Vito VOLTERRA and Enrico VOLTERRA

ON THE DISTORTIONS
OF ELASTIC BODIES

(THEORY AND APPLICATIONS)

Translated by D. H. Delphenich

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PREFACE

Vito Volterra was born in Ancona in 3 May 1860, and the present year then marks the centennial anniversary of his birth. *L'Accademia nazionale dei Lincei*, to which he was elected in 1888, published his celebrated paper "Sopra le funzioni che dipendono da altre funzioni," in which he founded functional analysis, one year later, and has planned a commemorative ceremony for next November that will unite the representatives of numerous academies and universities in Rome. The year 1960 has seen the appearance of the fourth and penultimate volume of the distinguished publication of the *Mathematical Works of Vito Volterra* ([1]).

It is good that the same year 1960 has seen the publication of the present volume in France. We owe that to our friend and colleague Henri Villat, who has welcomed the manuscript into the excellent collection of the *Mémorial des Sciences mathématiques* and the present edition can be thus assured to be published by the house of Gauthier-Villars, just like all of the other books that Vito Volterra published in France.

From the beginning of this Century, profound bonds of friendship were established between Vito Volterra and the French scholars Émile Borel, Paul Painlevé, Jean Perrin, Aimé Cotton, and Paul Langevin, to cite only the ones that have gone. Those bonds were further reinforced in the years 1914-1919 by converging preoccupations and common work on the scientific problems of the era.

I was, I believe, in 1912 that Vito Volterra gave a course at the Sorbonne for the first time. He returned to it quite often afterwards and it is precisely the part of his work that we have thus learned about by his direct teaching. Some other bonds of affectionate admiration were thus established whose memory is precious to us.

Vito Volterra presented the theory that was the subject of Chapters I to III in the present volume in some conferences that were held at the Institut Henri Poincaré and that Abbot P. Costabel has collected and edited with the view of producing a publication that the events of 1940 had delayed.

I was particularly happy that Enrico Volterra accepted the task of reviewing the conference manuscript by his father and adding some desirable complements after a very long delay in publication. He was particularly qualified to do that by his beautiful work on the questions of elasticity and plasticity, which is work that is both theoretical and experimental and is driven by practical applications.

In the introduction to the volume, Enrico Volterra has very precisely specified the place that the Volterra "distortions" occupy in a series of studies that extend or generalize the notions thus-introduced. His presentation leads up to the most current questions in the mechanics of solids. In Chapter IV, he has developed some applications to some practical problems in construction. The discussion there of the very important role that is played by elasto-plastic deformation and deformations that are imposed *a priori* in order for a structure to perform well is especially precise.

Joseph PÉRÈS

([1]) *Opere matematiche* (Memorie e Note), published by the Accademia Nazionale dei Lincei with support from the national research council.

FOREWORD

In 1938, my father, Vito Volterra, proposed that Costabel and myself should collaborate on the preparation of the present work. Costabel wrote up the conference talks that my father had previously given at l'Institut Henri Poincaré in Paris. At the same time, I prepared the text that was concerned with the applications of distortions in practical constructions. However, events that took the form of the war prevented the completion of this volume.

The incomplete manuscript was preserved by Professor Joseph Pérès, Member of the Institute. In 1956, Pérès insisted that I should assume the responsibility of revising, completing, and updating the manuscript.

The successful completion of this volume was also made possible thanks to the Graduate School of the University of Texas, who provided a subsidy.

On this occasion, I would like to thank Abbot Costabel for the editing of my father's talks. My thanks also go out to the "London Institute of Civil Engineers" and to the "London Institution of Structural Engineers" for the permission that they have graciously given for me to reproduce my drawings that were first published in their journals.

I would also like to thank Professor Henri Villat, Member of the Institute, for having accepted this volume into his Collection and the house of Gauthier-Villars for having made this publication appear on the occasion of the centenary of my father's birth.

Finally, I would like to express my gratitude to Professor Joseph Pérès for having encouraged me to revise the first three chapters and write the rest of this work. I especially appreciated his Preface, as well as the assistance that he was so kind as to give me.

Enrico Volterra

Ariccia, 20 June 1960.

TABLE OF CONTENTS

INTRODUCTION

1. – J. Weingarten is generally considered to be the initiator of the theory of elastic distortions. In a paper that he published in February 1901 in the *Rendiconti* of the Italian National Academy *dei Lincei* [**1**], he emphasized that there can exist cases in which elastic solids are not in a natural state when they are not subject to any external forces (so they are free from volume or surfaces forces), but are subject to a state of internal stress.

In order to prove that phenomenon, it will suffice to consider the case of an elastic ring with a transverse section cut through it, such that the two sections of the cut have been reattached. After having been cut, the ring will be in the natural state, but after that operation, it will be in a state of stress that is not due to any external forces. One can arrive at the same conclusion if one imagines that one has forcibly introduced a rigid wedge into the cut in that same ring.

2. – Vito Volterra started with Weingarten's observations and developed his theory of distortions in multiply-connected elastic bodies in a series of papers that were published in the *Rendiconti* of the National Academy *dei Lincei* in the years 1905-1906 [**2**], [**3**], [**4**], [**5**], [**6**], [**7**], [**8**] ([1]). He published that theory, after he had organized it into a systematic fashion, in 1907 in the *Annales de l'École Normale Supérieure* in a paper that was entitled "Sur l'équilibre des corps élastiques multiplement connexes" [**9**]. Certain results that were obtained by Cesàro [**10**] were also included in that paper.

([1]) In 1905, Vito Volterra wrote the following letter to Weingarten while sending the latter a preprint of Volterra's first paper on the theory of elastic distortions:

Herrn Dr. Prof. J. Weingarten
Dreikonigstr. 38
Freiburg i. B. (Germany)

Rome, 27 February 1905
Via in Lucina, 17

My dear professor,

It is an honor for me to send you a preprint of a note that I just published in the *Rendiconti de l'Accademia die Lincei* on a topic in elasticity.

I started with some considerations that you had developed in your note in 1901, and I have tried to clarify a question that you left hanging: Are there cases in which the elastic body is simply-connected?

You might remember that I had some doubts about that since 1901. I have now solved the question in the negative, when the deformation has the property that I call "regularity."

Last Sunday, I presented a second note as a continuation of the one that I sent you, and I intend to present a third note on the same subject, as well.

I hope that you will be content that I have continued a line of research in which you published a very interesting work.

…Dear Sir, allow me to express my highest regard and sentiments of respectful amity for you.

VITO VOLTERRA

Volterra began his work by posing the following question: Other than the rings and other multiply-connected bodies, do there exist simply-connected bodies that are subject to a state of internal stress without being subject to volume or surface forces? With the aid of a simple analytical observation, he showed that such cases cannot exist if one supposes that the continuity of the deformation also extends to the first and second derivatives of the elements that characterize that deformation.

Upon supposing that the state of an elastic solid is called "regular" when the six components of the deformation are finite, monodromic, and continuous functions with first and second-order derivatives that are likewise finite and monodromic, Volterra proved the theorem that if an elastic body occupies a finite, simply-connected (i.e., *acyclic*) space and it is subjected to only regular deformations then it will be in the natural state if it is free of volume and surface forces. On the contrary, if the solid body occupies a multiply-connected (i.e., *cyclic*) space then it might also not be in the natural state; i.e., it might present internal stresses even when it is not subject to volume and surface forces, even though its deformation is regular.

One can find the counterpart to that phenomenon in hydrodynamics [**11**]: If a vessel with fixed, rigid walls contains an incompressible liquid with no vortices in its interior then the liquid must be at rest if the volume that it occupies is simply-connected. On the contrary, motion of the liquid is possible if the space occupied by the liquid is multiply-connected. Irrotational motion in hydrodynamics corresponds to regular deformation in elasticity.

3. – Volterra gave a very complete discussion of the deformations (which he called *distortions*) to which the theory is applicable in the case of hollow cylinders and some systems of thin beams.

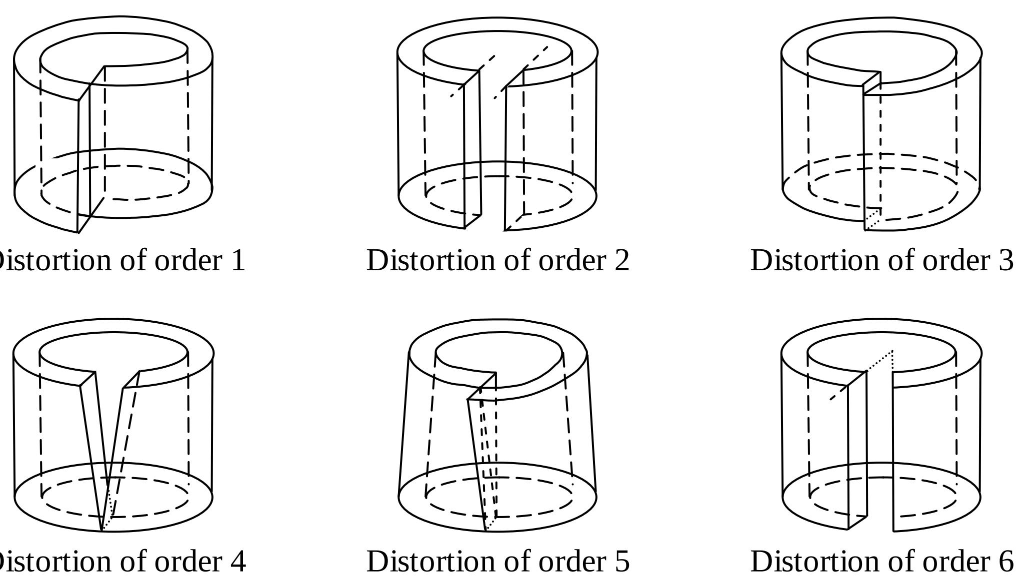

Figure 1.

The six possible distortions in the case of hollow cylinders are illustrated in Fig. 1

In each of those six cases, Volterra determined the deformation tensor and the stress tensor completely, and showed that in some particular cases, the distribution of stress in solids is produced in a fashion that could not have predicted by pure intuition.

4. – One of the particularly important results that Volterra obtained consisted of a reciprocity theorem between two elastic distortions in multiply-connected elastic bodies. Another reciprocity theorem was stated in 1912 by Colonetti [**12**] [**13**], [**14**], [**15**], which he called his *second reciprocity theorem*, and it established a relation between a state of elastic coaction and a state of elastic deformation that is due to the action of external forces. The two reciprocity theorems above have some important applications in mechanics.

In 1938, Vito Volterra, while preparing the publication in this volume, gave his reciprocity principle in the most general form that is presented here. Betti's reciprocity theorem, Colonetti's reciprocity theorem, and the Volterra's previous theorem are particular cases of that more general reciprocity theorem ([1]).

5. – Volterra was not content to have simply established the mathematical laws of distortion of multiply-connected elastic bodies. He also wished to verify his theory with the aid of experiments.

In 1907, the house of Pirelli in Milan constructed some hollow rubber cylinders in which different types of distortions had been created. From an examination of the form that was taken by the bases and surface of the cylinders after applying the distortion, it was indeed possible to qualitatively verify some of the results that had been found theoretically.

Similarly, in 1907, some hollow gelatin cylinders that were subjected to elastic distortion were examined by passing polarized light through them in the physics laboratory of the University of Geneva by Doctor Rolla [**17**].

In 1909, some more precise experiments on transparent models of hollow gelatin cylinders in which some distortions of the type that Volterra studied had been produced were conducted by Corbino and Trabacchi in the physics laboratory of the University of Rome [**18**], [**19**], [**20**]. Those experiments completely confirm the theoretical results that were found by Volterra.

The experiments that were done at the University of Geneva and the University of Rome have historical value because they represent one of the first applications of the phenomenon of accidental anisotropy to the study of the state of stress that exists inside of an elastic body; i.e., one of the first applications of photoelasticity (see § IX).

6. – The distortions that Volterra described, and which now bear his name, satisfy the following conditions:

([1]) Those results were communicated by Vito Volterra to Professor Joseph Pérès in a letter from Ariccia that was dated 11 October 1938 and were incorporated in pages 177, 178, and 179 of volume three of Volterra's *Complete Works*, in which his paper "Sur l'équilibre des corps élastiques multiplement connexes" was reproduced [**16**].

a. The deformation tensor (and consequently, the stress tensor, as well) is continuous.

b. The first and second derivatives of the deformation tensor are also continuous.

All of the stress components, which are assumed to be continuous, are represented by monodromic functions at the cut locations. Meanwhile that hypothesis imposes a restriction on the displacements, since they are coupled to the stresses by the equations of elasticity. The components of the displacement can be represented by polydromic functions, but the cyclic constants of those functions must have a certain form. Indeed, one can show that for monodromic stresses, the displacements that one encounters upon crossing the cut are defined only up to a displacement of the type that is called a "rigid-body" motion.

It is obvious that the distortions of the type that Volterra studied do not cover all of the possible cases in which an elastic solid body, which is not necessarily multiply-connected, is subjected to an internal stress state in the absence of volume and surface forces. Some cases of more general deformations were considered by Somigliana ([1]) and some other authors [**21**], [**22**], [**23**], [**24**], [**25**], [**26**], [**27**].

([1]) The following letter, which Somigliana sent to Volterra in March 1905, shows that Somigliana had begun to take an interest in elastic distortions after Volterra's work on the subject.

In effect, Volterra's theory of distortions had inspired Somigliana's later work on the subject.

Turin, 25 March 1905

My dearest Volterra,

I read your last two notes on the deformations of multiply-connected bodies with much interest, and also with a certain amount of satisfaction, since I saw that you appealed to my formulas. Now, *à propos* of that, I shall communicate an observation to you that is very simple, albeit not strictly connected with the problem that you studied. Your method for reducing problems that relate to the deformations that are produced by discontinuities along the surfaces of cuts when they are subjected to rigid relative displacements to ordinary problems is also valid when those displacements are arbitrary. Of course, in that case, the six deformation components will no longer be generally continuous. In fact, if one considers the displacements to be represented by the surface integrals that contain the value of u, v, w in my formulas then those integrals will be discontinuous and will have discontinuities that are proportional to u, v, w (as I proved in the case where $u = l + qz - ry$, …), even when u, v, w are arbitrary. In addition, they will give rise to pressures that are *continuous* upon crossing the surface. That property was proved in my note "Sulla rappresentazione dei campi di forza." It is equivalent to the continuity of the normal derivatives of the potential function of an inhomogeneous double layer.

It is therefore also possible in that case to determine a regular deformation that eliminates the surface pressures by means of a fictitious body, as in your method.

I do not know if that observation might interest you, but since it occurred to me spontaneously when I read your work, I wanted to communicate it to you.

I send you a thousand cordial regards and repeat my congratulations to you on your recent nomination to the Senate.

Your friend,
CARLO SOMIGLIANA

Professor Vito Volterra
Senator of the Kingdom
Via in Lucina, 17, Rome

The most general distortion can be produced by making a cut into the interior of a (not-necessarily-multiply-connected) elastic body and elastically deforming the two faces of the cut by applying forces to them. The faces of the cut are then reattached and the external forces eliminated, which will leave a continuous solution of an arbitrary, elastic displacement inside the body. A hypothesis that is generally less restrictive is made for the stresses: Only the components of the stresses that act on the surface of the cut must be continuous. The others can be discontinuous, and everything can be represented by polydromic functions.

Some other common types of distortions are the ones that are produced in an elastic body by non-uniform temperature distributions. For example, consider an elastic solid that is not necessarily multiply-connected, and suppose that a point of the solid is raised to a high temperature. That point will dilate more than the neighboring points, but its dilatation will be resisted by its neighboring elements. The presence of those neighboring elements, which are now at a much lower temperature, will constitute an obstacle for the elements at a higher temperature and the reciprocal actions among them will be the cause of a state of stress in the absence of external forces. That stress state can be defined to be the resultant of two successive and distinct operations:

Under the first operation, each element is considered to be ideally isolated in such a fashion that the changes in temperature can freely produce the corresponding dilatations (which are well-defined functions of those variations in temperature). That first operation will produce discontinuities in the material; i.e., superpositions or voids in the interior of the material.

Under the second operation, the surfaces of the different elements that initially coincide and have been displaced later must be once more made to coincide by producing a deformation of an exclusively-elastic nature that time, which along with the stress state that accompanies it and justifies it, will be defined completely by the fact that it must recreate the continuity and the connectivity of the solid, which is the continuity and connectivity that was temporarily destroyed.

If one supposes that the deformations of elastic solids are very small (i.e., that the elastic displacement that characterize them are always negligible in comparison with the dimensions of the elastic body) then the following theorem of Colonetti will be true [**28**], [**29**], [**30**], [**31**]:

The tensions that characterize the equilibrium state of a body are the ones that give a minimum for the function:

$$I = \Phi + \int_V [\tau_{11}\,\bar{\gamma}_{11} + \tau_{22}\,\bar{\gamma}_{22} + \tau_{33}\,\bar{\gamma}_{33} + \tau_{12}\,\bar{\gamma}_{12} + \tau_{23}\,\bar{\gamma}_{23} + \tau_{13}\,\bar{\gamma}_{13}]\,dV$$

with respect to all values of the same function that are compatible with the system of inelastic deformations and applied forces.

In the equation above, Φ represents the elastic potential energy in the solid, $\bar{\gamma}_{ik}$ are the components of the imposed deformation, and τ_{ik} are the components of the stress.

Colonetti's theorem is a generalization of a theorem from the classical mathematical theory of elasticity. When one supposes that the imposed deformation $\bar{\gamma}_{ik}$ is zero, the theorem above will express the condition that the function Φ is a minimum.

That is the theorem of minimum work of deformation, which was first formulated by Menabrea more than a century ago.

7. – The word "distortion," which Volterra introduced in order to describe a discontinuity in the displacement in a multiply-connected elastic body, was translated by Love [**32**] into the English word "dislocation." Today the English word "dislocation" was been generally adopted in place of the older term "distortion," and it includes the most general types of distortion, as well as the particular types of distortions that Volterra discussed. In general, the word "dislocation" characterizes a line in an elastic solid across which the displacement is subject to a sudden discontinuity.

The term "dislocation" took on a particular significance in modern solid-state physics in the year 1934, when G. I. Taylor explained the mechanism of plastic deformation in crystals by his "theory of dislocations" [**33**], [**34**].

According to that theory, when a solid of a crystalline nature is deformed, it is highly unlikely that one layer of atoms will glide over the other in such a manner that all of the atoms shift at the same time. It is much more probable that the deformation will begin at one extremity and propagate across the glide layer. If that is the case then on no particular glide layer at each moment during the deformation, one must have a line (viz., a dislocation) that separates that part of the layer in which the atoms glide from the part in which they do not. Taylor proposed a model for the arrangement of the dislocations in a crystal that is tempered by a plastic deformation, and he could evaluate the elastic stresses in crystals by applying Volterra's theorem.

The theory of "dislocations" in solid-state physics has developed very rapidly in recent years. Books and articles have appeared that were written by Nabarro (1952) [**35**], Cottrell (1953) [**36**], Orowan (1953) [**37**], Read (1953) [**38**], Mott (1956) [**39**], Seeger (1955) [**40**], J. M. and W. G. Burgers (1956) [**41**], Friedel (1956) [**42**]. With the "theory of dislocations," it is now possible to explain not only the phenomena of plastic deformation, but also the growth of crystals, diffusion and precipitation in solids, surface phenomena, and chemical reactions.

8. – Another field of application of the theory of elastic dislocations is geophysics.

In relation to that study, one can mention the recent work of the Russian geophysicists and the work that was done at the University of Toronto by Steketee [**43**] and others in which some of the qualitative considerations on orogenesis that were proposed by Wilson [**44**] and Scheidigger [**45**] have been placed upon a more quantitative basis. That gives one some hope that the theory of dislocations will shed some light upon the displacements and the energy of the stresses and deformations that are associated with fractures, faults, and earthquakes. It is not impossible that some of the new developments of solid-state physics can have some importance in the treatment of problems such as the accumulation of stress that precede earthquakes or the deformation and extension of fractures and faults.

Nonetheless, although in some works on that subject that have appeared recently, many authors have called those sorts of dislocations "Volterra dislocations," those dislocations do not seem to exhibit the characteristics that are generally associated with Volterra dislocations and must be considered to be dislocations of a more general type instead.

9. – The classical Volterra theory of the equilibrium of multiply-connected elastic bodies will be presented in the first three chapters of this volume, and some applications of the theory of distortions to practical construction will be discussed in the fourth chapter.

After giving the theory of how one traces out lines of influence in the statically-indeterminate systems, which is based upon the second reciprocity theorem, the problems of the theory of elasto-plastic deformation and systematic deformations will be discussed.

The last two problems are intimately linked to each other, and have taken on great importance in modern constructions [**46**]. The solution to the first one will permit us to extend our knowledge of the stress state and the deformations of structures when the elastic limit of the material has been exceeded and the material is in a condition that we call "plastic."

