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On the theory of multiplicities and the calculus of variations

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1. In some preceding publications ([1]), I was led, in the study of the propagation of a disturbance as a wave according to the law of enveloping waves (Huygens's principle), to restate two general problems in the calculus of variation:

a. The study of the minimum of a definite integral of the given (homogeneous) form:

$$J = \int F(x_1, \ldots, x_n, dx_1, \ldots, dx_n), \tag{1}$$

taken along an arc of a variable curve with given extremities.

b. The study of the minimum value that is taken by a variable t at the extremity of an arc of a variable curve with a given origin and extremity, and whose value is given at the origin of this same arc and satisfies a given differential equation:

$$dt = F(t \mid x_1, \ldots, x_n \mid dx_1, \ldots, dx_n). \tag{2}$$

These two functionals represent, in fact, the duration of the propagation along the arc of the curve considered, from its origin to its extremity, in a regime that is permanent in the former case and variable in the latter.

The essential fact that one confirms is that the minimum is given by the arcs of the trajectories of propagation, at least for sufficiently neighboring extremities, when the arc considered pierces the elementary waves that issue from its successive points (and in the sense that the disturbance is propagated along that curve) at the points in whose neighborhood these waves are concave towards to their respective origins.

In the first of the two articles recalled, I deduced this result from a consideration of the second variation; in the second one, I also employed a direct method that was more rigorous, and which is equivalent to the methods of Weierstrass and Hilbert, and from which, the study of the propagation of waves follows almost intuitively.

([1]) *Sur l'interprétation mécanique des transformations de contact infinitésimales (Bulletin de la Soc. math. de France,* t. XXXIV, 1906); *Essai sur la propagation par ondes (Annales de l'Éc. Normale sup.,* 3rd series, t. XXVI, 1909).

In his thesis, and later on, in his memoir on the strong maxima and minima ([1]), Carathéodory has made use of a geometric representation in which one is involved, in the case $n = 2$, and for the first of these two problems, with the wave curves that are called *indicatrices.* In his *Leçons sur le Calcul des variations* ([2]), which we shall refer to often in this article, Hadamard used an analogous geometrical representation: It is what one calls the *figurative* of the problem. Moreover, he introduced its polar reciprocal, the *figuratrix*. He did that in the general problems to which he gave the names of *Lagrange problem* and *Mayer problem*, and which differ from the preceding ones in that the variables $x_1, \ldots, x_n$ are linked by some given differential relations:

$$(3) \qquad F_h(x_1, \ldots, x_n \mid dx_1, \ldots, dx_n) = 0 \qquad (h = 1, 2, \ldots, \alpha),$$

in the case of the integral (1) (the Lagrange problem), and:

$$(4) \qquad F_h(t \mid x_1, \ldots, x_n \mid dx_1, \ldots, dx_n) = 0 \qquad (h = 1, 2, \ldots, \alpha),$$

in the case of equation (2) (the Mayer problem).

However, for these two authors, the purpose was only to illustrate some analytical results, whereas from our viewpoint, the elementary waves – or the wave multiplicities that are their homothetic transforms – are at the root of the problem and completely dominate it. For example, it is what one finds imposed absolutely at the introduction of the partial differential equation of Hamilton-Jacobi that defines, for us, the families of waves that issue from the disturbance simultaneously from the various points of that multiplicity, as well as the introduction of the canonical system, which is the complete translation of the general motion of propagation whose essential character is to be a displacement of the contact elements of the waves.

2. In the following pages, we treat the *Lagrange problem* from the same point of view. The elementary waves and the wave multiplicities that correspond to them are $n - 1$-dimensional multiplicities here, whose point-like support contains $\infty^{n-1-\alpha}$ points. However, if one wishes to have true propagation then one is led to assume that the *tangential support* of these multiplicities – i.e., the system of planes of their contact elements – contains ∞^{n-1} planes ([3]). Now, this amounts to assuming that the problem of the calculus of variations is an *ordinary problem* ([4]), a hypothesis that is also imposed in the calculus of variations for different reasons.

In our preceding articles, we have shown the importance of the tangential viewpoint in the problem considered; it is equivalent to the consideration of Hadamard's figuratrix. Here again, it is the representation of the wave multiplicity by its tangential support that

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([1]) *Ueber die discontinuerlichen Lösungen der Variations-Rechnung* (Göttingen, 1904);
Ueber die starken Maxima und Minima bei einfachen Integralen (Math. Annalen, v. LXII, 1906). Carathéodory said that his *indicatrices* were no different from the *wave surfaces* that one uses in optics, but he made no use of this analogy. Moreover, the memoirs of Carathéodory were not available to me at the moment when I published my preceding articles.

([2]) Paris, 1910. Hadamard taught at the Collège de France on the calculus of variation starting in 1902.

([3]) Without this, the disturbances produced at an arbitrary point would not propagate, if a definite orientation is chosen for the contact elements at that point. I will return to this point in a later work.

([4]) Cf., Hadamard, *loc. cit.*, pp. 239, 267, 268.

is fundamental to our method. It is quite remarkable that it permits one to avoid the objection of du Bois-Reymond on the premature introduction of the second derivative into any question in which one does not assume its existence, *a priori*. Indeed, it is the canonical system of Hamilton that first presents itself in order to define the extremals of the problem.

One also avoids any very delicate discussions that are necessary in order to justify the use of Lagrange multipliers, because here they present themselves in order to define the choice that one has to make between the various contact elements that are associated with each point on the successive elementary waves that are encountered by an extremal, and one sees, in addition, the true significance of these multipliers.

As far as the conditions for a minimum are concerned, we limit ourselves to establishing sufficient conditions by generalizing the method that was already employed in our article in the *Annales de l'École Normale*, 1909. It again translates into the condition for the concavity of the wave multiplicities towards their origins, which does not seem to have been stated for that problem previously.

In order to facilitate our presentation, we have recalled in part one the principles of the geometry of multiplicities that will employ. There, one will find, in addition to the double representation by means of the point-like support and tangential support, the definition and study of the concavity, whether at a contact element or in the domain of such an element. In order to discuss this concavity, one is led to consider the quadratic form:

$$\varpi = \sum_{i=1}^{n} dp_i dq_i ,$$

where $p_1, \ldots, p_n$ are the coordinates of a point of a contact element and $q_1, \ldots, q_n$ are the direction coefficients ([1]) of its plane. The study of this quadratic form comes down to that of another quadratic form in which the givens of the representation – whether point-like or tangential – of the multiplicity come into play. The notions of "osculating element" and "asymptotic variation" for the most general multiplicities depend upon that same form ϖ. The fact that they vanish identically characterizes the linear multiplicities.

3. The consideration of waves, where the integral J represents a duration of propagation, demands that the differential element be positive. As one knows, one comes to this case – viz., the one in which that element changes sign along the arc of the curve considered – upon adding a conveniently chosen total differential to F: This amounts to performing a projective transformation that is the dual of a translation on the wave multiplicity (which may always be defined). From the viewpoint of the statement of these results, it is more natural when one restricts oneself to the consideration of elementary waves: The minimum comes about if there is concavity when F is positive and convexity when F is negative.

([1]) In order to better exhibit the dualistic character of all of the considerations that come into play, we have imposed the condition $\sum_{i=1}^{n} p_i q_i = 1$ on these $2n$ coordinates. However, nothing essential will change if one employs homogeneous coordinates.

The process of representing wave multiplicities, which we have systematically employed, uses the tangential equation in the form that is solved with respect to the homogeneous coordinates. In reality, this entails no essential restriction, because the condition of possibility that comes about is the one that also presents itself in the application of the existence theorems for integrals of a differential system in differential equations for the extremals, when it is written in the form that was given by Lagrange.

Nevertheless, we reserve the demonstration that our method of defining equations for the problem in the calculus of variations is not related to the particular mode of representation, but only to the sign of the differential element, for another work. That demonstration may also be extended to the case of multiple integrals.

In conclusion, we remark that the analytical character of the differential equations of the problem that result from our argument consists of the one that expresses that the variation of the curvilinear integral:

$$\int \sum_{i=1}^{n} q_i dx_i \tag{5}$$

is null, the variables $x_1, \ldots, x_n ; q_1, \ldots, q_n$ being coupled by just one relation:

$$G(x_1, \ldots, x_n \mid q_1, \ldots, q_n) = 0. \tag{6}$$

One may, in turn, attach the study of the Lagrange problem to the reduction of a Pfaff expression. Here, we limit ourselves to the indication of that new method that is susceptible to being extended to non-ordinary problems, the Mayer problem, and also problems of the extrema of multiple integrals.

I. – REMARKS ON THE GEOMETRY OF MULTIPLICITIES.

1. In an n-dimensional space, we denote the Cartesian coordinates of an arbitrary point by $p_1, \ldots, p_n$, and agree to reduce the equation of an arbitrary plane to the form:

$$\sum_{i=1}^{n} q_i X_i = 1, \tag{1}$$

$X_1, \ldots, X_n$ being the current point-like coordinates. The given coordinates of a *contact element* will then be $2n$ numbers $p_1, \ldots, p_n$; $q, \ldots, q_n$ that are linked by the relation:

$$\sum_{i=1}^{n} p_i q_i = 0. \tag{2}$$

One finds that the contact elements whose plane passes through the origin and the ones whose point is at infinity are excluded ([1]).

An $n - 1$ dimensional *multiplicity* (M) will be defined by a system (S) of $n + 1$ equations between the coordinates of a current contact element; this system is subject to the double condition of having equation (2) as a consequence and one or the other of the following two Pfaff equations ([2]):

$$\sum_{i=1}^{n} q_i dp_i = 0, \tag{3}$$

$$\sum_{i=1}^{n} p_i dq_i = 0, \tag{4}$$

which are equivalent when one takes equation (2) into account.

Now, one may eliminate $q_1, \ldots, q_n$ between the equations of the system (S). One thus obtains a certain number of the $\alpha + 1$ equations in $p_1, \ldots, p_n$ that are independent of them. Likewise, one eliminates $p_1, \ldots, p_n$ and obtains $\beta + 1$ equations in $q_1, \ldots, q_n$ that are independent of them. The two partial systems thus obtained:

$$F_h(p_1, \ldots, p_n) = 0 \qquad (h = 0, 1, 2, \ldots, \alpha), \tag{5}$$

$$G_k(q_1, \ldots, q_n) = 0 \qquad (k = 0, 1, 2, \ldots, \beta), \tag{6}$$

([1]) If one has to consider one of them then one may reduce to the normal case, respectively, by one of the following transformations (translation and correlative translation):

$$p_i' = p_i + a_i, \qquad q_i' = \frac{q_i}{1+\sum_i a_i q_i} \qquad (i = 1, 2, \ldots, n);$$

$$q_i' = q_i + b_i, \qquad p_i' = \frac{p_i}{1+\sum_i b_i p_i} \qquad (i = 1, 2, \ldots, n).$$

([2]) One generally speaks only of equation (3). Our exposition has the goal of making appear the absolutely symmetric role of two groups of coordinates $p_1, \ldots, p_n$ and $q_1, \ldots, q_n$.

define what one may call the *point-like support* and the *tangential support*, respectively, of the multiplicity, namely, the locus of points and the family of planes, respectively, that define part of the contact elements of that multiplicity.

The one or the other of these supports defines the multiplicity entirely. We recall the reason for the point-like support, for example.

To say that equations (5) express the result of the elimination of $q_1, \ldots, q_n$ between the equations of the system (S) is equivalent to saying that the equations $dF_h = 0$ ($h = 0, 1, 2, \ldots, \alpha$) express all of the independent relations in $dp_1, \ldots, dp_n$ that result from the differentiation of the equations of the system (S). Equation (3) may only then be a consequence of equations $dF_h = 0$ ($h = 0, 1, 2, \ldots, \alpha$). Upon expressing this fact, one obtains $n - \alpha - 1$ homogeneous linear equations in $q_1, \ldots, q_n$ that are independent and, along with (2), which is not homogeneous, constitute a system (Σ) of $n - \alpha$ linear equations that are independent in $q_1, \ldots, q_n$.

Having said this, let ($p_1, \ldots, p_n$) be a point of the support (5). From the definition of support, there is at least one contact element of (M) that is associated with this point, and the coordinates $q_1, \ldots, q_n$ that succeed in defining such an element satisfy the linear system (Σ), by virtue of the definition of multiplicities and the preceding explanations. If we would therefore like to determine all of the contact elements of (M) that are associated with the point considered, we may deduce from (Σ), the expressions for $n - \alpha$ of the unknowns q_i as functions of the other ones – $q_1, \ldots, q_\alpha$, for example – which will amount to calculating them in such a manner as to satisfy the equations obtained by substituting these expressions in equations (S). These equations in $q_1, \ldots, q_\alpha$ are not incompatible, and, moreover, they might only be identities, because otherwise the system (S) will be equivalent to a system that is formed from more than $n + 1$ independent equations, and the multiplicity (M) will have dimension at least $n - 1$.

The contact elements of (M) are therefore defined entirely by equations (5) and the system (Σ) ([1]). As for system (Σ), it is formed from equation (2) and the equations obtained upon eliminating the auxiliary unknowns λ_h ($h = 0, 1, 2, \ldots, a$) between the equations:

(7) $$q_i = \frac{\partial f_0}{\partial p_i} \qquad (i = 1, 2, \ldots, n),$$

where f denotes the function:

(8) $$f_0 = \sum_{h=0}^{\alpha} \lambda_h F_h$$

of the independent variables $p_1, \ldots, p_n$; $\lambda_0, \lambda_1, \ldots, \lambda_\alpha$.

It is, in general, preferable to maintain equations (7) such that the general contact element of (M) is thus found to be expressed by means of the $\alpha + 1$ parameters λ_h and the $n - \alpha - 1$ parameters that the current point of the point-like support depend upon. The parameters are coupled by the relation:

([1]) The following theorem results from this: *All of the planes that are associated with the same point and define contact elements of* (M) *define a linear system*, and, in turn, correlatively: *All of the points of the same plane that define contact elements of* (M), *along with that plane, constitute a linear variety*.

(9) $$\sum_{i=1}^{n} p_i \frac{\partial f_0}{\partial p_i} = 1,$$

which results from (2).

One remarks that this relation excludes the case where the system (5) is homogeneous, because the left-hand side of (9) is then annulled for any point of support. Indeed, in this case all of the contact elements of (*M*) are found exclusively by condition (2).

All of these reasons and results apply to the tangential support. One thus has another form for the equations of the multiplicity by associating equations (6) with the equations:

(10) $$p_i = \frac{\partial g_0}{\partial q_i} \qquad (i = 1, 2, \ldots, n),$$

where one has set:

(11) $$g_0 = \sum_{k=0}^{\beta} \mu_k G_k ,$$

and the equation of condition:

(12) $$\sum_{i=1}^{n} q_i \frac{\partial g_0}{\partial q_i} = 1.$$

It finally results from this that equations (6) result from the elimination of $p_1, \ldots, p_n$; $\lambda_0, \lambda_1, \ldots, \lambda_\alpha$ between the equations (5), (7), (9), and that equations (5) result from the elimination of $q_1, \ldots, q_n$; $\mu_0, \mu_1, \ldots, \mu_\beta$ between the equations (6), (10), (12).

2. The preceding formulas are simplified if one gives a particular form to the equations of support that is equivalent to the use of polar coordinates. For example, we occupy ourselves with the point-like support and suppose first of all that $\alpha = 0$. Any point of this support is found on a certain ray that issues from the origin, whose direction parameters we denote by $\alpha_1, \ldots, \alpha_n$, and whose length we denote by ρ. It is found to be defined by the equation:

(13) $$F_0(\rho a_1, \ldots, \rho a_n) = 0.$$

One may consider it as defining $1/\rho$ to be a function of $a_1, \ldots, a_n$ in a neighborhood of the point in question, because, by excluding the contact elements whose plane passes through the origin we have excluded the hypothesis that $\rho = 0$. One thus obtains, for this domain, an equation of the form:

(14) $$\frac{1}{\rho} = F(a_1, \ldots, a_n).$$

The function *F* remains positive in this domain. One may observe, moreover, that it remains definite for the values $ma_1, \ldots, ma_n$, where *m* is sufficiently close to 1, because they are also as close as one desires to the initial values, and, from the form of (13), one sees that the value of ρ that is deduced by continuity from the first one is therefore ρ/m.

Therefore, the value of $1/\rho$ is m/ρ; i.e., the function F is (*positively*) homogeneous of degree *one* ([1]); one may apply the Euler identity to it and its partial derivatives will have the known (*positive*) homogeneities.

All of this remains true if one considers, more generally, $a_1, \ldots, a_n, \rho$ to be coordinates that are linked to the coordinates $p_1, \ldots, p_n$ by just the relations:

(15) $$p_i = \rho\, a_i \qquad (i = 1, 2, \ldots, n),$$

without imposing any restriction on $a_1, \ldots, a_n$ other than that they have the same sign as the $\alpha_1, \ldots, \alpha_n$, respectively. One may likewise assume that $a_1, \ldots, a_n$ have values that are close as one desires to $p_1, \ldots, p_n$, and as a result, one will deduce from equation (14), the equation:

(16) $$\frac{1}{\rho} = F\left(\frac{p_1}{\rho}, \cdots, \frac{p_n}{\rho}\right) = \frac{1}{\rho} F(p_1, \ldots, p_n),$$

since $1/\rho$ will be as close to *one* as one desires.

As a consequence, the equation of support will appear in the form:

(17) $$0 = F_0 \equiv F(p_1, \ldots, p_n) - 1,$$

where F has the indicated property of homogeneity. It is clear that this equation, from the manner by which we arrived at it, might represent only a portion of the point-like support that is encountered at a point by at most one arbitrary ray that issues from the origin.

