loci of the points M and N will be orthogonal trajectories of the generators. Now, the orthogonal trajectories will intersect equal segments on the generators of the segments. It will then result that if one considers two positions (*P*), (*P′*) and the corresponding positions MN, $M'N'$ then $MN = M'N'$. *The correspondence between any two of the planes* (*P*) *that are determined by the orthogonal trajectories of that family of planes will transform any curve of one of the planes into an equal curve.* In particular, the planes (*P*) contain planar lines of curvature, so *all of the* *planar lines of curvature of the surface* (*S*) *will be equal. It will then be generated by the motion of a planar curve of invariable form.* In order to arrive at its definition, it will suffice to know the motion of its plane (*P*).

In order to do that, recall equations (2):

$$\frac{d\xi}{ds} - \frac{\eta}{R} + 1 = 0, \qquad \frac{d\eta}{ds} + \frac{\xi}{R} = 0 \tag{2}$$

and integrate them. First, consider the equations without a right-hand side:

$$R\frac{d\xi}{ds} - \eta = 0, \qquad R\frac{d\eta}{ds} + \xi = 0.$$

Introduce the arc length σ of the spherical indicatrix of (*A*) and set:

$$d\sigma = \frac{ds}{R}. \tag{3}$$

The equations will then become:

$$\frac{d\xi}{d\sigma} - \eta = 0, \qquad \frac{d\eta}{d\sigma} + \xi = 0,$$

which is a system of linear equations with no right-hand side and constant coefficients, and whose general integral is:

$$\xi = A \cos \sigma + B \sin \sigma, \qquad \eta = - A \sin \sigma + B \cos \sigma. \tag{4}$$

We then pass on to the system with a right-hand side:

$$\frac{d\xi}{d\sigma} = \eta - R, \qquad \frac{d\eta}{d\sigma} = - \xi. \tag{5}$$

Consider *A*, *B* to be functions of φ, according to the method of the variation of constants, and seek to satisfy the system (5). It will become:

$$\frac{d\xi}{d\sigma} = \eta + \frac{dA}{d\sigma}\cos\sigma + \frac{dB}{d\sigma}\sin\sigma = \eta - R, \qquad \frac{d\eta}{d\sigma} = - \xi - \frac{dA}{d\sigma}\sin\sigma + \frac{dB}{d\sigma}\cos\sigma = -\eta;$$

i.e.:

$$\frac{dA}{d\sigma}\cos\sigma + \frac{dB}{d\sigma}\sin\sigma = - R, \qquad \xi - \frac{dA}{d\sigma}\sin\sigma + \frac{dB}{d\sigma}\cos\sigma = 0.$$

Hence:

$$\frac{dA}{d\sigma} = - R\cos\sigma, \qquad \frac{dB}{d\sigma} = - R\sin\sigma,$$

or upon reintroducing s from formula (3):

$$\frac{dA}{ds} = -\cos\sigma, \qquad \frac{dB}{ds} = -\sin\sigma,$$

and

$$A = -\int \cos\sigma \cdot ds, \qquad B = -\int \sin\sigma \cdot ds.$$

Set:

(6) $$x_0 = \int \cos\sigma \cdot ds, \qquad y_0 = \int \sin\sigma \cdot ds,$$

so

$$A = -x_0, \qquad B = -y_0.$$

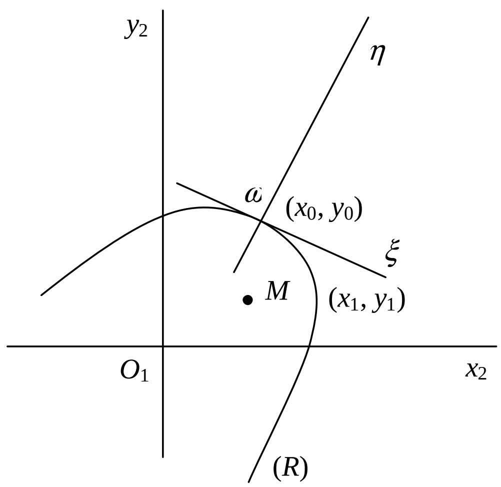
We then have a particular integral:

$$\xi = -x_0 \cos\sigma - y_0 \sin\sigma, \qquad \xi = x_0 \sin\sigma - y_0 \cos\sigma,$$

and if x_1, y_1 denote two arbitrary constants then the general integral will be:

(7) $$\xi = (x_1 - x_0)\cos\sigma + (y_1 - y_0)\sin\sigma, \quad \eta = -(x_1 - x_0)\sin\sigma + (y_1 - y_0)\cos\sigma.$$

These are the formulas that define the orthogonal trajectories of the planes (P). *They suppose that one has performed the three quadratures* (3) *and* (6).



