"Sur les équations différentielles de la dynamique réduites au plus petit nombre possible de variables," J. de math. pures et appl. **14** (1849), 201-224.

On the differential equations of dynamics, reduced to the smallest possible number of variables

By JULES VIELLE

Translated by D. H. Delphenich

Let $L = 0$, $M = 0$, $N = 0$, etc., be the condition equations that exist between the coordinates x, y, z, x', etc., of the various points of a system in motion, which are given by the nature of each problem. Let n denote the number of material points, and let i denote the number of those condition equations. One can infer the values of i of the coordinates as functions of the other $3n - i$, or more generally, one can imagine that the $3n$ coordinates are replaced by known functions of the $3n - i$ new variables θ, φ, ψ, etc., and one knows that the general formula of dynamics:

$$\sum\left[\left(X - m\frac{d^2x}{dt^2}\right)\delta x + \left(Y - m\frac{d^2y}{dt^2}\right)\delta y + \left(Z - m\frac{d^2z}{dt^2}\right)\delta z\right] = 0 \tag{1}$$

will split into $3n - i$ distinct equations under that transformation, which will suffice to determine the values of θ, φ, ψ, etc., as functions of time.

In the second part of *Mécanique analytique*, Lagrange gave a remarkable form to those differential equations. What characterizes that form and makes it eminently appropriate to the solution of most problems is that the differential terms that it includes are provided by just one function T, which is nothing but one-half the sum of the *vis vivas* of all of the points of the system.

Meanwhile, those equations, however useful, are omitted from the most widely-used treatises on mechanics.

Lagrange's proof is not complete. Indeed, it supposes that the expression for each *differential* dx as a function of the new variables θ, φ, ψ, etc., differs from the corresponding *variation* δx by only the change of d into δ. Now, that will be quite true when the function of the variables θ, φ, ψ, etc., that x represents does not contain time t explicitly. However, it will no longer be true when x is an explicit function of t and θ, φ, ψ, etc., say:

$$x = f(t, \theta, \varphi, \psi, \ldots) .$$

The differential dx will then include the term $(df / dt)\, dt$, which has no correspondent in the variation δx, since one must keep t constant when one makes that variation.

That is why we believe that we would be doing something useful and instructive by representing Lagrange's analysis and giving it all of the desirable rigor. At the same time, we will simplify the algorithm, and then apply those equations to various problems in dynamics that will then produce very simple solutions.

Therefore, let θ, φ, ψ, etc., be the variables, when reduced to the smallest-possible number, while x, y, z, x', etc., are known functions that are deduced from the condition equations:

$$L = 0, \quad M = 0, \; N = 0 .$$

Since the latter can contain time t explicitly, that variable will generally enter into the functions that x, y, z, x', etc., represent. Set:

$$\frac{d\theta}{dt} = \theta', \qquad \frac{d\varphi}{dt} = \varphi', \qquad \frac{d\psi}{dt} = \psi', \qquad \dots$$

One will have expressions for $\frac{dx}{dt}$, $\frac{dy}{dt}$, $\frac{dz}{dt}$, δx, δy, δz of the form:

$$(2) \qquad \left\{ \begin{aligned} \frac{dx}{dt} &= \alpha + a\,\theta' + a_1\,\varphi' + \cdots, \quad & \delta x &= a\,\delta\theta + a_1\,\delta\varphi + \cdots, \\ \frac{dy}{dt} &= \beta + b\,\theta' + b_1\,\varphi' + \cdots, \quad & \delta y &= b\,\delta\theta + b_1\,\delta\varphi + \cdots, \\ \frac{dz}{dt} &= \alpha + a\,\theta' + a_1\,\varphi' + \cdots, \quad & \delta z &= c\,\delta\theta + c_1\,\delta\varphi + \cdots, \end{aligned} \right.$$

in which α, β, γ represent the partial derivatives of x, y, z with respect to t. Those quantities, as well as a, b, c, a_1, b_1, c_1, etc., are known functions of t, θ, φ, ψ, etc.

Having said that, before substituting the values of $x, y, z, \frac{dx}{dt}, \dots, \delta x$, etc., in equation (1), one transforms the trinomial:

$$\left(\frac{d^2x}{dt^2}\,\delta x + \frac{d^2y}{dt^2}\,\delta y + \frac{d^2z}{dt^2}\,\delta z \right)$$

into two groups that contain first-order differentials and applies the symbol d to one of them, while applying the symbol δ to the other. To that effect, one will have:

$$\frac{d^2x}{dt^2}\,\delta x + \frac{d^2y}{dt^2}\,\delta y + \frac{d^2z}{dt^2}\,\delta z = \frac{d}{dt}\left(\frac{dx}{dt}\,\delta x + \frac{dy}{dt}\,\delta y + \frac{dz}{dt}\,\delta z \right) - \tfrac{1}{2}\,\delta\left(\frac{d^2x}{dt^2} + \frac{d^2y}{dt^2} + \frac{d^2z}{dt^2} \right) ,$$

in such a way that equation (1) will take the form:

$$(3)\quad \sum m\left[\frac{d}{dt}\left(\frac{dx}{dt}\,\delta x+\frac{dy}{dt}\,\delta y+\frac{dz}{dt}\,\delta z\right)-\tfrac{1}{2}\,\delta\left(\frac{d^2x}{dt^2}+\frac{d^2y}{dt^2}+\frac{d^2z}{dt^2}\right)\right]-\sum(X\,\delta x+Y\,\delta y+Z\,\delta z)=0\,.$$

We shall now calculate the two quantities:

$$\left(\frac{dx}{dt}\,\delta x+\frac{dy}{dt}\,\delta y+\frac{dz}{dt}\,\delta z\right),\qquad \tfrac{1}{2}\left(\frac{d^2x}{dt^2}+\frac{d^2y}{dt^2}+\frac{d^2z}{dt^2}\right)$$

separately. However, since one must finally equate the set of terms that multiply each of the independent variations $\delta\theta$, $\delta\varphi$, etc., to zero, and those variables obviously enter in the same manner into the calculations, moreover, we shall address only the composition of the coefficient of $\delta\theta$.