That exceeding of the elastic limit happens much more frequently than one might suppose, even in structures that are calculated with the greatest case and the highest precision, and asserts itself by the appearance of permanent deformations or deformations that do not disappear when the external forces that produced them cease to act.

The second problem is that of producing internal stresses that are favorable to stability inside of the structure itself with the aid of distortions that are created artificially.

A precise examination of the state of tension and deformation that exists in a statically-indeterminate metallic structure, part of which is in a plastic state, shows that the effect of the plastic deformation is the same as the effect that is produced by a rigid displacement. That hypothesis seems to be confirmed by hypothesis. Nonetheless, that does not imply that the distortion that ensues in the structure must be of the classical Volterra type, because it can have a more general type.

However, in certain cases, it can actually coincide with the classical distortion. For example, consider the important case of an arch that is fixed at its abutments. In that particular case, the foundation that is added to the ground will establish the ring, and the distortion that ensues naturally in the structure when the elastic limit is attained, or the artificial distortion that can be produced by force in the ring in order to produce a stress state that is favorable to the stability of the structure, will coincide with a distortion of Volterra type.

In the case of practical applications in which one must deal with distortions of the most general type, it is also always possible to utilize the methods that are derived from the classical theory of elastic distortions that Vito Volterra gave us.

CHAPTER I

GENERAL PRINCIPLES

I. – Review of the elements of the theory of elasticity.

1. – No entirely rigid bodies exist in nature. The bodies that one calls *solid* will, in fact, take on noticeable deformations when one applies sufficiently-large forces to them. Elasticity is the property that they possess of going back to their original forms in a manner that is more or less perfect when one suppresses the stresses that produced the deformation. Certain substances *preserve* the deformation that was imposed upon them: They are called *perfectly plastic.*

Perfect plasticity is the opposite property to perfect elasticity. They are two extreme states that are not realized rigorously by any body.

2. – The study of elastic phenomena can be carried out by starting from molecular or atomic hypotheses and utilizing the results of the theories of matter. We shall omit any hypothesis on the constitution of an elastic body from our study and consider such a body to be a continuous medium and rapidly review the principles that drop out of the mathematical study of its equilibrium.

Let a body that occupies a certain volume S be bounded by a boundary σ, which can be composed of one or more surfaces. The external forces that act upon it at an instant t can be classified by two categories:

1. The ones that act upon the volume elements dS and S, and which have the form $F\,dm = F\rho\,dS$, in which ρ is the density of the medium in the element dS.

2. The ones that act upon the surface elements $d\sigma$ of σ and that are assumed to have the form $T\,d\sigma$.

We shall call the vector F the *volume force;* i.e., the force that acts upon the element dm per unit mass and call the vector T the *surface tension.*

On the other hand, one can define a force that relates to each surface element $d\sigma$ whose normal is supposed to be oriented inward to the continuous medium and is the resultant of the actions of the material elements that are contiguous to $d\sigma$ on the negative side of the normal upon the material elements that are contiguous to the positive side. We call it the *elementary internal stress* that is exerted on the negative face of the $d\sigma$ and denote it by $T\,d\sigma$.

Assume that there exists a state, which one calls *natural*, of the elastic body for which all of the internal stresses are zero. Upon starting from that natural state, one can deform the body by applying volume forces and surface tensions. Each point P of the medium will then experience a displacement (u, v, w), which we shall regard as infinitely small, and which will take it to P'.

3. – Suppose, for the moment, that u, v, w are known as functions of the coordinates x, y, z of the point P and direct one's attention to the geometric transformation that takes a small volume element around P to a homologous element around P'.

It is classical that this transformation can be decomposed into a *displacement* and a *pure deformation;* i.e., a deformation that does not alter three rectangular directions of the medium and which results from the superposition of the three dilatations that are performed parallel to those axes.

Recall that the displacement is defined by a translation whose components are u, v, w and a rotation with components are p, q, r :

$$p = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right), \qquad q = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right), \qquad r = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right). \tag{1}$$

The deformation of just he element (dx, dy, dz) will have the components:

$$\left\{\begin{aligned} \delta\, dx &= \gamma_{11}\, dx + \tfrac{1}{2}\gamma_{12}\, dy + \tfrac{1}{2}\gamma_{13}\, dz, \\ \delta\, dy &= \tfrac{1}{2}\gamma_{21}\, dx + \gamma_{22}\, dy + \tfrac{1}{2}\gamma_{23}\, dz, \\ \delta\, dz &= \tfrac{1}{2}\gamma_{31}\, dx + \tfrac{1}{2}\gamma_{32}\, dy + \gamma_{33}\, dz, \end{aligned}\right. \tag{2}$$

in which one has:

$$\left\{\begin{gathered} \gamma_{11} = \frac{\partial u}{\partial x}, \quad \gamma_{22} = \frac{\partial v}{\partial y}, \quad \gamma_{33} = \frac{\partial w}{\partial z}, \\ \gamma_{32} = \gamma_{23} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \quad \gamma_{31} = \gamma_{31} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}, \\ \gamma_{21} = \gamma_{12} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}. \end{gathered}\right. \tag{3}$$

The quantities γ_{rs} are the characteristic functions of the deformation, since by definition they are the components of a tensor upon which the new configuration of the volume element considered will depend (viz., the *deformation tensor*). There will be displacement without deformation when the characteristic functions are zero.

4. – On the other hand, one knows that the internal stresses at each point P are characterized by the quantities t_{11} , t_{12} , t_{13} , t_{21} , t_{22} , t_{31} , t_{31} , t_{32} , t_{33} , which satisfy the equalities:

$$t_{12} = t_{21}\,, \qquad t_{23} = t_{32}\,, \qquad t_{31} = t_{13}\,,$$

and are the components of a tensor (viz., the *stress tensor*).

The internal stress or unit tension T that is exerted upon the negative face of each element of the surface ds with its center at P will have components that are deduced from t_{rs} by the formulas:

(4)
$$\begin{cases} T_x = t_{11}\cos nx + t_{12}\cos ny + t_{13}\cos nz, \\ T_y = t_{21}\cos nx + t_{22}\cos ny + t_{23}\cos nz, \\ T_z = t_{31}\cos nx + t_{32}\cos ny + t_{33}\cos nz, \end{cases}$$

in which n is the oriented normal to the element $d\sigma$.

5. – Finally, if X, Y, Z are the components of the internal force that acts upon the point P per unit mass then if ρ is the density at P then the equations of elastic equilibrium equations that Cauchy gave are the following ones:

(5)
$$\begin{cases} \dfrac{\partial t_{11}}{\partial x} + \dfrac{\partial t_{12}}{\partial y} + \dfrac{\partial t_{13}}{\partial z} = \rho X, \\ \dfrac{\partial t_{21}}{\partial x} + \dfrac{\partial t_{22}}{\partial y} + \dfrac{\partial t_{23}}{\partial z} = \rho Y, \\ \dfrac{\partial t_{31}}{\partial x} + \dfrac{\partial t_{32}}{\partial y} + \dfrac{\partial t_{33}}{\partial z} = \rho Z. \end{cases}$$

6. – In order to link the volume forces and the new state of the body, it remains for us to establish some relations between the internal stresses and the characteristic functions of the deformation.

One is led to suppose that there exists an elastic force potential that relates to each element dS of the deformed medium and whose expression is:

(6)
$$E\,(\gamma_{11}\,,\ \gamma_{22}\,,\ \gamma_{33}\,,\ \gamma_{23}\,,\ \gamma_{31}\,,\ \gamma_{12})\ dS,$$

in which E is a quadratic form in the variables that figure in it, which is a form that is always negative and will vanish only when all of the quantities γ_{rs} are zero.

If the form E is known then the components t_{rs} of the stress tensor can be expressed as functions of the γ_{rs} by means of the relations:

(7)
$$t_{rs} = \frac{\partial E}{\partial \gamma_{rs}}\,.$$

Recall that the expression that was posed for the elastic potential implies Hooke's law, which is verified by experiment and is obvious in our equations. The internal forces are linear functions of the deformations that produce them.

7. – The preceding considerations immediately give the solution to a first problem: *Find the forces that can produce a known state of deformation when they are applied to an elastic solid.*

Indeed, if the γ_{rs} are known then the relations (7) will determine the t_{rs} , and equations (5) will imply the volume forces. Finally, the surface tensions must be chosen in such a manner that they will equilibrate the corresponding internal tensions.

However, the fundamental problem of the theory of elasticity is the inverse problem to the preceding one: *Knowing the forces, find the deformation.*

The solution will again depend upon equations (5), in which one replaces the t_{rs} with their values in (7). However, this time, the unknowns are u, v, w, and one will have a system of second-order partial differential equations for determining them. Some simplifications present themselves for homogeneous media (i.e., ones with constant density ρ) and for isotropic media (i.e., ones whose structure is the same in all directions around each point).

In order to write out those fundamental equations, we shall confine ourselves to the case of a body that is both *homogeneous* and *isotropic*.

The potential in this case is equal to:

$$P = \tfrac{1}{2} L\,\Theta^2 + K\,\psi, \tag{8}$$

upon setting:

$$\Theta = \gamma_{11} + \gamma_{22} + \gamma_{33}\,,$$

$$\psi = \gamma_{11}^2 + \gamma_{22}^2 + \gamma_{33}^2 + \tfrac{1}{2}[\gamma_{23} + \gamma_{31} + \gamma_{12}]^2,$$

in which Θ is the *cubic dilatation.* L and K are constant, negative quantities.

One will then infer that:

$$\left\{\begin{aligned} t_{11} &= L\,\Theta + 2K\,\gamma_{11}, & t_{23} &= K\,\gamma_{23}, \\ t_{22} &= L\,\Theta + 2K\,\gamma_{22}, & t_{31} &= K\,\gamma_{31}, \\ t_{33} &= L\,\Theta + 2K\,\gamma_{33}, & t_{12} &= K\,\gamma_{12}, \end{aligned}\right. \tag{10}$$

and the equations of elastic equilibrium will be written:

$$\left\{\begin{aligned} K\Delta^2 u + [L+K]\frac{\partial\Theta}{\partial x} &= \rho X, \\ K\Delta^2 v + [L+K]\frac{\partial\Theta}{\partial y} &= \rho Y, \\ K\Delta^2 w + [L+K]\frac{\partial\Theta}{\partial z} &= \rho Z, \end{aligned}\right. \tag{11}$$

in which the symbol Δ^2 represents the operator $\dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$ (or Laplacian).

Those partial differential equations are equations of elliptic type. Their integration will introduce biharmonic functions; i.e., ones that satisfy the equation $\Delta^2\,(\Delta^2) = 0$.

In order to obtain the deformation that the body experiences, one will be led to find some functions u, v, w of x, y, z that verify equations (11) and satisfy the boundary conditions.

Those conditions are the ones that the functions u, v, w must verify on the surface of the body. They can take different forms since one can know either the displacements on the surface of the body or the tensions.

8. – A fundamental theorem is the uniqueness theorem for the solution: Namely, that the boundary conditions define just one possible deformation (for convenient values of L and K.

One sometimes regards that theorem as obvious from the physical viewpoint, but it is necessary to prove it rigorously. One arrives at it by generalizing the argument for equations (11) that permits one to treat the analogous question for the Laplace equation.

We shall not develop the general methods of integrating the equations of elasticity here. We shall point out only the elegant method of Almansi [**47**] for the problem of the elastic sphere.

However, we must insist upon the question of the uniqueness of solutions. We shall see that the study of that question is closely linked with that of the connectivity of the space that fills the body and the theory of monodromic and polydromic functions.

As a consequence, the theory of elasticity for multiply-connected bodies is presented in a manner that is different from the theory in simply-connected bodies as a result of the role that the polydromic solutions then play.

II. – Some theorems on equilibrium.

1. – We rapidly recall some well-known definitions: A three-dimensional domain with just one piece is called *simply-connected* or *acyclic* if any closed line that one can trace inside of it can be reduced to a point by continuous deformation without leaving the domain. If that condition is not satisfied then the domain will be *multiply-connected* or *cyclic.*

Suppose that a cyclic domain becomes acyclic by making just one transverse cut. One then says that the space is *doubly-connected*.

If two cuts are necessary in order for the domain to become acyclic then the domain will be called *triply-connected*, and so on.

The notion of connectivity can be generalized to the case of a space of more than three dimensions, moreover. We remark incidentally that in the case of domains in three-dimensional space, there is good reason to consider several types of connection: e.g., the surface connectivity and the linear connectivity. It is the second one that will be of interest here.

2. – The connectivity of space, which plays such an important role in the study of elastic equilibrium, also enters into some other questions of mathematical physics. In order to give an example, it will suffice to recall the theorem of hydrodynamics that says that when *a fluid in which there are no vortices occupies a closed space that is bounded by fixed walls, it will necessarily be at rest if the space is simply-connected, but can be very well in a state of motion if the space is multiply-connected.*

For a fluid that fills up a ring with axis *Oz*, one will then have a possible motion by taking the function:

$$\varphi = \arctan \frac{y}{x}$$

to be the velocity potential.

That velocity potential is a polydromic function (or one with several determinations). When the point x, y, z has crossed a closed cycle around the z-axis, the function will take on its initial value, increased by 2π.

The partial derivatives:

$$\frac{\partial \varphi}{\partial x} = -\frac{y}{x^2 + y^2}, \qquad \frac{\partial \varphi}{\partial y} = \frac{x}{x^2 + y^2},$$

as well as the successive derivatives, are finite, continuous, and monodromic, except along the *Oz* axis ($x = y = 0$).

3. – Before going further, recall the classical theorems of Gauss and Stokes:

GAUSS'S THEOREM. – *Let S be a three-dimensional domain whose boundary is* σ, *and let X, Y, Z be three monodromic function that are finite and continuous in S, along with their first derivatives. One will have:*

$$\int_S \left(\frac{\partial X}{\partial x} + \frac{\partial Y}{\partial y} + \frac{\partial Z}{\partial z} \right) dS = -\int_\sigma [X \cos nx + Y \cos ny + Z \cos nz] \, d\sigma,$$

in which n denotes the normal to σ *that points into S.*

One can further say that the flux of the vector (X, Y, Z) that crosses σ is equal to the integral over S of the divergence of the vector (X, Y, Z).

STOKES'S THEOREM. – *If* σ *is a surface with boundary s, and X, Y, Z are three monodromic functions that are finite and continuous, along with their first derivatives, then one will have:*

$$\int_\sigma \left\{ \left(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z} \right) \cos nx + \left(\frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x} \right) \cos ny + \left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y} \right) \cos nz \right\} d\sigma$$

$$= \int_s [X\, dx + Y\, dy + Z\, dz].$$

If the sense of traversal that is taken around s has the same direction as the positive normal to σ then one can also say that the rotational flux of *X*, *Y*, *Z* across σ is equal to the circulation (or the work done) by the vector (*X*, *Y*, *Z*) along *s*.

Those two theorems are valid only in the case where one considers functions that are finite, continuous, and *monodromic*.

That is the profound reason for the difference between the solutions to the problem of elasticity for different connectivities.

4. – We shall account for it very simply.

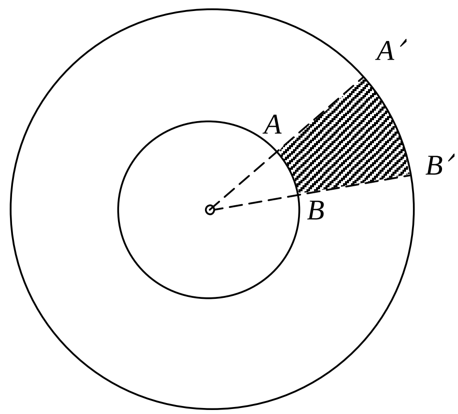

Figure 2.

One often states the theorem that says that *an elastic body that is not subject to any external action (volume forces vanish, along with the tensions or pressures on the surface) is in the natural state. All of the coefficient* γ_{rs} *will be zero.* It will then follow immediately that *the deformation will be determined in a unique fashion when one is given the external stresses.*

Those statements are exact only in the case of an elastic body that occupies a simply-connected volume.

In order to show that they can break down in the other cases, it will suffice to imagine a body that forms a ring (viz., torus) from which one has removed a very thin radial wedge and then soldered the two boundaries of the cut *AA′*, *BB′* (Fig. 2).

After soldering, the body will be subject to internal stresses, but it will nonetheless support no external action.

One might think that one must necessarily find a discontinuity or singularity in the deformation at the location where the soldering was made, but one can show that no such thing is produced and that nothing in the deformation will permit one recover the location where the cut was made.

The solution to that contradiction with the classical proof of the theorem that was just stated is found in an application of Gauss's theorem that one utilizes in order to show that when one starts from equations (4) and (5), the deformation will be zero when *X*, *Y*, *Z*, T_x, T_y, T_z are zero.

Now, Gauss's theorem cannot be applied in the case of the preceding ring, since the displacements *u*, *v*, *z* will be polydromic functions after soldering. Indeed, two points that belong to the sections AA' and BB', respectively, and are brought into contact by soldering will have taken on displacements whose difference represents precisely the corresponding size of the fissure. That is why if one starts from the point M_0 on AA' and continuously follows the displacement (u, v, w) along a cycle that is described in the ring then one will return to the starting point on the other side of the cut with values of u, v, w that are different from the initial values.

One might even be tempted to apply the preceding remark to an acyclic body and intuitively consider that upon subtracting a very thin slice from such a body and then bringing the boundaries of the cut back together, one will have a state of equilibrium with no external forces, but with stresses and deformations in the body that vary with no discontinuities.

Such a conclusion would be false. Indeed, we shall show that if one supposes that the deformation is *regular – i.e., that* γ_{11} , γ_{22} , γ_{33} , γ_{23} , γ_{31} , γ_{12} *are finite, continuous, monodromic functions, along with their derivatives (of first and second order) – then the polydromy of the displacements can be present only if the body has a cyclic form.*

5. – In order to do that, we must first look for relations that will permit us to pass from deformations to the displacements. We shall follow the elegant method of Cesàro [**10**] in order to establish the formulas that Vito Volterra gave on that subject.

Let s be a curve in the body with extremities A_0, A_1 ; let x_0, y_0, z_0 be the coordinates of A_0, and let x_1, y_1, z_1 be the coordinates of A_1 .

The values of u, v, w at the point A_1 are given as functions of their values at the point A_0 by the obvious formula:

$$u_1 = u_0 + \int_s \left[\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz\right],$$

and the analogous formulas for v_1, w_1 .

In order to transform these formulas, get the values of the first partial derivatives of u, v, w from the relations (1) and (3); it will then follow that:

$$u_1 = u_0 + \int_s [\gamma_{11}\, dx + \tfrac{1}{2}\gamma_{12}\, dy + \tfrac{1}{2}\gamma_{13}\, dz] + \int_s [q\, dz - r\, dy] .$$

Upon denoting the values of p, q, r at the point A_0 by p_0 , q_0 , r_0 , a simple integration by parts will give:

$$\int_s [q\, dz - r\, dy] = q_0\, [z_1 - z_0] - q_0\, [z_1 - z_0] + \int_s [(z_1 - z)\, dq - (y_1 - y)\, dr] .$$

On the other hand, one will have:

$$\frac{\partial q}{\partial x} = \frac{\partial \gamma_{11}}{\partial z} - \frac{1}{2}\frac{\partial \gamma_{12}}{\partial z}, \qquad \frac{\partial r}{\partial x} = \frac{1}{2}\frac{\partial \gamma_{12}}{\partial x} - \frac{\partial \gamma_{11}}{\partial y},$$

and analogous formulas for:

$$\frac{\partial q}{\partial y} \text{ and } \frac{\partial r}{\partial y}, \qquad\qquad \frac{\partial q}{\partial z} \text{ and } \frac{\partial r}{\partial z}.$$

As a result:

$$u_1 = u_0 + q_0\,(z_1 - z_0) - r_0\,(y_1 - y_0) + \int_s (\xi\,dx + \eta\,dy + \zeta\,dz),$$

with:

$$(12)\qquad \left\{\begin{aligned}
\xi &= \gamma_{11} + (y_1 - y)\left[\frac{\partial\gamma_{11}}{\partial y} - \frac{1}{2}\frac{\partial\gamma_{12}}{\partial x}\right] + (z_1 - z)\left[\frac{\partial\gamma_{11}}{\partial z} - \frac{1}{2}\frac{\partial\gamma_{13}}{\partial x}\right],\\
\eta &= \tfrac{1}{2}\gamma_{21} + (y_1 - y)\left[\frac{1}{2}\frac{\partial\gamma_{21}}{\partial y} - \frac{\partial\gamma_{22}}{\partial x}\right] + (z_1 - z)\left[\frac{1}{2}\frac{\partial\gamma_{21}}{\partial z} - \frac{1}{2}\frac{\partial\gamma_{23}}{\partial x}\right],\\
\zeta &= \tfrac{1}{2}\gamma_{31} + (y_1 - y)\left[\frac{1}{2}\frac{\partial\gamma_{31}}{\partial y} - \frac{1}{2}\frac{\partial\gamma_{32}}{\partial x}\right] + (z_1 - z)\left[\frac{1}{2}\frac{\partial\gamma_{31}}{\partial z} - \frac{\partial\gamma_{33}}{\partial x}\right].
\end{aligned}\right.$$

One will get analogous formulas for v_1, w_1, so finally:

$$(\mathrm{I})\qquad \left\{\begin{aligned}
u_1 &= u_0 + q_0\,(z_1 - z_0) - r_0(y_1 - y_0) + \int_s (\xi\;dx + \eta\;dy + \zeta\;dz),\\
v_1 &= v_0 + r_0\,(x_1 - x_0) - p_0(z_1 - z_0) + \int_s (\xi'\,dx + \eta'\,dy + \zeta'\,dz),\\
w_1 &= w_0 + p_0(y_1 - y_0) - q_0(x_1 - x_0) + \int_s (\xi''\,dx + \eta''\,dy + \zeta''\,dz),
\end{aligned}\right.$$

in which ξ', η', ζ', ξ'', η'', ζ'' are expressed as functions of the γ_{rs} in a manner that is analogous to the expressions for ξ, η, ζ.