Here, this ray is subject only to at most the condition that it sweep out only an (n-dimensional) portion of the space considered.

If, on the contrary, one now supposes that $\alpha > 0$ then its direction must satisfy α conditions that one may write in the form:

(18) $$F_h(p_1, \ldots, p_n) = 0 \qquad (h = 1, 2, \ldots, \alpha).$$

The functions $F_0, F_1, \ldots, F_\alpha$ having been chosen, one immediately sees that condition (9) is equivalent to $\lambda_0 = 1$. The multiplicity (M) is thus defined, for the portion in question of the point-like support, by the equations:

(19) $$F(p_1, \ldots, p_n) = 1,$$

(20) $$F_h(p_1, \ldots, p_n) = 0 \qquad (h = 1, 2, \ldots, \alpha),$$

(21) $$q_i = \frac{\partial f}{\partial p_i}, \qquad f = F + \sum_{h=1}^{\alpha} \lambda_h F_h \qquad (i = 1, 2, \ldots, n).$$

One might remark that f is homogeneous of degree *one* with respect to all of the variables $p_1, \ldots, p_n; \lambda_1, \ldots, \lambda_\alpha$, and always from the positive viewpoint. It then results that the

([1]) If it happens that it is not initially defined for all of the values of $ma_1, \ldots, ma_n$ ($m > 0$) then the property of homogeneity will permit us to *prolong* it to all of these values. However, the negative values of m remain excluded.

tangential support will be deduced from only the equations (20) and (21); i.e., without involving equation (19).

One will remark that one may replace equation (19) by equation (2), in the preceding system, due to the homogeneity of the functions F and F_h .

The tangential support will be susceptible, in turn, to an analogous representation that will be applied to a portion of that support such that it contains only a plane that meets an arbitrary ray that issues from the origin orthogonally. The equations of that support will then be of the form:

(22) $$G(q_1, \ldots, q_n) = 1,$$

(23) $$G_k(q_1, \ldots, q_n) = 0 \qquad (k = 1, 2, \ldots, \beta),$$

G and G_k being *positively* homogeneous; G has degree one and the G_k have degree zero ([1]). Moreover, in order to have the multiplicity (M), or at least the part of it that corresponds to that portion of the tangential support, one must add to equations (22) and (28), the equations:

(24) $$p_i = \frac{\partial g}{\partial q_i}, \qquad g = G + \sum_{k=1}^{\beta} \mu_k G_k \qquad (i = 1, 2, \ldots, n).$$

Since g is *positively* homogeneous of degree one in $q_1, \ldots, q_n$; $\mu_1, \ldots, \mu_\beta$, the point-like support here is defined by just equations (23) and (24). Finally, (22) may be replaced by equation (2).

3. Suppose that the multiplicity (M) considered depends upon one or more parameters, and let a be one of them; this parameter will figure in the functions F, F_h, G, G_k . The derivatives of these functions are then coupled by one simple relation that one obtains by differentiating the identity (2) totally with respect to all of the variables in question, including the parameter a. Equations (3) and (4) are no longer verified, and one obtains only:

(25) $$\sum_{i=1}^{n} q_i dp_i + \sum_{i=1}^{n} p_i dq_i = 0;$$

i.e., due to the formulas (21) and (26):

(26) $$\sum_{i=1}^{n} \frac{\partial f}{\partial p_i} dp_i + \sum_{i=1}^{n} \frac{\partial g}{\partial q_i} dq_i = 0.$$

On the other hand, due to (19), (20), (22), (23), one has the identities:

(27) $$f = 1, \quad g = 1,$$

([1]) The following results subsist, as far as their essentials are concerned, and the arguments are modified only slightly, when one supposes that the degree of homogeneity of the F_k and the G_k is equal to *one*. This hypothesis is often more convenient in applications.

which one may differentiate. Since one has:

(28) $$\frac{\partial f}{\partial \lambda_h} = F_h \qquad (h = 1, 2, \ldots, \alpha); \qquad \frac{\partial g}{\partial \mu_k} = G_k \qquad (k = 1, 2, \ldots, \beta),$$

the terms in $d\lambda_h$ and $d\mu_k$ do not enter in, and what remains is:

(29) $$\frac{\partial f}{\partial a} da + \sum_{i=1}^{n} \frac{\partial f}{\partial p_i} dp_i = 0,$$

(30) $$\frac{\partial g}{\partial a} da + \sum_{i=1}^{n} \frac{\partial g}{\partial q_i} dq_i = 0.$$

By comparison with (26), one then obtains the stated relation:

(31) $$\frac{\partial f}{\partial a} + \frac{\partial g}{\partial a} = 0.$$

4. Here is another remark that we will make use of:

Suppose that the tangential support is n – 1-dimensional, which, for us, will be the most important case; equations (23) do not exist then and g reduces to G. Due to their homogeneity, equations (24) thus give a parametric representation for the point-like support by means of the parameters $q_1, \ldots, q_n$, where it is only the ratios that come into play. Due to this circumstance, we may likewise suppose that these parameters verify equation (22).

One then has that $(p_1, \ldots, p_n; q_1, \ldots, q_n)$ is a contact element of the multiplicity (M) that is associated with the point $(p_1, \ldots, p_n)$ of the point-like support. However, it is not necessarily the only one. In order to find all of them, one must search for all of the systems $(p_1, \ldots, p_n; y_1, \ldots, y_n)$ that satisfy the conditions:

(32) $$\sum_{i=1}^{n} y_i p_i = 1, \qquad \sum_{i=1}^{n} y_i dp_i = 0.$$

The p_i are given by equations (24), and among their differentials, one may consider $dq_1, \ldots, dq_n$ to be independent, because it now only comes down to using the parametric representation for the point-like support, for which one will have make an abstraction of (22), due to the homogeneity of formulas (24). The equations of the problem are then:

(33) $$\sum_{i=1}^{n} y_i \frac{\partial G}{\partial q_i} = 1,$$

(34) $$\sum_{i=1}^{n} y_i \frac{\partial^2 G}{\partial q_i \partial q_j} = 0 \qquad (j = 1, 2, \ldots, n).$$

In order to solve them, one first determines the general solution to equations (34) and one disposes of the arbitrary factor that appears in it in such a manner as to satisfy the condition (33).

Therefore, if the Hessian of G, which is identically null, due to the homogeneity of G, is only of rank one then the system (33), (34) has only one solution, which is:

(35) $$y_i = q_i \qquad (i = 1, 2, \ldots, n),$$

and the contact element ($p_1, \ldots, p_n; q_1, \ldots, q_n$) is the only one that contains the point of support ($p_1, \ldots, p_n$). This is the case when equations (24) give just one relation between $p_1, \ldots, p_n$; i.e., the point-like support is itself $n - 1$-dimensional.

If, on the contrary, the Hessian of G is of rank $\alpha + 1$ ($\alpha > 0$) – i.e., if the point-like support is $n - \alpha - 1$-dimensional – then this support is represented by the system (19), (20), and the result that is expressed by formulas (21) shows that the general solution of equations (33), (34) is written, with α arbitrary $u_1, \ldots, u_\alpha$:

(36) $$y_i = \frac{\partial \bar{f}}{\partial p_i}, \qquad \bar{f} = F + \sum_{h=1}^{\alpha} u_h F_h \qquad (i = 1, 2, \ldots, n).$$

Now, equations (19), (20), (21), in turn, have, as a consequence, equations (22) and (24). Therefore, equations (19), (20), and (36), which differ only by a change of notations, will first have, as a consequence:

(37) $$\bar{G} \equiv G(y_1, .., y_n) = 1,$$

in which, to abbreviate, we write $\bar{G}$ for the function G, when it is written in terms of the letters $y_1, .., y_n$, in place of the $q_1, .., q_n$; moreover:

(38) $$p_i = \frac{\partial \bar{G}}{\partial y_i} \qquad (i = 1, 2, .., n).$$

As a result, any solution of the system (33), (34) satisfies the system that is defined by equation (37) and the equations:

(39) $$\frac{\partial \bar{G}}{\partial y_i} = \frac{\partial G}{\partial q_i} \qquad (i = 1, 2, \ldots, n).$$

Suppose, moreover, that the multiplicity (M) depends upon a parameter a as in no. 3. Due to formulas (36) and (24), the y_i may be considered to be well-defined functions:

(40) $$p_i = Q_{i,n}(q_1, .., q_n) + \sum_{h=1}^{\alpha} u_h Q_{i,h}(q_1, \cdots, q_n) \qquad (i = 1, 2, \ldots, n)$$

of the variables $q_1, .., q_n$; $u_1, .., u_\alpha$, which constitute the general solution of the system (33), (34), and these functions turn, not only these equations, but also equations (37) and (39), into identities. In the present case, these identities are also valid at a, which likewise enters as a parameter into the coefficients Q of formula (40). One thus obtains equations (33) and (37) by differentiation with respect to a:

$$\sum_{i=1}^{n} y_i \frac{\partial^2 G}{\partial q_i \partial a} da + \sum_{i=1}^{n} \frac{\partial G}{\partial q_i} dy_i = 0, \tag{41}$$

$$\frac{\partial \bar{G}}{\partial a} da + \sum_{i=1}^{n} \frac{\partial \bar{G}}{\partial y_i} dy_i = 0; \tag{42}$$

thus, upon taking equations (39) into account, one has the formula:

$$\frac{\partial \bar{G}}{\partial a} = \sum_{i=1}^{n} y_i \frac{\partial^2 G}{\partial q_i \partial a}, \tag{43}$$

which is a new consequence of equations (33) and (34).

5. In the application of the study of multiplicities to the calculus of variations, the multiplicity (M) will be given for us by equations (19), (20), which define its point-like support, and we then make use of the representation (22), (23), (24), which is based on the tangential support. We must therefore examine under what sort of condition this second mode of representation is legitimate in the domain of a contact element of a multiplicity.

More generally, take a point-like support that has $n - \alpha - 1 = \gamma$ dimensions and is represented by the arbitrary parametric equations:

$$p_i = \varphi_i(t_1, \dots, t_\gamma) \qquad (i = 1, 2, \dots, n). \tag{44}$$

The contact elements ($p_1, \dots, p_n; q_1, \dots, q_n$) that are associated with an arbitrary point of this support are defined by the equations:

$$\sum_{i=1}^{n} q_i \frac{\partial \varphi_i}{\partial t_l} = 0 \qquad (l = 1, 2, \dots, \gamma), \tag{45}$$

which is equivalent to the condition (3), and by equation (2), which is written here as:

$$\sum_{i=1}^{n} q_i \varphi_i = 1. \tag{46}$$

In order to arrive at the representation of the tangential support in the form (22), (23), we give direction coefficients ($b_1, \dots, b_n$) to the normal to the plane of the contact element, and we seek to determine that element (cf., no. 2, where analogous considerations are applied to the point-like support). We must then pose:

$$q_i = b_i\,\sigma \qquad (i = 1, 2, \ldots, n), \tag{47}$$

and seek to obtain $1/\sigma$ as a function of $b_1, \ldots, b_n$ by eliminating the parameters $t_1, \ldots, t_\gamma$. To that effect, we have made use of the equations that are obtained by substituting the values (47) in equations (45) and (46). If, to abbreviate, we set, while considering $b_1, \ldots, b_n$ to be given constants:

$$\bar{\varphi}(t_1, \cdots, t_\gamma) \equiv \sum_{i=1}^{n} b_i \varphi_i\,, \tag{48}$$

then this gives the equations:

$$\frac{\partial \bar{\varphi}}{\partial t_i} = 0 \qquad (l = 1, 2, \ldots, \gamma) \tag{49}$$

and:

$$\frac{1}{\sigma} = \varphi, \tag{50}$$

and we may replace the latter with the combination:

$$\frac{1}{\sigma} = \bar{\varphi} - \sum_{l=1}^{\gamma} t_l \frac{\partial \bar{\varphi}}{\partial t_l} = \varphi_h\,. \tag{51}$$

Since one then has the identity:

$$d\varphi_0 + \sum_{l=1}^{\gamma} t_l \frac{\partial \bar{\varphi}}{\partial t_l} = 0, \tag{52}$$

one sees that φ_0 is a function of those derivatives $\partial\varphi / \partial t_l$ that are independent. As a result, everything depends uniquely on the functional determinant of these derivatives – i.e., on the Hessian of the function $\bar{\varphi}$. Then again, if one studies what happens in the domain of a contact element $(p_1, \ldots, p_n; q_1, \ldots, q_n)$ of the multiplicity then, from the nature of the Hessian of the function of $t_1, \ldots, t_\gamma$, one has:

$$\varphi = \sum_{i=1}^{n} q_i \varphi_i\,, \tag{53}$$

where $q_1, \ldots, q_n$ are considered to be constants ([1]).

([1]) One arrives at this same result more quickly by confirming that equations (45) determine the points of the support where a tangent plane is parallel to the plane:

$$\sum_{i=1}^{n} q_i X_i = 0.$$

For a general contact element of the multiplicity (M), this Hessian will be of a certain rank β ($\beta \geq 0$), and in the domain of that element one will have the representation (22), (23), (24) of (M). Furthermore, such a representation will cease to be applicable only for the contact elements that raise the rank of the Hessian considered.

6. One arrives at a more geometric statement if one observes that this Hessian is the discriminant of the quadratic form in $dt_1, \ldots, dt_\gamma$ that one obtains by calculating the differential expression $\varpi \equiv \sum_{i=1}^{n} dp_i dq_i$ corresponding to an arbitrary displacement of the multiplicity when starting with one of its elements. Indeed, one has:

$$\sum_{i=1}^{n} dp_i dq_i = \sum_{i=1}^{n} dq_i \sum_{l=1}^{\gamma} \frac{\partial \varphi_i}{\partial t_l} dt_l = \sum_{l=1}^{n} dt_l \sum_{i=1}^{n} \frac{\partial \varphi_i}{\partial t_l} dq_i . \tag{54}$$

However, the differentiation of the identities (45) thus gives:

$$\sum_{i=1}^{n} \frac{\partial \varphi_i}{\partial t_l} dq_i + \sum_{i=1}^{n} q_i \, d\frac{\partial \varphi_i}{\partial t_l} = 0 \qquad (l = 1, 2, \ldots, \gamma); \tag{55}$$

in such a way that one has:

$$\varpi \equiv \sum_{i=1}^{n} dp_i \, dq_i = -\sum_{i=1}^{n} q_i \sum_{l=1}^{\gamma} \sum_{m=1}^{\gamma} \frac{\partial^2 \varphi_i}{\partial t_l \partial t_m} dt_l \, dt_m , \tag{56}$$

and the Hessian considered is precisely the discriminant of the quadratic form thus obtained.

Since the multiplicity (M) is n – 1-dimensional, this quadratic form $\sum_{i=1}^{n} dp_i dq_i$ will be written in the most general form by means of n – 1 independent differentials, and must be considered to be a quadratic form in n – 1 variables. By the mode of representation that we employed, we will have thus obtained that form in the case $\alpha = 0$ ($\gamma = n - 1$); i.e., in the case where the point-like support has the maximum number of dimensions. One sees, moreover, that if $\beta = 0$ – i.e., if the tangential support also has the maximum number of dimensions – then the rank of its determinant is null, and that quadratic form is of general class.

In the general case, the rank of its discriminant – while always considering it to be a quadratic form in n – 1 variables – is $\alpha + \beta$, and the form is the sum of ($n - 1 - \alpha - \beta$) independent squares. Of course, this supposes that one is dealing with a variation ($dp_1, \ldots, dp_n$; $dq_1, \ldots, dq_n$) that is performed by starting with a contact element of (M).

The representation (22), (23), (24) [*and then also, by reason of symmetry, the representation* (19), (20), (21)] *may cease to be valid only in the domain of the contact* *elements that raise the rank of the discriminant of the form* $\varpi \equiv \sum_{i=1}^{n} dp_i dq_i$. *We say that such elements are* EXCEPTIONAL.

7. This statement, which summarizes the preceding discussion, is easy to apply to all of the modes of representation of the multiplicity. In order to solve the question that was posed at the beginning of this paragraph, we apply it to the case where the multiplicity is given in the form (19), (20), (21). The quadratic form to consider is then:

$$\varpi \equiv \sum_{i=1}^{n} dp_i dq_i = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial p_i \partial p_j} dp_i dp_j + \sum_{i=1}^{n}\sum_{h=1}^{n} \frac{\partial F_h}{\partial p_i} dp_i d\lambda_h ; \tag{57}$$

however, one must suppose that the variables $dp_1, \ldots, dp_n$ are coupled by the equations of condition that are obtained by differentiating equations (19) and (20), namely:

$$\sum_{i=1}^{n} \frac{\partial F_h}{\partial p_i} dp_i = 0 \qquad (h = 1, 2, \ldots, \alpha), \tag{58}$$

$$\sum_{i=1}^{n} \frac{\partial F}{\partial p_i} dp_i = 0, \tag{59}$$

which reduces the number of variables $dp_1, \ldots, dp_n$; $d\lambda_1, \ldots, d\lambda_\alpha$ to $n - 1$. Indeed, one must observe that we pass over, as *singular*, the points of the point-like support for which these equations (58), (59) cease to be independent in the $dp_1, \ldots, dp_n$, because for such points the normal condition of solubility for the system (19), (20) ceases to apply.

As for the quadratic form (57), it reduces, upon taking (58) into account, to the form:

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial p_i \partial p_j} dp_i dp_j , \tag{60}$$

and we have to determine the rank of its discriminant – i.e., the difference between the number of variables (which is $n - 1$, here) and the number of independent equations that are obtained by equating to zero the partial derivatives of the form with respect to these variables.