One will then have:

$$\frac{dx}{dt}\,\delta x+\frac{dy}{dt}\,\delta y+\frac{dz}{dt}\,\delta z=(H+P\,\theta'+Q\,\varphi'+\cdots)\,\delta\theta+\cdots,$$

$$\tfrac{1}{2}\left(\frac{d^2x}{dt^2}+\frac{d^2y}{dt^2}+\frac{d^2z}{dt^2}\right)=G+H\,\theta'+\tfrac{1}{2}P\,\theta'^2+Q\,\theta'\,\varphi'+\cdots$$

upon setting:

$$\begin{aligned}\alpha^2+\beta^2+\gamma^2&=G\,,\\ \alpha\,a+\beta\,b+\gamma\,c&=H\,,\\ a^2+b^2+c^2&=P\,,\\ \alpha\,a_1+\beta\,b_1+\gamma\,c_1&=Q\,,\\ \cdots\cdots\cdots&\cdots\end{aligned}$$

If one differentiates the first expression by d and the second one by δ then one will have (upon observing that $\dfrac{d\cdot\delta\theta}{dt}=\delta\cdot\dfrac{d\theta}{dt}=\delta\cdot\theta'$):

$$\frac{d}{dt}\left(\frac{dx}{dt}\,\delta x+\frac{dy}{dt}\,\delta y+\frac{dz}{dt}\,\delta z\right)=\delta\theta\times\frac{d}{dt}(H+P\,\theta'+Q\,\varphi'+\cdots)+(H+P\,\theta'+Q\,\varphi'+\cdots)\,\delta\theta'+\cdots,$$

$$\tfrac{1}{2}\,\delta\left(\frac{d^2x}{dt^2}+\frac{d^2y}{dt^2}+\frac{d^2z}{dt^2}\right)$$
$$=\left(\frac{dG}{d\theta}+\theta'\,\frac{dH}{d\theta}+\tfrac{1}{2}\,\theta'^2\,\frac{dP}{d\theta}+\theta'\,\varphi'\,\frac{dQ}{d\theta}+\cdots\right)\delta\theta+(H+P\,\theta'+Q\,\varphi'+\cdots)\,\delta\theta'+\cdots$$

If one substitutes those values in equation (3) then *the terms that are multiplied by* $\delta\theta'$ *will disappear by themselves*, and it is that reduction that accounts for the success of the method. All that will remain for the coefficient of the variation $\delta\theta$ is:

(4) $$\sum m\left[\frac{d}{dt}(H + P\,\theta' + Q\,\varphi' + \cdots) - \left(\frac{dG}{d\theta} + \theta'\,\frac{dH}{d\theta} + \tfrac{1}{2}\,\theta'^2\,\frac{dP}{d\theta} + \theta'\,\varphi'\,\frac{dQ}{d\theta} + \cdots\right)\right] .$$

If one now lets T be one-half the sum of the *vis vivas* of the system:

$$T = \tfrac{1}{2}\sum m\left[\left(\frac{dr'}{dt}\right)^2 + \left(r'\,\frac{d\psi}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right] = \sum m\left(G + H\,\theta' + \tfrac{1}{2}\,P\,\theta'^2 + Q\,\theta'\,\varphi' + \cdots\right)$$

then one will see as a result that the expression (4) is nothing but:

$$\frac{d}{dt}\left(\frac{dT}{d\theta'}\right) - \frac{dT}{d\theta} .$$

As for the part of the left-hand side of equation (3) that includes the applied forces (X, Y, Z), namely:

$$\sum(X\,\delta x + Y\,\delta y + Z\,\delta z) ,$$

its transformation as a function of the variables θ, φ, ψ, etc., can be performed by a simple substitution, and one will then know the coefficient of $\delta\theta$ that this implies in each particular case. In the usual applications in dynamics, there exists a *force function*, i.e., one has:

$$\sum(X\,dx + Y\,dy + Z\,dz) = dV ,$$

V is a function of θ, φ, ψ, etc., and as a result:

$$\sum(X\,\delta x + Y\,\delta y + Z\,\delta z) = \delta V = \frac{dV}{d\theta}\,\delta\theta + \cdots$$

The complete coefficient of $\delta\,\theta$ in equation (3) will then be:

$$\frac{d}{dt}\left(\frac{dT}{d\theta'}\right) - \frac{dT}{d\theta} - \frac{dV}{d\theta} .$$

By definition, equation (3) will split into the following ones:

$$
(5) \qquad \begin{cases}
\dfrac{d}{dt}\left(\dfrac{dT}{d\theta'}\right) - \dfrac{dT}{d\theta} - \dfrac{dV}{d\theta} = 0, \\[2ex]
\dfrac{d}{dt}\left(\dfrac{dT}{d\varphi'}\right) - \dfrac{dT}{d\varphi} - \dfrac{dV}{d\varphi} = 0, \\[2ex]
\dfrac{d}{dt}\left(\dfrac{dT}{d\psi'}\right) - \dfrac{dT}{d\psi} - \dfrac{dV}{d\psi} = 0 \cdots
\end{cases}
$$

There are many of those equations as variables θ, φ, ψ, etc., upon which the position of the system depends at each instant.

When the function T does not include t explicitly, one will immediately obtain a first integral of equations (5) by multiplying them by $d\theta$, $d\varphi$, $d\psi$, respectively, and upon adding the products, that integral:

$$
(6) \qquad T - V = \text{constant}
$$

will imply the principle of *vis viva*.

Liouville made a remarkable application of those equations (*Journal of Mathématiques*, t. XI) to various cases in which the equations of motion of a material point can be integrated: Notably, in the problem of the motion of a point that is attracted to two fixed centers, in which Euler arrived at a separation of the variables only by a very laborious calculation, one finds that it can be reduced to quadratures with a high degree of simplicity. The same method will also solve the case that Lagrange imagined of a third fixed center of attraction that is placed at the midpoint of the line that joins the other two with no further pain.

In order to show the ease that those equations contribute to the solution of problems, we shall present some applications. We shall first treat two problems that were posed in the last few years for the *Concours d'agrégation*.

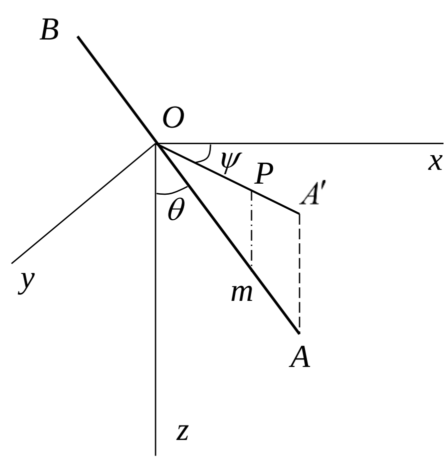

Figure 1.