6. – Suppose that the line s is a closed cycle.

If the elastic body occupies an acyclic region then one can regard s as the contour of a surface σ that is situated in the interior of the body and transform the integrals that enter into formulas (12) by Stokes's theorem, and thus replace them with the surface integrals:

$$\int_\sigma \left\{\left[\frac{z_1 - z}{2}C - \frac{y_1 - y}{2}B\right]\cos nx - \left[\frac{y_1 - y}{2}A + (z_1 - z)F\right]\cos ny - \left[\frac{z_1 - z}{2}A + (y_1 - y)G\right]\cos nz\right\} d\sigma,$$

$$\int_\sigma \left\{-\left[\frac{x_1 - x}{2}B + (z_1 - z)E\right]\cos nx - \left[\frac{z_1 - z}{2}C - \frac{x_1 - x}{2}A\right]\cos ny - \left[(x_1 - x)G + \frac{z_1 - z}{2}B\right]\cos nz\right\} d\sigma,$$

$$\int_\sigma \left\{-\left[(y_1 - y)E + \frac{x_1 - x}{2}C\right]\cos nx - \left[(x_1 - x)F - \frac{y_1 - y}{2}C\right]\cos ny - \left[\frac{x_1 - x}{2}A - \frac{y_1 - y}{2}G\right]\cos nz\right\} d\sigma,$$

respectively. A, B, C, E, F, G have the following significance:

$$A = \frac{\partial}{\partial x}\left[\frac{\partial \gamma_{31}}{\partial y} + \frac{\partial \gamma_{12}}{\partial z} - \frac{\partial \gamma_{23}}{\partial x}\right] - 2\frac{\partial^2 \gamma_{11}}{\partial z\,\partial y},$$

$$B = \frac{\partial}{\partial y}\left[\frac{\partial \gamma_{12}}{\partial z} + \frac{\partial \gamma_{21}}{\partial x} - \frac{\partial \gamma_{31}}{\partial y}\right] - 2\frac{\partial^2 \gamma_{22}}{\partial x\,\partial z},$$

$$C = \frac{\partial}{\partial z}\left[\frac{\partial \gamma_{23}}{\partial x} + \frac{\partial \gamma_{31}}{\partial y} - \frac{\partial \gamma_{12}}{\partial z}\right] - 2\frac{\partial^2 \gamma_{33}}{\partial y\,\partial x},$$

$$E = \frac{\partial^2 \gamma_{32}}{\partial y\,\partial z} - \frac{\partial^2 \gamma_{23}}{\partial z^2} - \frac{\partial^2 \gamma_{33}}{\partial y^2},$$

$$F = \frac{\partial^2 \gamma_{13}}{\partial z\,\partial x} - \frac{\partial^2 \gamma_{33}}{\partial x^2} - \frac{\partial^2 \gamma_{11}}{\partial z^2},$$

$$G = \frac{\partial^2 \gamma_{21}}{\partial x\,\partial y} - \frac{\partial^2 \gamma_{11}}{\partial y^2} - \frac{\partial^2 \gamma_{22}}{\partial x^2}.$$

One immediately verifies that:

$$A = B = C = E = F = G = 0. \tag{II}$$

These are well-known formulas, moreover, and they are due to Barré de Saint-Venant. The integrals in the formulas (12) will be zero then, and one will have:

$$u_1 = u_0 , \qquad v_1 = v_0 , \qquad w_1 = w_0 .$$

If the deformation is regular and the body is acyclic then the displacements will necessarily be monodromic.

However, if the body is cyclic then a closed cycle s is not necessarily the contour of a surface σ that belongs to the space that is occupied by the body, so one cannot apply Stokes's theorem, and the displacements can be polydromic.

7. – We can then state the following propositions:

An elastic body that has an acyclic form and a regular deformation can be brought to the natural state by finite, continuous, and monodromic displacements.

If the elastic body has a cyclic form and the deformation is always regular then in order to return to the natural state, it can sometimes be necessary to make one or more cuts and remove some parts of the body, and the displacements are not obligated to be monodromic.

Finally, if the external forces are known then the regular deformation of an acyclic body will be well-defined. That of a cyclic body will not be, except in the case where one knows in advance that one can return to the natural state by monodromic displacements.

The proof of the latter proposition follows from the one that one makes in order to prove that an elastic body that is not subject to external forces is found in the natural state, which is a proof that implicitly supposes that the displacements are finite, continuous, and monodromic, and that the deformation is regular.

We shall study the case of the cyclic body more closely.

8. – When the body is cyclic, one can make it acyclic by making some cuts. One easily infers the discontinuities in the displacements along the cuts from formulas (I).

We first take a doubly-connected body that can be made acyclic by one cut.

If the values of the displacements at two points α and β that are contiguous on one side and the other of the cut are:

$$u_\alpha, \quad v_\alpha, \quad w_\alpha \quad \text{and} \quad u_\beta, \quad v_\beta, \quad w_\beta,$$

respectively, then formulas (I), in which one takes:

$$x_0 = x_1, \qquad y_0 = y_1, \qquad z_0 = z_1,$$

where x_1, y_1, z_1 denote the common coordinates of the points α and β, will give:

$$u_\beta - u_\alpha = \int_{(\alpha\beta)} [\xi\, dx + \eta\, dy + \zeta\, dz],$$

and the analogous expressions for $(v_\beta - v_\alpha)$, $(w_\beta - w_\alpha)$, where $(\alpha\beta)$ is a line of integration that is situated entirely in the space that is occupied by the body.

One can isolate the terms that depend upon y_1, z_1 in ξ, η, ζ, as given by the preceding equations (12), and upon denoting the remaining terms by ξ_0, η_0, ζ_0, one will have:

$$u_\beta - u_\alpha = \int_{(\alpha\beta)} [\xi_0\, dx + \eta_0\, dy + \zeta_0\, dz] + z_1 \int_{(\alpha\beta)} dq - y_1 \int_{(\alpha\beta)} dr\,.$$

The calculations that were made in the preceding number and the formulas of Barré de Saint-Venant show immediately that the integral of:

$$[\xi_0\, dx + \eta_0\, dy + \zeta_0\, dz]$$

will be zero when it is taken along a closed curve in a simply-connected space.

One deduces by an argument that is very simple and classical that when the integral of that expression is taken along $(\alpha\beta)$ in the case that we are currently dealing with, it will be independent of the pair (α, β). *The same thing will be true for all pairs of points* α, β *that are separated by the cut.* We denote the integral by l.

The two integrals $\int_{(\alpha\beta)} dq$ and $\int_{(\alpha\beta)} dr$, which we denote by the values q and r, will be the same constants, regardless of the pair (α, β) that one envisions. It finally follows that:

$$u_\beta - u_\alpha = l + q\,z_1 - z\,y_1,$$

along with analogous formulas for $v_\beta - v_\alpha$ and $w_\beta - w_\alpha$.

One can easily generalize this to the case of a cyclic body with a higher order of connectivity by saying that for each cut the values of the discontinuities in the displacements will have the form:

$$\text{(III)} \qquad \left\{ \begin{aligned} u_\beta - u_\alpha &= l + qz - ry, \\ v_\beta - v_\alpha &= m + rx - pz, \\ w_\beta - w_\alpha &= n + py - qx; \end{aligned} \right.$$

l, m, n, p, q, r are the *characteristic constants of each cut,* and x, y, z are the coordinates of the pair (α, β).

That being the case, in bodies that are made acyclic by cuts, the displacements will be finite, continuous, and monodromic, and will satisfy relations (III). That is why one can state the following theorem:

If a cyclic body is deformed regularly then the deformation will be well-defined by the external forces and the constants of each cut.

That theorem completely specifies the difference between acyclic bodies and cyclic bodies.

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CHAPTER II

DISTORTIONS

III. – Definition. The corresponding problems of equilibrium.

1. – Formulas (III) immediately give the physical significance of the six characteristic constants of each cut.

Indeed, make some cuts in the material along the chosen sections in the deformed cyclic body and let it return to its natural state. If certain parts of the body get superimposed by that operation then we shall suppress those surplus parts.

Formulas (III) will then show that *the material elements that are placed on one side of the cut and the other and that previously adhered to each other will exhibit a relative displacement that results from a translation and a rotation that are the same for all pairs of adjacent molecules on the same cut.*

Upon taking the origin of the coordinates to be the center of reduction of the displacements, the constants of the cut will be the components of the translation and the rotation around the coordinate axes.

2. – Now consider things from a different viewpoint: Take a cyclic body in the natural state. In order to convert it into a state of tension, one can perform the operation that is inverse to the preceding one, i.e.:

1. *Section it in order to make it acyclic.*

2. *Displace the two faces of each cut* with respect to each other in such a manner that *the relative displacements of all pairs of elements (that adhered to each other before the cut separated) result from the same rotation and the same translation.*

3. *Re-establish the connectivity and continuity along each cut* by removing or adding the matter that would be necessary and soldering the parts that are in contact.

We say that the set of those operations constitutes a *distortion* and that the six characteristic constants of the cut can be called *the characteristics of the distortion.*

One sees that a distortion will lead to a state of equilibrium of the elastic body with internal deformation, which is a remarkable property of bodies with a cyclic form.

3. – The formulas that led us to those results show, on the one hand, that the cuts that can be deduced from each other by a continuous deformation have the same characteristic constants. It will then follow that the characteristics of a distortion are not linked with a well-defined cut, but only depend upon the deformation of the body and its geometric nature.

It is natural to call two cuts *equivalent* when they transform into each other by a continuous transformation. A distortion is known when the characteristics and the relevant cut or an equivalent one are given.

One can further say in a much more suggestive manner that in a cyclic body whose deformation is regular and which is subject to a certain number of distortions, an inspection of the deformation cannot in any way reveal the locations of the cuts and the ensuing distortions that were performed.

The number of independent distortions to which an elastic body can be subjected is obviously equal to the order of connectivity of the space that is occupied by the body minus one.

4. – The problems that pose themselves naturally here are the following ones:

1. Can one take the characteristics of the distortion arbitrarily and get a regular deformation of the body while supposing that the external actions are zero?

2. If the distortions are known then what is the state of deformation for arbitrary given external forces?

The solution to those problems is facilitated by the following theorem:

If one takes an arbitrary set of distortions in a multiply-connected, isotropic, elastic body then one can calculate an infinite number of regular deformations that correspond to those distortions and are equilibrated by external surface forces (which we denote by T) that for a system of vectors that is equivalent to zero (i.e., a zero resultant and a zero moment about a point).

We shall ultimately return to the proof of this theorem, but we shall first show how it permits one to study the problems that were posed above.

In order to see that the external forces on an isotropic body will be zero when it is given a distortion that corresponds to a state of equilibrium, from the preceding theorem, it will suffice to establish that the surface forces T that are applied to the contour of the body when it is not subject to any distortion will determine a state of regular deformation that equilibrates those forces.

Indeed, let Γ be the deformation that relates to the given distortion and the forces T that are found to act upon the boundary of the body, and let Γ' be the deformation that is determined by the forces $-T$ when the body is not subject to any distortion. The deformation Γ'' that results from Γ and Γ' will then correspond to the given distortion and zero external forces.

It remains for us to know whether Γ' exists at all and how one can obtain it. That is a problem in ordinary elastic equilibrium, since there is no distortion. On the other hand, the system of forces T is equivalent to zero, by hypothesis. Those forces then fulfill the fundamental conditions that are necessary for the existence of the deformation Γ', and one knows that except for certain singularities (which are pointless to specify here) in the geometric form of the elastic body, that existence is established.

The first problem that was posed is thus found to have been solved:

For a cyclic, isotropic body that is not subject to external actions, one can give the distortions *arbitrarily,* and the latter will correspond to *a state of equilibrium with no external forces and a regular deformation.*

5. – Now suppose that a distortion is given and that the body is subject to given non-zero external forces.

If the body is always supposed to be isotropic then one will get the solution to the problem if one superimposes the deformation Γ that is determined by the distortion and the external forces T with the deformation with no distortions that is determined by the given external forces and the external forces $-T$.

Hence, the second question that was posed is solved in theory.

By definition, the theorem that was stated at the beginning of the preceding number will permit one to eliminate the distortions and reduce the questions that are attached to it to questions of ordinary elastic equilibrium.

6. – The case of an anisotropic body is a bit different. One easily sees that the state of deformation Γ is equilibrated by some surface forces T and *volume forces.* Even in that case, it will then be easy to eliminate the distortions and to then reduce it to a problem of ordinary elastic equilibrium.

7. – It remains for us to prove the fundamental theorem that was stated in no. **4**. We shall first imagine a simple special case.

Let σ be an area in the xz-plane that is simply-connected and finite and does not intersect Oz. Turn the xz-plane around Oz and suppose that the area σ deforms without changing its nature and without intersecting Oz during that motion, but in such a way that it will return to its original configuration after it completes a circuit.

We thus generate a doubly-connected annular solid. Suppose that it is filled with an isotropic, homogeneous elastic substance. Impose the most-general distortion upon that body that is obtained means of a cut through a plane that passes through Oz and study the deformation.

The way that one generates the body will suggest the probable form of the solution. If l, m, n, p, q, r are the characteristic constants of the distortion then the following functions:

$$\frac{1}{2\pi}(l + qz - ry)\arctan\frac{y}{x},$$

$$\frac{1}{2\pi}(m + rx - pz)\arctan\frac{y}{x},$$

$$\frac{1}{2\pi}(n + py - qx) \arctan \frac{y}{x}$$

will be biharmonic, and due to the polydromy of arctan y / x, they will be polydromic and have the polydromy that corresponds to the distortion with characteristics l, m, n, p, q, r.

However, those functions do not satisfy the equations of elasticity, namely:

$$(13) \qquad \begin{cases} K\Delta^2 u + (L+K)\dfrac{\partial\Theta}{\partial x} = 0, \\ K\Delta^2 v + (L+K)\dfrac{\partial\Theta}{\partial y} = 0, \\ K\Delta^2 w + (L+K)\dfrac{\partial\Theta}{\partial z} = 0 \end{cases}$$

(in the case of isotropy and zero volume forces).

We shall then add some well-defined monodromic functions to the preceding three functions in such a fashion that we will obtain three functions that do verify equations (13).

We can take those monodromic functions to be functions of the form:

$$f_1 \log (x^2 + y^2), \qquad f_2 \log (x^2 + y^2), \qquad f_3 \log (x^2 + y^2),$$

in which f_1 , f_2 , f_3 are polynomials of degree one in x, y, z. An identifications will easily give the values of the coefficients of f_1 , f_2 , f_3 as functions of l, m, n, p, q, r and L, K.

By definition, one will then get the following expressions of the desired form:

$$(14) \qquad \begin{cases} u = \dfrac{1}{2\pi}\left\{(l + qz - ry)\arctan\dfrac{y}{x} + \dfrac{1}{2}\left(-m + pz + \dfrac{rK}{L+2K}x\right)\log(x^2 + y^2)\right\}, \\ v = \dfrac{1}{2\pi}\left\{(m + rx - pz)\arctan\dfrac{y}{x} + \dfrac{1}{2}\left(l + qz + \dfrac{rK}{L+2K}y\right)\log(x^2 + y^2)\right\}, \\ w = \dfrac{1}{2\pi}\left\{(n + py - qx)\arctan\dfrac{y}{x} + \dfrac{1}{2}(px + qy)\log(x^2 + y^2)\right\}. \end{cases}$$

That deformation is indeed regular and answers the statement of the theorem to be established. One effortlessly calculates the surface stresses T that correspond to them.

8. – We shall now give the general proof of the theorem in question.

We start from Somigliana's [**48**] fundamental functions:

$$u_1 = \frac{1}{r} + \frac{\alpha}{2}\frac{\partial^2 r}{\partial x^2}, \qquad v_1 = \frac{\alpha}{2}\frac{\partial^2 r}{\partial x\,\partial y}, \qquad w_1 = \frac{\alpha}{2}\frac{\partial^2 r}{\partial x\,\partial z},$$

$$u_2 = \frac{\alpha}{2}\frac{\partial^2 r}{\partial y\,\partial x}, \qquad v_2 = \frac{1}{r} + \frac{\alpha}{2}\frac{\partial^2 r}{\partial y^2}, \qquad w_2 = \frac{\alpha}{2}\frac{\partial^2 r}{\partial y\,\partial z},$$

$$u_3 = \frac{\alpha}{2}\frac{\partial^2 r}{\partial z\,\partial x}, \qquad v_1 = \frac{\alpha}{2}\frac{\partial^2 r}{\partial z\,\partial y}, \qquad w_1 = \frac{1}{r} + \frac{\alpha}{2}\frac{\partial^2 r}{\partial z^2},$$

in which r denotes the distance between two points (x, y, z) and (ξ, η, ζ). Those functions are symmetric with respect to the pairs of variables x, ξ ; y, η ; z, ζ ; they are singular only for:

$$x = \xi, \quad y = \eta, \quad z = \zeta.$$

Each term in those functions u_s , v_s , w_s (s = 1, 2, 3) verifies equations (13) or the equations that one deduces from them by substituting ξ, η, ζ for x, y, z in all of space (except for the indicated singularity). When u_s , v_s , w_s are considered to be functions of x, y, z or ξ, η, ζ, they can then be taken to be the components of the displacements of the points of a homogeneous, isotropic medium with no volume forces.

Let $d\sigma$ be a surface element around the point (ξ, η, ζ) then, and let X_s , Y_s , Z_s be the components of the unit stresses that correspond to the displacements u_s , v_s , w_s and are exerted upon the part of the elastic medium that is placed on the positive side of the normal to $d\sigma$ by the part that is placed on the negative side of the same normal. The calculation of X_s , Y_s , Z_s is immediate [cf., Chap. I, formulas (4) and (7)].

Now, if u_0 , v_0 , w_0 are integrals of equations (13) that are regular in the domain S that is bounded by a surface Σ, and if X_0 , Y_0 , Z_0 are components of the corresponding tension then the Somigliana formulas will give:

$$(15)\quad \begin{cases} \dfrac{1}{4\pi K}\displaystyle\int_\Sigma (X_1\,u_0 + Y_1\,v_0 + Z_1\,w_0)\,ds - \dfrac{1}{4\pi K}\displaystyle\int_\Sigma (X_0\,u_1 + Y_0\,v_1 + Z_0\,w_1)\,ds = u_0(x, y, z), \\ \dfrac{1}{4\pi K}\displaystyle\int_\Sigma (X_2\,u_0 + Y_2\,v_0 + Z_2\,w_0)\,ds - \dfrac{1}{4\pi K}\displaystyle\int_\Sigma (X_0\,u_2 + Y_0\,v_2 + Z_0\,w_2)\,ds = v_0(x, y, z), \\ \dfrac{1}{4\pi K}\displaystyle\int_\Sigma (X_3\,u_0 + Y_3\,v_0 + Z_3\,w_0)\,ds - \dfrac{1}{4\pi K}\displaystyle\int_\Sigma (X_0\,u_3 + Y_0\,v_3 + Z_0\,w_3)\,ds = w_0(x, y, z), \end{cases}$$

in which one supposes that the point (x, y, z) is interior to the domain S and that ξ, η, ζ represent the coordinates of the points of Σ, and one takes the normal to Σ to point into the domain S. When the point (x, y, z) is external to S, the right-hand sides of (15) will be zero.