We recall how one resolves questions of this type: Let Q be a canonical form in m variables $X_1, \ldots, X_m$ that are coupled by independent linear relations $L_1 = 0, \ldots, L_p = 0$. One may make a linear change of variables such that one has identically $L_1 \equiv X_1, \ldots, L_p \equiv X_p = 0$, and since the equations of condition are then $X_1 = 0, \ldots, X_p = 0$ it then remains for us to consider the form:

$$\bar{Q} = Q(0, \ldots, 0, X_{p+1}, \ldots, X_m), \tag{61}$$

which gives the system:

$$\frac{\partial \bar{Q}}{\partial X_i} = 0 \qquad (i = p + 1, \ldots, m), \tag{62}$$

which is equivalent to the system:

$$(63)\qquad \begin{cases} \dfrac{\partial Q}{\partial X_i} = 0 & (i = p+1, \cdots, m), \\ X_h = 0 & (h = 1, 2, \cdots, p). \end{cases}$$

Now, if one considers the quadratic form in $m + p$ variables:

$$(64)\qquad S \equiv Q + \sum_{h=1}^{p} Y_h L_h \equiv Q + \sum_{h=1}^{p} Y_h X_h ,$$

and if one equates its partial derivatives to zero then one obtains the system:

$$(65)\qquad \begin{cases} \dfrac{\partial Q}{\partial X_i} + Y_h = 0 & (h = 1, 2, \cdots, p), \\ \dfrac{\partial Q}{\partial X_i} = 0 & (i = p+1, \cdots, m), \\ X_h = 0 & (h = 1, 2, \cdots, p), \end{cases}$$

which obviously contains p independent equations more than the system (63), into which the variables $Y_1, \ldots, Y_p$ do not enter. The rank of the discriminant of $\bar{Q}$ is thus the same as the tank of the discriminant of S, in which the $m + p$ are independent.

We apply this result to the form (60). Upon setting, to abbreviate the notation, $dp_i = P_i$ ($i = 1, 2, \ldots, n$), we will have to consider the quadratic form:

$$(66)\qquad \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial p_i \partial p_j} P_i P_j + \sum_{i=1}^{n} \sum_{h=1}^{\alpha} \frac{\partial F_h}{\partial p_i} P_i Y_h + \sum_{j=1}^{n} \frac{\partial F}{\partial p_i} P_i Y ,$$

which gives the system:

$$(67)\qquad \sum_{j=1}^{n} \frac{\partial^2 f}{\partial p_i \partial p_j} P_j + \sum_{h=1}^{\alpha} \frac{\partial F_h}{\partial p_i} Y_h + \frac{\partial F}{\partial p_i} Y = 0 \quad (i = 1, 2, \ldots, n),$$

$$(68)\qquad \sum_{j=1}^{n} \frac{\partial F_h}{\partial p_j} P_j = 0 \qquad (h = 1, 2, \ldots, \alpha),$$

$$(69)\qquad \sum_{j=1}^{n} \frac{\partial F}{\partial p_j} P_j = 0.$$

Multiply the equations (67) and (68) by p_i and λ_h, respectively, and add them. Since f is homogeneous of degree one with respect to p_i and λ_h, and since one has:

$$\frac{\partial F_h}{\partial p_j} = \frac{\partial^2 f}{\partial \lambda_h \partial p_j},$$

we obtain, upon also taking into account the homogeneity of F_h and F, and finally equation (19), the simple combination:

(70) $$Y = 0.$$

The system considered is therefore equivalent to the one that one obtains by adding equations (69) and (70) to the system:

(71) $$\sum_{j=1}^{n} \frac{\partial^2 f}{\partial p_i \partial p_j} P_j + \sum_{h=1}^{\alpha} \frac{\partial F_h}{\partial p_i} Y_h = 0 \quad (i = 1, 2, \ldots, n),$$

(72) $$\sum_{j=1}^{n} \frac{\partial F_h}{\partial p_j} P_j = 0 \qquad (h = 1, 2, \ldots, \alpha).$$

Equations (69) and (70) are obviously independent of each other. Equation (70) might not be a consequence of equations (71) and (72), and the same is true for equation (69), because it does not admit the solution:

$$P_j = p_j \qquad (j = 1, 2, \ldots, n), \qquad Y_h = \lambda_h \qquad (h = 1, 2, \ldots, \alpha),$$

which satisfies the system (71) and (72) for the reasons of homogeneity that were used before.

The system (67), (68), (69) thus has a number of independent equations that is higher than both of the independent equations of the system (71), (72) combined. Now, this latter system is, up to notation, the one that one obtains by equating to zero the partial derivatives of the quadratic form that constitutes the right-hand side of the identity (57). Since that form contains $\alpha + 1$ variables less than ([1]) the form (66), we conclude that *the rank of the discriminant of the form* $\varpi \equiv \sum_{i=1}^{n} dp_i dq_i$ *is greater by* $\alpha - 1$ *units than the rank of the Hessian of the function f. It will likewise be greater by* $\beta - 1$ *than the rank of the Hessian of the function g.*

Therefore, *the exceptional elements of the multiplicity* (*M*), *which are defined by equations* (19), (20), (21), *for which the correlative mode of representation* (22), (23), (24) *might no longer be valid, are the ones that raise the rank of the Hessian of the function f.*

([1]) One must recall that (66) must be considered as depending upon the variables $P_i = dp_i$ ($i = 1, 2, \ldots, n$), $d\lambda_h$ ($h = 1, 2, \ldots, \alpha$), Y_h ($h = 1, 2, \ldots, \alpha$) and Y. Thus, if the system (71), (72) contains exactly $n + \alpha - (\beta + 1)$ independent equations then the rank of the Hessian of f is $(\beta + 1)$, and the rank of discriminant of (66) is $n + 2\alpha + 1 - (n + \alpha - \beta + 1) = \alpha + \beta$.

8. The study of the form $\varpi \equiv \sum_{i=1}^{n} dp_i dq_i$ is equivalent to that of the second order projective properties of multiplicities. It will permit one to classify, in a more precise manner, the exceptional elements that we just considered by means of the number of units by which they elevate the rank of that form ϖ.

For example, $n = 3$ in the case of ordinary space. For non-developable surfaces $\alpha = 0$, $\beta = 0$, and ϖ is a general form in two variables. The exceptional elements, for which ϖ is a perfect square, correspond to the parabolic points of support; the ones for which ϖ is identically null correspond to tangent planes for which the distance of a neighboring point of contact of such a plane is an infinitesimal of greater than second order.

For the developable surfaces $\alpha = 0$, $\beta = 1$, and ϖ reduces, in general, to a form in just one variable; it becomes identically null for the exceptional points where the order of contact of the tangent plane with the surface is raised.

For only one plane, one has:

$$\alpha = 0, \quad \beta = \alpha,$$

and the form is identically null.

For a curve, one has $\alpha = 1$, and the case of $\alpha = 1$, $\beta = 1$ comes about only if the curve is a line. One thus has, in general, $\beta = 0$, and ϖ is a form that is reducible to just one square; it is identically null for the exceptional contact elements that are formed from a point of the curve and its osculating plane.

For a line, $\alpha = 1$, $\beta = 1$, and ϖ is identically null.

Finally, for a point ϖ is again identically null.

In the case where n is arbitrary, the rank $(\alpha + \beta)$ of the discriminant is necessarily equal to at most $n - 1$. This is obvious, *a priori*, because the $(\alpha + 1)$ equations of the point-like support and the $(\beta + 1)$ equations of tangential support are independent, and the total number of equations between $p_1, \ldots, p_n; q_1, \ldots, q_n$ that define an $n - 1$-dimensional multiplicity is $n + 1$. We remark that this is equivalent to saying that the sum of the numbers of dimensions of the two supports is equal to at least $n - 1$.

One may show that if the preceding inequalities are changed into equalities – i.e., *if the form ϖ is identically null – then the multiplicity is linear.*

Indeed, refer to formula (66). We may suppose that one has chosen the coordinates p_i to be the parameters $t_1, \ldots, t_\gamma$, γ, for example, $p_1, p_2, \ldots, p_\gamma$. The $q_1, q_2, \ldots, q_\gamma$ then disappear from the form (56), while the equations of condition (45) can be solved precisely with respect to these coordinates. In order to express the idea that (56) is identically null, one does not therefore need to take the equations of condition into account, and one obtains the result that all of the second derivatives of the functions $(\varphi_{\gamma+1}, \varphi_{\gamma+2}, \ldots, \varphi_n)$ are identically null. These functions are linear and, as a consequence, the same is true for the equations of point-like support and, also as a consequence, that of the tangential support, from the duality of all of our considerations.

One verifies, moreover, that if the point-like support is:

$$p_{\gamma+h} = \sum_{l=1}^{\gamma} a_{hl}\, p_l a_h \qquad [h = 1, 2, \ldots, (\alpha + 1)] \tag{73}$$

then the tangential support is given by equations (45), which are:

$$q_l + \sum_{h=1}^{\alpha+1} a_{hl} q_{\gamma+h} = 0, \tag{74}$$

and by equation (46), which reduces to:

$$\sum_{h=1}^{\alpha+1} a_h q_{\gamma+h} = 1. \tag{75}$$

We remark that, from the remarks that were made above, one may state the result thus obtained: *The only multiplicities for which the sum of the numbers of dimensions of the two supports is equal to* $n - 1$ *are the linear multiplicities; the two supports of a multiplicity are linear at the same time if that multiplicity is* $n - 1$*-dimensional*.

On a nonlinear multiplicity, the form ϖ may thus be annulled identically only for the exceptional elements, which one might call the *osculating elements.*

9. Consider a contact element for the multiplicity (M) for which the form $\varpi = \sum_{i=1}^{n} dp_i dq_i$ is neither identically null nor reducible to just one square ([1]). There are then variations ($dp_1, \ldots, dp_n$; $dq_1, \ldots, dq_n$) that one performs by starting with an element that annuls the form ϖ. They are what one might call *asymptotic variations;* however, it is intended that one will rule out the ones for which all of the dp_i or all of the dq_i are null.

If one considers, more especially, the point-like support then an asymptotic variation will correspond to an *asymptotic direction* on that support. However, if β is non-null then that asymptotic direction will inversely correspond to an infinitude of asymptotic variations, because the q_i are expressed by formulas (21), the dq_i depend upon the dp_i , and the $d\lambda_h$, and the $d\lambda_h$ remain arbitrary here, since they disappear from the formula (57) when one takes (59) into account. On the contrary, the asymptotic variation is entirely determined by the asymptotic direction when the tangential support is $n - 1$-dimensional.

If one considers the tangential support then an asymptotic variation will correspond to an *asymptotic characteristic*, namely, the $n - 2$-dimensional linear multiplicity that is defined by two equations:

$$\sum_{i=1}^{n} q_i X_i = 1, \qquad \sum_{i=1}^{n} dq_i X_i = 0. \tag{76}$$

One then sees immediately that an asymptotic variation is defined by the association of a direction in the point-like support and a characteristic of the tangential support such that the characteristic contains the direction. Moreover, one sees the reciprocal in the same equations at the same time. It is, of course, intended that the direction and the characteristic considered are supposed to be furnished by the same variation and contact

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([1]) The case of a one-dimensional point-like or tangential support is excluded by this [*see* equation (56)].

element considered on the multiplicity. We again remark that any n – 2-dimensional linear multiplicity that is contained in the plane of the contact element and passes through the point of that element may be considered to be a characteristic of that element.

10. The form $\varpi = \sum_{i=1}^{n} dp_i dq_i$ intervenes once more in the notions of the *concavity* and *convexity* of a multiplicity, which play an essential role in the calculus of variations. In order to define that notion, one may adopt either of two correlative viewpoints: Let (p_1, …, p_n; q_1, …, q_n) be a contact element (E) of the multiplicity and let (p'_1, …, p'_n; q'_1, …, q'_n) be a neighboring contact element (E') of the same multiplicity, which will be arbitrary in the domain of (E).

From the first viewpoint, the multiplicity will be *concave at* (E) *towards the origin* if the point of (E') is constantly on the same side as the origin with respect to the plane of (E). From the second viewpoint, the multiplicity will be concave at (E) towards the origin if the point of (E) is always on the same side as the origin with respect to the plane of (E'). This will therefore translate, according to the adopted definition, into one or the other inequality:

$$\sum_{i=1}^{n} p'_i q_i - 1 < 0, \tag{77}$$

$$\sum_{i=1}^{n} p_i q'_i - 1 < 0. \tag{78}$$

The two definitions are, moreover, equivalent if one considers the *concavity in a region* of the multiplicity: The elements (E), (E') are then two arbitrary elements that are different, but sufficiently close to that domain, and play a symmetric role in the question.

The concavity in a region, no matter what the support – point-like or tangential – is thus expressed by inequalities of the same nature as the ones that define the multiplicity.

We return to the concavity at an element and consider, for example, the inequality (77). The p'_i are then deduced from the p_i upon given arbitrary infinitely small increases to the parameters that define the particular element (E) considered. If we order the left-hand side of (77) in increasing powers then the term of degree zero, which is $\sum_{i=1}^{n} p_i q_i$ - 1, is null, due to the condition (2), and the set of terms of first degree, which is $\sum_{i=1}^{n} p_i dq_i$, is null due to equation (3); finally, the set of terms of second degree is, as would result from the differentiation of the equation (3):

$$\sum_{i=1}^{n} q_i d^2 p_i = - \sum_{i=1}^{n} dp_i dq_i = - \varpi. \tag{79}$$

One obtains the same result for the left-hand side of (78).

As a consequence, *the one or the other concavity is realized if the form ϖ takes only positive values for any variation performed on the contact element considered.* We intend this to means that the variation neither annuls all of the dp_i nor all of the dq_i . *The one or the other concavity may be realized only if the form ϖ takes on only positive or null values.*

One naturally passes from concavity to *convexity* upon changing the signs in all of the preceding.

If one of the two supports of the multiplicity is one-dimensional then one sees from formula (56) that at an element that is neither singular nor osculating the multiplicity is either convex or concave towards the origin. On the contrary, if the two supports are at most one-dimensional then the concavity and the convexity must be considered as exceptional cases. *There is concavity or convexity if all of the asymptotic variations are imaginary*, because then the form ϖ may not be annulled or change its sign, and the preceding stated sufficient condition is found to be satisfied.

11. That same sufficient condition demands that in order for it to be verified, at least one of the supports must be n – 1-dimensional. Indeed, refer to formula (56) and the considerations of no. 6. If the tangential support is not n – 1-dimensional then the right-hand side of (56) decomposes into at least $\gamma = n - \alpha - 1$ squares; upon equating these squares to zero, one defines one or more real asymptotic displacements on the point-like support. Now, if the point-like support is also less than n – 1-dimensional then, in addition to the dt_l , some arbitrary quantities $d\lambda_h$ (h = 1, 2, …, α) enter into the expressions for the dq_i , as we remarked in no. 9. The real asymptotic displacements obtained thus correspond to real asymptotic variations in which the dq_i , not just the dp_i, are not all null; i.e., the asymptotic variation is effective.

We suppose first of all that the tangential support is n – 1-dimensional, and limit ourselves to considering only exceptional contact elements. The sufficient condition is found to be equivalent to the following one: The form ϖ is the sum of $n - 1 - \alpha = \gamma$ independent positive squares. When expressed by the formula (56) and considered as a form in $dt_1, \dots, dt_\gamma$, ϖ is thus a positive definite form.

Next, consider ϖ in the form (57) or, more precisely, consider the form $\bar{\varpi}$ in the $n + \alpha$ independent variables $dp_1, \dots, dp_n$; $d\lambda_1, \dots, d\lambda_\alpha$ that constitute that expression. We know, from no. 7, that in order for ϖ to be the sum of $n - 1 - \alpha = \gamma$ independent squares, it is necessary and sufficient that the form $\bar{\varpi}$ be the sum of $n + \alpha - 1 = \gamma + 2\alpha$ independent squares; in other words, the Hessian of f is null, but not of all of its minors of rank one.

In order to express the second part of the condition, perform the reduction of ϖ into squares. Upon setting:

$$U_h = \sum_{i=1}^{n} \frac{\partial F_h}{\partial p_i} dp_i \qquad (h = 1, 2, \dots, \alpha), \tag{80}$$

and letting $V_1, \dots, V_{n-\alpha}$ denote some other linear forms in $dp_1, \dots, dp_n$ that define, along with the forms $U_1, \dots, U_\alpha$, a system of n independent forms, one may write the quadratic form $\bar{\varpi}$ as:

(81) $$\bar{\varpi} = \sum_{h=1}^{\alpha} U_h d\lambda_h + \sum_{h=1}^{\alpha} U_h U'_h + Q(V_1, \ldots, V_{n-\alpha}),$$

where Q is a quadratic form and where the U'_h are linear forms in $U_1, \ldots, U_\alpha$; $V_1, \ldots, V_{n-\alpha}$. Thus, upon setting:

(82) $$W_h = d\lambda_h + U'_h \qquad (h = 1, 2, \ldots, \alpha),$$

we will have:

(83) $$\bar{\varpi} = \sum_{h=1}^{\alpha} U_h W_h + Q(V_1, \ldots, V_{n-\alpha}).$$

Since $\bar{\varpi}$ reduces to $\gamma + 2\alpha$ independent squares, Q will be the sum of only γ squares ($\gamma = n - \alpha - 1$), in such a way that one finally has:

(84) $$\bar{\varpi} = \sum_{h=1}^{\alpha} U_h W_h + \sum_{l=1}^{\gamma} \varepsilon_h T_l^2 ,$$

where the U_h , W_h, T_l are independent real forms and the ε_l are equal to (+1) or (–1).