Problem I: *Determine the motion of a massive line that is free to turn in space around one of its points, which is supposed to be fixed.*

Take the origin to be the fixed point O and take the z-axis to be vertical in the direction of gravity. Here, the variables that determine the position of the system at each instant are two in number, namely, the angle $AOz = \theta$, that the rod AB makes with the z-axis, and the angle $A'Ox = \psi$ that its horizontal projection makes with the x-axis. Let M be the mass of the rod, which we do not suppose to be homogeneous, while a is the distance from its center of gravity to the point O, r is the radius vector Om to an arbitrary point m, and r' is the projection OP of r. One first seeks the functions T and V.

One immediately has:

$$V = M\,a\,g\cos\theta .$$

For the function T, one will find:

$$T = \tfrac{1}{2}\sum m\left[\left(\frac{dr'}{dt}\right)^2 + \left(r'\frac{d\psi}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right] = \tfrac{1}{2}\sum m\,r^2\,(\theta'^2 + \sin^2\theta\cdot\psi'^2)$$
$$= \tfrac{1}{2}(\theta'^2 + \sin^2\theta\cdot\psi'^2)\,M\,(a^2 + K^2)\ ,$$

in succession. $M\,K^2$ denotes the moment of inertia of the rod with respect to an axis that is perpendicular to its direction and passes through the center of gravity.

Having said that, equation (6) will first give:

$$\theta'^2 + \sin^2\theta\cdot\psi'^2 = C + \frac{2g}{a + K^2/a}\cos\theta\ , \tag{7}$$

in which C is an arbitrary constant.

It will then suffice to associate that equation with one of equations (5): Since T and V do not contain the variable φ, one prefers to choose the following one:

$$\frac{d}{dt}\left(\frac{dT}{d\psi'}\right) - \frac{dT}{d\psi} - \frac{dV}{d\psi} = 0\ ,$$

which reduces to:

$$\frac{d}{dt}\left(\frac{dT}{d\psi'}\right) = 0\ ,$$

so

$$\frac{dT}{d\psi'} = \text{constant},$$

or rather:

$$\sin^2\theta\cdot\psi' = C'\ , \tag{8}$$

in which C' is a second arbitrary constant.

Equations (7) and (8) solve the proposed problem.

If the heavy rod is reduced to a material point that is at a distance of l from the point O then the equations of its motion will be deduced from the preceding two by setting:

$$K = 0 \quad \text{and} \qquad a = l\,.$$

It follows from this that the heavy rod moves around the point O like a simple pendulum of length:

$$l = a + \frac{K^2}{a}\,.$$

The problem that we address is then reduced to the conical pendulum.

One infers the values of dt and $d\psi$ as functions of θ from equations (7) and (8):

$$dt = \frac{\sin\theta\, d\theta}{\sqrt{\left(C + \frac{2g}{l}\cos\theta\right)\sin^2\theta - C'^2}}\,, \quad dy = \frac{C'\, d\theta}{\sin\theta\sqrt{\left(C + \frac{2g}{l}\cos\theta\right)\sin^2\theta - C'^2}}\,.$$

Those formulas are reducible to elliptic functions, and consequently can by integrated only by approximation. When the rod is homogeneous, one will easily get the length of the corresponding simple pendulum, because if L denotes the length of the rod, and one supposes, for simplicity, that the fixed point is one of the extremities then one will find that:

$$l = a + \frac{K^2}{a} = \frac{\int r^2 dr}{a\,L} = \tfrac{2}{3}L\,.$$

Therefore, the length of the simple pendulum is 2/3 the length of the rod.

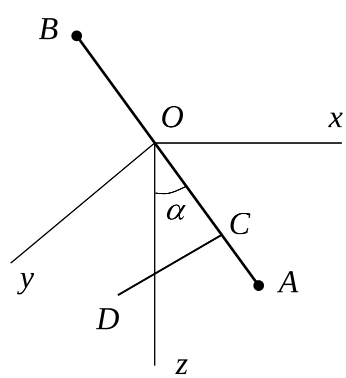

Figure 2.

The constants C and C' are determined from the initial data. Let α be the angle *AOz* that the rod makes with the vertical when the rod is in its initial position. *CD* is the direction of the impact that must be given to it, and one can suppose that it is perpendicular to the rod. The rod will then begin to turn in the plane *OCD*, because there is no reason for the motion to begin on one side of the plane, rather than the other: Furthermore, one can consider that plane to be the plane of the two principle axes relative to the point *O*. One will then apply the formula for the angular velocity of rotation, since there will be a fixed axis that is perpendicular to the plane *OCD* at *O*. Let ω be that angular velocity, let μv be the intensity of the impact, and f is the distance *OC*, so one will have:

$$\omega = \frac{\mu v f}{M\,(a^2 + K^2)} .$$

Now, the initial velocity v of an arbitrary point M of the rod is $r\ \omega$. Therefore, if one replaces the left-hand side of equation (7), which is nothing but v^2 / r^2, with the value above for ω^2 then one will have the equation:

$$\omega^2 = c + \frac{2g}{l}\cos\alpha \qquad \left(l = a + \frac{k^2}{a}\right), \tag{9}$$

which will determine the constant c.

As for the constant c', from equation (8), one has:

$$c' = \sin^2\alpha \left(\frac{d\psi}{dt}\right)_0 .$$

Now, $\sin\alpha\left(\frac{d\psi}{dt}\right)_0$ represents the initial velocity of the horizontal projection of the point on the rod whose distance from the fixed point is unity. Upon denoting the angle between the direction *CD* and a perpendicular to the plane *zOA* by ε, one will have:

$$\sin\alpha\left(\frac{d\psi}{dt}\right)_0 = \omega\cos\varepsilon .$$

It will then result that:

$$c' = \omega\cos\varepsilon\sin\alpha . \tag{10}$$

We shall say nothing about the case in which the rod deviates very little from the vertical. One then neglects the very small quantities of order three in θ and α, and the values of θ and ψ will be obtained as functions of time in finite form by the formulas for the conical pendulum.

What conditions must the initial impact fulfill *in order for the rod to describe a right cone* around the vertical?