Finally, take:

$$(16)\qquad u_0 = l + qz - ry, \qquad v_0 = m + rx - pz, \qquad w_0 = n + py - qx$$

in the relations (15), in which l, m, n, p, q, r are constants; the corresponding stresses will obviously be zero, and the integrals:

$$U = \frac{1}{4\pi K}\int_\Sigma (X_1 u_0 + Y_1 v_0 + Z_1 w_0)\,d\sigma\,,$$

$$V = \frac{1}{4\pi K}\int_\Sigma (X_2 u_0 + Y_2 v_0 + Z_2 w_0)\,d\sigma\,,$$

$$W = \frac{1}{4\pi K}\int_\Sigma (X_3 u_0 + Y_3 v_0 + Z_3 w_0)\,d\sigma$$

will be equal to:

$$l + qz - ry, \qquad m + rx - pz, \qquad n + py - qx,$$

respectively, if (x, y, z) is interior to S and zero if (x, y, z) is exterior to S. They will then be discontinuous upon crossing Σ, while the corresponding Γ_{rs} :

$$\Gamma_{11} = \frac{\partial U}{\partial x}, \qquad \Gamma_{12} = \frac{\partial V}{\partial x} + \frac{\partial U}{\partial y}, \qquad \ldots$$

will be zero identically in all space.

9. – Divide the surface Σ into two parts σ and σ' then and define u, v, w, and u', v', w' by integrals that are analogous to the ones that give U, V, W, but are extended over the open surfaces σ and σ', respectively.

Upon considering u, v, w, for example, it is clear that:

1. They are finite, continuous, and monodromic functions of x, y, z, along with their derivatives of arbitrary order at any point in space that does not belong to σ.

2. They verify equations (13) and thus define the displacements of a homogeneous, isotropic, elastic medium that is not subject to any volume forces.

3. Upon crossing σ, u, v, w will experience the same discontinuity as U, V, W, in such a way that if one distinguishes the values of those functions interior and exterior to the domain S by i and e, resp., then will have:

$$\text{(17)} \qquad \begin{cases} u_i - u_e = l + qz - ry, \\ v_i - v_e = m + rx - pz, \\ w_i - w_e = n + py - qx. \end{cases}$$

4. By contrast, the γ_{rs} (as well as the Γ_{rs}) that are calculated by starting from u, v, w will remain continuous upon crossing σ, in such a way that u, v, w are regular in all space, except at most on the contour of σ.

Finally, if one substitutes the values (16) of u_0, v_0, w_0 in the integrals that give u, v, w and arrange the right-hand sides with respect to l, m, n, p, q, r then one will arrive at the following theorem:

$$(18)\begin{cases} A_{i1}^{(\sigma)} = \dfrac{1}{4\pi K}\displaystyle\int_\sigma X_i\,d\sigma, & A_{i2}^{(\sigma)} = \dfrac{1}{4\pi K}\displaystyle\int_\sigma Y_i\,d\sigma, \quad A_{i3}^{(\sigma)} = \dfrac{1}{4\pi K}\displaystyle\int_\sigma Z_i\,d\sigma, \\ B_{i1}^{(\sigma)} = \dfrac{1}{4\pi K}\displaystyle\int_\sigma (\eta Z_i - \zeta Y_i)\,d\sigma, & B_{i2}^{(\sigma)} = \dfrac{1}{4\pi K}\displaystyle\int_\sigma (\zeta X_i - \xi Z_i)\,d\sigma, \quad B_{i3}^{(\sigma)} = \dfrac{1}{4\pi K}\displaystyle\int_\sigma (\xi Y_i - \eta X_i)\,d\sigma, \end{cases}$$

$$(19)\qquad \begin{cases} u = A_{11}^{(\sigma)} l + A_{12}^{(\sigma)} m + A_{13}^{(\sigma)} n + B_{11}^{(\sigma)} p + B_{12}^{(\sigma)} q + B_{13}^{(\sigma)} r, \\ v = A_{21}^{(\sigma)} l + A_{22}^{(\sigma)} m + A_{23}^{(\sigma)} n + B_{21}^{(\sigma)} p + B_{22}^{(\sigma)} q + B_{23}^{(\sigma)} r, \\ w = A_{31}^{(\sigma)} l + A_{32}^{(\sigma)} m + A_{33}^{(\sigma)} n + B_{31}^{(\sigma)} p + B_{32}^{(\sigma)} q + B_{33}^{(\sigma)} r, \end{cases}$$

in which l, m, n, p, q, r are arbitrary constants. One can regard u, v, w as the components of the displacement of an indefinite elastic medium that is isotropic, homogeneous, and has a deformation that is regular in all space, except at most on the contour of σ. That medium is in equilibrium with no volume forces. Finally, the displacements have discontinuities on σ that are given by the preceding equations (17). *In the case where σ is an open surface, when one calculates the quantities u, v, w by means of formulas* (I) *of the previous chapter upon starting with the characteristics of the deformation, they will be polydromic, and the lines of separation (diramation) will be composed of the contour σ and the polydromy will be defined by formulas* (17).

10. – In order to establish the theorem in no. **4**, imagine a body S whose order of connectivity is $n + 1$ and render it simply-connected by means of n cuts along the surfaces σ_1, σ_2, …, σ_n. Some formulas that are analogous to equations (19), but in which the arbitrary constants l_i, m_i, n_i, p_i, q_i, r_i, and the functions $A_{rs}^{(\sigma)}$, $B_{rs}^{(\sigma)}$, that define a regular transformation in S for each cut appear, will correspond arbitrary distortions along the cuts. The corresponding volume forces will obviously be zero. On the contrary, the external forces that act on the boundary of S will not be zero, in general; however, since S is in equilibrium, they will form a system that is equivalent to zero.

IV. – The energy of deformation in the case of distortions. Stresses.

1. – Recall the essential difference between elastic equilibrium in a simply-connected body and in a multiply-connected one. In the latter case, one can determine a state of stress in the body in the absence of any external force by means of the distortions.

The essential problem of the theory of distortions is simply to determine the stress states that are due to given distortions, along with the corresponding deformations.

In order to make the solution of that problem easier, along with the ones that will be posed later, we shall briefly present some general remarks on the energy of deformation.

We have already seen that the internal tensions at each point of an elastic solid are characterized by the six quantities t_{rs} that are obtained as functions of the characteristics of deformation γ_{rs} and by the intermediary of the potential function E (γ_{rs}), which is homogeneous and of second degree, by means of the formulas:

$$t_{rs} = \frac{\partial E}{\partial \gamma_{rs}}\,; \tag{7}$$

one can then write:

$$E = \tfrac{1}{2}\sum t_{rs}\,\gamma_{rs}\,. \tag{7′}$$

The energy of the system will then be:

$$\mathcal{E} = -\tfrac{1}{2}\int_S \sum t_{rs}\,\gamma_{rs}\,dS\,,$$

in which S represents the space that is occupied by the body.

The case that is of interest to us in the one in which S is multiply-connected. In that case, one can transform the preceding integral directly by means of Gauss's theorem with no precautions, because the components u, v, w of the displacement are polydromic functions. We shall specify them.

2. – Recall a formula from integral calculus that is directly associated with Gauss's theorem:

If U and V are two monodromic functions then one will have:

$$\int_S V\frac{\partial U}{\partial x}\,dS = -\int_S U\frac{\partial V}{\partial x}\,dS - \int_\sigma UV\cos nx\,d\sigma$$

for a volume S that has a boundary σ, where n denotes the interior normal to σ.

On the contrary, in the case where the function U is polydromic, a very simple argument will show that one has:

$$\int_S V\frac{\partial U}{\partial x}\,dS = -\int_S U\frac{\partial V}{\partial x}\,dS - \int_\sigma VU\cos nx\,d\sigma + \int_\omega V[U_\beta - U_\alpha]\cos \nu x\,d\omega\,,$$

upon denoting the cuts that render S simply-connected by ω, an area element of those cuts by $d\omega$, and the corresponding oriented normal by ν, and finally, denoting the values of U at a point on the positive or negative side of the normal ν to the cut by U_α and U_β, respectively.

3. – It is the latter formulas and the analogous formulas for the derivatives at y and z that we shall apply in order to transform $\mathcal{E}$, since $\mathcal{E}$ is a sum of integrals such as:

$$-\int t_{11}\frac{\partial u}{\partial x}\,dS\,,$$

and the functions u, v, w are polydromic in the case envisioned.

Imagine that the cuts ω_1, ω_2, …, ω_n that render S simply-connected have been made and let ν_i denote the oriented normal at a point of the cut ω_i, let $u_\alpha^{(i)}$, $v_\alpha^{(i)}$, $w_\alpha^{(i)}$ denote the values of u, v, w on the cut on the positive side of the normal, and let $u_\beta^{(i)}$, $v_\beta^{(i)}$, $w_\beta^{(i)}$ denote the values on the other side.

Under the stated transformation of $\mathcal{E}$, one will then have:

$$\mathcal{E} = \tfrac{1}{2}\int_S \sum u\left(\frac{\partial t_{11}}{\partial x}+\frac{\partial t_{12}}{\partial y}+\frac{\partial t_{13}}{\partial z}\right)dS \tag{20}$$

$$+\ \tfrac{1}{2}\int_\sigma \sum u\,(t_{11}\cos nx+t_{12}\cos ny+t_{13}\cos nz)\,d\sigma$$

$$+\ \tfrac{1}{2}\sum_{i=1}^{n}\ \int_\sigma \sum (u_\alpha^{(i)}-u_\beta^{(i)})\,(t_{11}\cos n_i x+t_{12}\cos n_i y+t_{13}\cos n_i z)\,d\omega_i$$

Let l_i, m_i, n_i, p_i, q_i, r_i denote the characteristics of the distortion that relates to the cut ω_i, and let X_i, Y_i, Z_i represent the components of the force of unit tension that is exerted on each area element of that cut.

One first has:

$$u_\alpha^{(i)}-u_\beta^{(i)} = l_i + q_i\,z - r_i\,y,$$
$$v_\alpha^{(i)}-v_\beta^{(i)} = m_i + r_i\,x - p_i\,z,$$
$$w_\alpha^{(i)}-w_\beta^{(i)} = n_i + p_i\,y - q_i\,x.$$

Moreover, if the external forces are zero then the first two integrals in the expression of ($\mathcal{E}$) will obviously be zero and the expression will reduce to:

$$\mathcal{E} = \tfrac{1}{2}\sum_{i=1}^{n}\int_{\omega_i}[(l_i-r_i\,y+q_i\,z)X_i+(m_i+r_i\,x-p_i\,z)Y_i+(n_i+p_i\,y-q_i\,x)Z_i]\,d\omega_i$$

$$= \tfrac{1}{2}\sum_{i=1}^{n}\Bigg[\,l_i\int_{\omega_i}X_i\,d\omega_i+m_i\int_{\omega_i}Y_i\,d\omega_i+n_i\int_{\omega_i}Z_i\,d\omega_i$$

$$+\ p_i\int_{\omega_i}(-Y_i\,z+Z_i\,y)\,d\omega_i+q_i\int_{\omega_i}(X_i\,z-Z_i\,x)\,d\omega_i+r_i\int_{\omega_i}(Y_i\,x-X_i\,y)\,d\omega_i\Bigg].$$

If one sets:

$$L_i = \int_{\omega_i} X_i \, d\omega_i , \qquad M_i = \int_{\omega_i} Y_i \, d\omega_i , \qquad N_i = \int_{\omega_i} Z_i \, d\omega_i ,$$

$$P_i = \int_{\omega_i} (Z_i \, y - Y_i \, z) \, d\omega_i , \qquad Q_i = \int_{\omega_i} (X_i \, z - Z_i \, x) \, d\omega_i , \qquad R_i = \int_{\omega_i} (Y_i \, x - X_i \, y) \, d\omega_i$$

then one will finally have:

$$\mathcal{E} = \tfrac{1}{2} \sum_{i=1}^{n} (L_i \, l_i + M_i \, m_i + N_i \, n_i + P_i \, p_i + Q_i \, q_i + R_i \, r_i) . \tag{21}$$

One can write that formula in a simpler manner by letting $s_1 , s_2 , s_3 , \ldots, s_{6n}$ denote the $6n$ characteristics of the distortions, and let $E_1 , E_2 , E_3 , \ldots, E_{6n}$ denote the coefficients that correspond to them in the expression (16); one will then get:

$$\mathcal{E} = \tfrac{1}{2} \sum_{i=1}^{6n} E_i \, s_i . \tag{21′}$$

If there exist external forces that are applied to the elastic body, in addition to the distortions, then if one lets:

$$\rho X \, dS, \qquad \rho Y \, dS, \qquad \rho Z \, dS$$

denote the volume forces and lets:

$$X_\sigma \, d\sigma, \qquad Y_\sigma \, d\sigma, \qquad Z_\sigma \, d\sigma$$

denote the surface forces then:

$$\frac{\partial t_{11}}{\partial x} + \frac{\partial t_{12}}{\partial y} + \frac{\partial t_{13}}{\partial z} = \rho X, \qquad \frac{\partial t_{21}}{\partial x} + \frac{\partial t_{22}}{\partial y} + \frac{\partial t_{23}}{\partial z} = \rho Y, \qquad \frac{\partial t_{31}}{\partial x} + \frac{\partial t_{32}}{\partial y} + \frac{\partial t_{33}}{\partial z} = \rho Z,$$

$$\begin{aligned} X_\sigma &= t_{11} \cos nx + t_{12} \cos ny + t_{13} \cos nz , \\ Y_\sigma &= t_{21} \cos nx + t_{22} \cos ny + t_{23} \cos nz , \\ Z_\sigma &= t_{31} \cos nx + t_{32} \cos ny + t_{33} \cos nz , \end{aligned}$$

and the energy of the system will be obtained by adding:

$$\int_S \rho \, (Xu + Yv + Zw) \, dS + \int_\sigma (X_\sigma u + Y_\sigma v + Z_\sigma w) \, d\sigma$$

to the right-hand side of the preceding equation (21′).

4. – We call the distortion that corresponds to characteristic quantities s_i that are all zero, except for one, which is equal to unity, an *elementary distortion*.

Suppose that the latter is s_h , and let E_{ih} denote the corresponding values of the coefficients E_i . One sees that in the general case where the distortions have characteristics s_1 , s_2 , s_3 , …, s_{6n} , one will have:

$$E_i = \sum_h E_{ih}\, s_h \, .$$

(The expressions for the coefficients E_i that were given above indeed show that those coefficients are linear, homogeneous functions of the characteristics of the cuts.) One will then have, in turn:

$$E = \tfrac{1}{2} \sum_l \sum_h E_{ih}\, s_i\, s_h \, . \tag{22}$$

It is interesting to specify the significance of the quantities E_i and E_{ih} .

To that effect, observe that L_i , M_i , N_i are the components of the resultant force and P_i, Q_i , R_i are the components of the resultant couple of the stresses that act upon the section ω_i when one takes the origin of the axes to be the center of reduction.

We then let L_i , M_i , N_i , P_i , Q_i , R_i *denote the stresses that act in the section* ω_i , and in general, we will say that E_1 , E_2 , E_3 , …, E_{6n} are the *stresses that correspond to the distortions* s_1 , s_2 , s_3 , …, s_{6n} , resp. E_{ih} is called *the stress of order i that is induced by the distortion of order h* [i.e., it is defined by $s_i = 0$ ($i \neq h$), $s_h = 1$]. Even more simply, *the coefficients* E_{ih} *can be called the stress coefficients.*

5. – One can now give a fundamental property of the coefficients E_{ih} .

Green proved a fundamental proposition in potential theory by applying Gauss's theorem. However, the application of Green's method is not limited to the case of potentials, since it can be extended to considerable number of cases. Vito Volterra showed that it can be extended to all problems that depend upon the calculus of variations. Now, the problems in elasticity, like all problems in mechanics, are attached to questions in the calculus of variations. That is why one has a theorem in elasticity that is analogous to Green's.

It was given for the first time by Betti, who made some remarkable applications in the integration of problems in elastic equilibrium.

Now, if the displacements are polydromic then since Gauss's theorem no longer applies, Betti's theorem will no longer be applicable, either. One sees that even in that case, upon recalling Green's fundamental idea, one will be led to an interesting law of reciprocity for the stress coefficients.

Suppose that we apply two distortions to an elastic body in succession that are characterized by the values $s_1', \ldots, s_{6n}'$; $s_1'', \ldots, s_{6n}''$ of the characteristics.

Let γ_{rs}' and γ_{rs}'' be the values of γ_{rs} that correspond to two deformations and let E' and E'' be the values of the elastic potential. From a well-known theorem on homogeneous functions of degree two, one will have:

$$\int_S \sum \frac{\partial E'}{\partial \gamma'_{rs}} \gamma''_{rs}\, dS = \int_S \sum \frac{\partial E''}{\partial \gamma''_{rs}} \gamma'_{rs}\, dS\,,$$

and if one transforms those integrals after rendering the volume S acyclic by cuts, while always regarding the faces of the cuts as belonging to the boundary of S, then one will find that:

$$\sum_{i=1}^{6n} E''_i s'_i = \sum_{i=1}^{6n} E'_i s''_i\,,$$

in which E'' and E' are the values of the quantities E that correspond to two distortions.

One then deduces that:

$$\sum_i \sum_h E_{ih}\, s'_i\, s''_h = \sum_i \sum_h E_{ih}\, s''_i\, s'_h\,. \tag{23}$$

One can further say that:

If two systems of distortions in a multiply-connected body generate two systems of stresses then the sum of the products of the stresses of the first system of distortions with the characteristics of the second system will be equal to the sum of the products of the stresses in the second system of distortions with the characteristics of the first system.

Since the quantities s' and s'' are arbitrary, one must necessarily have:

$$E_{ih} = E_{hi} \tag{24}$$

for all values of the indices i and h, and conversely, those equalities will imply the relation (23) as a consequence. The reciprocity theorem that we just obtained can then be stated as follows:

The stress of order i that is induced by the elementary distortion of order h is equal to the stress of order h that is induced by the elementary distortion of order i,

or more simply:

The values of the stress coefficients do not change under a transposition of the indices.

6. – Given the numerous applications of the reciprocity theorem, it would not be pointless to examine them further from another viewpoint.

Take two arbitrary sections σ_1 and σ_2 of the elastic body, which are sections that can also coincide, and perform a distortion that is consistent with a relative translation T_1 along the direction h_1 of the elements of the two faces of σ_1. Then determine the projection S_2 along the direction h_2 of the resultant of the stresses that act in the section

σ_2 . On the contrary, in place of the preceding distortion, imagine another distortion that consists of a translation T_2 along the direction h_2 and determine the projection S_1 along h_1 of the resultant of the stresses that act in the section σ_1 .

The reciprocity theorem gives:

$$S_2\, T_2 = S_1\, T_1 \quad \text{or} \qquad \frac{S_2}{T_1} = \frac{S_1}{T_2},$$

so the projections of the two stresses along the directions of the two translations are proportional to the values of the translations themselves.

One will get an analogous theorem by replacing the translation T_1 with the rotation T_1 around a line h_1 provided that one replaces the projection S_1 of the resultant of the stresses that act in the elements of σ_1 with the moment of those stresses with respect to the line h_1 .

Finally, one will get a third statement that is analogous to the first two by similar substitutions that bear upon T_2 and S_2 .

7. – Finally, we remark that from the expression (22) for the energy, the equality of the coefficients E_{ih} and E_{hi} will imply that:

$$E_i = \frac{\partial E}{\partial s_i}. \tag{25}$$

Upon introducing the coefficients e_{ih} of the adjoint form, the energy will take the following expression:

$$E = \tfrac{1}{2}\sum_i \sum_h e_{ih}\, E_i\, E_h\ . \tag{26}$$

8. – We have already seen that if one is given a deformation of a multiply-connected system then the distortions will not be specific to the cut that was performed, but the distortions that correspond to equivalent cuts will be equal. We shall complete that proposition by showing that the same thing will be true for the stresses that correspond to two equivalent cuts.

Let σ_1, σ_2 be two equivalent cuts. One passes from one to the other by a continuous deformation such that σ_1 sweeps out a portion S_1 of the elastic body under that deformation.

Consider the solid S_1 . It is in equilibrium under the action of only the stresses that act in σ_1 and σ_2 . The equality of the stresses will indeed result, and like the distortions, those stresses will depend upon only the geometric nature of the space that is occupied by the body.

The first problem that one can propose to solve in order to arrive at the solution to the general problem that was posed at the beginning of this paragraph will then be:

Begin given the 6*n distortions, determine the* 6*n stresses when one supposes that the external forces are zero.*

That amounts to the determination of the stress coefficients.

9. – It is interesting to specify how one can generalize the reciprocity theorem in no. **5** [formula (23)] when volume forces and surface stresses intervene.

Imagine two more cases of equilibrium in a body, one of which involves the distortions s_1', s_2', …, s_n', the volume forces:

$$\rho X' dS, \qquad \rho Y' dS, \qquad \rho Z' dS,$$

and the surface forces:

$$X'_\sigma\, d\sigma, \qquad Y'_\sigma\, d\sigma, \qquad Z'_\sigma\, d\sigma,$$

and the other of which involves the distortions s_1', s_2', …, s_n', and the volume and surface forces:

$$\rho X'' dS, \qquad \rho Y'' dS, \qquad \rho Z'' dS,$$

$$X''_\sigma\, d\sigma, \qquad Y''_\sigma\, d\sigma, \qquad Z''_\sigma\, d\sigma,$$

resp.