If one takes into account conditions (58), (59) then the T_l will become new linear forms T'_l in only the $dp_1, \ldots, dp_n$, and since the U_h will null, and we will obtain γ independent squares for ϖ, these T'_l will be independent forms, in such a way that the reduced form of ϖ is:

(85) $$\varpi = \sum_{l=1}^{\gamma} \varepsilon_l T_l'^2 .$$

The sufficient condition for the concavity thus demands that the ε_l are all equal to (+1), and if one refers to formula (84) then one sees that this is equivalent to saying that $\bar{\varpi}$ decomposes into:

$$\alpha + \gamma = n - 1$$

positive squares and α negative squares. We thus arrive at the following conclusion:

The sufficient condition for the concavity is expressed, in the case of $\beta = 0$, *by the fact that the quadratic form in* $n + \alpha$ *variables:*

(86) $$\bar{\varpi} = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial p_i \partial p_j} dp_i dp_j + \sum_{i=1}^{n} \sum_{h=1}^{\alpha} \frac{\partial F_h}{\partial p_i} dp_i d\lambda_h$$

decomposes into a sum of $(n + \alpha - 1)$ *independent real squares, where* α *of them have the* (–) *sign and* $(n - 1)$ *of them have the* (+) *sign.*

We pass to the case where the tangential support is (n – β – 1)-dimensional (β > 0), but where, as a consequence, the point-like support is (n – 1)-dimensional. The function f reduces to F, and the $d\lambda_h$ disappear of the form $\bar{\varpi}$, which is simply:

$$\bar{\varpi} = \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 F}{\partial p_i \partial p_j} dp_i dp_j . \tag{87}$$

Since the rank of the discriminant of ϖ is then β, the Hessian of F is, from no. 7, of rank β – (α – 1) = β + 1, and decomposes into (n – β – 1) independent squares, like the form ϖ itself ([1]).

Thus, if one has put $\bar{\varpi}$ into the form:

$$\bar{\varpi} = \sum_{m=1}^{\delta}\varepsilon_m T_m^2 \qquad (\delta = n - \beta - 1), \tag{88}$$

one sees that, when one takes into account (59), the T_m become linear forms in only n – 1 independent variables, and one will have:

$$\varpi = \sum_{m=1}^{\delta}\varepsilon_m T_m'^2 . \tag{89}$$

The sufficient condition for concavity in the case α = 0 *is therefore that the form* $\bar{\varpi}$ *contains only positive squares.*

We add that the same mode of reasoning will prove, more generally, that *the form* $\bar{\varpi}$ *has* 2α *independent squares more than the form* ϖ, *with* α *positive squares and* α *negative squares.* Thus, *in order for* ϖ *to have only positive squares, it is necessary and sufficient that* $\bar{\varpi}$ *have only a negative squares.*

Finally, *a necessary condition for the concavity is that the form* ϖ *contains only positive squares and, in turn, that the form* $\bar{\varpi}$ *contain only* α *negative squares.*

One naturally obtains results that are entirely similar by starting with the tangential representation (22), (23), (24) of the multiplicity. The form $\bar{\varpi}$ will be replaced by the form:

$$\bar{\varpi} = \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial^2 g}{\partial p_i \partial p_j} dq_i dq_j + \sum_{i=1}^{n}\sum_{k=1}^{\beta}\frac{\partial G_k}{\partial q_i} dp_i d\mu_k , \tag{90}$$

and the number α by the number β.

([1]) In a general manner, one concludes from no. 7 that *in any case* $\bar{\varpi}$ *contains* 2α *independent squares more than* ϖ. Here, α = 0, so the number of squares is the same in the two forms.

II. – THE LAGRANGE PROBLEM. FORMULATION IN EQUATIONS.

12. The problem in question is that of the study of the conditions for an extremum for a curvilinear integral in n-dimensional space:

$$\int F(x_1, \ldots, x_n \mid dx_1, \ldots, dx_n), \tag{1}$$

that is taken along an arc of the curve whose extremities might be subjected to given conditions of a diverse nature, which themselves may also be restricted to satisfying a first-order system of differential equations:

$$F_h(x_1, \ldots, x_n \mid dx_1, \ldots, dx_n) = 0 \qquad (h = 1, 2, \ldots, \alpha). \tag{2}$$

We suppose that the arc of the curve is given parametrically by means of a variable u that varies from 0 to 1, for example, and always increases.

The $F(x_1, \ldots, x_n \mid y_1, \ldots, y_n)$ is, as one knows ([1]), a (*positively*) homogeneous function of the arguments $y_1, \ldots, y_n$. We suppose, moreover, that $F(x_1, \ldots, x_n \mid dx_1, \ldots, dx_n)$ takes on only positive values ([2]), at least for the curves that we will consider, and the differentials $dx_1, \ldots, dx_n$ correspond to an arbitrary positive increase du. $F_h(x_1, \ldots, x_n \mid y_1, \ldots, y_n)$ are homogeneous of degree zero, and at least positively, with respect to the arguments $y_1, \ldots, y_n$. We then set:

$$F(x_1, \ldots, x_n \mid dx_1, \ldots, dx_n) = \omega\, du; \tag{3}$$

ω will be a positive variable that is defined by that equation, and we will have to study the extremum conditions for the integral:

$$J = \int_0^1 \omega\, du. \tag{4}$$

Furthermore, we replace the equations of condition (2) and (3) with the following system, which is equivalent to it, due to the conditions of homogeneity that we assumed:

$$dx_i = \omega p_i\, du \qquad (i = 1, 2, \ldots, n), \tag{5}$$

$$F(x_1, \ldots, x_n \mid p_1, \ldots, p_n) = 1, \tag{6}$$

$$F_h(x_1, \ldots, x_n \mid p_1, \ldots, p_n) = 0, \qquad (h = 1, 2, \ldots, a). \tag{7}$$

In other words, we introduce, along with the variable ω, which corresponds to an integral element, the variables $p_1, \ldots, p_n$, which correspond to a curve element, and we see that the problem is characterized by the nature of the multiplicity (6), (7), which is

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([1]) Cf., HADAMARD, *Leçons sur le Calcul des variations*, t. I, pp. 80.

([2]) This restriction, as we will show later on (no. 20), may be raised. Cf., HADAMARD, *loc. cit.*, pp. 384.

found to be associated with each point ($x_1, \ldots, x_n$) of space. Due to formulas (5), we must interpret $p_1, \ldots, p_n$ as the coordinates of a point of the same space with respect to a new system of axes that are parallel to the original coordinate axes, and have the point $x_1, \ldots, x_n$ for its origin: This amounts to saying that in the original system of coordinates, this same point will have the coordinates $x_1 + p_1, \ldots, x_n + p_n$. The point-like multiplicity ([1]) thus introduced will be called the *wave multiplicity*, which has the point ($x_1, \ldots, x_n$) for its origin.

Consider the points of space as being affected by modifications or disturbances, which then propagate, step by step, according to the following law: The disturbance that is present at a point and a certain instant is present after an infinitesimal time dt at all points of the *elementary wave* that one obtains by constructing the homothety of the wave multiplicity that has the point in question for its origin, which is also the pole of the homothety, and the homothety ratio is dt. This amounts to saying that the possible displacements of the disturbance by starting with each point ($x_1, \ldots, x_n$) during the time dt are given by the general formulas:

$$dx_i = p_i \, dt \qquad (i = 1, 2, \ldots, n), \tag{8}$$

where $p_1, \ldots, p_n$ must satisfy the equations (6) and (7).

One then sees from formulas (5) that the differential element of the integral J represents the time that the disturbance takes in propagating from a point of the curve considered to an infinitely close point that follows along the same curve. As a result, the integral itself represents the duration of propagation of the disturbance from the origin to the extremity of that curve, when one supposes that one prohibits any propagation of that disturbance outside of the points of the curve itself. This is then what we shall call the *duration of propagation of the disturbance along the curve.*

If, for example, the problem posed is a problem of finding a minimum then it comes down to the determination of the curves along which the propagation considered moves the fastest.

13. The method that we shall present consists of considering the wave multiplicity to be a multiplicity of contact elements and employing the formulas that were obtained in the first part of this article in order to transform the system (5), (6), (7): The variables $x_1, \ldots, x_n$ must be considered to be simple parameters that appear in the formulas in question, such as the parameter a considered in nos. 3 and 4.

We may thus introduce the new variables $\lambda_1, \ldots, \lambda_\alpha$, and the formulas:

$$q_i = \frac{\partial f}{\partial p_i} \qquad (i = 1, 2, \ldots, n), \tag{9}$$

where f is the function:

$$f = F + \sum_{h=1}^{\alpha} \lambda_h F_h \, ; \tag{9'}$$

([1]) If $\alpha > 0$ then it is a multiplicity that is just $n - \alpha - 1$-dimensional from the point-like viewpoint, which is the only one that we will consider at the moment.

in such a way that ($p_1, \ldots, p_n$; $q_1, \ldots, q_n$) are the coordinates of an arbitrary contact element of the wave multiplicity (cf., nos. 1 and 2).

Since the Hessian of f is of rank 1 here, if one suppose that the problem is *ordinary* ([1]) then the tangential support is $n - 1$-dimensional. One may thus, under the conditions ([2]) that were discussed in nos. 6 and 7, employ formulas of the form:

$$G(x_1, \ldots, x_n \mid q_1, \ldots, q_n) = 1, \tag{10}$$

$$p_i = \frac{\partial G}{\partial q_i} \qquad (i = 1, 2, \ldots, n) \tag{11}$$

to represent the wave multiplicity, where the first one results from simply the elimination of the p_i and the λ_h between the equations (7) and (8) (cf., nos. 1 and 2).

We are then reduced to the following *canonical problem:*

PROBLEM A. – *Find* $2n + 1$ *functions* $x_1, \ldots, x_n$; $q_1, \ldots, q_n$; ω *of one variable* u *that satisfy the equations of condition:*

$$G(x_1, \ldots, x_n \mid q_1, \ldots, q_n) = 1, \tag{10}$$

$$dx_i = \omega \frac{\partial G}{\partial q_i} du \qquad (i = 1, 2, \ldots, n), \tag{11}$$

and such that one has an extremum for the integral:

$$J = \int_0^1 \omega\, du\,. \tag{4}$$

The functions $x_1, \ldots, x_n$ *are, moreover, subject to certain conditions at the limits* ($u = 0$, $u = 1$) *of the integration interval, and the function* ω *may be positive in that interval.*

A necessary condition for the extremum is, as one knows, that δJ be null for any system of variations $\delta x_1, \ldots, \delta x_n$; $\delta q_1, \ldots, \delta q_n$, ω that satisfies equations of condition that are obtained by taking the variations of the two sides of equations (10) and (12), and are, moreover, such that $\delta x_1, \ldots, \delta x_n$ is annulled at the limits of the interval of integration.

It is this condition that we shall first seek to express.

14. In order to avoiding having to introduce the condition $\delta G = 0$, we introduce an unknown function by setting:

$$\gamma_i = q_i\, \gamma_0 \qquad (i = 1, 2, \ldots, n). \tag{13}$$

([1]) Cf., HADAMARD, *loc. cit.*, pp. 239, 267, 268.

([2]) In fact, the restrictions that they impose on the problem are in the nature of the question itself, as we will confirm later on (no. 20).

Due to the homogeneity of G, since the function γ is assumed to be positive the system (10), (12) will be replaced by the system:

(14) $$\gamma_0 = G(x_1, \ldots, x_n \mid \gamma_1, \ldots, \gamma_n) \equiv G',$$

(15) $$dx_i = \omega \frac{\partial G'}{\partial \gamma_i} du \qquad (i = 1, 2, \ldots, n).$$

Equation (14) then has no other effect than determining γ_0, the equation that one deduces by taking the variations has no other utility than determining $\delta\gamma_0$, and none of these quantities γ_0, $\delta\gamma_0$ enter into the relations between $\delta x_1, \ldots, \delta x_n$; $\delta\gamma_1, \ldots, \delta\gamma_n$; ω, $x_1, \ldots, x_n$; $\gamma_1, \ldots, \gamma_n$; ω, u. In order to express the idea that δJ is null, we will then have to take into account only equations (4), (15), and the ones that one obtains by taking the variations of the two sides of each of these equations – i.e.:

(16) $$\delta J = \int_0^1 \delta\omega\, du ,$$

(17) $$\frac{d\,\delta x_i}{du} = \omega \sum_{j=1}^n \frac{\partial^2 G'}{\partial\gamma_i \partial x_j}\delta x_j + \omega \sum_{j=1}^n \frac{\partial^2 G'}{\partial\gamma_i \partial\gamma_j}\delta\gamma_j + \frac{\partial G'}{\partial\gamma_i}\delta\omega \qquad (i = 1, 2, \ldots, n).$$

In order to express that the δx_i are annulled for $u = 0$, $u = 1$, we shall seek to calculate them upon considering $\delta x_1, \ldots, \delta x_n$; $\delta\omega$ to be known; i.e., to integrate the system (17).

According to the most elementary method, we consider the corresponding homogeneous system:

(18) $$\frac{dz_i}{du} = \omega \sum_{j=1}^n \frac{\partial^2 G'}{\partial\gamma_i \partial x_j} z_j \qquad (i = 1, 2, \ldots, n).$$

Let:

(19) $$z_i = z_{ki} \qquad (k = 1, 2, \ldots, n;\ i = 1, 2, \ldots, n)$$

be n independent solutions of that system. Applying the method of variation of constants, we may then set:

(20) $$\delta x_i = \sum_{k=1}^n \xi_k z_{ki} \qquad (i = 1, 2, \ldots, n),$$

which gives the system:

(21) $$\sum_{k=1}^n z_{ki} \frac{d\xi_k}{du} = A_i \qquad (i = 1, 2, \ldots, n),$$

upon setting, to abbreviate:

(22) $$A_i = \omega \sum_{j=1}^n \frac{\partial^2 G'}{\partial\gamma_i \partial\gamma_j}\delta\gamma_j + \frac{\partial G'}{\partial\gamma_i}\delta\omega \qquad (i = 1, 2, \ldots, n).$$

Now, it is obvious from formulas (20) that the δx_i are annulled simultaneously when they are based upon the ξ_k , and only in this case. The condition that is imposed on the δx_i then amounts to the following one: The functions ξ_k , which are solutions of (21) that are annulled for $u = 0$, are also annulled $u = 1$.

In order to calculate these functions ξ_k , we put the system (21) into the solved form:

$$\text{(23)} \qquad \frac{d\xi_k}{du} = \sum_{i=1}^{n} v_{ki} A_i \qquad (k = 1, 2, \ldots, n).$$

The functions v_{ki} are defined by the equations:

$$\text{(24)} \qquad \sum_{i=1}^{n} z_{ki} v_{ji} = \begin{cases} 0 & \text{for } k \neq j \\ 1 & \text{for } k = j \end{cases} \qquad (i = 1, 2, \ldots, n).$$

They constitute what one calls the *adjoint system* to the system of the z_{ki} and the formulas:

$$\text{(25)} \qquad v_i = v_{ki} \qquad (i = 1, 2, \ldots, n;\ k = 1, 2, \ldots, n)$$

define n independent solutions of the *adjoint linear system* to the system (18), namely:

$$\text{(26)} \qquad \frac{dv_i}{du} + \omega \sum_{j=1}^{n} \frac{\partial^2 G'}{\partial \gamma_i \partial x_j} v_j = 0 \qquad (i = 1, 2, \ldots, n).$$

One then has, for the desired functions ξ_k :

$$\text{(27)} \qquad \xi_k = \int_0^n \sum_{i=1}^{n} v_{ki} A_i \, du \qquad (k = 1, 2, \ldots, n),$$

and the conditions that they are subjected to are obtained by regarding formulas (22):

$$\text{(28)} \qquad \int_0^1 \sum_{i=1}^{n} v_{ki} \left\{ \omega \sum_{j=1}^{n} \frac{\partial^2 G'}{\partial \gamma_i \partial \gamma_j} \delta\gamma_j + \frac{\partial G'}{\partial \gamma_i} \delta\omega \right\} du = 0 \qquad (k = 1, 2, \ldots, n).$$

Moreover, all that remains for us to do is only for us to write that the integral (16) is null for any choice of functions $\delta\gamma_1, \ldots, \delta\gamma_n$; $\delta\omega$ that satisfy the conditions (28). The condition ([1]) is thus that there exist n constants $c_1, \ldots, c_n$ such that one has the identity:

$$\text{(29)} \qquad \delta\omega = \sum_{k=1}^{n} c_k \left[\sum_{i=1}^{n} v_{ki} \left(\omega \sum_{j=1}^{n} \frac{\partial^2 G'}{\partial \gamma_i \partial \gamma_j} \delta\gamma_j + \frac{\partial G'}{\partial \gamma_i} \delta\omega \right) \right].$$

([1]) This condition is well-known, at least, in an equivalent form. We shall return later on to its statement and proof (*see* no. 23).

This decomposes into:

(30) $$\sum_{i=1}^{n} y_i \frac{\partial G'}{\partial \gamma_i} = 1,$$

(31) $$\sum_{i=1}^{n} y_i \frac{\partial^2 G'}{\partial \gamma_i \partial \gamma_j} = 0 \qquad (j = 1, 2, \ldots, n),$$

when one sets:

(32) $$y_i = \sum_{k=1}^{n} c_k v_{ki} \qquad (i = 1, 2, \ldots, n).$$

Now, when $c_1, \ldots, c_n$ are arbitrary constants, the right-hand sides of the latter formulas are the general solution to the system (26). The condition found is then that there exist a solution $v_i = y_i$ ($i = 1, 2, \ldots, n$) of the system (26) that satisfies equations (30) and (31).