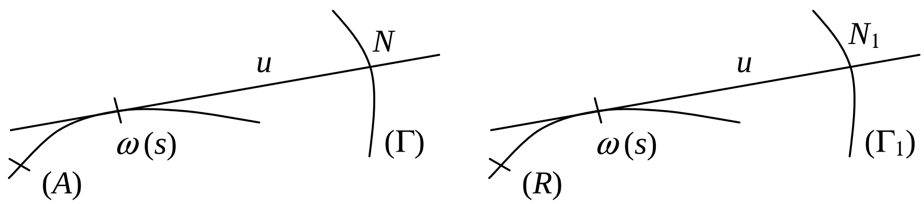
Let us interpret these results geometrically:

When the preceding formulas are solved for x_1, y_1, they will have:

(8) $$x_1 = x_0 + \xi\cos\sigma - \eta\sin\sigma, \qquad y_1 = y_0 + \xi\sin\sigma + \eta\cos\sigma.$$

Take two fixed axes $O_1 x_1$, $O_1 y_1$ in the plane (P) and construct the curve (R) that is the locus of the point (x_0, y_0) with respect to those axes. The curve (R) is the curve of the plane (P) that has the same radius of curvature as the edge of regression (A). For each value of s, the point (x_0, y_0) will occupy a position ω on the curve (R), and σ will be the angle between the tangent to (R) at ω with $O_1 x_1$. Consider a system of axes $\omega\xi\eta$, where the axis $\omega\xi$ is the tangent to (R) that corresponds to the sense in which ω is displaced. σ

is the angle between $\omega\xi$ and $O_1\,x_1$. ξ, η, which are functions of s, are the coordinates of a point M that is fixed with respect to the system $x_1\,O_1\,y_1$, when taken with respect to the axes $\xi\omega\eta$, and x_1, y_1 are the coordinates of that same point with respect to the axes $x_1\,O_1\,y_1$. In order to get the orthogonal trajectory, it is sufficient to carry the plane (P) in space on the osculating plane to the curve (A), while the lines in the plane, which are called $\omega\xi$ and $\omega\eta$, coincide with the tangent $\omega\xi$ and the principal $\omega\eta$ normal to (A), respectively. During that motion, the curves (R) and (A) will coincide at all of their points in succession. The radii of curvature are the same in magnitude and sign, so the centers of curvature will coincide. If s varies then the curve (R) will roll on the curve (A), and any point M that is invariably coupled with the curve (R) will describe the orthogonal trajectory. *The motion of the plane P will then be obtained by making the planar curve* (R) *roll on the curve* (A) *in such a fashion that the plane P will coincide with the osculating plane to the curve* (A) *at each instant.* One can say that *the plane P rolls on the developable that it envelops*, as we shall explain.



Consider the edge of regression (A) and a tangent $\omega\xi$. In order to develop that curve onto a plane, one must [Chap. V, § 4] construct the planar curve whose radius of curvature at each point has the same expression as a function of arc length as that of the curve (A); it is precisely the curve (R). The position of a point N on the developable is defined by the arc s, which fixes the position of the point ω on (A), and by the segment $\omega N = n$. The point N_1 that corresponds to N in the development is determined by the same values of s, u. The generators of the developable are developed along the tangents to the curve (R). Consider a curve (Γ) on the developable and the corresponding curve (Γ_1) in the plane. The homologous arcs on those two curves are equal, in such a way that any curve that is traced on the plane will roll on the corresponding curve of the developable. One can imagine that one has rolled a deformable planar leaf onto the developable. The motion of the plane (P) will then consist of unrolling that leaf in such a fashion that it remains constantly tensed. An arbitrary point of the leaf will describe an orthogonal trajectory of the tangent planes to the developable. In some way, we will then obtain the *involute surface of a developable* by the generalization of the process that gives the involute of a planar curve.

Finally, we shall examine the motion of the plane (P) from the kinematic viewpoint. From (1) and (2), we have:

$$\frac{dX}{ds} = -\frac{\alpha''}{T}\,\eta, \qquad \frac{dY}{ds} = -\frac{\beta''}{T}\,\eta, \qquad \frac{dZ}{ds} = -\frac{\gamma''}{T}\,\eta,$$

and in turn, the projections of the velocity of the point M onto the axes $\xi\eta\zeta$, which are invariably coupled with the plane (P), are:

$$V\xi = \sum \alpha \frac{dX}{ds} = 0, \qquad V\eta = \sum \alpha' \frac{dY}{ds} = 0, \qquad V\zeta = \sum \alpha'' \frac{dZ}{ds} = -\frac{1}{T}\eta .$$

The instantaneous motion of the plane (*P*) is a rotation around $\omega\xi$, which is tangent to (*A*), such that the instantaneous rotation is – 1 / *T*. *The osculating plane* (*P*) *rolls on the curve* (*A*) *while turning around the tangent with an angular velocity that is equal to* – 1 / *T*.