An easy calculation shows that equation (23) is replaced with the following one:

$$\sum_{i=1}^{6n} E_i'' s_i' + \int_S \rho (X'' u' + Y'' v' + Z'' w')\, dS + \int_\sigma (X''_\sigma u' + Y''_\sigma v' + Z''_\sigma w')\, d\sigma$$

$$= \sum_{i=1}^{6n} E_i' s_i'' + \int_S \rho (X' u'' + Y' v'' + Z' w'')\, dS + \int_\sigma (X'_\sigma u'' + Y'_\sigma v'' + Z'_\sigma w'')\, d\sigma .$$

When there are only distortions and the forces are absent, one will fall back on the results of the preceding numbers. When only the forces exist, and the distortions are absent, one will recover Betti's theorem. Finally, when one considers two states of equilibrium, one of which has no distortions and the other of which has no applied forces, one will get Colonetti's theorem ([1]).

([1]) Colonetti obtained that theorem in 1912 [**12**], where he established it directly and independently of the preceding propositions. In a paper in 1907 [**9**] that was published in the *Annales scientifiques de l'École Normale Superéure* and in the Notes that preceded it, Vito Volterra considered only the case of distortions alone (preceding nos. 5-7). Colonetti called his theorem the *second reciprocity principle*, and he made a large number of applications of it. We shall return to it in the last chapter.

V. – Distortions and stresses in a symmetric cyclic body.

1. – We shall not pursue the general study of the problem of distortions. We shall now study the results that one obtains with some particular distortions that are performed on symmetric, cyclic bodies by using the preceding principles and results.

We shall achieve that goal without taking recourse to the integration of the equations of elasticity, but simply by examining the expression for energy.

Here, we shall once more verify that a deeper study is quite necessary in this type of questions. Upon being guided by our intuition, we will be led to think, for example, that in the case of the ring when one removes a very thin radial wedge and then solders the two faces of the cut together, the body will take on a state of regular deformation such that the actions that are exerted on the two soldered faces are *tensions*. *That is not true*, and we will see that in the new state of the ring, there is always a tensed part (external boundary) and a compressed part (internal boundary); moreover, the sum of the forces of tensions is equal to the sum of the forces of compression.

2. – Before giving the proof of that interesting proposition, we shall make a general study of the stresses in a symmetric, cyclic body.

Let us first give some definitions: The quantities E_i enter into the expression for energy (21′):

$$E = \frac{1}{2}\sum_{i=1}^{6n} E_i\, s_i \,.$$

We shall call E_i the *conjugate stress* to the characteristic s_i of the distortion.

We know that the distortion that is applied to each cut is composed of a relative translation and a rotation of the faces of the cut, which is a translation and a rotation that have well-defined components when one has chosen the coordinate axes and their origin. If we use that origin as the center of reduction for the actions that are exerted on the elements of one face of the same cut then we will find a resultant force and a resultant couple by proceeding as if those actions were exerted upon a rigid system.

The preceding definition indicates that the components of that resultant force are the conjugate stresses to the components of the translation, and the components of the resultant couple are the conjugate stresses to the components of the rotation. If the distortion is elementary then only one of the characteristics will be non-zero. The components of the force or couple that is conjugate to that characteristic will be called the conjugate stress to the elementary distortion.

3. – A solid of revolution can be generated by the revolution of a connected planar surface (viz., the generating surface) around an axis in its plane. Let n be the order of connectivity of the generating surface: If its axis is external then the order of connectivity of the solid will be $n + 1$, but if, on the contrary, the axis constitutes one part of the contour of the generating surface then the order of connectivity of solid will be equal to n.

Make the generating surface simply-connected by means of n – 1 linear cuts. Under rotation, those cuts will generate just as many surfaces that can be considered to be section of the solid. In the second case, those sections will suffice to make the solid simply-connected, whereas in the first case, it will again be necessary to make a transverse cut that coincides with one of the positions of the generating surface.

The last cut (or any equivalent one) will be said to be *of the first kind*, while the other cuts (or their equivalents) will be said to be *of the second kind.*

For example, take a simply-connected surface that is external to the axis of revolution or else a surface that is doubly-connected, but bounded by the axis (Fig. 3*a* and 3*b*). In one and the other case, one will get a solid with an order of connectivity that is equal to 2.

In order to make the body simply-connected, we shall make a cut *of the first kind* in the former case, and a cut *of the second kind* in the latter.

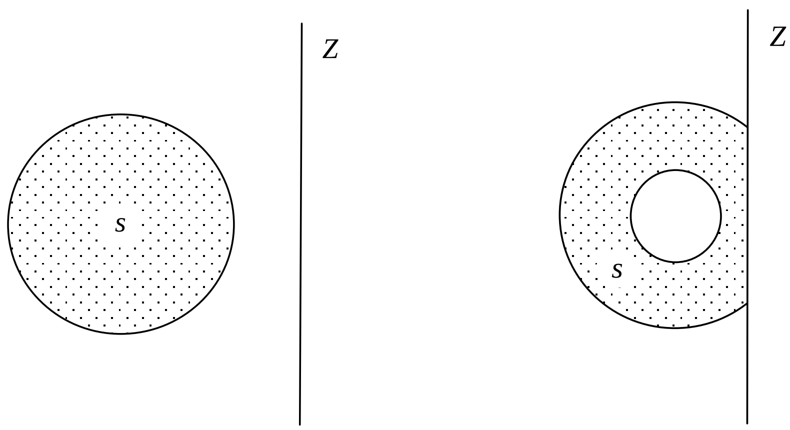

Figure 3*a*. Figure 3*b*.

We say that the first body is *doubly-connected of the first kind* and that the second one is *doubly-connected of the second kind.*

4. – Consider a symmetric elastic body that is doubly-connected of the first kind such that the symmetry is not limited in form, but also extends to the constitution of the elastic matter under the hypothesis of isotropy. We shall study the energy of that material system by supposing that a distortion has been performed along a cut σ that is made along one is the position of the generating surface.

Take the coordinate origin O to be along the axis of revolution, and take that axis to be Oz.

Let:

$$s_1 = l, \qquad s_2 = m, \qquad s_3 = n, \qquad s_4 = p, \qquad s_5 = q, \qquad s_6 = r$$

be the characteristics of the distortion.

The energy of the system has the preceding expression:

(22) $$E = \tfrac{1}{2}\sum_{i=}^{6}\sum_{h=1}^{6} E_{ih}\, s_i\, s_h\ ,$$

which we shall simplify by some symmetry considerations.

Apply the given distortion to a section σ' that makes an angle θ with σ. Since the two sections are equivalent, the energy will be the same: It will then follow that it must not be modified when one applies the distortion:

$$s_1' = s_1 \cos\theta + s_2 \sin\theta, \qquad s_2' = -s_1 \cos\theta + s_2 \sin\theta, \qquad s_3' = s_3\,,$$
$$s_4' = s_4 \cos\theta + s_5 \sin\theta, \qquad s_5' = -s_4 \cos\theta + s_5 \sin\theta, \qquad s_6' = s_6$$

to the section ω, instead of the given distortion.

As a consequence, the expression:

$$E = \tfrac{1}{2}\sum_{i=}^{6}\sum_{h=1}^{6} E_{ih}\, s_i'\, s_h'$$

must be independent of θ, or:

$$\frac{dE}{d\theta} = 0.$$

Upon performing the calculations, one will find that E reduces to:

$$E = \tfrac{1}{2} E_{11}(s_1^2 + s_2^2) + E_{33}\, s_3^2 + E_{44}(s_4^2 + s_5^2) + E_{55}\, s_6^2$$
$$+ 2E_{14}\,(s_1\, s_4 + s_2\, s_5) + 2E_{24}\,(s_2\, s_4 - s_1\, s_5) + E_{36}\, s_3\, s_6\,.$$

The distortion that is due to a unique rotation around the Oz axis, or a distortion of order 6, gives a deformation such that the actions that are exerted upon σ are normal to σ (in order to see that it will suffice to utilize the symmetry with respect to σ).

Upon taking the origin to be the center of reduction of those actions, one will see that one obtains a resultant force that is normal to σ and a couple whose axis is parallel to σ. One will then have:

$$E_{16} = E_{36} = E_{56} = 0.$$

Upon envisioning the elementary distortion of order 2 – i.e., the one that is due to a translation that is parallel to Oy – one will see, by means of an analogous argument, that:

$$E_{12} = E_{32} = E_{52} = 0.$$

On the other hand, E_{14}, which is equal to E_{24}, as the formula that was written above will show, will also be equal to E_{52}, namely, zero, and the energy will be written:

$$E = \tfrac{1}{2}[E_{11}(s_1^2 + s_2^2) + E_{33}\, s_3^2 + E_{44}(s_4^2 + s_5^2) + E_{66}\, s_6^2 + 2E_{24}(s_2\, s_4 - s_1\, s_5)]\,.$$

The coefficient E_{11} is never zero, because if that were not true then the energy that corresponds to a distortion of order 1 or 2 could be zero. However, that would be absurd.

From the preceding, one easily deduces that the actions that σ is subject to under the effect of a distortion of order 2 are equivalent to a single force, since the reduction to the origin provides a force E_{22} that is parallel to Oy and a couple of moment E_{24} and axis Ox. That single force will cut Oz at a point Ω.

It is natural to take that point to be the center of reduction; i.e., to displace O to Ω.

One will then have the new system of axes $E_{24} = 0$, and as a result:

$$E = \tfrac{1}{2}[E_{11}(s_1^2 + s_2^2) + E_{33}\, s_3^2 + E_{44}(s_4^2 + s_5^2) + E_{66}\, s_6^2] \; . \tag{27}$$

In the case of symmetric bodies that are doubly-connected of the second kind, an argument that is analogous to the one that was just made will show that the energy E is expressed in the same fashion as in formula (27).

The privileged point of the axis Ω will be called the *central point* of the axis of revolution.

5. – It is not without interest to point out that we just repeated some calculations and arguments that are known in hydrodynamics for this question of elasticity. The expression (22) for energy can indeed be compared to the one for the *vis viva* of an indefinite liquid with no vortices in whose interior one finds a symmetric solid body.

6. – If one takes into account the principle of equivalent cuts then one will immediately deduce the following theorem from formula (27):

In a symmetric, doubly-connected body, each elementary distortion will generate just one stress that is the conjugate stress when one takes the central point of the axis to be the center of reduction.

That will imply the corollaries:

If the distortion is a relative translation of the elements of the two faces of the cut then the total stress that is generated will be a force that passes through the central point. If the distortion is a rotation around an axis that passes through the central point then the total stress will be a couple.

We add one last remark of a general order:

If a symmetric body has a plane of symmetry that is perpendicular to its axis then the central point will be the intersection of that axis with the symmetry plane.

7. – Recall the case of a symmetric ring that is a doubly-connected body of the first kind. Apply a distortion of order 6 to it.

From the preceding, the stress will reduce to a couple that has the symmetry axis for its axis; hence, the actions that a face of the cut is subjected to will have a resultant that is generally zero. Now, one can realize that distortion of order 6 by removing a thin radial wedge and soldering the two boundaries of the cut that was made.

One will then recover the result that was stated above: One part of the soldered joint is compressed, while the other is tensed, and the sum of the stresses of traction is equal to the sum of the compressive stresses.

One can complete those results in a very unexpected way by adding that the moment of the forces of tension exceed that of the forces of compression by the quantity E_{66} . Hence, the tensed part is the external part, while the compressed part is the internal one.

8. – Now consider the case of a distortion of order 2. The stress is then a force that is normal to the symmetry axis, which meets that axis at the central point, and there are once more compressed parts and other ones that are tensed, but the tensions will exceed the pressures by precisely the quantity E_{11}, which represents the preceding stress. Moreover, the moment of the former is equal to the moment of the latter with respect to the symmetry axis.

One sees here that the internal part is tensed, while the external part is compressed.

9. – The preceding two cases (distortion of order 6 or 2) correspond to removing a slice from the ring whose thickness is proportional to the distance x from the symmetry axis or whose thickness is constant, respectively.

Upon removing a slice whose thickness is given by:

$$s_2 - s_6\, x$$

and then soldering the two boundaries of the slice together, one will generate a stress that is normal to the section as a result, and its line of action will be at a distance of:

$$h = \frac{s_6}{s_1}\frac{E_{66}}{E_{11}}$$

from the symmetry axis. By a convenient choice of ratio s_6 / s_2 , one can do that in such a way that the line of action will be at an arbitrary distance from the symmetry axis.

Finally, imagine a distortion of order 3 that is obtained by sliding the two faces of the cut with respect to each other in the sense of the symmetry axis. From the general results, it is clear that the corresponding stress will have a line of action that is the symmetry axis. The elements of one face of the cut will shifted in the sense of the slide that was made, while the other one will be shifted in the opposite sense. Moreover, the moment of the former actions will be equal to the moment of the latter with respect to an axis that is normal to the section and meets the symmetry axis.

The other special cases will give rise to some considerations and conclusions that are analogous to the ones that we have just developed and formulated. We shall not dwell upon that, though.

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CHAPTER III.

SPECIAL CASES AND EXPERIMENTAL VERIFICATION

VI. – Distortions in a hollow cylinder of revolution.

1. – In order to get experimental confirmation of some of the results that were obtained, one can work with solids that are made of rubber, since it is easy to obtain appreciable deformations with it. The chosen form will be that of a hollow cylinder of revolution, and in order to compare the results of calculation and experiments as completely as possible, let us summarize the complete study of the distortions of such a body.

In such a study, which will complete the results that were obtained in the preceding paragraph by using only energetic considerations, one will follow the general method that was developed in Section III (no. **4**). The calculations will simplify in the case of an elastic body that forms a cylinder, moreover.

The symmetry around the axis indicates that the distortions of order 1 and 2 and the ones of order 4 and 5 will reduce to each other. It will then suffice to study the distortions of order 2, 3, 5, and 6.

2. – We shall first treat the simplest case, which is that of a distortion of order 6.

Formulas (14) of Section III, in which one sets:

$$l = m = n = 0, \qquad p = q = 0, \qquad r = 2\pi\alpha,$$

will give the displacements that correspond to a distortion of order 6 when the cylinder is subjected to external actions that are, as one can verify, uniform along the cylindrical surfaces that form the lateral contour of the body and along its bases.

One can easily eliminate the first of those actions – viz., the lateral tensions – by composing the displacements (14) with the displacements:

$$u = \lambda \frac{x}{r^2} + \mu\, x, \qquad u = \lambda \frac{y}{r^2} + \mu\, y, \qquad u = 0$$

and choosing the constants λ and μ conveniently. In fact, the latter formulas are suitable to the displacements that are produced by lateral actions.

One then arrives at the following displacements:

$$u = -\alpha\left[y \arctan\frac{y}{x} - \frac{1}{2}\frac{K}{L+2K}\, x \log r^2 + \frac{L+K}{L+2K} R_1^2 R_2^2 \frac{\log R_1^2 - \log R_2^2}{R_1^2 - R_2^2}\frac{x}{r^2} \right.$$
$$\left. + \frac{x}{2}\left(1 + \frac{K}{L+2K}\frac{R_1^2 \log R_1^2 - R_2^2 \log R_2^2}{R_1^2 - R_2^2}\right)\right],$$

$$v = -\alpha\left[-x\arctan\frac{y}{x} - \frac{1}{2}\frac{K}{L+2K}\,y\log r^2 + \frac{L+K}{L+2K}R_1^2R_2^2\frac{\log R_1^2 - \log R_2^2}{R_1^2 - R_2^2}\frac{y}{r^2}\right.$$
$$\left. + \frac{y}{2}\left(1 + \frac{K}{L+2K}\frac{R_1^2\log R_1^2 - R_2^2\log R_2^2}{R_1^2 - R_2^2}\right)\right],$$

$$w = 0,$$

which correspond to a distortion of order 6 (which is due to a radial fissure of angular opening $2\pi\alpha$), while the cylinder is subjected to only actions on its bases that keep those bases *planar* and *at their original distance.* R_1 and R_2 denote the radii of the lateral cylindrical surfaces.

In order to specify those actions, one calculates the characteristics t_{rs} of the stresses, which is immediate. One then obtains $t_{21} = t_{12} = 0$, which proves that the forces that act upon the bases are normal to them, and on the other hand:

$$P_\omega = t_{33} = \frac{\alpha LK}{L+2K}\left(1 + \log r^2 - \frac{R_1^2\log R_1^2 - R_2^2\log R_2^2}{R_1^2 - R_2^2}\right). \tag{28}$$

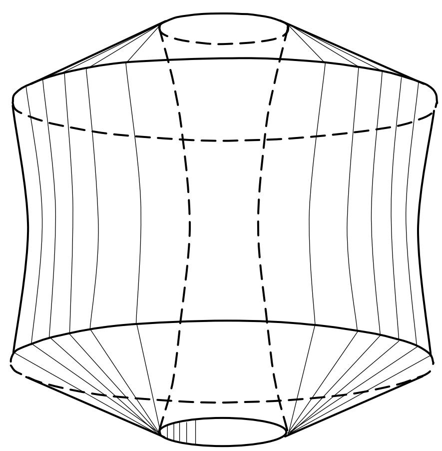

Figure 4.

One can then state the following proposition:

A hollow cylinder of revolution that is subject to a distortion of order 6 will preserve it base planes and their original distance with the aid of forces that act normally to those bases and are given (per unit area) by formula (28).

The distribution of those forces, which is easy to study, will vanish for just one value of r, which will have the expression:

$$\rho = \frac{R_1 + R_2}{2}\left[1 + \left(\frac{R_1 - R_2}{R_1}\right)^2\right],$$

when one neglects powers of:

$$\left(\frac{R_1 - R_2}{R_1}\right)^2$$

that are higher than the second.

A simple calculation shows that t_{33} can be written:

$$P_\omega = t_{33} = \frac{2\alpha LK}{L + 2K}\log\frac{r}{\rho},$$

or, upon introducing the modulus of elasticity E, the Poisson coefficient η, and the angular opening $\theta = 2\pi\alpha$ of the radial fissure that is made:

$$P_\omega = t_{33} = \frac{E\eta}{1-\eta^2}\frac{\theta}{2\pi}\log\frac{r}{\rho},$$

so t_{33} will be positive when r is less than ρ and negative when r is greater than ρ.

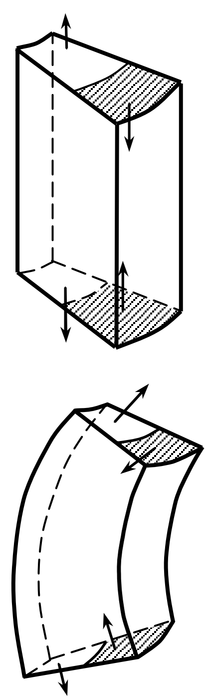

Figure 5.

Now, by the manner itself by which one calculated t_{33} in the form (28), the action that t_{33} represents will point from the exterior of the cylinder to its interior when t_{33} is positive and from the interior to the exterior when t_{33} is negative.

It will then result that on each base, one will have a compressed region and a tensed region that are separated by a circle of radius ρ [which is very close to $(R_1 + R_2) / 2$, in general], and the compressed region will be the internal part, while the tensed region will be the external part.

Now, suppress the actions on the bases and look for the form that the cylinder will then take by virtue of the single distortion with no external forces.

The internal part, which is free of compressions, will be raised; on the contrary, the external part will be lowered, since it is no longer stretched.

Figures 4 and 5 show the form that is taken by the body and the mechanism that schematizes the flexure of each radial wedge of the cylinder.

3. – We shall now study the case of the distortion of order 2, which is obtained by means of a fissure of uniform size. The deformation is then more noticeable and more singular than before, and the calculations to which one is led in its study are more complicated. We shall summarize them.

The general formulas (14), in which one sets:

$$l = n = p = q = r = 0,$$

give the displacements that correspond to a distortion of order 2 when the body is subject to surface actions, namely:

$$u = -\frac{1}{2}\frac{m}{2\pi}\log(x^2 + y^2),$$

$$v = \frac{m}{2\pi}\arctan\frac{y}{x}.$$

As in the preceding case, we can first eliminate the lateral actions by once more applying the general method of Section III. We indicate only the result: If R_1 and R_2 have the same significance as above then the displacements:

$$u = \frac{m}{2\pi}\left\{\frac{K}{L+2K}\log r + \frac{L+K}{2[L+2K]}\left(r^2 - \frac{R_1^2 R_2^2}{R_1^2 + R_2^2}\right)\frac{\partial^2 \log r}{\partial x^2}\right.$$
$$\left. + \frac{1}{2[L+2K][R_1^2 + R_2^2]}[(3L+5K)\,y^2 + (L+K)\,x^2]\right\},$$

$$v = \frac{m}{2\pi}\left\{\arctan\frac{y}{x} + \frac{L+K}{2[L+2K]}\left(r^2 - \frac{R_1^2 R_2^2}{R_1^2 + R_2^2}\right)\frac{\partial^2 \log r}{\partial x\,\partial y} - \frac{L+3K}{(L+2K)(R_1^2 + R_2^2)}\,xy\right\},$$

$$w = 0$$

will correspond to a distortion of order 2 (that is due to a fissure of uniform size *m*) when the cylinder is subjected to only actions on the bases that are forces capable of preserving their planes and the original distance.