If one now recalls the variables $q_1, \ldots, q_n$, by means of equations (13), since the $\partial G / \partial q_i$ and $\partial^2 G / \partial q_j \partial x_i$ are homogeneous of degree zero and the $\partial^2 G / \partial q_j \partial q_i$ are homogeneous, then one obtains the desired condition in the following form:

In order for the variation of the integral J to be null under the conditions that were assumed, it is necessary and sufficient that there exist n auxiliary functions $y_1, \ldots, y_n$ that satisfy the equations:

(33) $$\frac{dy_i}{du} = -\,\omega \sum_{j=1}^{n} \frac{\partial^2 G}{\partial q_j \partial x_i} y_j \qquad (i = 1, 2, \ldots, n),$$

(34) $$\sum_{i=1}^{n} y_i \frac{\partial G}{\partial q_i} = 1,$$

(35) $$\sum_{i=1}^{n} y_i \frac{\partial^2 G}{\partial q_i \partial q_j} = 0 \qquad (j = 1, 2, \ldots, n).$$

15. However, this result can be transformed if one takes into account the remarks of no. 4. We first examine the simplest case: that of the *free extremum* ([1]). The point-like support of the wave multiplicity is $n - 1$-dimensional, since there are no equations of condition. Equations (34) and (35) then admit $y_i = q_i$ ($i = 1, 2, \ldots, n$) as their only solution, and since one has identically:

(36) $$\frac{\partial G}{\partial x_i} = \sum_{i=1}^{n} q_i \frac{\partial^2 G}{\partial x_i \partial q_j} \qquad (i = 1, 2, \ldots, n),$$

([1]) Cf., HADAMARD, *loc. cit.*, pp. 41.

the condition obtained is that the functions $x_1, \ldots, x_n$; $q_1, \ldots, q_n$; ω, *which, by hypothesis, satisfy equations* (10) *and* (12), *also satisfy the equations:*

$$(37) \qquad dq_i = -\,\omega \frac{\partial G}{\partial x_i}\, du \qquad (i = 1, 2, \ldots, n).$$

One remarks that it results from this that the functions q_i are themselves differentiable, since they are identical to the functions y_i that are, by their definition itself, derivatives. By virtue of the relations (8), (11), (12), this is equivalent to saying that $x_1, \ldots, x_n$ *are derivatives of second order.* The hypotheses that are implicitly necessitated by our reasoning are only those of the continuity of the derivatives of first order, as well as the existence and continuity of the partial derivatives of the function G that appeared in our calculations.

Equations (12) and (37) leave arbitrary the choice of the function ω, which one must address, due to the indeterminacy of the parametric representation that was adopted. *One will thus have to integrate the canonical system:*

$$(38) \qquad \begin{cases} dx_i = \dfrac{\partial G}{\partial q_i} dt & (i = 1, 2, \cdots, n), \\ dq_i = \dfrac{\partial G}{\partial x_i} dt & (i = 1, 2, \cdots, n), \end{cases}$$

when one takes initial values that satisfy (10). Since equations (38) admit the first integral G = const., equation (10) will then be verified for any system of functions x_i, q_i that is thus determined. When t varies by increasing, the point $(x_1, \ldots, x_n)$ will describe a curve that is called an *extremal in a definite sense* – namely, the sense of propagation – and the *positive* variation of t from one point to another of the curves is the duration of the propagation of a disturbance along the arc of the curve considered. This amounts to saying that:

$$(39) \qquad dt = \omega\, du,$$

as one sees by comparing the two systems (38) and (12), (37), and to saying that $\omega\, du$ is the differential element of the integral J (cf., no. 12). The variation of that duration of propagation between two arbitrary points of an extremal is null when one replaces the arc of the extemal that goes from one point to the other, in the sense of propagation, by another arc of the curve that is infinitely close and has the same origin and extremity.

16. One must observe that if an extremal curve is known, with the sense of propagation along that curve, then the values of $q_1, \ldots, q_n$ at each point of that curve are determined completely, with no new integration, and the variable t, which corresponds to the duration of the propagation from one point to the other on that curve, is obtained by a quadrature.

Geometrically, this amounts to saying that at each point of the curve the direction of the tangent that corresponds to it, in the sense of propagation, pierces the wave multiplicity that has that point for its origin at a well-defined point, and thus determines a contact element ($p_1, \ldots, p_n$; $q_1, \ldots, q_n$). Moreover, the corresponding elementary wave must pass through the infinitely close point on the curve, also taken in the positive direction of the tangent, while the homothety ratio dt of that elementary wave and the wave multiplicity is also known.

Analytically, this fact results from formulas (39), (3), (5), and (8), which give:

(40) $$dt = F(x_1, \ldots, x_n \mid dx_1, \ldots, dx_n),$$

(41) $$p_i = \frac{dx_i}{dt} \qquad (i = 1,2, \ldots, n),$$

(42) $$q_i = \frac{\partial F(x_1, \cdots, x_n \mid dx_1, \cdots, dx_n)}{\partial p_i} \qquad (i = 1, 2, \ldots, n).$$

This leads one to search for a differential system which defines the extremal curves without passing to the auxiliary variables $q_1, \ldots, q_n$.

The solution results immediately from the comparison of the equations (38), (41), (42), and the equation (31) of no. 3, which gives:

(43) $$\frac{\partial F}{\partial x_i} + \frac{\partial G}{\partial x_i} = 0 \qquad (i = 1, 2, \ldots, n).$$

One thus obtains the well-known *differential system of the extremals:*

(44) $$\begin{cases} d\,\dfrac{\partial F(x_1, \cdots, x_n \mid dx_1, \cdots, dx_n)}{\partial dx_i} - \dfrac{\partial F(x_1, \cdots, x_n \mid dx_1, \cdots, dx_n)}{\partial dx_i} = 0 \\ (i = 1, 2, \cdots, n). \end{cases}$$

17. We now pass to the general case of the *constrained extremal*.

The point-like support of the wave multiplicity is then $n - \alpha - 1$-dimensional since one must account for equations (2). Each point of that support correspond to ∞^{α} contact elements, whose coordinates ($p_1, \ldots, p_n$; $y_1 .., y_n$) are defined precisely as functions of them and one of the ($p_1, \ldots, p_n$; $q_1 .., q_n$) by equation (34) and (35) [equations (33) and (34) of no. 4].

Upon taking into account equations (37), (39), and (43) of no. 4, one sees that if one sets, for the sake of neatness:

(45) $$\bar{G} = G(x_1, \ldots, x_n \mid y_1, .., y_n),$$

then one may add to the system thus found, which is defined by equations (10), (12), (33), (34), (35), the equations:

(46) $$\bar{G} = 1,$$

(47) $$\frac{\partial \bar{G}}{\partial y_i} = \frac{\partial G}{\partial q_i} \qquad (i = 1, 2, \ldots, n),$$

which are consequences of them, and that one may replace equations (33) by the following ones:

(48) $$dy_i = -\,\omega \frac{\partial \bar{G}}{\partial x_i} du \qquad (i = 1, 2, \ldots, n).$$

Equations (47) then permit us to write equations (12) in the form:

(49) $$dx_i = \omega \frac{\partial \bar{G}}{\partial y_i} du \qquad (i = 1, 2, \ldots, n).$$

Finally, equations (34), (35) express only that ($p_1, \ldots, p_n$; q_1 .., q_n) and ($p_1, \ldots, p_n$; y_1 .., y_n) are two contact elements that are associated with the point ($p_1, \ldots, p_n$) of the wave multiplicity, and due to equations (47) the coordinates of this point may be written as either $p_i = \partial G / \partial q_i$ ($i = 1, 2, \ldots, n$) or $p_i = \partial \bar{G} / \partial y_i$ ($i = 1, 2, .., n$). One may, consequently, further replace these equations by the ones that one deduces by exchanging the y_i and the q_i; i.e., with the equations:

(50) $$\sum_{i=1}^{n} q_i \frac{\partial \bar{G}}{\partial y_i} = 1,$$

(51) $$\sum_{i=1}^{n} q_i \frac{\partial^2 \bar{G}}{\partial y_i \partial y_j} = 0 \qquad (j = 1, 2, \ldots, n),$$

and the ones that they entail, in turn, as consequences of equations (10) and (47).

Eventually, what remains is the system that is composed of equations (46), (48), (49), (50), (51), and one may further, as in no. 15, and with the benefit of the same observations, substitute for equations (48) and (49), the canonical system that one deduces by introducing the variable t by means of equation (39).

Equations (50) and (51) are verified by the functions $q_i = y_i$ ($i = 1, 2, \ldots, n$), because the first one reduces to (46), and the other ones are identities for $q_i = y_i$ ($i = 1, 2, \ldots, n$). From this, it results that if $y_1, \ldots, y_n$ is a system of auxiliary functions that correspond to a solution ($x_1 \ldots, x_n$; $q_1, \ldots, q_n$; ω) of the problem of the null variation of J then ($x_1 \ldots, x_n$; $y_1, \ldots, y_n$; ω) is also a solution to the same problem. One thus sees a particular class of solutions to the problem appear that we may call *canonical solutions*, since they are defined by the canonical system (38), and any extremal curve enters into at least one canonical solution. From this, one finds, as in no. 15, *the existence of the second derivatives of* x_i *with respect to any extremal.*

18. Imagine an extremal curve and a canonical solution that includes it. The tangent to the extremal at a point M that points in the direction of propagation determines a point

(P) of the wave multiplicity that has (M) for its origin. The canonical solution serves to fix one of the ∞^{α} contact elements of the wave multiplicity at this point. Finally, equations (50), (51) signify only that in any solution – canonical or not – that includes the same extremal the values of $q_1, \ldots, q_n$ correspond to any of the contact elements in question. For an arbitrary solution that includes a well-defined extremal, the functions $q_1, \ldots, q_n$ are thus coupled to the functions $x_1, \ldots, x_n$ by just the condition that at each point of the extremal they provide the plane of one of the contact elements that we just defined. On the contrary, a particular of these contact elements enter into the canonical solutions.

In order to exhibit this choice, one may, by means of formulas (8), introduce the α indeterminates $\lambda_1, \ldots, \lambda_\alpha$ that the contact elements in question depend upon. One thus has to write that these values:

$$q_i = \frac{\partial f}{\partial p_i} \qquad \left(f = F + \sum_{h=1}^{\alpha} \lambda_h F_h \right) \qquad (i = 1, 2, \ldots, n), \tag{9}$$

where one supposes [no. 16, equation (41)]:

$$p_i = \frac{dx_i}{dt} \qquad (i = 1, 2, \ldots, n), \tag{41}$$

satisfy the canonical system (38). Taking into account equations (31) of no. 3, one thus obtains the *Lagrange system*, where the λ_h are nothing but the *Lagrange multipliers*, namely:

$$\frac{d}{dt}\frac{\partial f}{\partial p_i} - \frac{\partial f}{\partial x_i} = 0 \qquad (i = 1, 2, \ldots, n). \tag{51}$$

In order to define the extremals, one must add to this system the equations of condition (2); i.e., with the present notations:

$$F_h(x_1, \ldots, x_n \mid p_1, \ldots, p_n) = 0 \qquad (h = 1, 2, \ldots, \alpha). \tag{52}$$

A question suggests itself naturally: How many canonical solutions correspond to the same extremal? The defining formulas of the preceding section show that the auxiliary functions $y_1, \ldots, y_n$ are the same for all systems of functions $q_1, \ldots, q_n$ that correspond to the same extremal, since all of these systems of functions $q_1, \ldots, q_n$ satisfy equations (50), (51), as long as the $y_1, \ldots, y_n$ correspond to one of them. One may thus discuss the search for functions $y_1, \ldots, y_n$ by means of equations (33), (34), (36) (no. 14) upon considering the $x_1, \ldots, x_n$; $q_1, \ldots, q_n$, ω to be functions of the u knowns. The linear form of these equations shows that the general solution will be of the form:

$$y_i = y_{0,i} + \sum_{j=1}^{\gamma} \rho_j (y_{j,i} - y_{0,i}) \qquad (i = 1, 2, \ldots, n), \tag{53}$$

in which the ρ_j are arbitrary and the $y_{j,i}$ ($j = 0, 1, 2, \ldots, \gamma$) are ($\gamma + 1$) particular solutions.

One also concludes this from equations (51), and naturally γ is at most equal to α.

Suppose that the general equations of the extremal that are obtained by integration of (38):

(54) $$x_i = \varphi_i(t - t_0 \mid x_1^0, \ldots, x_n^0;\ q_1^0, \ldots, q_n^0) \qquad (i = 1, 2, \ldots, n),$$

in which the initial values satisfy equation (10), depend, by abstraction from the constant t_0, on $2n - \delta - 1$ arbitrary essential constants; equations (52), and consequently, the set of solutions of the system (38), depend upon $2n - \delta - 1$ constants. One thus has $2n - \delta + \gamma - 1 = 2n - 1$; i.e., $\gamma = \delta$. One remarks that under the same conditions the extremal *curves* depend upon $2n - \delta - 2 = 2n - \gamma - 2$ essential arbitrary constants. In order that one may make an extremal curve pass through any two arbitrarily chosen points, it is necessary that $\gamma = 0$, and this is sufficient if one limits oneself to a convenient domain. In that case, and only in that case, only one canonical solution corresponds to each extremal.

We summarize the results obtained in the following statement:

The systems of functions $x_1, \ldots, x_n$; $q_1, \ldots, q_n$; ω of the variable u that satisfy the equations of condition (10) *and* (12), *and annul the variation of the integral* (4) *may be divided into two classes: First, there are the systems of functions (canonical solutions) that are obtained by taking ω to be an arbitrary positive function of u and then determining $x_1, \ldots, x_n$; $q_1, \ldots, q_n$ by means of the canonical system* (38), *combined with equation* (39). *These solutions are the only ones in the problem of a free extremum – i.e., the Hessian of G, considered to be a function $q_1, \ldots, q_n$, is of rank* 1. *Moreover, in this case there is only one canonical solution that provides each curve* (through $x_1, \ldots, x_n$) *or extremal that solves the problem.*

On the contrary, if the Hessian of G is of rank $\alpha + 1$ ($\alpha > 0$) then to each canonical solution ($x_1, \ldots, x_n$; $q_1, \ldots, q_n$; ω) correspond to ∞^α other solutions that furnish the same extremal: They are obtained by replacing $q_1, \ldots, q_n$ with any of the solutions $q_1', \ldots, q_n'$ of the system:

(55) $$\sum_{i=1}^n q_i' \frac{\partial G}{\partial q_i} = 1,$$

(56) $$\sum_{i=1}^n q_i' \frac{\partial^2 G}{\partial q_i \partial q_j} = 0 \qquad (j = 1, 2, \ldots, n).$$

In the general case, for which there are ∞^{2n-2} extremals, each of them is furnished by just one canonical solution. If, on the contrary, there are $\infty^{2n-\gamma-2}$ extremals ($\gamma > 0$) then ∞^γ canonical solutions define a subset of the ∞^α solutions that correspond to any of them by means of equations (54) *and* (55) ([1]).

([1]) It is clear, from the initial form (1) of the integral J, that it depends only upon the arc of the extremal, and not on those of the ∞^α solutions at $q_1, \ldots, q_n$ that one may associate with them in the latter case. This also results from the fact that δJ is constantly null when, leaving the arc of the extremal fixed, one varies the system of functions $q_1, \ldots, q_n$, provided that it always represents one of these ∞^α solutions.

19. Suppose that the integral J is taken along an extremal arc and makes that arc vary without leaving its extremities fixed. δJ is then non-null, in general, and it is given, as one knows, by a fundamental formula that is called the *formula at the limits* ([1]). It is easy to deduce the calculations of no. 14 from this.

Indeed, formula (29) is true, by hypothesis. With the notations (22) and taking into account equations (23), it may be written:

$$\delta\omega = \sum_{k=1}^{n} c_k \sum_{i=1}^{n} v_{ki} A_i = \sum_{k=1}^{n} c_k \frac{d\xi_k}{du}. \tag{57}$$

From this, one deduces, by integration, if one lets the index *zero* denote the values that correspond to the origin of the arc of the curve considered and lets the index *one* denote the ones that correspond to the extremity, that:

$$\delta J = \sum_{k=1}^{n} c_k (\xi_k' - \xi_k^0). \tag{58}$$

Now, one infers from equations (20), upon solving them, that:

$$\xi_k = \sum_{i=1}^{n} v_{ki} \delta x_i \qquad (k = 1, 2, \dots, n), \tag{59}$$

and one has, as a result, upon taking into account formulas (32), that:

$$\sum_{k=1}^{n} c_k \xi_k = \sum_{i=1}^{n} y_i \delta x_i. \tag{60}$$

Therefore, if we replace the letters $y_1, \dots, y_n$ with the letters $q_1, \dots, q_n$, which must correspond to one of the particular solutions to the canonical system that was at issue in the final statements of no. 18, then we obtain the formula at the limits in the form ([2]):

$$\delta J = \left[\sum_{i=1}^{n} q_i \delta x_i \right]_0^1. \tag{61}$$

We remark that the integral J is itself written in the analogous form:

([1]) Cf., HADAMARD, *loc. cit.*, pp. 246.

([2]) If there are only $\infty^{2n-\gamma-2}$ extremal curves then formula (61) is furnished by each of the systems of functions (53), and one has γ independent relations of the form $\left[\sum_{i=1}^{n} Q_i \delta x_i \right]_0^1 = 0$ between variations of $x_1, \dots, x_n$.