In that case, one will always have:

$$\theta = \alpha ,$$

and consequently:

$$\frac{d\theta}{dt} = 0 .$$

From the preceding value of $d\theta / dt$, one then concludes that:

$$\sin^2\alpha\left(c + \frac{2g}{l}\cos\alpha\right) - c'^2 = 0 .$$

However, that equation expresses only the idea that $d\theta / dt$ is zero for $\theta = \alpha$, i.e., with the origin of motion: It must be combined with the condition that $d\,\frac{d\theta}{dt} = 0$, which will give:

$$\frac{g}{\cos\alpha} = \frac{c'^2}{\sin^2\alpha} .$$

It remains for us to substitute the values of the constants c and c' that are inferred from equations (9) and (10) in those two equations. The first one will then reduce to:

$$\cos\varepsilon = 1 , \qquad \text{so} \qquad \varepsilon = 0 ,$$

and the second one will reduce to:

$$\omega^2 = \frac{g\sin^2\alpha}{l\cos\alpha} .$$

$\varepsilon = 0$ means that the impact must be directed perpendicular to the vertical plane that contains the rod. The equation:

$$\omega^2 = \frac{g\sin^2\alpha}{l\cos\alpha}$$

will permit us to assign the magnitude of impact that corresponds to a given angle α. Conversely, if the impact is given then one can conclude the angle α by solving the equation:

$$\cos^2\alpha + \frac{\omega^2 l}{g}\cos\alpha - 1 = 0 ,$$

which will always admit *one* real value for $\cos\alpha$ that is much smaller than 1. Finally, equation (8) shows that the right cone will describe a uniform motion. The velocity of that motion will be:

$$\frac{d\psi}{dt} = \frac{\omega}{\sin\alpha}.$$

Let k denote that velocity and replace ω with its previous value. We will then have:

$$k = \sqrt{\frac{g}{l\cos\alpha}}.$$

One sees that once the angle α is given, that velocity k will be determined independently of the impact. No matter what that velocity is, it will have a minimum value $\sqrt{g/l}$, below which k cannot fall. However, that limit will never be attained, because that would require α to become zero. The rod would then be vertical, and since ω would also be zero, there would be no impact. The rod would then be at rest.

In order to complete the solution to the problem, it remains for us to calculate the variable pressure P that supports the fixed point. That study leads to some resultants that are worth pointing out. As one knows, the method consists of imagining that one applies a force to O that is equal and opposite to the pressure. One can regard the rod as becoming free then and apply the six equations of equilibrium of a solid body. Let X_1, Y_1, Z_1 be the components of the pressure P, and let x_1, y_1, z_1 be the coordinates of the center of gravity of the rod. The lost forces are easily expressed as functions of x_1, y_1, z_1, and the sum of their components parallel to each axis are:

$$-M\frac{d^2x}{dt^2}, \qquad -M\frac{d^2y}{dt^2}, \qquad -M\left(\frac{d^2z}{dt^2} - g\right).$$

From that, the first three equations of equilibrium, which include only the components X_1, Y_1, Z_1 of the pressure, will be:

$$(11)\qquad \begin{cases} X_1 + M\dfrac{d^2x_1}{dx} = 0, \\[2ex] Y_1 + M\dfrac{d^2y_1}{dx} = 0, \\[2ex] Z_1 + M\left(\dfrac{d^2z_1}{dx} - g\right) = 0. \end{cases}$$

Moreover, one will have:

$$x_1 = a\sin\theta\cos\psi, \qquad y_1 = a\sin\theta\sin\psi, \qquad z_1 = a\cos\theta.$$

Those equations will yield the three components of the pressure as functions of the variables θ and ψ, which are themselves known functions of t, from the preceding formulas. However, the direct calculation of the values of X_1, Y_1, Z_1 will become quite complicated. One can simplify those calculations immensely by appealing to the three moment equations. Those equations, into which we introduce the coordinates x_1, y_1, z_1, are:

$$(12) \qquad \begin{cases} x_1 \dfrac{d^2 y_1}{dx} - y_1 \dfrac{d^2 x_1}{dx} = 0, \\ y_1 \dfrac{d^2 z_1}{dx} - z_1 \dfrac{d^2 y_1}{dx} = \dfrac{g\,a\,y_1}{l}, \\ z_1 \dfrac{d^2 x_1}{dx} - x_1 \dfrac{d^2 z_1}{dx} = -\dfrac{g\,a\,x_1}{l}. \end{cases}$$

They will reduce to two distinct ones.

When equations (11) and (12) are combined, one will deduce with no difficulty that:

$$Y_1\,x_1 - X_1\,y_1 = 0 \qquad \text{and} \qquad Z_1\,y_1 - Y_1\,z_1 = M\,g\,y_1\left(1 - \frac{a}{l}\right).$$

Those equations exhibit two properties of the pressure:

1. The components X_1, Y_1 are proportional to x_1, y_1. That implies that *the pressure is always included in the vertical plane that contains the rod*.

2. The components Z_1, Y_1 are not proportional to the coordinates z_1, y_1 (since l or $a + k^2 / a$ is essentially different from a). It will then follow that *the pressure does not point along the rod.*

In addition, the preceding two equations will permit one to express X_1 and Y_1 as functions Z_1 *without differential coefficients:*

$$Y_1 = \left[Z_1 - M\,g\left(1 - \frac{a}{l}\right)\right]\frac{y_1}{z_1}, \quad X_1 = \left[Z_1 - M\,g\left(1 - \frac{a}{l}\right)\right]\frac{x_1}{z_1}.$$

One then concludes that:

$$P^2 = X_1^2 + Y_1^2 + Z_1^2 = Z_1^2 + \left[Z_1 - M\,g\left(1 - \frac{a}{l}\right)\right]^2 \frac{\sin^2\theta}{\cos^2\theta}.$$

Everything then comes down to determining Z_1 as a function of θ. To that effect, equation (11) will give:

$$Z_1 = M\left[g + a\cos\theta\cdot\left(\frac{d\theta}{dt}\right)^2 + a\sin\theta\,\frac{d^2\theta}{dt^2}\right].$$

One already knows $\frac{d\theta}{dt}$ as a function of θ. One can then infer $\frac{d^2\theta}{dt^2}$, and when one makes that substitution, that will give:

$$Z_1 = M\left[g\left(1-\frac{a}{l}\right)+a\,c\cos\theta+\frac{3a\,g}{l}\cos^2\theta\right].$$