One can further calculate the characteristics of the tensions t_{rs} here. One will have $t_{21} = t_{12} = 0$; the forces that act upon their bases will be normal. One also verifies that the external actions are zero on the lateral surfaces of the cylinder.

The study of the distribution of forces that act on the base is carried out here with the help of the cubic dilatation:

$$\theta = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{mKx}{\pi(L+2K)}\left[\frac{1}{r^2} - \frac{2}{R_1^2 + R_2^2}\right].$$

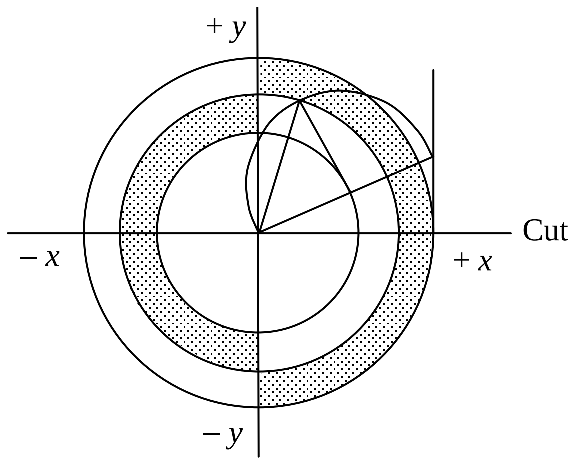

Figure 6.

Figure 6 represents the compressed and dilated regions within the limitation that involves the circle of radius:

$$\sqrt{\frac{R_1^2 + R_2^2}{2}}.$$

The shaded regions represent the compressed parts ($\theta < 0$) and the regions that are not shaded represent the dilated parts ($\theta > 0$).

Let us now study the form that is taken by the elastic cylinder while we always suppose that its bases are kept planar and at their original distance apart.

In order to do that, it will suffice to look for the deformation of the bases in their planes: Formulas (23) permit one to calculate the values of the displacements *u*, *v* on the circumferences σ_1, σ_2 of radii R_1, R_2 that form the original contour of the bases.

One confirms, by means of a simple decomposition, that if one neglects the second-order infinitesimals then the displacement of each point of σ_1 will be a translation that is parallel to *Oy* and proportional to the arc of σ_1 that is included between the origin of the arcs that are situated on *Ox* and the point itself. The same thing will be true for σ_2.

Figure 7 represents the original contours σ_1 , σ_2 in fine lines, and the contours that are obtained by the displacements are represented by thick lines.

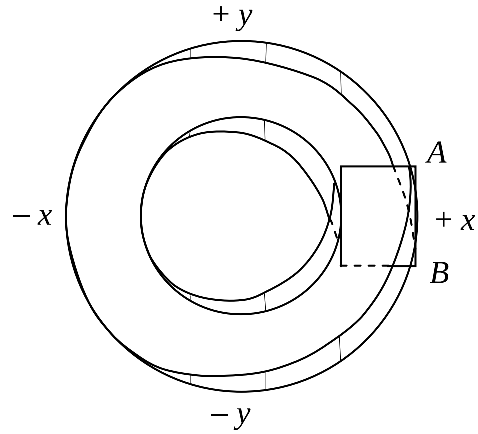

Figure 7.

Since the boundaries *A* and *B* are soldered to each other under the operation that was performed on the cylinder, one will see immediately the form that will be taken by the body after the distortion when one keeps its bases planar and at their original distance apart.

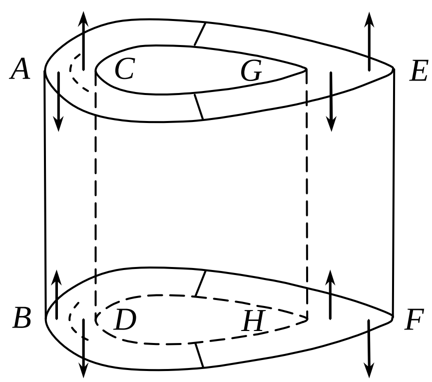

Figure 8.

Figure 8 shows that form, as well as the regions in which one must exert normal actions, which we can express as a function of the modulus of elasticity *E* and the Poisson coefficient η :

(29) $$P = t_{33} = -\frac{m}{2\pi}\frac{E\eta x}{1-\eta^2}\left[\frac{1}{r^2} - \frac{2}{R_1^2 + R_2^2}\right].$$

Now suppress those actions. In order to get the form that is taken on by the cylinder, one can study what happens for each very thin radial wedge by applying the general method that we gave: There will be flexure from the raising of the compressed parts and the lowering of the tensed parts. That will give the form of the cylinder (when it is subject to a distortion of order 2 with no external force).

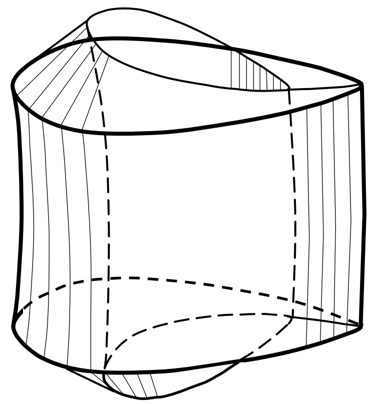

Figure 9.

4. – We shall rapidly pass over the cases of distortions of order 3 and 5. The distortions of order 3 are obtained by making a meridian section in the cylinder, sliding the two faces of the cut with respect to each other in the direction of the axis, and then soldering.

The state of deformation is easily calculated by starting from the general formulas (14). One will later see the form that the body takes under the influence of only that distortion (Fig. 9).

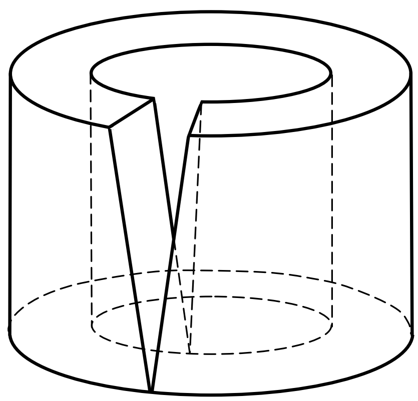

Figure 10.

The cases of distortions of order 4 or 5 can be reduced to the case of a distortion of order 2 by some simple integrations. We shall content ourselves by simply stating that property. The distortion of order 4 can be obtained by means of a wedge-shaped fissure, as is indicated in Fig. 10. After soldering the two faces of the cut, the body will take on a form that one can imagine, as in the preceding cases.

5. – All of the results are in perfect accord with experiments. Moreover, ironworkers indeed know about these deformation phenomena and take the necessary precautions when they have to shrink tubes.

However, it is appropriate to make some precise verifications. They were accomplished thanks to the courtesy of Mr. Jona, an engineer in Milan, who prepared some hollow rubber cylinders at the Pirelli factory that were subject to the various types of distortion. Those cylinders were cut, and then the two boundaries of the cut were brought back together and soldered in such a manner as to realize the various types of distortions. All of the peculiarities that were predicted by the theory were then verified.

Figures 11 through 16 (†) reproduce some photographs of plaster castings of cylinders after distortion, which are casts that are preserved in the collection of models at l'Institut Henri Poincaré. One can compare Figures 16 and 12, which give the casts for the distortions of order 6 and 2, respectively, with the drawings in Figures 4 and 8. Figure 14 represents the distortion of order 4 from two different sides. Finally, Figures 11, 13, 15 are concerned with the dislocations of order 1, 3, 5, respectively. All three of them correspond to an axial cut, and one then displaces the two faces of the cut by either sliding them normal to the cylinder axis, or by sliding along that axis, or by a rotation around the perpendicular to the two faces that is drawn through the center of the cylinder.

VII. – Cyclic system of pliable elastic elements.

1. – We previously posed (§ IV) a fundamental problem in the theory of distortions of multiply-connected elastic bodies: *Given the distortions in an elastic system, determine the stresses.* In the present paragraph, we shall present the principles of the solution to that problem in a case that presents a special interest. It is the case of a system that is composed of an assemblage of pliable elastic elements. We shall now define what we mean by that.

2. – First consider a rectilinear rod whose transverse dimensions (i.e., of a cross-section) are very small with respect to its length.

Let A and B be the material elements that constitute the extremities of that rod. When it is deformed, the relative displacements of A and B will generally be very large with respect to the pure deformation of the particles of A and B themselves, in such a way that one can imagine A and B approximately as rigid elements whose relative displacement results from a translation and a rotation.

We shall suppose, moreover, that the relative displacements of A and B are such that one can neglect the powers of the components of the preceding rotations and translations that are higher than the first.

Now suppose that the external forces reduce to some forces that are applied at only A and B and that the system is in equilibrium. Under that hypothesis, imagine a transverse section σ that divides the rod into two parts S_a (on the side of A) and S_b (on the side of B) and reduce the elastic action that S_b exerts on S_a along σ by taking the center of reduction to be an arbitrary point O. It is clear that when one keeps the point fixed and changes the section σ in no particular way then the elements of reduction will be independent of that section. Furthermore, they will be equal to the resultant and couple, respectively, that

(†) Translator: The cited photographs were not available to me at the time of translation, although one might simply confer the original version of this article for those plates.

give the reduction of the forces that are applied to B and opposite to the resultant and couple that one obtains by starting from the forces that are applied at A, while O is always the center of reduction.

To simplify the calculations, assume that the rod is isotropic and that in its natural state it has the form of a cylinder of revolution of height l and radius R. Take the origin O to be the center of the base that is adjacent to the element A, while the z-axis is the axis of the cylinder. Choose O to be the center of reduction, and let $X_1^{(ab)}$, $X_2^{(ab)}$, $X_3^{(ab)}$ be the components of the result of the external actions that are applied to B, while $X_4^{(ab)}$, $X_5^{(ab)}$, $X_6^{(ab)}$ are the components of the resultant couple of those actions. Finally, let $x_1^{(a)}$, $x_2^{(a)}$, $x_3^{(a)}$ denote the components of the translation that A experiences when it starts from the natural state, and let $x_4^{(a)}$, $x_5^{(a)}$, $x_6^{(a)}$ denote the components of the rotation of that element. Let $x_1^{(b)}$, $x_2^{(b)}$, …, $x_6^{(b)}$ be the analogous quantities for the element B. The relative displacement of B with respect to A will be defined by the differences:

$$x_1^{(b)} - x_1^{(a)}, \; x_2^{(b)} - x_2^{(a)}, \; \ldots, \; x_6^{(b)} - x_6^{(a)} .$$

It is easy to obtain the relations between the quantities $X_i^{(ab)}$ and $x_i^{(b)} - x_i^{(a)}$. For example, upon using the Saint-Venant method, one will get:

$$(30) \qquad \left\{ \begin{aligned} x_1^{(b)} - x_1^{(a)} &= \frac{1}{E}\frac{l^2}{\mu}\left[-X_3^{(ab)} + \frac{2}{3}X_1^{(ab)}l\right], \\ x_2^{(b)} - x_2^{(a)} &= \frac{1}{E}\frac{l^2}{\mu}\left[\;\; X_4^{(ab)} + \frac{2}{3}X_2^{(ab)}l\right], \\ x_3^{(b)} - x_3^{(a)} &= 0; \end{aligned} \right.$$

$$(31) \qquad \left\{ \begin{aligned} x_4^{(b)} - x_4^{(a)} &= \frac{1}{E}\frac{l^2}{\mu}\left[2X_4^{(ab)} + X_2^{(ab)}l\right], \\ x_5^{(b)} - x_5^{(a)} &= \frac{1}{E}\frac{l^2}{\mu}\left[2X_5^{(ab)} - X_1^{(ab)}l\right], \\ x_6^{(b)} - x_6^{(a)} &= \frac{2(1+\eta)}{E}\frac{l}{\mu}X_6^{(ab)}, \end{aligned} \right.$$

in which E represents the modulus of elasticity, η, Poisson's elasticity constant, and $\mu = \pi R^4 / 2$ is the moment of inertia of the cross-section of the small rod with respect to its center. In those formulas, one supposes that the quantities $X_i^{(ab)}$ have the same order and that one has suppressed the higher-order infinitesimals.

One sees that one can *arbitrarily* choose the three components of the relative rotation of B with respect to A, as well as the two components of the relative translation in the direction normal to the rod. One can always find a system of external forces that are capable of generating such a relative displacement. However, the relative translation in the direction of the axis has, on the contrary, the same order as negligible quantities.

Furthermore, it is easy to modify the conditions of the system slightly in such a way as to make that translation noticeable.

Suppose, for example, that the external forces are applied to two small loops (*coulants*) that can slide along the rod in the longitudinal direction and are maintained by agencies (*ressorts*) such as stresses X_3 that have the same order of magnitude as the ones that produce the flexures and torsions of the rod and lead to relative displacements of the loops along the rod that have the same order as the flexures and torsions. If one supposes that the loops are at the extremities of the rod and that one calls them A and B then the preceding relations (30) and (30′) will persist with the single difference that the three equations (30) will become:

$$x_3^{(b)} - x_3^{(a)} = m\,X_3^{(ab)},$$

in which m is a positive coefficient with the same order as the coefficients of the $X_i^{(ab)}$ in the other equations.

The elastic energy of the deformed system is:

$$H = \tfrac{1}{2}\sum_{i=1}^{6}(x_i^{(b)} - x_i^{(a)})\,X_i^{(ab)},$$

and upon replacing the $(x_i^{(b)} - x_i^{(a)})$ with their values that one infers from the relations (30) and (30′), one will find a form for H that is quadratic with respect to the $X_i^{(ab)}$:

$$H = \frac{1}{E}\frac{l}{\mu}\left\{\frac{1}{3}(X_1^{(ab)}l)^2 + (X_2^{(ab)}l)^2 + \frac{m}{2}(X_3^{(ab)})^2 + (X_4^{(ab)})^2 \right.$$
$$\left. +\ (X_4^{(ab)})^2 + (1-\eta)(X_6^{(ab)})^2 - X_1^{(ab)}X_5^{(ab)}l + X_2^{(ab)}X_4^{(ab)}l\right\}.$$

This will be a positive-definite form if m is non-zero. If m is zero then it will be a positive form that can vanish without $X_3^{(ab)}$ having to be zero; however, it cannot be zero unless the $X_i^{(ab)}$ (i = 1, 2, 3, 4, 5, 6) are all zero. In any case, *it will then suffice that just one of the quantities* $x_i^{(b)} - x_i^{(a)}$ *should be zero in order for H to also be zero.*

3. – One can imagine an infinitude of other cases in which bodies of widely-varied form have properties that are analogous to the ones that we just obtained in a very simple case. In order for us to assume a completely-general viewpoint, we imagine *in abstracto* some bodies to which we attribute the properties in question *absolutely*, namely:

1. There exist two particles A and B in the body, which we call *extremities,* whose deformations are negligible with respect to the relative translations and rotations that they are subjected to.

2. If one supposes that the body is in equilibrium under the action of external forces that are applied to just A and B then the components of the relative translation and rotation of B with respect to A are linear functions of the components of the resultant force and couple of the external actions that are exerted on B.

3. The elastic energy in the deformed body, which is always positive, can vanish only if the relative displacement of B with respect to A is zero.

We say that bodies that enjoy those properties are *pliable elastic elements.* One can sort them into two classes:

1. *The freely-pliable elastic elements,* for which the arbitrarily-given relative displacement (B, A) corresponds to external actions that one can determine completely (type: rod with loops).

2. *Pliable elements that are subject to constraints;* i.e., such that the components of the relative displacement (B, A) are not independent, but are linked by one or more linear relations (type: simple rod without loops).

We shall always let:

$$X_1^{(ab)},\ X_2^{(ab)},\ X_3^{(ab)},\ X_4^{(ab)},\ X_5^{(ab)},\ X_6^{(ab)}$$

denote the components of the resultant force and couple of the external actions on B.

By virtue of equilibrium, the quantities $-X_i^{(ab)}$ are the external actions that are applied at A. We represent them by $X_i^{(ab)}$. We further say that $X_i^{(ab)}$ is the stress of order i that acts in the element AB.

The characteristics $x_i^{(b)} - x_i^{(a)}$ of the relative displacement of B with respect to A are coupled with the stresses by the relations of the following form:

$$x_i^{(b)} - x_i^{(a)} = \sum_s A_{is}^{(ab)} X_s^{(ab)}, \tag{32}$$

in which the quantities $A_{is}^{(ab)}$ depend upon only the nature of the elastic body and its initial form. We call those quantities the *direct constants of the element;* when one changes the roles of the extremities A and B, one will obviously have:

$$A_{is}^{(ab)} = A_{is}^{(ba)}.$$

Finally, we note that the $A_{is}^{(ab)}$ define a tensor and that we can establish transformation relations under a change of axes quite easily.

If the elements are *freely-pliable* then one can invert the preceding formulas (32), so:

$$X_s^{(ab)} = \sum_i a_{is}^{(ab)}[x_i^{(ab)} - x_i^{(ab)}]. \tag{33}$$

We call the coefficients $a_{is}^{(ab)}$ the *inverse constants* to the element *AB*.

The elastic energy of the element will be given by:

$$H = \tfrac{1}{2}\sum_{i=1}^{6} X_i^{(ab)}(x_i^{(b)} - x_i^{(a)}) = \tfrac{1}{2}\sum_{i=1}^{6}\sum_{s=1}^{6} A_{is}\, X_i^{(ab)} X_s^{(ab)}.$$

The energy will a *positive* form, and it will be *definite* in the case of freely-pliable elements; it will not be definite if the pliable element is subject to constraints.

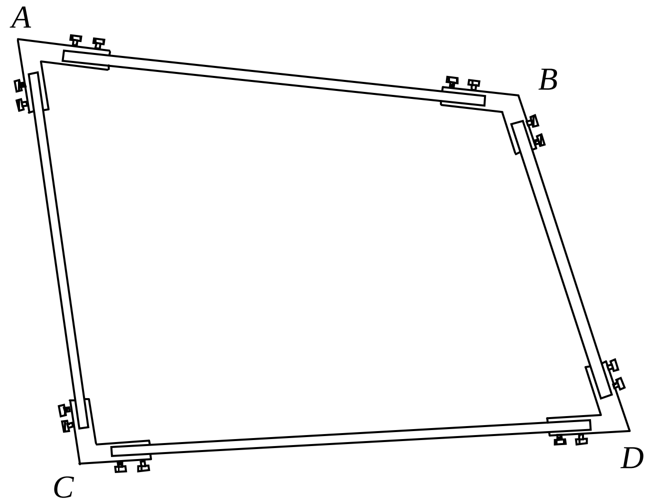

Figure 17.

4. – We shall now address the question that is of interest to us. We construct a cyclic elastic system by forming an assemblage of pliable elements that are rigidly connected to each other by only their extremities, and in such a manner that they are all in the natural state. Example: the flexible quadrilateral *ABCD* in Fig. 17.

We shall study the effect of distortions on such a system.

Suppose, in general, that we have a cyclic system that is composed of *n* elements that are coupled at their extremities into *m* nodes, and that they are not subject to any external actions.

Make a cut in any of the elements and perform a distortion after making that cut. We shall see how the system is deformed and the stresses that will be induced.

The equations upon which the solution to that question depends are immediate. Consider an element *AB* of the system and let $\alpha_1^{(ab)}$, $\alpha_2^{(ab)}$, $\alpha_3^{(ab)}$, $\alpha_4^{(ab)}$, $\alpha_5^{(ab)}$, $\alpha_6^{(ab)}$ denote the characteristics of the distortions that *AB* is subject to (which will be zero if that element is not subject to distortions).

Upon preserving the notations of no. **3**, we can immediately write out the relations:

(*D*) $$x_i^{(b)} - x_i^{(a)} - \alpha_i^{(ab)} = \sum_s A_{is}^{(ab)} X_s^{(ab)} \qquad (i = 1, 2, \ldots, 6),$$

and we will have six analogous equations for each element.

On the other hand, at a node *A* at which it ends and which is coupled rigidly to the extremities of the elements *AB*, *AC*, *AD*, …, equilibrium will give the six relations:

(*E*) $$X_i^{(ab)} + X_i^{(ac)} + \cdots + X_i^{(al)} + \cdots = 0 \qquad (i = 1, 2, \ldots, 6),$$

and one will have six analogous relation for each node.