(62) $$J = \int_0^1 \sum_{i=1}^n q_i dx_i \, ,$$

because one has, from equations (12):

(63) $$\sum_{i=1}^n q_i dx_i = \sum_{i=1}^n q_i \frac{\partial G}{\partial q_i} \omega du = G \; \omega du,$$

and, as a consequence, due to (10):

(64) $$\omega du = \sum_{i=1}^n q_i dx_i \, .$$

One may likewise consider formula (62) to be an application of formula (61). It suffices to vary the extremity of the arc of the extremal past the origin of that arc, which one keeps fixed, up to the point of the extremal that one must take to be the defining extremity of the arc considered by making it describe all of the arc in question at that point. Formula (61) is then constantly:

(65) $$\delta J = \sum_{i=1}^n q_i \delta x_i \, ,$$

and formula (62) results by integration.

We return later on to the circumstances that give rise to formula (62) (*see* nos. 25, *et seq.*).

20. We now discuss the restrictions that we imposed on the function F in nos. 12 and 13.

We first address the legitimacy of the representation of the wave multiplicity by means of equations of form (10) and (11). From the conclusions of the discussion that was made in nos. 6 and 7, one must overlook the contact elements in the neighborhoods where the rank of the Hessian of f (or of F, in that case $\alpha = 0$) is raised.

Now, as far as the system (44) is concerned, which is the one that one comes to mind in the case of a free extremum ($\alpha \equiv 0$), when one reverts to the viewpoint of just the variables $x_1, \ldots, x_n$, one sees that this Hessian is nothing but the determinant of the coefficients of the second differentials in equations (44). One will thus be led to make the same hypothesis in order to affirm the existence of integrals to that system under non-singular conditions. We thus have every right to say (making note of no. 20) that the restriction thus introduced is not artificial, but is in the nature of the question itself.

The same is true for the more general case ($\alpha > 0$) of a constrained extremum, which leads us to the system (55), (56). Upon seeking to solve this system with respect to the second derivatives of the x_i and the first derivatives of the λ_h , one will be led to differentiate equations (56), and the determinant that this defines, like the determinant of the homogeneous linear system one obtains, will be the Hessian of f. One will thus arrive at the same conclusion on the nature of that primary restriction.

The other restriction is the condition that we imposed on $F(x_1, \ldots, x_n\ dx_1, \ldots, dx_n)$ – viz., that it be positive. Suppose, on the contrary, that the function F is not constantly positive along the curve on which one proposes to examine whether the variation (1) is null. One may then always find an auxiliary function $H(x_1, \ldots, x_n)$ that is defined at all of the points of the curve considered, and in the domain of these points – i.e., at all points of a certain domain (D) that contains the curve in its interior – admits continuous partial derivatives in that domain, and where the value on the curve is an increasing function of the parameter u that serves to represent the curve. One may likewise give an arbitrary value to that function at various points of the curve; for example, it might be the value of the parameter u itself. $\sum_{i=1}^{n} \frac{\partial H}{\partial x_i} \frac{dx_i}{du}$ then has the value 1 on this curve. On the other hand, $F\left(x_1, \cdots, x_n, \frac{dx_1}{du}, \cdots, \frac{dx_n}{du}\right)$ is supposed to be continuous and finite on the curve; its absolute value is then less than a fixed number M. If one then sets:

$$\overline{F} = F(x_1, \ldots, x_n \mid dx_1, \ldots, dx_n) + M \sum_{i=1}^{n} \frac{\partial H}{\partial x_i} dx_i \tag{66}$$

then the quotient $\overline{F} / du$ will be, on the curve considered, greater than a fixed positive number, and will remain positive on the curves that one deduces by continuous variations that relate to the points and tangents. Now, if one wishes to obtain the necessary conditions for the variation to be null then it suffices to consider such variations of the integration curve.

To abbreviate the notation, set:

$$M \frac{\partial H}{\partial x_i} = H_i , \tag{67}$$

in such a way that the function (66) is:

$$\overline{F} = F(x_1, \ldots, x_n \mid dx_1, \ldots, dx_n) + \sum_{i=1}^{n} H_i\, dx_i , \tag{68}$$

and consider ([1]) the integral:

$$\overline{J} = \int_0^1 \overline{F} \equiv J + M[H(x_1, \cdots x_n)]_0^2 \equiv \int_0^1 \overline{\varpi}\, \overline{d}u . \tag{69}$$

([1]) This transformation is, at its basis, the one that was utilized by Carathéodory, and which also served for Hadamard (cf., *loc. cit.*, pp. 385). However, these authors introduced the hypothesis that the function ε of Weierstrass had a constant sign, while that hypothesis does not enter in here. Also, he examined only the case $\alpha = 0$, $n = 2$.

It is clear that $\delta\bar{J} = \delta J$ when the extremities of the integration arc remain fixed. We thus have to express that $\delta\bar{J}$ is null, and all of the preceding reasons and results will still apply for this. What remains is for us to see what this gives relative to the original givens.

21. For this, compare the wave multiplicity in both cases at the same point of the curve considered, while the origin of the coordinates is consequently placed at that point. The coordinates of the new point of the curve that correspond to a positive increase *du* of the parameter are $dx_1, \ldots, dx_n$, and, in the former case they satisfy the equations:

$$\text{(70)} \qquad \begin{cases} F(x_1, \cdots, x_n \mid dx_1, \cdots, dx_n) = \omega\, du, \\ F_h(x_1, \cdots, x_n \mid dx_1, \cdots, dx_n) = 0 \qquad (h = 1, 2, \cdots, \alpha), \end{cases}$$

and, in the latter case, analogous equations:

$$\text{(71)} \qquad \begin{cases} \bar{F}(x_1, \cdots, x_n \mid dx_1, \cdots, dx_n) = \bar{\omega}\, du, \\ F_h(x_1, \cdots, x_n \mid dx_1, \cdots, dx_n) = 0 \qquad (h = 1, 2, \cdots, \alpha). \end{cases}$$

To abbreviate the notation, set:

$$\text{(72)} \qquad X_i\, du = dx_i \qquad (i = 1, 2, \ldots, n),$$

and cease writing the letters $x_1, \ldots, x_n$ in the functions considered, since they are constant parameters. By virtue of formulas (5), the wave multiplicity is defined, for $\bar{J}$, by the equations:

$$\text{(73)} \qquad \begin{cases} \bar{F}(X_1, \cdots, X_n) = \bar{\omega}, \\ F_h(X_1, \cdots, X_n) = 0, \quad (h = 1, 2, \cdots, \alpha), \\ \qquad X_i = \bar{\omega}\bar{p}_i \qquad (i = 1, 2, \cdots, n), \end{cases}$$

in which $\bar{p}_1, \ldots, \bar{p}_n$ are the coordinate of the point that corresponds to the point $dx_1, \ldots, dx_n$ of the elementary wave. Moreover, if ω is positive then one will have completely similar formulas for *J*:

$$\text{(74)} \qquad \begin{cases} F(X_1, \cdots, X_n) = \omega, \\ F_h(X_1, \cdots, X_n) = 0, \quad (h = 1, 2, \cdots, \alpha), \\ \qquad X_i = \omega p_i \qquad (i = 1, 2, \cdots, n). \end{cases}$$

These formulas show that when ω is annulled the corresponding point of the wave multiplicity goes to infinity. We may consider them as the definition of the wave multiplicity in any case where ω is positive or negative.

The two wave multiplicities correspond to each other point by point by means of formulas (73) and (74). We seek the formulas for that correspondence. Due to (68), (72), (74), one has:

(75) $$\bar{\varpi} = \omega + \sum_{i=1}^{n} H_i X_i = \omega\left(1 + \sum_{i=1}^{n} H_i p_i\right),$$

which gives the formulas:

(76) $$\bar{p}_i = \frac{p_i}{1 + \sum_{i=1}^{n} H_i p_i} \qquad (i = 1, 2, \dots, n).$$

We seek the coordinates $q_1, \dots, q_n$; $\bar{q}_1, \dots, \bar{q}_n$ for the corresponding contact elements. For (74), ω being a constant, we have to write that $\sum_{i=1}^{n} q_i dX_i$ is a linear and homogeneous combination of the dF and dF_h . Therefore:

(77) $$q_i = \lambda_0 \frac{\partial F}{\partial X_i} + \sum_{h=1}^{\alpha} \lambda_h \frac{\partial F_h}{\partial X_i} \qquad (i = 1, 2, \dots, n).$$

The condition $\sum_{i=1}^{n} q_i p_i = 1$ gives $\lambda_0 = 1$. One thus has, upon once more setting, as in no. 1:

(78) $$f = F + \sum_{h=1}^{\alpha} \lambda_h F_h ,$$

the formulas:

(79) $$q_i = \frac{\partial f}{\partial X_i} \qquad (i = 1, 2, \dots, n).$$

For the quantities $\bar{q}_i$, one will have analogous formulas:

(80) $$\bar{q}_i = \frac{\partial \bar{f}}{\partial X_i}, \qquad \bar{f} = \bar{F} + \sum_{h=1}^{\alpha} \bar{\lambda}_h F_h \qquad (i = 1, 2, \dots, n).$$

Since the λ_h, $\bar{\lambda}_h$ are entirely arbitrary, one may impose the condition that they be equal:

$$\lambda_h = \bar{\lambda}_h \qquad (h = 1, 2, \dots, \alpha),$$

respectively, which then gives the law of correspondence for the contact elements of the two wave multiplicities. It is expressed by equations (76) and the equations:

(81) $$\bar{q}_i = q_i + H_i \qquad (i = 1, 2, \ldots, n).$$

One sees that it is nothing but the *dual transformation to a translation* [cf., no. 1, note ([1])].

Before returning to the differential equations for the Lagrange problem, we again indicate how one generalizes the formulas that were obtained in nos. 2 and 3 for the multiplicity that was defined by equations (74). As we just saw, the tangential support is defined by equations (79). Here, we suppose, as in no. 13, that the Hessian of f is of rank 1; i.e., the elimination of $X_1, \ldots, X_n, \lambda_1, \ldots, \lambda_\alpha$ between equations (79) and the equations of condition $F_h = 0$ ($h = 1, 2, \ldots, \alpha$), furnishes just one equation, which is reducible to the form:

(82) $$G(x_1, \ldots, x_n \mid q_1, \ldots, q_n) = 1.$$

On the one hand, one has:

(83) $$\sum_{i=1}^{n} q_i X_i = \omega, \qquad \sum_{i=1}^{n} q_i dX_i = 0;$$

thus, one further has:

(84) $$\sum X_i \, dq_i = 0.$$

One concludes that the X_i are proportional to the $\partial G / \partial q_i$, and one confirms, due to (83) and (82), that the ratio of proportionality is ω. One thus has:

(85) $$X_i = \omega \frac{\partial G}{\partial q_i} \qquad (i = 1, 2, \ldots, n),$$

and as a result, one again has formulas (11):

(86) $$p_i = \frac{\partial G}{\partial q_i} \qquad (i = 1, 2, \ldots, n).$$

Upon recalling the calculations of no. 3, but while introducing the X_i in place of the p_i, one finally verifies that equation (31) of no. 3 will be replaced by the following one:

(87) $$\frac{\partial f(x_1, \cdots, x_n \mid X_1, \cdots, X_n)}{\partial x_i} + \omega \frac{\partial G(x_1, \cdots, x_n \mid q_1, \cdots, q_n)}{\partial x_i} = 0 \qquad (i = 1, 2, \ldots, n).$$

22. Now let:

(88) $$\frac{dx_i}{d\theta} = \frac{\partial \Gamma}{\partial \bar{q}_i}, \qquad \frac{d\bar{q}_i}{d\theta} = -\frac{\partial \Gamma}{\partial x_i}, \qquad d\theta = \bar{\varpi} \; du \qquad (i = 1, 2, \ldots, n),$$

so the canonical system is obtained by equating the variation of $\bar{J}$ to zero. The equation:

(89) $$\Gamma(x_1, \cdots, x_n \mid \bar{q}_1, \cdots, \bar{q}_n) = 1$$

is the tangential equation of the corresponding wave multiplicity.

We have to make the change of variables that is defined by formulas (76) and (81), in such a way that G is defined by the equation:

(90) $$\Gamma(x_1, \ldots, x_n \mid q_1 + H_1 G, \ldots, q_n + H_n G) = G,$$

because in order to obtain equation (82), one must solve it with respect to the homogeneity variable in the equation of the original wave multiplicity, which results from (89) by the change of coordinates (81) (cf., no. 2).

If one next differentiates the identity relation (90) then one obtains:

(91) $$\frac{\partial \Gamma}{\partial \bar{q}_i} + \sum_{j=1}^{n} \frac{\partial \Gamma}{\partial \bar{q}_j} H_j \frac{\partial G}{\partial q_i} = \frac{\partial G}{\partial q_i} \qquad (i = 1, 2, \ldots, n),$$

(92) $$\frac{\partial \Gamma}{\partial x_i} + \sum_{j=1}^{n} \frac{\partial \Gamma}{\partial \bar{q}_j} \frac{\partial H_j}{\partial x_i} G + \sum_{j=1}^{n} \frac{\partial \Gamma}{\partial \bar{q}_j} H_j \frac{\partial G}{\partial x_i} = \frac{\partial G}{\partial q_i} \qquad (i = 1, 2, \ldots, n).$$

We remark that one further has:

(93) $$\bar{p}_j = \frac{\partial \Gamma}{\partial \bar{q}_j} \qquad (j = 1, 2, \ldots, n),$$

and as a result, upon taking into account (76) and (75):

(94) $$1 - \sum_{j=1}^{n} H_j \frac{\partial \Gamma}{\partial \bar{q}_j} = 1 - \sum_{j=1}^{n} H_j \bar{p}_j = \frac{1}{1 + \sum_{j=1}^{n} H_j p_j} = \frac{\omega}{\bar{\omega}} .$$

Equations (91) and (92) may, in turn, be written, by observing that $\frac{\partial H_j}{\partial x_i} = \frac{\partial H_i}{\partial x_j}$, and that $\bar{\omega}\bar{p}_j = dx_j / du$:

(95) $$\omega \frac{\partial \Gamma}{\partial \bar{q}_i} = \omega \frac{\partial G}{\partial q_i} \qquad (i = 1, 2, \ldots, n),$$

(96) $$\bar{\omega} \frac{\partial \Gamma}{\partial x_i} + \sum_{j=1}^{n} \frac{\partial H_i}{\partial x_j} \frac{dx_j}{du} = \omega \frac{\partial G}{\partial x_i} \qquad (i = 1, 2, \ldots, n).$$

Finally, if one takes into account equations (88), then what remains is:

(97) $$dx_i = \omega \frac{\partial G}{\partial q_i} du \qquad (i = 1, 2, \ldots, n),$$

(98) $$-\, d\overline{q}_i + dH_i = \omega \frac{\partial G}{\partial x_i} du \qquad (i = 1, 2, \ldots, n).$$

Due to (81), the latter reduce to:

(99) $$dq_i = -\,\omega \frac{\partial G}{\partial x_i} du \qquad (i = 1, 2, \ldots, n).$$

One thus recovers the canonical system that was deduced from equation (82).

It is convenient to observe that due to formulas (83) the $\partial G / \partial q_i$ will be infinite, in general, for $\omega = 0$; however, the canonical system may, in the same manner that we just obtained, be transformed in such a way that this apparent difficulty disappears.

As for the systems (44) or (41), (51), (52), one will again deduce them from the canonical system upon referring to formulas (87). One will also obtain them more immediately by starting with the analogous system that relates to the integral $\overline{J}$. Indeed, one has, with the variables $x_1, \ldots, x_n$; $p_1, \ldots, p_n$:

(100) $$\frac{\partial \overline{f}}{\partial x_i} = \frac{\partial f}{\partial x_i} + \sum_{j=1}^{n} \frac{\partial H_j}{\partial x_i} p_j = \frac{\partial f}{\partial x_i} + \sum_{j=1}^{n} \frac{\partial H_i}{\partial x_j} p_j \qquad (i = 1, 2, \ldots, n).$$

Thus, due to (41):

(101) $$\begin{cases} \dfrac{\partial \overline{f}}{\partial x_i} = \dfrac{\partial f}{\partial x_i} + \dfrac{dH}{dt} & (i = 1, 2, \cdots, n), \\[2ex] \dfrac{\partial f}{\partial p_i} = \dfrac{\partial f}{\partial p_i} + H_i & (i = 1, 2, \cdots, n), \end{cases}$$

and, in turn:

(102) $$\frac{d}{dt}\frac{\partial \overline{f}}{\partial p_i} - \frac{\partial \overline{f}}{\partial x_i} = \frac{d}{dt}\frac{\partial f}{\partial p_i} - \frac{\partial f}{\partial x_i} \qquad (i = 1, 2, \ldots, n),$$

which shows clearly that one comes down to the system in question.

23. The theorem that permits us to express, as in no. 14, that δJ is null for any system of variations $\delta\gamma_1, \ldots, \delta\gamma_n$, $\delta\omega$ that satisfy the conditions (28) is, at its basis, the theorem that serves to found the classical method of multipliers in the isoperimetric problems.

However, since it is presented in a particular form in our method, it will not be pointless to give the corresponding precise statement, with its proof ([1]).

It refers to definite integrals of the form:

$$1 = \int_0^1 L\,du\,, \tag{103}$$

where L is a linear form with respect to indeterminate functions $w_1, \ldots, w_n$ of u, in which the coefficients of that form are given functions of u.