As a result:

$$P^2 = M^2\left[g^2\left(1-\frac{a}{l}\right)^2+a^2c^2+2\,a\,g\,c\left(1+\frac{2a}{l}\right)\cos\theta+\frac{3a\,g}{l}\left(2+\frac{a}{l}\right)\cos^2\theta\right].$$

The inclination γ of the pressure on the vertical will be likewise known as a function of θ, since:

$$\cos\gamma = \frac{Z}{P}.$$

The magnitude and direction of the pressure depend upon only the variable θ explicitly, and not on ψ. When the angle θ is constant, i.e., in the case where the rod describes a right cone, the pressure will be constant, and its direction will likewise describe a right cone around the vertical. The general value of P in that case reduces to:

$$P = M\,g\sqrt{1+\frac{a^2}{l^2}\tan^2\alpha}.$$

One has:

$$Z_1 = M\,g,$$

moreover, so:

$$\cos\gamma = \frac{1}{\sqrt{1+\frac{a^2}{l^2}\tan^2\alpha}},$$

or rather:

$$\tan\gamma = \frac{a}{l}\tan\alpha.$$

Since a / l is < 1, the angle at the center of right cone that the pressure describes will be smaller than that of the right cone that the rod describes.

The latter formulas, which relate to the case of a right cone, will be obtained directly and in a much simpler manner when one remarks that the magnitudes and directions of the *effective forces* are then known immediately, since each point m describes a circle whose radius ρ is r sin α with a velocity of:

$$v = r\,\omega = r \sin\alpha \sqrt{\frac{g}{l\cos\alpha}}\ .$$

The effective force $m v^2 / \rho$ will then have the value $(m\, r\, g / l) \tan\alpha$, and since its direction does not vary from one point to another, the resultant of the effective forces is equal to their sum:

$$\sum \frac{mv^2}{\rho} = M\, a \frac{g}{l} \tan\alpha\ .$$

When that resultant is taken in the opposite sense and composed with weight $M\, g$, it will give the values for the pressure P that was found before.

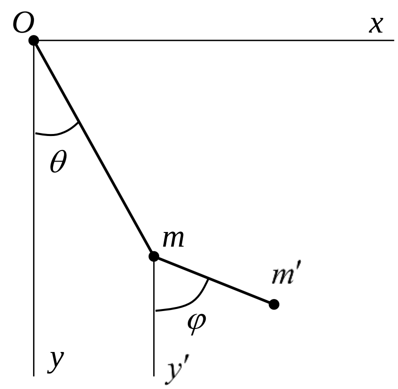

Figure 3.

Problem II: *Determine the motion of a flexible and inextensible string that is suspended from a fixed point O and loaded at two material points with weights m, m'. One supposes that at the origin of the motion, the two material points depart from the vertical without leaving a vertical plane that passes through the fixed point and are then left to themselves with no initial velocity.*

The oscillatory motion of each point will obviously take place in the vertical plane yOx that passes through the initial position of the string. Let $Om = a$, $mm' = b$. The two unknowns of the problem are the angles $mOy = \theta$, $m' m\, y' = \varphi$ that the two portions of the string make with the vertical.

One has:

$$x = a \sin\theta,$$
$$y = a \cos\theta,$$

for the point m, and:

$$x' = a \sin\theta + b \sin\varphi,$$
$$y' = a \cos\theta + b \cos\varphi,$$

for m'. One easily concludes then that:

$$V = (m+m')\,g\,a\cos\theta + m'g\,b\,\cos\varphi\,,$$
$$T = \tfrac{1}{2}\,[(m+m')\,a^2\cdot\theta'^2 + m'\,b^2\varphi'^2 + 2\,m'a\,b\cos(\varphi-\psi)\cdot\theta'\,\varphi']\;.$$

When those values are substituted in equations (5), that will produce the two equations of motion:

$$(13)\quad \begin{cases} (m+m')\,a\,\dfrac{d^2\theta}{dt^2} + m'\,b\cos(\varphi-\theta)\,\dfrac{d^2\varphi}{dt^2} - m'b\sin(\varphi-\theta)\,\dfrac{d\varphi}{dt}\left(\dfrac{d\varphi}{dt}-\dfrac{d\theta}{dt}\right) \\[2ex] \qquad - m'\,b\sin(\varphi-\theta)\left(\dfrac{d\theta}{dt}\right)^2\dfrac{d\varphi}{dt} + (m+m')g\sin\theta = 0\,, \\[2ex] b\,\dfrac{d^2\varphi}{dt^2} + a\cos(\varphi-\theta) - a\sin(\varphi-\theta)\,\dfrac{d\theta}{dt}\left(\dfrac{d\varphi}{dt}-\dfrac{d\theta}{dt}\right) + a\sin(\varphi-\theta)\,\dfrac{d\theta}{dt}\left(\dfrac{d\theta}{dt}\right)^2 + g\sin\varphi = 0. \end{cases}$$

One can replace one of them by the *equation of vis viva*, which is only first-order. However, we shall not do that, because the latter equation does not lend itself to the determination of the *small oscillations* as well, which we shall now address.

If the oscillations are assumed to be very small then one neglect the squares of θ, φ, $\dfrac{d\theta}{dt}$, $\dfrac{d\varphi}{dt}$, and the products of those variables. Equations (13) then reduce to the following two:

$$(m+m')\,a\,\frac{d^2\theta}{dt^2} + m'\,b\,\frac{d^2\varphi}{dt^2} + (m+m')\,g\,\theta\;=0\;,$$
$$a\,\frac{d^2\theta}{dt^2} + b\,\frac{d^2\varphi}{dt^2} + g\,\varphi\;=0\;.$$

The first one can be simplified further by taking the second one into account, and one finally has the system:

$$(14)\qquad \begin{cases} m\,a\,\dfrac{d^2\theta}{dt^2} + g\,(m+m')\,\theta - g\,m'\varphi = 0\,, \\[2ex] \qquad a\,\dfrac{d^2\theta}{dt^2} + b\,\dfrac{d^2\varphi}{dt^2} + g\,\varphi = 0. \end{cases}$$

Those equations are linear and have constant coefficients, whereas if one had employed the *vis viva* equation then one would have obtained an equation that is, in truth, first-order, but *nonlinear*, since it obviously includes the squares of $\dfrac{d\theta}{dt}$ and $\dfrac{d\varphi}{dt}$.