The surface (*S*) that is generated by the preceding motion is a *milling surface* or *Monge surface.*

Consider a curve (*C*) in the plane (*P*) that is invariably coupled with the system of axes $\omega\xi\eta$ and its development (*K*). During the motion of the plane (*P*), the curve (*C*) will generate a milling surface (*S*) with the developable on which the plane (*P*) rolls for one of the sheets of its development, and since the normals to (*C*), which are normal to (*S*), are tangent to (*K*), the second sheet of the development of (*S*) will be generated by the development (*K*) of the profile (*C*). It will also be a milling surface then. Hence, *one of the sheets of the development of a milling surface will be a developable, while the other one will be a milling surface.*

Special cases

Let us examine the special case in which the developable that is the envelope of the planes (*P*) is a cylinder or a cone.

1. If *the planes* (*P*) *envelop a cylinder* then the tangents to the orthogonal trajectory will be parallel to the planes of the cross-section, so the orthogonal trajectories will be the involutes of the cross-sections; they will be planar lines. *The two systems of lines of curvature of the surface will then be planar curves. The plane* (*P*) *rolls on the cylinder in such a fashion that its intersection with the plane of a cross-section will roll on that cross-section. One can further generate the surface by considering a family of parallel curves (that are the involutes of the cross-section of the cylinder here) in a plane and displacing each of those curves with a motion that is a translation perpendicular to the plane.*

2. Suppose that *the plane* (*P*) *envelopes a cone* with summit *A*, and consider an orthogonal trajectory that meets the plane (*P*) at *M*. The tangent at *M* is perpendicular to *AM*, whose orthogonal trajectory is a curve that is traced on the sphere with center *A*. Then cut the cone with a sphere of center *A* and radius *E*, let (*C*) be the intersection, and consider the circle (*S*) in the plane *P* with its center at *A* and a radius of *R*. *The plane P rolls on the cone in such a fashion that the circle* (*S*) *rolls on the curve* (*C*).

Other hypotheses. – Let us now seek to determine whether the two sheets of the development of a surface can be developables. The surface will then be a milling surface in two ways; the two systems of lines of curvature are planar curves. The orthogonal trajectories of the planes (*P*), which envelop one of the sheets of the development, must be planar, since they constitute one of the systems of lines of curvature. Let (P^1) be the plane of one of them. The planes (*P*) are all normal to a curve that is situated in (P^1);

they will then all be perpendicular to (P^1). Hence, if the planes (P) are not parallel then the planes (P^1) will all be parallel; the planes (P) will envelope a cylinder, and the planes (P^1) will be perpendicular to the generators of that cylinder, as well as the normals to the surface. The profile that is situated in a plane (P) and generates the milling surface will be a parallel to the generators of the cylinder. The surfaces thus-obtained will be cylinders then; the second sheet of the development will be a line that is pushed out to infinity.

If the planes (P) are parallel then one will arrive at the same conclusion, because the planes (P^1) envelope a cylinder.

The case that was assumed will then be impossible.

Suppose that one of the sheets of the development is a developable, while the other one is a curve. The surface is a milling surface that one obtains by the motion of a profile that is situated in the plane (P) that envelops the developable. The second sheet of the development will be generated by the development of the profile under this motion. In order for that to be a curve, it is necessary that the development of the profile must be a point, and therefore that the profile must be a circle. Imagine the sphere that has that profile for its great circle then; it is inscribed on the surface. *The surface is an envelope of spheres of constant radius.* It is a canal surface with constant circular section.

Conversely, any envelope of a family of equal spheres satisfies the preceding condition. Let a sphere have its center at a, b, c and a constant radius of r:

$$\sum (x-a)^2 - r^2 = 0.$$

The characteristic has:

$$\sum (x-a)\, da = 0$$

for the second equation. It is then a great circle of the sphere. The normals to the enveloping surface are in the plane of the circle. One of the sheets of the development will be the envelope of the planes of that circle. If we consider the locus of the centers of the sphere then the plane of its great circle will be constantly be normal. *The surface is generated by a circle of constant radius whose center describes a curve, and whose plane remains constantly normal to that curve.*

Finally, as a singular case, we again have the one in which *one of the sheets of the development is a line. The surface will then be one of revolution around that line.*