Consider m integrals of that form:

$$I_h = \int_0^1 L_h\,du \qquad (h = 1, 2, \ldots, m), \tag{104}$$

and associate the functions w_i with m arbitrarily chosen systems of determinations:

$$w_i = w_{ki} \qquad (i = 1, 2, \ldots, n;\ k = 1, 2, \ldots, m). \tag{105}$$

The integrals considered – viz., $I_1, \ldots, I_m$ – take on the corresponding numerical values:

$$I_h = I_{kh} \qquad (h = 1, 2, \ldots, m\ ;\ k = 1, 2, \ldots, m). \tag{106}$$

Consider the determinant formed from these numbers I_{kh} and examine the case where it is null for all of the systems of determination (105): There will then be a *principal* minor determinant that is not zero, while the minor determinants of higher degree are all null. One may suppose that this principal determinant is the one that corresponds to $k = 1, 2, \ldots, s$; $h = 1, 2, \ldots, s$. All of the determinants:

$$\begin{vmatrix} I_{11} & \cdots & I_{s1} & I_1 \\ \cdots & \cdots & \cdots & \cdots \\ I_{1s} & \cdots & I_{ss} & I_s \\ I_{1,s+l} & \cdots & I_{s,s+l} & I_{s+l} \end{vmatrix} \qquad (l = 1, 2, \ldots, m-s) \tag{107}$$

are then null, no matter what the functions w_i are. One thus has some equations with constant coefficients of the form:

$$I_{s+l} = \sum_{h=1}^{s} c_{lh} I_h \qquad (l = 1, 2, \ldots, m-s), \tag{108}$$

i.e.:

$$\int_0^1 \left(L_{s+l} - \sum_{h=1}^{s} c_{lh} L_h \right) du = 0 \qquad (l = 1, 2, .., m-s). \tag{109}$$

([1]) Cf., the analogous proof that was given by Hadamard (*loc. cit.*, pp. 196). However, the restriction that relates to *singular fields* does not apply here.

Since they are true for any w_i and the expressions under the $\int$ sign are linear forms with respect to w_i one concludes from this the identity relations in $w_1, \ldots, w_n$ (and u):

(110) $$L_{s+l} = \sum_{h=1}^{s} c_{lh} L_h \qquad (l = 1, 2, \ldots, m - s).$$

It results, in particular, that $I_{s+1}, \ldots, I_m$ are null for any choice of functions $w_1, \ldots, w_n$, which annuls $I_1, I_2, \ldots, I_s$.

24. Having said this, we propose to express the fact that the integral (103) is annulled for any choice of functions $w_1, \ldots, w_n$ that makes the integrals (104) null, and suppose first of all that the determinant of the quantities (106) is not null.

Furthermore, consider an $(m + 1)$-fold system of determinations in the w_i :

(111) $$w_i = w_{0,i} \qquad (i = 1, 2, \ldots, n),$$

and the corresponding values $I_{0,1}, \ldots, I_{0,m}$ of the integrals (104). One may determine the numbers μ_k by the equations:

(112) $$\sum_{k=1}^{m} \mu_k I_{k,h} - I_{0,h} = 0 \qquad (h = 1, 2, \ldots, m).$$

If one then takes the w_i to be the functions:

(113) $$w_i = \sum_{k=1}^{m} \mu_k w_{k,i} - w_{0,i} \qquad (i = 1, 2, \ldots, n)$$

then the values of the I_h , being equal to the left-hand side of equations (112), are null, and, by hypothesis, that of I is then also null.

If we denote the values of I that correspond to the choices (105) and (111) by $I_{1,0}, \ldots, I_{m,0}$, $I_{0,0}$ then one thus has the numerical equality:

(114) $$\sum_{k=1}^{m} \mu_k I_{k,0} - I_{0,0} = 0.$$

If one compares this with the equalities (112) that are also verified by the μ_k then one sees that the determinant:

(115) $$\begin{vmatrix} I_{0,0} & I_{1,0} & \cdots & I_{m,0} \\ I_{0,1} & I_{1,1} & \cdots & I_{m,1} \\ \cdots & \cdots & \cdots & \cdots \\ I_{0,m} & I_{1,m} & \cdots & I_{m,m} \end{vmatrix}$$

is null, while the minor of the first element is non-null, and that is true for any choice of $m + 1$ systems of determination for the w_i that are introduced. From the result of no. 23, one concludes from this an identity with constant coefficients of the same nature as (110):

$$L = \sum_{h=1}^{n} c_h L_h \,. \tag{116}$$

If the determinant of the quantities (106) is null then one may consider only the integrals $I_1, \ldots, I_s$ as giving the principal minor of that determinant. Because $I_{s+m}, \ldots, I_m$ are null whenever the preceding ones are null, it will suffice to express the notion that I is annulled whenever $I_1, \ldots, I_s$ are null. One thus again obtains an identity of the form (116) as a necessary condition, with:

$$c_{s+1} = c_{s+2} = \ldots = c_m = 0.$$

Since one has the identities (110) in this case, it is clear that one likewise has an infinitude of identities of the form (116), where the most general of them can be written, by means of one of them:

$$L = \sum_{h=1}^{m} c_h L_h + \sum_{l=1}^{m-s} \rho_l \left(L_{s+l} - \sum_{k=1}^{s} c_{ik} L_k \right), \tag{117}$$

in which the ρ_l are arbitrary.

Finally, in any case, the existence of an identity of the form (116) is sufficient for I to be null when the $I_1, \ldots, I_m$ are annulled simultaneously.

We thus obtain the stated theorem ([1]): *In order for the integral* (104) *to be annulled for any choice of functions* w_i *that simultaneously annul the integrals* (105), *it is necessary and sufficient that the linear forms of the indeterminates* w_i *that are denoted by* $L, L_1, \ldots, L_m$ *are linked by a relation of the form* (116), *which is an identity in* $w_1, \ldots, w_n$ *and* u, *and the letters* $c_1, \ldots, c_m$ *denote numerical constants.*

In the application of this theorem that was made in no. 14, the $\delta\gamma_1, \ldots, \delta\gamma_n$, $\delta\omega$ play the role of functions $w_1, \ldots, w_n$. The integral I is the integral (16) and the integrals I_h are the integrals (28). There exist γ relations that are analogous to the $(m - s)$ relations (110) in the case where the canonical system (38) furnishes only $\infty^{2n-\gamma-2}$ extremal curves (cf., the note in no. 19).

([1]) Since this theorem is deduced from the fundamental lemma of the calculus of variations (cf., HADAMARD, *loc. cit.*, pp. 64), it persists, like that lemma, if one imposes diverse restrictions on the functions $w_1, \ldots, w_n$ that relate to either their values at the limits or their analytical character.

III. – SUFFICIENT CONDITIONS FOR A WEAK EXTREMUM.

25. The preceding results that we found and summarized in the statement of no. 18 may be stated in a remarkable form, on the condition that, should the need arise, one limits oneself to canonical solutions. This restriction is, moreover, of little importance, since it does not prevent us from obtaining all of the curves that annul the variation of *J*. It permits us to suppose that the function $q_1, \ldots, q_n$ that one considers are differentiable.

Consider the integral:

$$H = \int_0^1 \sum_{i=1}^{n} q_i dx_i \,, \tag{1}$$

where one supposes that the functions $x_1, \ldots, x_n ; q_1, \ldots, q_n$ of the variable of integration u are linked by just one equation of condition:

$$G(x_1, \ldots, x_n \mid q_1, \ldots, q_n) = 1, \tag{2}$$

and demands that the variation δH be null. With what we have learned from the remark made above, we apply the classical procedure, which then gives us:

$$\delta H = \left(\sum_{i=1}^{n} q_i \delta x_i \right)_0^1 + \int_0^1 \sum_{i=1}^{n} (\delta q_i dx_i - \delta x_i dq_i) \,. \tag{3}$$

We further suppose that the extremities of the integration arc are fixed. The quantity that remains under the ∫ sign thus reproduces δG up to a factor, and one finds the canonical system:

$$\frac{dx_i}{\dfrac{\partial G}{\partial q_i}} = \frac{dq_i}{-\dfrac{\partial G}{\partial x_i}} = dt \qquad (i = 1, 2, \ldots, n), \tag{4}$$

which entails the formula:

$$dt = \sum_{i=1}^{n} q_i dx_i \,, \tag{5}$$

when one takes into account (2), and one recalls that G is homogeneous of degree 1 in $q_1, \ldots, q_n$.

The extremals of the problem A *are thus obtained by writing that the variation of the integral H is null, by means of just the equation of condition* (2), *when one supposes that the extremities of the integration arc are fixed.*

One sees, moreover, that if one varies only the functions $q_1, \ldots, q_n$ while leaving the integration arc fixed then one finds immediately, and consequently without any hypotheses on the differentiability of the function $q_1, \ldots, q_n$, the formula:

$$\delta_0 H = \int_0^1 \sum_{i=1}^{n} \delta q_i dx_i \,. \tag{6}$$

The condition for the variation $\delta_0 H$ to be null will thus be:

(7) $$dx_i = \frac{\partial G}{\partial q_i} dt \qquad (i = 1, 2, \ldots, n),$$

since the variations of the q_i will then be linked by just the equation:

(8) $$0 = \delta_0 G = \sum_{i=1}^{n} \frac{\partial G}{\partial q_i} \delta q_i ,$$

and one will again have formula (5).

Therefore, *the systems of formulas considered in the statement of problem* A *are the ones that one obtains by writing the condition of null variation in the following problem:*

PROBLEM B. – *Being given an arc of the curve that is represented by formulas:*

(9) $$x_i = \varphi_i(u), \qquad 0 \leq u \leq 1 \qquad (i = 1, 2, \ldots, n),$$

determine functions $q_1, \ldots, q_n$ of u that satisfy the equation:

(2) $$G(x_1, \ldots, x_n \mid q_1, \ldots, q_n) = 1,$$

and are such that they give an extremum for the integral:

(1) $$H = \int_0^1 \sum_{i=1}^{n} q_i dx_i .$$

One sees, moreover, upon referring to formula (62) of no. 19, which is the origin of the present remarks that *the integral J considered in problem* A *are the integral H of problem* B, *which corresponds to the case of the null variation* (for the same problem B.).

Meanwhile, in order to satisfy the condition $\omega > 0$ of problem A, it is necessary that problem B admit only functions $q_1, \ldots, q_n$ that verify the conditions:

(10) $$\sum_{i=1}^{n} q_i dx_i > 0.$$

If one interprets $q_1, \ldots, q_n$ as the components of a vector that has the point $(x_1, \ldots, x_n)$ for its origin and which describes the arc in question when u varies from 0 to 1 then this condition is equivalent to saying that this vector must be situated on the positive side of the tangent with respect to the normal plane $\sum_{i=1}^{n} (X_i - x_i) dx_i = 0$, or that it must make an acute angle with the positive direction of the tangent.

One must also observe that if the Hessian of G is of rank $(\alpha + 1)$, with $\alpha > 0$, then equations (7) have the same consequences as equations (2) of no. 12, namely:

(11) $$F_h(x_1, \ldots, x_n \mid dx_1, \ldots, dx_n) = 0 \qquad (h = 1, 2, \ldots, \alpha);$$

problem B is therefore possible in this case only if the given curve (9) is an integral curve of the Monge system.

26. In all of these cases, the preceding remarks introduce, and in the most natural manner, the Jacobi-Hamilton partial differential equation:

(12) $$G\left(x_1, \cdots, x_n \middle| \frac{\partial t}{\partial x_1}, \cdots, \frac{\partial t}{\partial x_n}\right) = 1.$$

Indeed, they show that one will obtain the extremals of problem A upon first determining functions $q_1, \ldots, q_n$ of $x_1, \ldots, x_n$ that satisfy (2) and annul the variation of the integral H, and then, upon determining $x_1, \ldots, x_n$ by means of equations (7), where one has substituted the functions of $x_1, \ldots, x_n$ that were thus found for the $q_1, \ldots, q_n$. Because the variation of H is then null, either when one varies the curve while preserving the functions we found for the q_i, or when one keeps the curves fixed and varies the q_j, it is therefore again null when one varies the curve and the q_i at the same time, because any variation of that general type is obtained by superposing two variations that belong to the two special categories considered, respectively.

Now, this calculation amounts to first taking the derivatives:

(13) $$q_i = \frac{\partial t}{\partial x_i} \qquad (i = 1, 2, \ldots, n)$$

of an integral of equation (12), namely:

(14) $$t = V(x_1, \ldots, x_n),$$

and then determining the *transversals* of the family of surfaces depending upon the parameter t, which are represented by equation (14); i.e., the curves that are defined by a property that shall recall.

Each point of M on any of the surfaces (14) is the origin of a wave multiplicity (cf., no. 12), and on that wave multiplicity there exists a point P whose coordinates are:

(15) $$X_i = x_i + \frac{\partial G}{\partial q_i} \qquad (i = 1, 2, \ldots, n),$$

in which the q_i have the values (13). This point is entirely defined by the fact that there exists, at each point, a plane that is tangent to the wave multiplicity and has the direction coefficients $\lambda q_1, \ldots, \lambda q_n$; i.e., it is parallel to the tangent plane at M of the surface (14) considered ([1]). The direction MP, whose direction coefficients are:

([1]) This point must be on the side of increasing V with respect to the tangent plane at M to the surface considered.

(16) $$p_i = \frac{\partial G}{\partial q_i} \qquad (i = 1, 2, \ldots, n),$$

is called *transversal* to the contact element $(x_1, \ldots, x_n ; \lambda q_1, \ldots, \lambda q_n)$ of the point M and is, in turn, transversal to the surface considered at M.

Having said this, the transversals of a one-parameter family of surfaces (to simplify, we are supposing that one and only surface of the family passes through each point of the space considered) are the curves whose direction, at each of their points, is transversal to the surface of the family that passes through this point.

One will remark that in the present case, due to the condition $\sum_{i=1}^{n} q_i dx_i > 0$, the transversals to the family of surfaces (14) must be assumed to point in the direction of increasing V.

27. Reciprocally, the preceding construction gives all of the extremals. Indeed, start with an initial surface (S_0) and each of its contact elements $(x_1^0, \ldots, x_n^0;\ q_1^0, \ldots, q_n^0)$, whose coordinates may be assumed to satisfy condition (2), and then associate the integral of the system (4), which satisfies the initial conditions $x_i = x_i^0$; $q_i = q_i^0$ $(i = 1, 2, \ldots, n)$. It will satisfy condition (2). At each point M_0 of (S_0), one will find it associated with the extremal curve that issues from M_0, and which is the locus that is described by the point M, whose coordinates are $(x_1, \ldots, x_n)$, when t varies by starting with the value $t = 0$. These curves, at least in the neighborhood of (S_0), will be such that one and only one of them passes through each point M of the space ([1]). Each point $(x_1, \ldots, x_n)$ will be associated with the value of the integral $\int \sum_{i=1}^{n} q_i dx_i$, which is taken from M_0 to M along the arc of the extremal that passes through M, since equations (4) entail equation (5): t is therefore a function $V(x_1, \ldots, x_n)$.

In order to find its total differential, it is necessary and sufficient to vary M in an arbitrary manner; it follows with its corresponding extremal, whose foot M_0 describes (S_0). If one applies ([2]) formula (3), upon remarking that one has:

$$\sum_{i=1}^{n} q_i^0\, \delta x_i^0 = 0,$$

then what remains is:

(17) $$\delta t = \sum_{i=1}^{n} q_i\, \delta x_i\ .$$

([1]) This will break down when (S_0) satisfies the (partial differential) equation $G(x_1^0, \ldots, x_n^0 \mid q_1^0, \ldots, q_n^0) = 0$, which is the singular case that we have implicitly passed over.

([2]) This method, which based on the formula at the limits, is employed by Hadamard (*loc. cit.*, pp. 169). The principle is due to Darboux (*Théorie des surfaces,* t. II, pp. 536).

The function t that we have constructed thus satisfies equations (13) and the partial differential equation (12). Moreover, when equations (7) are verified the extremals that have served to define that function are the transversals to the family of surfaces (14) that they correspond to.

Finally, it is clear that each extremal will be transversal to an infinitude of families of surfaces of the indicated type, because it corresponds (cf., nos. 16 and 18) to at least one canonical solution, and it will suffice that one of the surfaces of the family include one of the elements ($x_1, \ldots, x_n$; $q_1, .., q_n$) that is furnished by that solution in order that all of the other ones enter into the other surfaces of the family ([1]).

We further remark that one may attempt to replace the surface (S_0) with an n – 1-dimensional multiplicity whose point-like support has less than (n – 1) dimensions. The construction will further involve ∞^{n-1} canonical solutions, but if equation (12) has only $\infty^{2n-2-\gamma}$ characteristic curves (extremals of problem A) then it may happen that one obtains less than ∞^{n-1} extremals, and that consequently they do not fill up the space that neighbors (S_0). The definition of the function V will then break down. This will certainly happen, always upon supposing that $\gamma > 0$, if (S_0) is replaced by a point of space, and, more generally, if the support employed has less than γ dimensions.

Similarly, upon supposing that $\gamma = 0$, but $\alpha > 0$, and upon starting at a point of M_0 that is taken on the initial multiplicity, the extremals employed indeed sweep out an n-dimensional space, but that space will terminate at the point considered in a singular form, since the extremals that issue from that point do not point in all directions: It will not contain all of the points of a domain of M_0, nor, for that matter, those of the points that are on the same side of a surface that passes through M_0 and has a tangent plane at that point.

28. The families of surfaces that we just introduced are composed of successive states of the same wave (i.e., it is a locus of points that are disturbed at the same instant) when one considers the mode of propagation that was defined in no. 12. One must assume only that the state of a wave at time $t + dt$ can be deduced from the state of that wave at time t, up to infinitesimals of higher order, by taking the envelope of the elementary waves that issue from the various points of the multiplicity that includes the state of the wave at time t.