The ordinary method leads to the general integrals:

$$\theta \;=\; A_1\cos t\sqrt{r_1} + A_2\cos t\sqrt{r_2} + B_1\sin t\sqrt{r_1} + B_2\sin t\sqrt{r_2}\;,$$

$$\varphi = A_1\,\mu_1 \cos t\sqrt{r_1} + A_2\,\mu_2 \cos t\sqrt{r_2} + B_1\,\mu_1 \sin t\sqrt{r_1} + B_2\,\mu_2 \sin t\sqrt{r_2}\ .$$

A_1, A_2, B_1, B_2 are four arbitrary constants. r_1, r_2 are real and positive quantities that are roots of the equation of degree two in r :

(15) $$[(m+m')g - m\,a\,r](g - b\,r) - m'a\,g\,r = 0\ ,$$

and μ_1, μ_2 are corresponding values of μ that are inferred from the equation:

(16) $$\mu = \frac{a\,r}{g - b\,r}\ .$$

μ_1 and μ_2 have opposite signs because one of the roots of equation (15) is smaller than g / b, and the other one is greater than g / b since the hypothesis $r = g / b$ makes the left-hand side negative. Since one supposes that there are no initial velocities, one must have:

$$\frac{d\theta}{dt} = 0 \qquad \text{and} \qquad \frac{d\varphi}{dt} = 0$$

for $t = 0$, which implies that:

$$B_1 = 0 \qquad \text{and} \qquad B_2 = 0\ .$$

If one lets α and β be the initial values of θ and φ, resp., then one will have:

(17) $$A_1 + A_2 = \alpha\ , \qquad A_1\,\mu_1 + A_2\,\mu_2 = \beta\ ,$$

from which one can infer the values of the constants A_1 and A_2 . Having done that, the motion of the system will be determined by the formulas:

(18) $$\begin{cases} \theta = A_1 \cos t\sqrt{r_1} + A_2 \cos t\sqrt{r_2}\,, \\ \varphi = A_1\,\mu_1 \cos t\sqrt{r_1} + A_2\,\mu_2 \cos t\sqrt{r_2}\,, \end{cases}$$

In the case where the two material points are found at the origin of the motion along a straight line with the point of suspension, one will have:

$$\alpha = \beta\ ,$$

and as a result:

$$A_1 = \frac{\alpha\,(1-\mu_2)}{\mu_1 - \mu_2}\ , \qquad A_2 = \frac{\alpha\,(\mu_1 - 1)}{\mu_1 - \mu_2}\ .$$

Let $\mu_1 > 0$, so μ_2 will be < 0. In addition, it is easy to see that μ_1 will be > 1, in such a way that the values of the constants A_1, A_2 will be positive. If one demands that this condition must be fulfilled for *each of the oscillating points m and* m'*, as it is for a simple pendulum*, then one will set $A_1 = 0$ in equations (17), or rather $A_2 = 0$. Let $A_2 = 0$, so it will result that:

$$A_1 = \alpha \qquad \text{and} \qquad \mu_1 = \frac{\beta}{\alpha},$$

and as a result, equation (16) will give:

$$r_1 = \frac{\beta\, g}{a\,\alpha + b\,\beta}.$$

When that value is substituted in equation (15), it will lead to a relation between the masses m, m', the distances a, b, and the gaps α, β, namely:

$$(19) \qquad (m+m')\,a\,\alpha^2 + (m+m')\,(b-a)\,\alpha\,\beta - m'\,b\,\beta^2 = 0\,.$$

That is the desired condition: It will be impossible to fulfill if $\alpha = \beta$, because the hypothesis that $\alpha = \beta$ will reduce the preceding equation to $m\,b = 0$, so $m = 0$ or $b = 0$, which must say that the two material points reduce to just one. In that case, equations (18) will effectively give values to θ and φ that are constantly equal to each other:

$$\theta = \varphi = \alpha \cos t\sqrt{r_1}\,,$$

i.e., that the two material points will always remain along a straight line with the point of suspension, which is obviously impossible, at least unless they coincide.

When one is given the masses m, m', and the distances a, b, equation (19) will always produce a real value for the ratio α / β that is positive and less than 1. For example, let $a = b$, so one will have:

$$\alpha = \beta\sqrt{\frac{m'}{m+m'}}\,,$$

and then:

$$\mu_1 = \sqrt{\frac{m+m'}{m'}}\,,$$

and

$$r_1 = \frac{g}{a\left(1+\sqrt{\dfrac{m'}{m+m'}}\right)}\,.$$

In general, when the condition (19) is fulfilled, one will have:

$$\theta = \alpha \cos t\sqrt{r_1}\,, \qquad \varphi = \beta \cos t\sqrt{r_1}\,.$$

The durations of the small oscillations will then be the same for the two pendula, but their amplitudes will be different. They will be on the same side of the vertical at the same time, and they will coincide with it at the same time.

Problem III: *A circular wheel carries an annular channel around its circumference, in the interior of which one introduces a heavy homogeneous sphere m of very-small diameter that is equal to that of the channel. That wheel is supported by a horizontal plane AOB at a point B on its circumference, and at its center S by a fixed vertical axis SO that is inclined with respect to the plane of the wheel by a known angle of OSB =* α*. One supposes that one makes the wheel roll on the horizontal plane in such a manner that its support point describes a circle of radius OB with a constant velocity. The right cone whose axis is SO and whose half-angle at the center is* α *is then then touched in succession along its generators by the plane of the moving wheel.*

One demands to know what the laws of motion of the center of the sphere would be (when one ignores friction).