If we suppose that the *constants* A_{is} are known for each element, along with the *characteristics* $\alpha_i^{(ab)}$ of the distortion that relate to it, then we will have, by definition, 6*n* + 6*m* linear equations (*D*) and (*E*) in 6*n* + 6*m* unknowns x_i and X_i .

However, not all of equations (*E*) are independent, because if one adds corresponding sides of all the ones that correspond to the same index *i* then one will find:

$$\sum [X_i^{(ab)} + X_i^{(ba)}] = 0,$$

which is an identity. That proves that six of the equations (*E*) are consequences of the other ones, and we will have only 6*n* + 6*m* – 6 equations to determine 6*n* + 6*m* unknowns. That could have been predicted, since it is obvious *a priori* that one can choose the displacement at a node arbitrarily.

5. – We can now establish the following fundamental result:

In any cyclic system of pliable elements, when one is given the constants of the various elements and the distortion, that will determine completely the relative translations and rotations of all of the nodes, as well as the stresses that all of the elements of the system that are freely-pliable are subjected to.

To simplify, we suppose that the three components of the translation and the rotation that correspond to an arbitrarily-chosen node are zero. Suppose that the same system of values of the characteristics $\alpha_i^{(ab)}$ and coefficients $A_{is}^{(ab)}$ correspond to two systems of values of the $x_i^{(a)}$ and the $X_i^{(ab)}$, whose differences are denoted by $\xi_i^{(a)}$ and $\Xi_i^{(ab)}$, resp. Those latter quantities verify:

(34) $$\xi_i^{(b)} - \xi_i^{(a)} = \sum_{s=1}^{6} A_{is}^{(ab)} \, \Xi_s^{(ab)}$$

and

(35) $$\Xi_1^{(ab)} + \Xi_2^{(ab)} + \Xi_3^{(ab)} + \ldots = 0.$$

Upon multiplying the two sides of (34) by $\Xi_i^{(ab)}$ and the two sides of (35) by $\xi_i^{(a)}$ and adding the corresponding sides of all the equations that are obtained, the left-hand sides will vanish, so:

$$\sum_{a.b}\sum_{i=1}^{6}\sum_{s=1}^{6} A_{is}^{(ab)}\,\Xi_i^{(ab)}\,\Xi_s^{(ab)} = 0, \tag{36}$$

in which $\sum_{a,b,}$ is the sum of n terms that relate to all of the elastic elements of the system. Now, each of the forms:

$$\sum_{i=1}^{6}\sum_{s=1}^{6} A_{is}^{(ab)}\,\Xi_i^{(ab)}\,\Xi_s^{(ab)} \tag{37}$$

is positive, and from (36), they must be zero. It will finally follow that:

$$\xi_i^{(b)} - \xi_i^{(a)} = 0.$$

The components of the translations and rotations of the nodes in the two solutions that were envisioned cannot differ from each other. If the element *AB* is freely-pliable then the vanishing of the form (37) will imply the vanishing of the $\Xi_i^{(ab)}$. It will then follow that the stresses that relate to such an element *AB* cannot differ between the two solutions then.

The theorem thus-established then implies the following corollary:

In a cyclic system of freely-pliable elastic elements for which one knows the constants, the stresses are determined by the distortions, and one determines them by solving a system of first-degree equations.

While always supposing that the elements are freely-pliable, formula (34) and (35) will have no solutions other than:

$$\xi_i^{(a)} = 0, \qquad \Xi_i^{(ab)} = 0,$$

if we assume that the ξ are zero at a given node. It will then follow that equations (*D*) and (*E*) will be compatible for any $\alpha_i^{(ab)}$:

One can choose the distortions in an arbitrary fashion in a system of freely-pliable elements.

If the elements are not freely-pliable then it might happen that (34) and (35) are satisfied for values of the unknowns $\Xi_i^{(ab)}$ that are not all zero, or on the contrary, that they further imply that the $\Xi_i^{(ab)}$ are all zero. In the latter case, the distortions can be taken arbitrarily and the stresses will not be determined uniquely, while they will be in the former.

6. – It is interesting to remark that equations (*D*) and (*E*) present some close analogies with the Kirchhoff equations on the propagation of electric currents in a network of conductors. The components of the stresses are analogous to the current intensities, while the components of the displacements are analogous to the values of the potentials at the nodes. The characteristics of the distortions play the roles of electromotive forces and the constants of the elements play the roles of inverse electrical resistances. Equations (*D*) then take the form of Ohm's law. The only difference consists of the fact that we have six times as many equations in (*D*) and (*E*) as we would have in the case of electric currents. However, that is no obstacle to using the analogy that we just specified, and we will then possess a means of study that is quite fruitful and practical.

In particular, it is easy to imagine some cases that present themselves in a fashion that is analogous to the Wheatstone bridge in the electrical domain and then deduce from it a simple determination of the constants of elastic elements.

We also indicate that the principle of equivalent cuts will permit us to replace a distortion that is performed on a given cut with another one that is performed on a cut that is deduced from the first one by a continuous transformation, so that will give us a means of realizing the distortions in a system of pliable elements in practice: One performs distortions at the nodes, which can be accomplished by the manner itself by which one fixes one or the other of the extremities of the elements.

7. – To conclude this section, we shall establish some other interesting propositions that are concerned with pliable elements and systems that are composed of them.

The direct constants $A_{is}^{(ab)}$ *of an element verify the conditions:*

$$A_{is}^{(ab)} = A_{si}^{(ab)} .$$

In order to account for that, it will suffice to imagine two systems of stresses $X_i^{(ab)}$ and $\Xi_i^{(ab)}$ that correspond to the displacements $x_i^{(b)}$, $x_i^{(a)}$, and $\xi_i^{(b)}$, $\xi_i^{(a)}$, respectively. By virtue of Betti's principle, which we shall extend to pliable extended bodies, the works that are done by one system of stresses that is calculated for the deformation that is due to another one will be equal. Hence:

$$\sum_{i=1}^{6} (x_i^{(b)} - x_i^{(a)})\, \Xi_i^{(ab)} = \sum_{i=1}^{6} (\xi_i^{(b)} - \xi_i^{(a)})\, X_i^{(ab)} ,$$

so

$$\sum_{i=1}^{6} \sum_{s=1}^{6} A_{is}^{(ab)} X_s^{(ab)}\, \Xi_i^{(ab)} = \sum_{i=1}^{6} \sum_{s=1}^{6} A_{is}^{(ab)} \Xi_s^{(ab)}\, X_i^{(ab)} ,$$

which is the stated result.

The inverse constants obviously enjoy the same property.

8. – When one has n pliable elastic elements A_1, A_2, A_3, A_4, …, one can obviously combine them *in series;* i.e., in such a manner that two consecutive elements $A_{i-1} A_i$ and $A_i A_{i+1}$ have the common extremities A_i . One can also combine them by derivation (in which they are in parallel) by rigidly coupling all of the extremities A_i and all of the extremities B_i to one part. If one starts from the relations (32) or (33) then one will verify that:

The direct constants of an element that is composed in series are obtained from the sum of the corresponding constants of the components; on the other hand, for a parallel connection of freely-pliable elements, it will be the inverse constants that add together.

Those theorems are close to the well-known theorems in the resistance of conductors that is arranged in series or by derivation.

VIII. – Cyclic system of pliable planar elements.

1. – In the case of a pliable element AB that is located in a plane and is subjected to forces that are located in its plane, if one takes the plane in question to be the first coordinate plane then the characteristics $X_3^{(ab)}$, $X_4^{(ab)}$, $X_5^{(ab)}$ will be zero, and similarly for the differences $x_3^{(b)} - x_3^{(a)}$, $x_4^{(b)} - x_4^{(a)}$, $x_5^{(b)} - x_5^{(a)}$.

We now change the notations slightly, and let x, y denote the coordinate axes and adopt the letters $X^{(ab)}$, $Y^{(ab)}$, $M^{(ab)}$ in order to denote the characteristics $X_1^{(ab)}$, $X_2^{(ab)}$, $X_6^{(ab)}$ of the stresses and $x^{(a)}$, $y^{(a)}$, $r^{(a)}$, in order to denote the components $x_1^{(a)}$, $x_2^{(a)}$, $x_6^{(a)}$ of the displacement A, and finally, $x^{(b)}$, $y^{(b)}$, $r^{(b)}$, in order to denote the components of the displacement B. In general, we will have:

$$
\left\{
\begin{aligned}
x^{(b)} - x^{(a)} &= a_{11} X^{(ab)} + a_{12} Y^{(ab)} + a_{13} M^{(ab)}, \\
y^{(b)} - y^{(a)} &= a_{21} X^{(ab)} + a_{22} Y^{(ab)} + a_{23} M^{(ab)}, \\
r^{(b)} - r^{(a)} &= a_{31} X^{(ab)} + a_{32} Y^{(ab)} + a_{33} M^{(ab)},
\end{aligned}
\right. \tag{38}
$$

with $a_{rs} = a_{sr}$. Upon changing the origin, we will easily annul the coefficients a_{13} , a_{23} , and by modifying the orientation of the axes, we will likewise annul the a_{12} . For a convenient choice of axes, formulas (38) can then be reduced to the canonical form:

$$
\left\{
\begin{aligned}
x^{(b)} - x^{(a)} &= \lambda X^{(ab)}, \\
y^{(b)} - y^{(a)} &= \mu X^{(ab)}, \\
r^{(b)} - r^{(a)} &= \nu X^{(ab)}.
\end{aligned}
\right. \tag{39}
$$

The new coordinate origin is said to be the *center of the elastic element,* the coordinate axes are the *principal axes of that element,* and finally the coefficients λ and μ are called the *coefficients of traction,* while ν is the *coefficient of flexure.* It is easy to verify that *if the element admits two axes of symmetry* then they will be *the principal axes of the element.*

Suppose that the formulas are given that express the constants of an element relative to some arbitrary axes when one knows the coefficients of *traction* and *flexure.* If ξ and η denote the coordinates of the center with respect to some arbitrary axes (x, y) and θ denotes the angle between the two systems of axes (x, y) and (x', y') (the latter being the principal axes) then one will have immediately:

$$
(40)\qquad \left\{\begin{aligned}
x^{(b)} - x^{(a)} &= [\lambda\cos^2\theta + \mu\sin^2\theta + \nu\eta^2]\,X^{(ab)} \\
&\quad + [(\lambda-\mu)\sin\theta\cos\theta - \nu\xi\eta]\,Y^{(ab)} - \nu\,\eta M^{(ab)}, \\
y^{(b)} - y^{(a)} &= [(\lambda-\mu)\sin\theta\cos\theta - \nu\xi\eta]\,X^{(ab)} \\
&\quad + [\lambda\cos^2\theta + \mu\sin^2\theta + \nu\eta^2]\,X^{(ab)} + \nu\,\xi M^{(ab)}, \\
r^{(b)} - r^{(a)} &= -\nu\,\eta X^{(ab)} + \nu\,\xi\,Y^{(ab)} + \nu\,M^{(ab)}.
\end{aligned}\right.
$$

2. – Let an element be obtained by composing n planar elements in series then. Its direct constants are obtained by adding those of the components that are expressed by relations of the type (40), and it will then follow that:

The center of the composed elements will be the center of gravity of the centers of the component elements if one concentrates a mass that is equal to the coefficient of flexure on each of them.

The coefficient of flexure of an element that is composed in series is the sum of the coefficients of flexure of the various components.

If one draws unit segments through the center of the composed elements that are normals to the axes of each of the components, and if one concentrates a mass that is equal to the correspond coefficient of traction at the extremity of each of them, and if one places a mass that is equal to its coefficient of flexure at each of the centers of its components then the axes of inertia of that system of masses will be the principal axes of the composed element, and the principal moments of inertia will be the coefficients of traction.

3. – Now let a planar elastic body be doubly-connected and subject it some distortions that keep it planar. Since the x, y axes are situated on that plane, the six characteristics of a distortion will reduce to three: l, m, r (n, p, q being zero), and similarly, the stress characteristics will reduce to three: L, M, N, and one will have:

$$
\begin{aligned}
L &= E_{11}\,l + E_{12}\,m + E_{13}\,r, \\
M &= E_{21}\,l + E_{22}\,m + E_{23}\,r, \\
N &= E_{31}\,l + E_{32}\,m + E_{33}\,r.
\end{aligned}
$$

Here again, by a convenient choice of origin and the directions of the axes, one can reduce the preceding equations to the form:

$$L = E_{11}\, l, \qquad M = E_{22}\, m, \qquad N = E_{33}\, r.$$

There then exists a system of axes in the plane such that each elementary distortion will produce just one conjugate stress with respect to them.

If the planar system is composed of a series of pliable elements whose first and last extremities are coupled rigidly then the rules of no. **2** will give the axes that we spoke of in the preceding statement. The three stress coefficients are obtained by calculating the inverses of the coefficients of traction and flexure of the composed element.

The results that were just obtained cast a new light upon the analogy that was established above between the theory of distortions of systems of pliable elements and Kirchhoff's theory of the propagation of electrical currents. In the theorem of no. **2**, the result on the coefficient of flexure of a composed circuit corresponds to the proposition that says that resistances in series are additive. However, the rule that gives the coefficients of traction is more complicated and has no equivalent in the theory of electrical conduction. Moreover, there is nothing in the latter theory that corresponds to the *center* and the *principal axes* of pliable elements.

4. – Finally, in conclusion, imagine the case of n freely-pliable elements that are connected in *parallel*. Some simple calculations will exhibit the following theorem, which we shall be content to state:

If one draws unit segments through an arbitrary point that are parallel to the principal axes of the various elements and concentrates a mass that is equal to the inverse of the corresponding coefficient of flexure at the extremity of each segment then the axes of inertia of that set of masses will be parallel to the principal axes of the composed element and the principal moments of inertia will be the inverses $1 / \lambda$ *and* $1 / \mu$ *of its coefficients of traction.*

Upon considering those axes of inertia to be coordinate axes, the coefficients of $r^{(b)} - r^{(a)}$ *in the expressions for* $X^{(ab)}$ *and* $Y^{(ab)}$ *will be the coordinates* η *and* ξ*, respectively, of the center of the composed element, multiplied by* $1 / \lambda$ *and* $1 / \mu$*, respectively.*

IX. – Experimental verifications by the use of photoelasticity ([1]).

1. – Nonetheless, these verifications of the external form that will be taken by a body after distortion are not sufficient to justify the theory, because we still lack the distribution of internal stresses.

The measure of the deformations (i.e., the measure of certain dimensions of the body before and after) will provide us with a mean of the stresses along their finite lengths, but they will not permit us to go any further than that.

In order to penetrate the body and study the stress at each point, it is optics that we shall turn to, namely, the method that has received the name of *photoelasticity*, and we shall review its essentials, due to its present importance.

Isotropic, transparent solids can become temporarily anisotropic and birefringent when one applies sufficient external forces to them or when they are subject to internal stresses.

In 1813, Seebeck, and then Brewster in 1816, observed that phenomenon in layers of compressed glass. However, it was mostly in the last thirty years that the study of those phenomena and the technique of photoelasticimetry were developed, and in particular, thanks to Mesnager and Coker [**51**], [**52**], [**53**], [**54**].

For the moment, that technique will only apply in the case of a distribution of planar stresses. At each point, the stresses are defined by means of an ellipse that one calls the *Lamé ellipse.* The two axes of that ellipse are determined completely: Their directions are called the *principal directions* at the point considered, and one-half their magnitudes (or the magnitudes of the semi-axes) constitute what one calls the *principal stresses* (i.e., the stresses in the principal directions).

2. – Consider a thin slice of the transparent body, and let a light ray fall upon it that is normal to a point I. That ray will be subject to double refraction. Experiments permit us to state the following laws:

– The planes of polarization of the two emerging rays are defined by the principal directions.

– The difference between the paths that are followed by the two rays when they pass through the layer is proportional to:

([1]) Here, as in the rest of the volume, we have consistently sought to present the text in its original form, whenever possible.

However, it is obvious that the optical analysis of stresses has made great progress since Section IX was written some twenty years ago.

Not only has photoelasticity been extended from two dimensions to three, but it is now possible to study the state of stress beyond the elastic limit of the material by optical analysis, and today we also have photoplasticity, in addition to photoelasticity. Using optical analysis, it is possible to simultaneously study the stress state inside of an elastic solid that is subject to not only forces that are applied statically, but also ones that are applied dynamically.

Among the scholars in the United States that have contributed the most to the extraordinary progress in that field of research, one must mention T. J. Dolan, D. C. Drucker, A. J. Durelli, M. M. Frocht, L. E. Goodman, M. Hetenyi, M. M. Leven, R. D. Mindlin, W. M. Murray [**49**], [**50**].

1. The thickness of the layer.
2. The algebraic difference of the principal stresses.

Those laws permit one to experimentally study the stresses; i.e., to determine the direction and magnitude of the principal stresses at each point.

A. Principal directions. – If one places a transparent layer that is subjected to tangential stresses between a crossed polarizer and an analyzer that are traversed by white light then there will be darkness at all points where the principal directions are parallel to the planes of polarization of the polarizer and the analyzer.

The set of those points forms an *isocline*, viz., a curve whose principal directions have the same inclination.

– One can plot the isoclines in the laboratory photographically, which will then permit one to determine the principal directions at each point of the layer with a convenient installation.

B. Differences between principal tensions. – In white light, the loci of points with equal difference in path length for the emergent rays will define colored curves that are called *isochromes.* In order to avoid mixing them with isoclines, one can operate with circularly-polarized light. One will then obtain only the isochromes, along with a certain number of dark points that are singular points where the principal stresses are equal.

The isochromes will tell one directly what the distribution of the differences between principal stresses is.

It is also simple to utilize monochromatic radiation λ in place of white light. One will then obtain a network of black curves that correspond to path differences of:

$$\frac{\lambda}{2}, \frac{3\lambda}{2}, \frac{5\lambda}{2}, \ldots$$

They form a topographic surface that one can give by directly reading off the differences in principal stresses at each point, by virtue of the law that was stated above.

In order to get the path difference at each point, one can also utilize the method of compensation, either by means of a glass rod that is tensed or compressed parallel to one of the principal directions or with the aid of a Jamin or Babinet compensator.

C. Sum of the principal stresses. – If the differences in principal stresses are known then one can try to calculate another very simple function of those quantities, such as their sum, for example; it will suffice to determine it.

The theory shows that this sum is coupled to the thickness of the layer and the variation of that thickness by a simple formula.

The method then consists of measuring the variation of the thickness. To that effect, one can employ all of the resources of optics. Mesnager used the modifications of the interference fringes that were obtained from two planar mirrors, one of which was fixed on the elastic body.

Coker used a purely-mechanical apparatus that he called an "extensometer," which was based upon the noticeable displacement of a turning mirror that was linked with the body by an amplifying lever [**54**].

By definition, one will then get the magnitudes and directions of the principal stresses, and one will know the complete distribution of the stresses.

There exist some *purely-optical methods* for the experimental solution of the same problem that are different from the preceding ones and use interferometers. We shall not dwell upon those methods. They were studied by Favre and Fabry. One will find them discussed in the *Revue d'Optique* (1930), [**55**], [**56**], [**57**], [**58**].

3. – Although the techniques of photoelasticity have made great progress, they have not yet reached full maturity.

Here, we shall content ourselves by saying a few words about the results that have been obtained in the study of cylinders that are subject to distortions.

One must use a substance that is elastic and transparent. Corbino and Trabacchi employed gelatin for their experiments, which gave excellent images, but its use necessitated considerable experimental ingenuity.

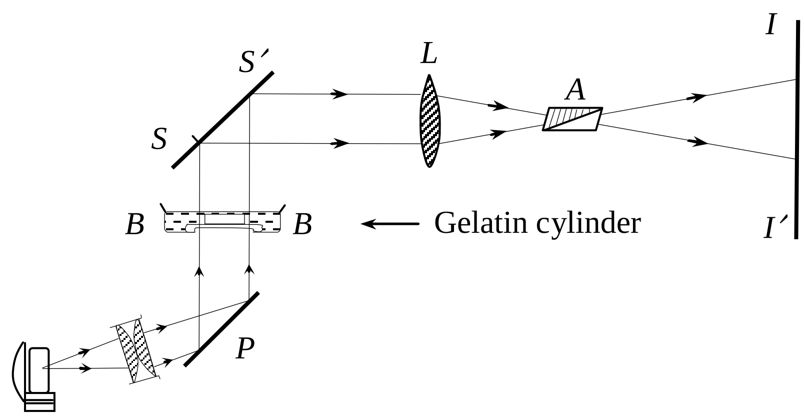

Figure 18.

They used a hollow gelatin cylinder of radii R_1 and R_2 and a small height, and that cylinder realized distortions of order 6 and 2, as we have explained.

We shall give some indications here about the results that were obtained in regard to the distribution of stresses by means of lines of equal differences in principal tensions.