On this point, we refer to our article in the *Annales de l'École Normale*, 3rd series, t. XXVI, 1909, pp. 405, because the interpretation in question is a consequence of the results in no. 5 of that article, and the reasoning that led up to it is based on the tangential representation of the elementary waves and does not cease to be applicable when the point-like support of the wave multiplicity has less than n – 1 dimensions, provided that the tangential support has exactly n – 1 dimensions.

Here again, one may conceive of extremals [which are characteristics of equation (12)] as the trajectories of propagation of the various contact elements.

([1]) The singular case that was pointed out in our preceding note may not present itself here, since all of the elements of a canonical solution satisfy equation (2).

However, in general, one must limit the extremal arc considered in order to obtain a family of surfaces (and transversals) such that one and only one of them passes through each point of the domain considered.

We would not like to reproduce that well-known argument here.

Cf., HADAMARD, *loc. cit.*, pp. 360, et seq.

Observe, moreover, that when one considers ($x_1, \ldots, x_n$; $q_1, .., q_n$) to be the homogeneous coordinates of a contact element, equations (4) define a one-parameter family group of contact transformations and, from the preceding discussion, provide the successive states of the same wave by the application of these various contact transformations to one of them. This was the viewpoint of our article in this *Bulletin* (t. XXXIV, 1906), but there we supposed that the Hessian of G was of rank 1, while we make no such restriction here in that regard.

The finite contact transformations of that group are defined by the correspondence that was established between each point of the space and the wave that resulted from a disturbance that is produced at that point after a definite time t.

In the case where equation (12) has only $\infty^{2n-2-\gamma}$ characteristic curves, there are $\infty^{2n-2-\gamma}$ of them passing through each point, in such a way that the particular point-like multiplicities that enter in here have only $n - 1 - \gamma$ dimensions. One is then involved with the contact transformations that are all defined for $\gamma + 1$ equations between the coordinates $x_1, .., x_n$ and $\bar{x}_1, \ldots, \bar{x}_n$ of the points of the two mutually transformed spaces.

29. The remarks of the preceding numbers, independently of their analytical interest, lead very simply to sufficient conditions for the extremum in problem A, by reducing it to the extremum of problem B. As we shall see, *one will have a minimum for* A *if one has a maximum for* B.

Indeed, imagine an extremal of A and two fixed points M_0 and M_1 on that extremal. Assume that these points are as close as one needs in order to associate the arc M_0M_1 of the extremal with a family of surfaces (14) that satisfies the conditions that were enumerated in no. 27, namely: There exists an n-dimensional portion of space that is continuous and all in one piece, such that through each of its points there passes one and only one surface (14). The arc M_0M_1 is completely interior to that space, meets each of the surfaces at no more than one point, and is transversal to each of them.

We let $\mathcal{E}$ denote the arc of the extremal considered, and let $\mathcal{C}$ denote another arc ([1]) that goes from M_0 to M_1 in the same portion of space, and differ from $\mathcal{E}$ by its points and tangents as little as one desires.

Since one may take the q_i to be derivatives $\partial V / \partial x_i$ (no. 26), the integral $J_{\mathcal{E}}$ is the integral of dV taken from M_0 to M_1, and does not change if one takes it along the arc $\mathcal{C}$. One thus has:

$$J_{\mathcal{E}} = J_{\mathcal{C}} = -\left(\int_{\mathcal{C}} \sum_{i=1}^{n} q_i dx_i - \int_{\mathcal{C}} \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} dx_i\right) = -(H_{\mathcal{C}} - H'_{\mathcal{C}}), \tag{18}$$

and one perceives in the right-hand side the difference between two integrals that relate to the curve $\mathcal{C}$ and have the same nature as the ones that enter into the statement of problem B, where the first of them corresponds, by hypothesis, to a system of functions $q_1, \ldots, q_n$ that annuls the variation of that integral. Moreover, the values of $q_1, .., q_n$ will also be as close as one pleases to the values:

([1]) Of course, in the case $\alpha > 0$ one must verify the Monge equation (II). (Cf., no. 25).

$$q_1' = \frac{\partial V}{\partial x_1}, \qquad \ldots, \qquad q_n' = \frac{\partial V}{\partial x_n}$$

at any point of $\mathcal{C}$, provided that $\mathcal{C}$ is sufficiently close to $\mathcal{E}$ (in terms of its points and tangents).

The arc $\mathcal{E}$ *will therefore furnish a weak minimum for J if the system* (q_1, …, q_n) *furnishes a maximum for* Π *taken along* $\mathcal{C}$.

30. Now, the discussion for the maximum in problem B is immediate, because one has:

(19) $$H_{\mathcal{C}} - H'_{\mathcal{C}} = \int_0^1 \sum_{i=1}^n (q_i - q_i')\, dx_i\,.$$

Upon setting, as in no. 12:

(20) $$dx_i = \omega p_i\, du \qquad (i = 1, 2, \ldots, n),$$

with [equation (64), no. 19)]:

(21) $$\omega\, du = \sum_{i=1}^n q_i dx_i\,,$$

one then has:

(20) $$H_{\mathcal{C}} - H'_{\mathcal{C}} = -\int_0^1 \sum_{i=1}^n (q_i' p_i - 1)\, \omega\, dx_i\,.$$

Since the element $\omega\, du$ is, by hypothesis, essentially positive, not only on $\mathcal{E}$, but also on the curves that are sufficiently close, one sees that in order for the left-hand side to be positive, it is necessary and sufficient that the condition:

(21) $$\sum_{i=1}^n q_i' p_i - 1 < 0$$

be verified on each of the wave multiplicities that has their origin at the various points of $\mathcal{C}$, and has the contact element (E) (p_1, …, p_n ; q_1, …, q_n) that enters into the integral $H_{\mathcal{C}}$.

If one refers to no. 10 then one sees that each of these multiplicities must be concave towards its origin at (E).

Therefore, *in order for the integral H to have a maximum for a system of functions* q_1, …, q_n *when it is taken along an arc of the curve* $\mathcal{C}$ *that satisfies the Monge equations* (11), *it is necessary and sufficient that the following conditions be satisfied:*

1. *The positive direction of the tangent at each point* (x_1, …, x_n) *of* $\mathcal{C}$ *pierces the wave multiplicity that has that point for its origin at a point to which there corresponds a contact element whose plane has the equation* $\sum_{i=1}^{n} q_i (X_i - x_i) = 1$.

2. *At that contact element, the wave multiplicity is concave towards its origin, in the sense that the point of that element is on the same side as the origin with respect to the planes of the neighboring contact elements.*

31. This condition will be satisfied for $\mathcal{C}$ if it is satisfied for $\mathcal{E}$ *in the domain* of the contact elements of the wave multiplicities, which results from the preceding construction when it is applied to $\mathcal{E}$ because then one may vary the curve infinitesimally without which the condition ceasing to be realized. It is always intended that the variation alters the curves and tangents infinitely little.

We thus arrive at the following conclusion:

An arc of the extremal in problem A *furnishes a minimum for the integral J* (*in the case where the extremities of the arc of integration remain fixed*) *if it satisfies the following conditions:*

1. *At least one of the wave families that is cut transversally by this extremal fills up a portion of the space that surrounds all sides of the arc considered in a regular manner* ([1]) *and satisfies, in addition, the following criterion:*

2. *Letting M be an arbitrary point of the extremal arc,* Π, *the tangent plane at the point M to the wave considered that passes through it; the positive direction of the tangent to the extremal at M pierces the wave multiplicity that has M for its origin at a point P, and it results from transversality that this point P, along with a plane that is parallel to the plane* Π, *forms a contact element* (*E*) *of that wave multiplicity. In the domain of that contact element* (*E*) *the wave multiplicity must be concave towards its origin.*

The developments of nos. 10 and 11 give one the means to verify analytically whether this criterion is found to be verified in the extended case. One may further involve the function *G* uniquely then.

We remark that in the case of the minimum that we just explained, the elementary waves that issue from the points of a wave of the family considered are convex towards their envelope, which constitutes the consecutive wave of that same family. The word "envelope" here thus has its etymological sense here, and, in a way, its physical one, too, and the picture that emerges is completely in agreement with the idea itself of propagation, that an arbitrary wave must be the *front* of the elementary waves that one considers to have produced it when one applies the principle of enveloping waves.

32. In the preceding remark, we have substituted the consideration of the elementary wave for that of the wave multiplicity. Since these two multiplicities are homothetic with respect to their common origin, this substitution is legitimate because they are, at the

([1]) This must say that one and only one surface of the family passes through each point.

same time, either convex or concave. It seems that the elementary wave makes the best image to envision, because it is more immediately associated with the differential element of the integral considered.

This is especially true in the case where the differential element $\omega\,du$ is susceptible to a change of sign on the arc of the curve considered, a case about which it remains for us to say a few words in conclusion.

For that, we refer to the transformations that were employed in nos. 20 and 21. Formulas (76) and (81) of no. 21, when applied simultaneously to two neighboring contact elements, give:

$$\sum_{i=1}^{n} \bar{p}_i \bar{q}_i' - 1 = \sum_{i=1}^{n} \frac{p_i (q_i' + H_i)}{1 + \sum_{j=1}^{n} H_j p_j} - 1 = \frac{\sum_{i=1}^{n} p_i q_i' + 1}{1 + \sum_{j=1}^{n} H_j p_j}. \tag{22}$$

Upon further taking into account formula (75) of no. 21, what remains is:

$$\bar{\omega}\left(\sum_{i=1}^{n} \bar{p}_i \bar{q}_i' - 1\right) = \bar{\omega}\left(\sum_{i=1}^{n} p_i q_i' - 1\right). \tag{23}$$

If the sufficient condition for the minimum is satisfied for the transformed integral with a positive differential element $\int \bar{\omega}\,du$ then the left-hand side of that identity (33) is negative. The same is then true for the right-hand side.

As a consequence, the second sufficient condition for the minimum translates into concavity or convexity of the wave multiplicity according to whether ω is positive or negative, respectively. However, the point that enters in here is, moreover, on the prositive or negative direction of the tangent according to whether ω is positive or negative, respectively. One may say that there is convexity towards the positive direction of the tangent.

It seems simplest to restrict oneself to the elementary wave:

$$\begin{cases} F(x_1, \cdots, x_n \mid X_1, \cdots, X_n) = \omega\,du, \\ F_h(x_1, \cdots, x_n \mid X_1, \cdots, X_n) = 0 \qquad (h = 1, 2, \cdots, \alpha), \end{cases} \tag{24}$$

for which the point to consider is always (in the system of coordinates that has the origin (x_1, .., x_n) of that wave for its origin):

$$X_i = dx_i \qquad (i = 1, 2, \ldots, n). \tag{25}$$

The values of $q_1, \ldots, q_n$ that one must associate at that point are always defined by the canonical solution considered, or furthermore, by the family of surfaces that is defined by equation (14), which corresponds to solutions of the Jacobi-Hamilton equation (12).

However, a fact that is worthy of note presents itself here when one applies the construction of no. 27 to find one of the solutions that corresponds to the extremal considered. When one passes a point of the extremal for which $\omega\, du$ is annulled upon changing sign, the value of t ceases to increase (for example) in order to begin decreasing. Past such a point N, it then repeats the values that it took on before. The surfaces that are obtained no longer cut the extremal arc at each of its points. At N, the corresponding surface is tangent to the arc of the extremal, since $\sum_{i=1}^{n} q_i dx_i$ is then null, and in the neighborhood of that particular surface the ones that cut the extremal cut it at two points: one on each side of N.

33. At such a point N, one does not need to preserve the latter side of the equation (= dt) in the canonical equations (4) for the definition itself of the extremal. Above all, as we remarked in no. 22, this canonical system is useless at that point in the form (4) because the derivatives of G will be, in general, infinite. However, this difficulty will disappear if one introduces the Jacobi-Hamilton equation in a less restrictive form than the form (12), which we have specialized only insofar as the theory is concerned.

Indeed, recall that this equation in the original form (2), defines the tangential support of the wave multiplicity, and, from the argument in no. 21, this support is obtained by eliminating the ratios of $X_1, \ldots, X_n$; $\lambda_1, \ldots, \lambda_n$ between the equations ([1]):

$$q_i = \frac{\partial F(x \mid X)}{\partial X_i} + \sum_{h=1}^{\alpha} \lambda_h \frac{\partial F_h(x \mid X)}{\partial X_i} \qquad (i = 1, 2, \ldots, n) \tag{26}$$

and the equations of condition:

$$0 = F_h(x \mid X) \equiv F_h(x_1, \ldots, x_n \mid X_1, \ldots, X_n) \qquad (h = 1, 2, \ldots, \alpha). \tag{27}$$

If the result of that elimination is obtained in an arbitrary form:

$$G(x_1, \ldots, x_n \mid q_1, \ldots, q_n) = 0, \tag{28}$$

then dG_0 and dG will be linked by an identity of the form:

$$dG \equiv M(x_1, \ldots, x_n \mid q_1, \ldots, q_n)\, dG_0 \tag{29}$$

for all of the elements $(x_1, \ldots, x_n \mid q_1, \ldots, q_n)$ that satisfy equation (2) or equation (28), which are assumed to be equivalent. The canonical system (4) and the equation (2) will then be replaced by equation (18), and the most general canonical system:

$$\frac{dx_1}{\dfrac{\partial G_0}{\partial q_1}} = \ldots = \frac{dx_n}{\dfrac{\partial G_0}{\partial q_n}} = \frac{dq_1}{-\dfrac{\partial G_0}{\partial x_1}} = \ldots = \frac{dq_n}{-\dfrac{\partial G_0}{\partial x_n}}. \tag{30}$$

([1]) The letters X_i denote the differentials dx_i here, instead of the derivatives dx_i / du; however, this changes nothing in the reasoning of no. 21, which, for brevity, we shall not repeat.

The canonical solutions will then be determined, and t will always depend upon the quadrature of the differential:

(31) $$dt = \sum_{i=1}^{n} q_i dx_i \,.$$

That differential, from equations (30), is annulled at the same time as the expression:

(32) $$D = \sum_{i=1}^{n} q_i \frac{\partial G_0}{\partial q_i},$$

and that expression intervenes precisely when one seeks to deduce G from G_0, since it amounts to solving the equation:

(33) $$G_0\left(x_1, \cdots, x_n \left| \frac{q_1}{G}, \cdots, \frac{q_n}{G} \right.\right) = 0$$

for G.

If one takes into account the fact that G must have the value 1 on the multiplicity considered then one deduces from that equation, by total differentiation:

(34) $$dG_0 = \left(\sum_{i=1}^{n} q_i \frac{\partial G_0}{\partial q_i} \right) dG\,,$$

which shows that D is the inverse of the coefficient M of the formula (29). This coefficient M must then be infinite for the exceptional contact elements that define the object of our discussion ([1]).

From another point of view, these elements are found to be exceptional under the same conditions as the elements that were omitted in no. 1, because the planes that figure in them pass through the origin of the coordinates (the origin of the elementary wave). However, the difficulty disappears here because the contact elements of the elementary wave have coordinates ($x_1, \ldots, x_n$; $q_1, \ldots, q_n$), which, from formulas (26) and (24), are now linked by the relation:

(35) $$\sum_{i=1}^{n} q_i X_i = \omega\, du,$$

([1]) It is easy to study these peculiarities in the examples. As a very simple case, one may take the following one: $\omega\, du = \sqrt{dx^2 + dy^2} - 2y\, dy$. One considers the extremal $y = x$ and the neighboring extremals $y = x + c$, which gives the family of parabolic waves $\frac{x+y}{\sqrt{2}} - y^2 = \text{const}$. The elementary waves are then conics that are simple to discuss.

By the process in no. 20, the integral $\int \omega\, du$ comes down to $\int \bar{\omega}\, du$, where $\bar{\omega}\; du = \sqrt{dx^2 + dy^2}$.

The exceptional point N of the extremal is $x = y = \sqrt{2}/2$, here.

which is equivalent to establishing the homogeneity of the coordinates $q_1, \ldots, q_n$, which we refrained from doing in no.1, in order to better indicate the duality between the two viewpoints – viz., point-like and tangential.

In summation: *The extremals of the Lagrange problem are given, without imposing the condition on the sign of the differential element of integration* (1) (no. 12), *by the solutions of the canonical system* (30) *that satisfy equation* (28). *It is the tangential equation of the elementary wave* (24), *where the coordinates* $(X_1, \ldots, X_n ; q_1, \ldots, q_n)$ *of a contact element of that wave are linked by the condition* (35). *An arc of the extremal thus obtained furnishes a minimum for the problem considered if the following* (sufficient) *conditions are satisfied:*

1. *By means of that extremal and* ∞^{n-1} *conveniently chosen neighboring extremals, construct a family of surfaces* (family of waves) *by the procedure of no.* 27 *that fills up, in a regular manner, a portion of the space that contains it and surrounds the arc of the extremal considered on all sides.*

2. *The elementary wave that has each point* $(x_1, \ldots, x_n)$ *of that arc for its origin and passes through the infinitely close consecutive point of that arc is concave or convex towards its origin* [*in the domain of the contact element of that elementary wave that contains that point and whose plane has the quantities* $(q_1, \ldots, q_n)$ *that are associated with* $(x_1, \ldots, x_n)$ *for its direction coefficients in the solution of the system* (28), (30) *that one considers*] *according to whether the differential element* $F(x_1, \ldots, x_n \mid dx_1, \ldots, dx_n) \equiv \omega\, du$ *is positive or negative, respectively.*