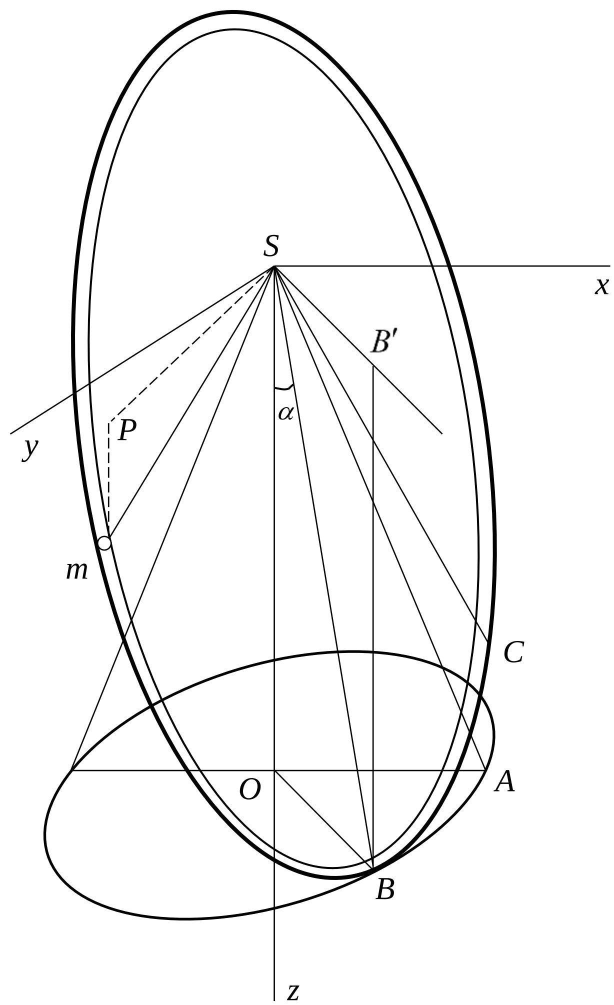

Figure 4.

A false solution to this problem was given in volume XIX of the *Annales de Gergonne*, page 360. The author took the expression for the effective accelerating force that animates the point m to be the quantity $r\frac{d^2\theta}{dt^2}$, in which r is the radius of the wheel, and θ is the angle that the radius Sm makes with a fixed radius that is traced in the plane of the wheel. However, there is a grave error in that whose effect is nothing less than to confuse the element of the *real trajectory that is described* by the moving body with the element of the circumference of the wheel.

Take the z-axis to be the vertical SO and make the zx-plane pass through the edge SA, along which the moving wheel touches the cone at the origin of the motion. At time t, let SB be the edge of contact, let SC be the position on the plane of the wheel that the radius that originally coincided with SA occupies at that point in time. Let Sm be the radius vector to the sphere: Its position is determined by the angles:

$$mSO = \varphi \qquad \text{and} \qquad PSx = \psi,$$

in which SP is the horizontal projection of Sm. However, those two angles are not independent of each other.

Let θ be the variable angle CSm. If one lets k denote the given constant velocity with which the angle AOB is described then one will have:

$$\text{angle } AOB = k\,t,$$

and since it results from the nature of the motion that:

$$\text{arc } AB = \text{arc } BC,$$

one can then conclude that:

$$\text{angle } BSC = k\,t \sin\alpha$$

and that:

$$\text{angle } mSB = \theta - k\,t \sin\alpha.$$

We shall denote that angle by ω, to abbreviate:

$$\omega = \theta - k\,t \sin\alpha. \tag{20}$$

Having said that, it is easy to express the angles φ and ψ as functions of ω. Indeed, the rectangular trihedron $SOmB$ will first give:

$$\cos\varphi = \cos\alpha\cos\omega. \tag{21}$$

Let SB' be the horizontal projection of the edge SB. One will have:

$$\psi = PSB' + B'S\,x = PSB' + BOA = PSB' + k\,t.$$

PSB' measures the dihedral angle that has an edge *SO* in the trihedron that was considered before. One will then have:

$$\tan\ PSB' = \frac{\tan\omega}{\sin\alpha}\ .$$

Consequently:

(22) $$\psi = k\,t + \text{arc}\left(\tan = \frac{\tan\omega}{\sin\alpha}\right)\ .$$

With the aid of formulas (21) and (22), the problem is reduced to the determination of ω as a function of time. Since we have *a function of just one variable* ω here, it will suffice to use one of the equations (5):

$$\frac{d}{dt}\left(\frac{dT}{d\omega'}\right) - \frac{dT}{d\omega} - \frac{dV}{d\omega} = 0\ ,$$

$$V = m\,g\,r\cos\varphi = mg\,r\cos\alpha\cos\omega\ ,$$

$$T = \tfrac{1}{2}m\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right] = \tfrac{1}{2}m\,r^2\left[\left(\frac{d\varphi}{dt}\right)^2 + \sin^2\left(\frac{d\psi}{dt}\right)^2\right].$$

If one replaces φ, $\dfrac{d\varphi}{dt}$, and $\dfrac{d\psi}{dt}$ with their values that are inferred from formulas (21) and (22) then one will find that the expression for T reduces to this very-simple formula:

$$T = \tfrac{1}{2}m\,r^2[\omega'^2 + 2\sin\alpha\cdot k\omega' + (1 - \cos^2\alpha\cos^2\omega)k^2]\ .$$

By definition, the equation of motion is:

$$r\frac{d^2\omega}{dt^2} - r\cos^2\alpha\cos\omega\sin\omega\,k^2 + g\cos\alpha\sin\omega = 0\ .$$

Upon multiplying by 2 $d\omega$, one integrates once, and one will have:

(23) $$r\left(\frac{d\omega}{dt}\right)^2 + k^2\,r\cos^2\alpha\cos\omega - 2g\cos\alpha\cos\omega + c = 0\ ,$$

in which c is an arbitrary constant that one will determine from the initial circumstance. Finally, the calculation of ω as a function of t is reduced to a quadrature:

(24) $$t = \int_{\omega_0}^{\omega_1} \frac{\pm\sqrt{r}\cdot d\omega}{\sqrt{2g\cos\alpha\cos\omega - k^2\, r\cos^2\alpha\cos\omega - c}}\,.$$

That integral can be calculated only by approximation, except for certain values of c. If one sets $k = 0$, which amounts to supposing that the wheel is immobile, then ω will coincide with θ, and one will recover the ordinary formula for the simple pendulum, except that gravity g is replaced by its component $g \cos \alpha$ in the plane of the wheel.

Since ω is known as a function of t by the preceding integral, equations (20), (21), and (22) will give θ, φ, and ψ as functions of that same variable. The elimination of t from the expressions for φ and ψ then produce the polar equation of the conical surface that is described by the radius vector of the sphere. If one transforms the latter equation into rectangular coordinates by means of the relations:

$$z = r \cos \varphi, \qquad \tan \varphi = \frac{y}{x}$$

then that transform, combined with the equation:

$$x^2 + y^2 + z^2 = r^2,$$

will determine the curve of double curvature that is described by the center of the sphere in space. Finally, the velocity of the sphere in its trajectory will be likewise known as a function of time by the formula:

$$v^2 = r^2\left[\left(\frac{d\omega}{dt}\right)^2 + 2k\sin\alpha\,\frac{d\omega}{dt} + (1 - \cos^2\alpha\cos^2\omega)\,k^2\right].$$

Consider the particular case in which one has:

$$\omega = 0 \quad \text{and} \quad \frac{d\omega}{dt} = 0$$

at the origin of the motion, or rather, what amounts to the same thing:

$$\theta = 0 \quad \text{and} \quad \frac{d\theta}{dt} = k \sin \alpha\,.$$

That is what will happen if the sphere is originally at A in the channel, and that it receives an initial velocity that is precisely equal to the one with which the radius SA describes the circumference of the wheel. Under that hypothesis, equation (23) will give:

$$c = 2\,g \cos \alpha - k^2 r^2 \cos^2 \alpha\,,$$

and that value, when substituted in (24), will lead to the integral:

$$t = \sqrt{r}\int\frac{d\omega}{\sqrt{(1-\cos\omega)[k^2 r\cos^2\alpha\,(1+\cos\omega)-2g\cos\alpha]}}\ ,$$

which one can obtain in finite form, because upon setting:

$$\tan\tfrac{1}{2}\omega = u\ , \quad k^2 r\cos^2\alpha - g\cos\alpha = a\ , \qquad g\cos\alpha = b\ .$$

That integral will become:

$$t = \sqrt{r}\int\frac{du}{u\sqrt{a-bu^2}} = \sqrt{\frac{r}{a}}\,l\left(\frac{-\sqrt{a-bu^2}+\sqrt{a}}{u}\right),$$

and upon replacing u its value, one will have:

$$c'e^{t\sqrt{\frac{a}{r}}} = \frac{-\sqrt{a-b\tan^2\frac{1}{2}\omega}+\sqrt{a}}{\tan\frac{1}{2}\omega}\ .$$

In order to determine c', one sets $t = 0$ and $\omega = 0$ in that equation, so one concludes that $c' = 0$, and consequently that $\tan\frac{1}{2}\omega$ *will be constantly zero.*

Hence, in the initial state that one supposes, the sphere will not leave the horizontal plane, and it will describe the circle *AB* with a constant velocity that is equal to $k\,r\sin\alpha$.

The formula (24) will again be integrated completely if one supposes that the initial state is such that the constant c is zero and that the coefficient of $\cos\omega$ under the radical must be equal to that of $\cos^2\omega$. Let θ_0 and $\frac{d\theta_0}{dt}$ be the initial values of θ and $\frac{d\theta}{dt}$. The hypothesis that we have adopted implies the two conditions:

$$\left(\frac{d\theta_0}{dt}-k\sin\alpha\right)^2 = k^2\cos^2\alpha\cos\theta_0(1-\cos\theta_0)\ ,$$

$$2\,g = k^2\,r\cos\alpha\ ,$$

so the expression for dt will then take the form:

$$dt = \pm\sqrt{\frac{r}{h}}\,\frac{d\omega}{\sqrt{2\cos\omega\,(1-\cos\omega)}}\ ,$$

in which we denote $g\cos\alpha$ by h, to abbreviate.

If one makes:

$$\tan \tfrac{1}{2}\omega = u$$

then it will become (upon first taking the + sign, which supposes that ω increases with t):

$$dt = \sqrt{\frac{r}{h}}\,\frac{du}{u\sqrt{1-u^2}},$$

whose integral is:

$$\frac{1-\sqrt{1-u^2}}{u} = c'\,e^{t\sqrt{\frac{h}{r}}}.$$

For $t = 0$, one has:

$$\tan \tfrac{1}{2}\omega = u = \tan \tfrac{1}{2}\theta_0,$$

so:

$$c' = \frac{1-\sqrt{1-\tan^2 \tfrac{1}{2}\theta_0}}{\tan \tfrac{1}{2}\theta_0}.$$

In order for that value to be real, it is necessary that the angle θ_0 must be equal to at most $\pi / 2$, and then the constant will be smaller than 1, or equal to at most 1, for $\theta_0 = \pi / 2$.

Solve the preceding integral for u and replace that variable by $\tan \tfrac{1}{2}\omega$. We will have:

$$\tan \tfrac{1}{2}\omega = \frac{2c'\,e^{t\sqrt{\frac{h}{r}}}}{c'^2\,e^{2t\sqrt{\frac{h}{r}}}+1}.$$

Upon taking the – sign in the expression for dt, one will arrive at the same integral.

The value of $\tan \tfrac{1}{2}\omega$ can be written $\dfrac{2M}{M^2+1}$ (upon setting $c'\,e^{t\sqrt{\frac{h}{r}}} = M$). The derivative of that quantity with respect to M is $\dfrac{2(1-M^2)}{(M^2+1)^2}$. Consequently, as long as one has $M < 1$, $\tan \tfrac{1}{2}\omega$ will be increasing with time. The variable ω will then attain a maximum for the value of t that satisfies the condition:

$$c'\,e^{t\sqrt{\frac{h}{r}}} = 1, \quad \text{hence} \quad t\sqrt{\frac{h}{r}} = l\left(\frac{1}{c'}\right)$$

(one saw above that $1/c'$ is greater than 1). At that moment:

$$\tan \tfrac{1}{2}\omega = 1,$$

and as a result:

$$\omega = \frac{\pi}{2} \; .$$

That maximum of ω agrees quite well with the differential expression:

$$dt = \sqrt{\frac{r}{h}} \frac{d\omega}{\sqrt{2\cos\omega(1-\cos\omega)}} \; .$$

As time continues to increase, ω will decrease since the derivative of $\tan\frac{1}{2}\omega$ will become negative and continue to decrease indefinitely without ω reducing to zero except for $t = \infty$.

Formula (20) will give the expression for θ as an explicit function of t :

$$\theta = k\, t \sin a + 2 \cdot \text{arc} \left\{ \tan \frac{2c' e^{t\sqrt{\frac{h}{r}}}}{c'^2 e^{2t\sqrt{\frac{h}{r}}} + 1} \right\} .$$

However, the search for the maxima and minima of that variable would be complicated.