Here is the experimental setup (Fig. 18): *B* is a bowl in which one has laid the gelatin cylinder that is subject to a distortion, *P* is a black polarizing mirror that emits a vertical sheaf of polarized light. After having crossed the cylinder, the light sheaf will be reflected horizontally by an ordinary mirror *S*, and it will be concentrated by a lens *L* onto a photographic screen *I*, while a Nicol analyzer is placed at the point of convergence *A*.

The Nicol analyzer is crossed with the polarizer. One will then get a photograph at *I* of black lines of equal differences in principal stresses.

In the case of a distortion of order 6, one will get the following figure (Fig. 19) (†), which will not change when one turns the cylinder around the axis. (The arms of the cross always correspond to the principal sections of the crossed polarizers.)

Corbino calculated that the radius of the black circle would be equal to:

$$r = R_1 R_2 \sqrt{\frac{\log R_1^2 - \log R_2^2}{R_1^2 - R_2^2}} .$$

One then finds some results that coincide with the results of the theory to a very high degree of precision.

In the case of a distortion of order 2, one will get figures that differ with the angle that the cut forms with a principal section of the polarizer.

One will get Fig. 20 when that angle is zero and Fig. 21 when that angle is 45°. In the former case, calculation will show that the black line has the equation in polar coordinates:

$$r^2 = \frac{1+\varepsilon^2}{2\varepsilon^2} R^2 \left[\sqrt{\cos^2 2\vartheta + (1 + 2\cos 2\vartheta)\frac{4\varepsilon^2}{(1+\varepsilon^2)^2}} - \cos 2\vartheta \right],$$

in which:

$$R = R_1 , \qquad \varepsilon = \frac{R_1}{R_2} .$$

The qualitative and quantitative comparison of the results of experiments with those of calculation will show complete accord.

The beautiful photographs that correspond to the preceding figures suffice to give some clear idea of that accord, and they constitute a nice result of photoelasticity.

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(†) Translator: Once again, Figures 19 through 21 were photographs, which were not available to me.

CHAPTER IV

THE APPLICATIONS OF DISTORTIONS TO CONSTRUCTION PRACTICE

X. – Tracing the lines of influence in statically-indeterminate systems.

1. – The search for lines of influence of hyperstatic unknowns in statically-indeterminate systems can be based upon the reciprocity theorem that Colonetti gave.

Consider an elastic body that occupies a volume V that is in equilibrium under the action of a given system of external forces.

Imagine that one has made a cut in the body that gives an arbitrary section of it. Subject the two faces of the cut to a displacement with respect to each other. Introduce a distortion whose characteristics are l, m, n, p, q, r. Let X, Y, Z denote the components of the unit tension that acts in each element, and let u, v, w be the displacements that define the configuration that is taken by the solid under the action of given external forces with components F_x, F_y, F_z ; P_x, P_y, P_z (volume forces and surface forces). One will then find that:

$$\int_V [F_x u + F_y v + F_z w] \, dV + \int_\Sigma [P_x u + P_y v + P_z w] \, d\Sigma$$

$$= \int_\Sigma [X(l + qz - ry) + Y(m + rx - pz) + Z(n + py - qx)] \, d\Sigma .$$

If one assumes that the characteristics of the distortion are constant then one will have:

$$(4) \qquad \left\{ \begin{aligned} & \int_V [F_x u + F_y v + F_z w] \, dV + \int_\Sigma [P_x u + P_y v + P_z w] \, d\Sigma \\ & = l \int_\Sigma X \, d\Sigma + m \int_\Sigma Y \, d\Sigma + n \int_\Sigma Z \, d\Sigma \\ & + p \int_\Sigma (Zy - Yz) \, d\Sigma + q \int_\Sigma (Xz - ZX) \, d\Sigma + r \int_\Sigma (Yx - Xy) \, d\Sigma. \end{aligned} \right.$$

Upon taking into account the fact that the integrals that relate to Σ are nothing but the characteristics of the system of stresses that develops in the body under the action of given external forces (namely, L, M, N, P, Q, R), one can conclude that:

The sum of the products of the six characteristics of the system of internal stresses that are developed in one section of an elastic body in equilibrium by the corresponding characteristics of a distortion is equal to the work done by the external forces that are applied to the body when it executes the change of configuration that the distortion gave rise to.

2. – The particular cases that are obtained when the relative displacement of the two faces of the cut reduces to a simple translation or a simple rotation are very interesting from the standpoint of applications. For a translation of unit magnitude in the direction of the X-axis (a direction that is completely arbitrary, moreover), one will have $l = 1$; $m = n = p = q = r = 0$, and equation (41) will become:

$$\int_V [F_x u + F_y v + F_z w] dV + \int_\Sigma [P_x u + P_y v + P_z w] d\Sigma = \int_\Sigma X \, d\Sigma, \tag{42}$$

which expresses the fact that:

The component along an arbitrary direction of the system of internal stresses that are developed in an elastic body in equilibrium in a given section is measured by the same number as the work done by the external forces that are applied to the body under deformation if, when one cuts the body along the section considered, one imposes a relative translation of unit magnitude in the direction considered to the two faces of the cut.

On the contrary, if one supposes that:

$$p = 1; \qquad l = m = n = q = r = 0$$

then one will have:

$$\int_V (F_x u + F_y v + F_z w) dV + \int_\Sigma (P_x u + P_y v + P_z w) d\Sigma = \int_\Sigma (Xy - Yz) d\Sigma; \tag{43}$$

i.e.:

The moment with respect to an arbitrary axis of the system of internal stresses that is developed in an elastic body in equilibrium in a given section is measured by the same number as the work done by the external forces that act upon it, when one cuts the body along the section, one imprints a relative rotation (of one with respect to the other) on the two boundaries of the cut that has unit magnitude around the axis considered.

3. – The analysis of the changes in configuration that are determined by a given relative displacement of the two faces of a cut that is made in an elastic body will take on a very special importance when the problem reduces to two dimensions.

One can then show that:

1. *If one applies two equal and opposite systems of forces to the two faces of the cut (that are arbitrary, moreover) then under the subsequent deformation of the system, the forces on the cut will subject it to a relative displacement around a point that is the antipole of the line of action of the resultant of the forces that are applied to each face with respect to a certain ellipse that takes the name of the* ***ellipse of relative elastic displacements.***

2. *The amplitude of that relative rotation is proportional to the moment of the resultant when it is taken about the center of the ellipse, and the coefficient of proportionality is a constant of the system that one continues to calls the "elastic*

weight," in analogy with what one generally does for the terminal elements in Culmann's theory of the ellipse of elasticity.

3. *If one imagines that the "elastic weight" is distributed over the given plane in such a way that it central ellipse of inertia coincides with the ellipse of relative elastic displacements then the relative displacement of two arbitrary points of the two faces of the cut that originally coincided will have a component along an arbitrary direction that is the product of the magnitude of the aforementioned resultant with the second-order moment of the elastic weight when it is taken with respect to the line of action of the resultant and with respect to the direction onto which one projects the displacement.*

It is not necessary in practice to repeat the construction of the ellipse for every section that one would like to examine. In most contemporary cases, one ellipse will be sufficient for every system of sections, which would result from the following theorem:

If two sections of a planar system are obtained by detaching one portion of that solid that is not connected with the rest of it and not subject to any external actions then the same elastic weight and the same ellipse of relative elastic displacements will correspond to both of those sections.

4. – In order to understand the importance of the preceding considerations, take a cylindrical beam with a rectilinear axis (viz., the X-axis) that is fixed by its two extremities A and O and acted upon by a force P that is normal to its axis – for example, parallel to the Y-axis (Fig. 22.a).

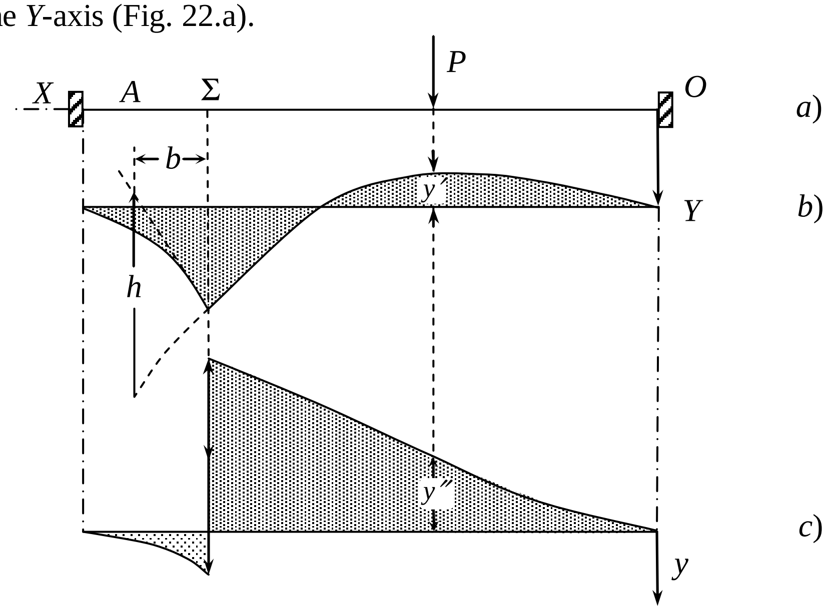


Figure 22.

The problem of equilibrium can be considered to have been solved if one succeeds in obtaining the characteristics of the state of stresses in an arbitrary cross-section Σ.

The force P will generally produce a moment of flexure M and a shearing stress T in such a section, and one can determine their magnitudes by imagining that one makes the usual cut along Σ and constrains the two faces of that cut to a relative rotation around the principal axis of inertia that is parallel to *OX* and a relative translation in the direction *OY*.

One then obtains two deformations of the type that is represented by the Figures 22.*b* and 22.*c* (of course, the scale of the ordinates is very different from that of the abscissa).

If the relative rotation of the two faces of the cut is equal to unity then the product of the magnitude of the applied force with the amplitude y of the deformation, when measured along the line of action of that force, must measure the moment of flexure M, according to formula (43).

In the figure, one sees that the rotation in question is measured by h / b. One will then have $Py'b / h$ for the measure of M.

Similarly, if the relative translation in the second deformation is measured by unity then, from formula (42), one will get the shear stress T by multiplying the magnitude P of the given force times the amplitude y'' of the deformation, when it is measured to the right of P. In Fig. 22.*c*, the translation is denoted by v, so the shear stress T will consequently be measured by Py''/ v .

The problem that was posed is then solved for any magnitude of the force P, and more generally, for any system of forces such that P acts upon the solid.

If the force P is constant in magnitude and displaces along the beam parallel to itself then the two deformation curves that we have traced out will provide the values of the moment of flexure and the shear stress in the chosen section Σ by simply reading off the ordinates, and those curves will be the influence curves of the characteristics of the internal stresses that relate to that section.

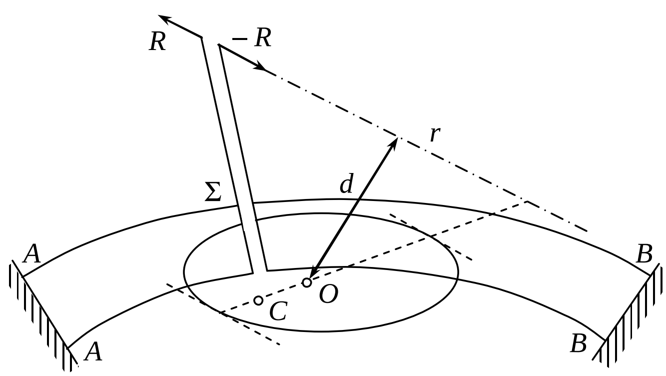


Figure 23.

5. – Consider the most general case of an elastic arch that is fixed at its two extremities A and B (Fig. 23). Imagine that a cut has been made along an arbitrary section Σ and constrain the two parts to take on a relative displacement with respect to each other – for example, a relative unit rotation around a well-defined point C. In order to do that, it will suffice to apply two equal and opposite systems of forces R to the cut, such that the line of action r of the resultant of the forces that act upon each of the faces will be the antipolar of the point C with respect to the ellipse of relative elastic displacements.

Since the amplitude of the rotation is measured by the product of the elastic weight W with the moment Rd of the aforementioned resultant with respect to the center O of the ellipse, one will get a unit rotation by the condition:

$$WR\,d = 1, \qquad \text{so} \qquad R = \frac{1}{Wd}.$$

The construction of the deformation of the geometric axis of the arch, when it is divided into two independent segments, no longer presents any difficulty. On the other hand, the deformation itself can be interpreted immediately as the line of influence of the moments of the system of internal stresses that are transmitted across the section considered Σ with respect to the arbitrary center C.

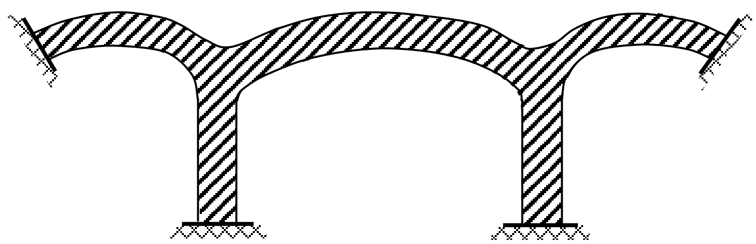


Figure 24.

6. – As G. E. Beggs [**59**], [**60**], [**61**] showed, one can utilize mechanical models and mechanisms in order to the trace out the line of influence. It will suffice to construct a model of a suitable scale of the elastic system under study and subject it to the desired distortions and plot the displacements of its various points directly.

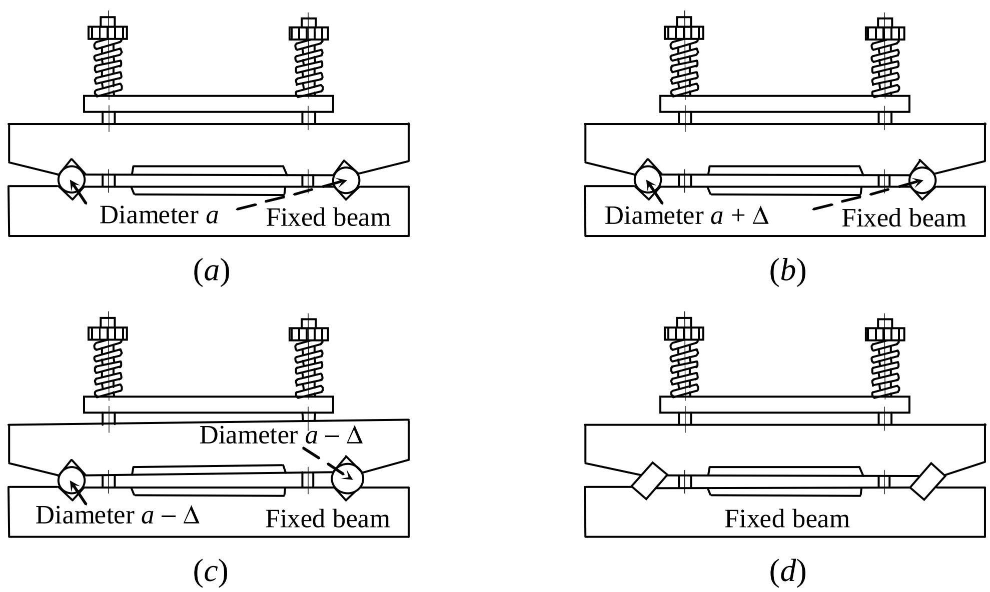


Figure 25.

Some excellent results can be obtained with celluloid models. The construction is then singularly facilitated by the fact that it is not even necessary to reproduce the scale of the body being studied. It will suffice that the various sections of the model have their moments of inertia (with respect to the neutral axis) proportional to the moments of inertia of the corresponding sections of the body, in such a way that the model can be reduced to a simple layer of constant thickness in all cases.

Suppose that one would like to study an elastic system with several sections to it; for example, the one that is represented by Fig. 24, which involves the determination of nine hyperstatic unknowns.

In order to study those unknowns, it will suffice to provide the sections with devices that permit one to realize the distortions simply and surely and in such a fashion that it is possible to separate an extreme section at will and to subject the body to a relative displacement with respect to the fixed reference system, to which the other supports are rigidly linked. In order to obtain such a result, one utilizes the device in Fig. 25, which consists essentially of two sturdy steel beams: The lower beam is fixed, while the other one, which is coupled to the extremity of the model that one would like to study, is strongly attracted to the first one by a convenient system of elastic connectors. The two beams present two pairs of V-shaped indentations that face each other, in which one can slide at will some rods of various forms and dimensions, in such a fashion as to determine different positions of the moving beam with respect to the fixed beam.

If the indentations and rods are carefully calibrated then one can determine a translation or a rotation of the extremity of the model, and the amplitudes of that translation and rotation will be as small as one desires, and they will be known perfectly.

In order to measure the displacements, one appeals to micrometric microscopes that one focuses on the various points *A* of the axis of the model (Fig. 26).

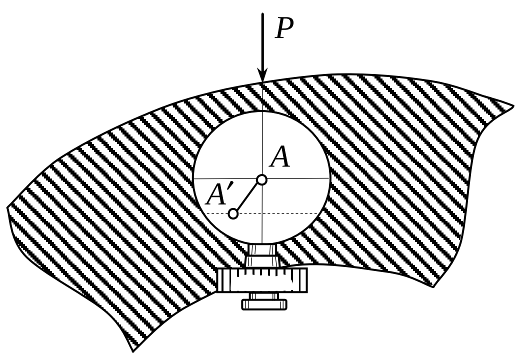


Figure 26.

One of the wires of the reticule (viz., the one that coincides with the axis of the micrometer) is placed in each instrument in such a manner that it will coincide with the line of action of the force *P* that acts upon the point being observed. The other wire is moved to the successive positions *A*, *A′* of the image of a point. One reads off the measurement of the displacements that is produced by each distortion from the difference of those two positions, or more precisely, the measurement of the projection of the displacement onto the direction of the force, which allows an immediate utilization, since that projection is necessary when one applies the second reciprocity principle.

XI. – Fundamentals of the theory of elasto-plastic deformations.

1. – In the calculation of hyperstatic structures that are subject to a given system of external forces, it will also be necessary to take into account the deformations that the resisting material can present when the internal stresses exceed the elastic limit: viz., *plastic deformations.* One knows that their introduction can be favorable to the stability of the construction by unloading that part of the construction that is overloaded and once more in the elastic domain, at the expense of some other parts that were initially less loaded and in which the stresses can contribute to the stability of the whole.

The most expert and enterprising engineers have always accounted for the tendency that more or less all of the materials that are employed in construction have of submitting to plastic adaptations, which is a tendency that has been compared to a natural healing process in the construction according to some fortunate formulas.

François Hennebique, one of the pioneers of reinforced concrete construction and one of the best modern engineers whose school of thought has defined the engineering principles of reinforced concrete since he was first hired, has taught that one should account for the plastic qualities of a material by adopting dimensions for the structure that are intentionally insufficient from the standpoint of the ordinary theory of elasticity. He was convinced that one can obtain a better use of the material by that process, and also when the material is its ideal state for submitting to plastic deformations without being damaged, while not delaying too much the removal of the reinforcement in the structure in order to realize the opportune plastic adaptations during the phase when the concrete is setting.

Danusso defined the basis for a theory of plastic deformations [**62**] in a conference talk that he presented in 1934 to the mathematics seminar at the Polytechnic School in Milan, and later Colonetti ([1]) [**63**], [**64**], [**65**], [**66**], [**67**], [**68**] developed a theory of elasto-plastic equilibrium that permits one to submit plastic subsidence (which is a condition that is effectively realized in a loaded structure) to a rigorous analysis and to specify in each case the special importance of the plastic deformation that the material can confidently support without any damage.

2. – In order to schematize the phenomena, and more precisely, the manner in which permanent deformations come about, the path to follow will be the following one: One must postulate the existence of a limiting stress beyond which the material passes from the elastic state to the plastic state, such that below that limit, the deformation is kept perfectly elastic (conforming to Hooke's law). On the contrary, above that limit, perfectly inelastic deformations are produced under constant load, which are superimposed with the elastic phenomena without altering or influencing its characteristics, moreover, and in particular, its tendency to vanish when the causes that produced them cease to act.

Now, that is in the nature of materials; for example, soft steel, whose effective behavior is approached under the aforementioned conditions to a high degree of approximation that is more than sufficient for the purposes of engineering.

([1]) One can find a complete bibliography of the work of Colonetti on this subject in his recent book that is entitled *L'équilibre des corps déformables*, Paris, Dunod, 1955.

It will then follow that if it so happens that the internal stresses attain the elastic limit in some part of the system then one must assume, with no hesitation, that the internal stresses in that part will immediately cease to increase and permanent deformations will begin to appear, which will stop only when the system attains an equilibrium state that is compatible with not only the given values of the applied external forces, but also with the fixed and known values that are taken by the internal stresses in the part of the system considered.

In that problem, we will be led to introduce the imposed deformations as the new unknowns, but at the same time, just as many of the original unknowns will have disappeared, or better, they will have taken on known values that the corresponding internal stresses.

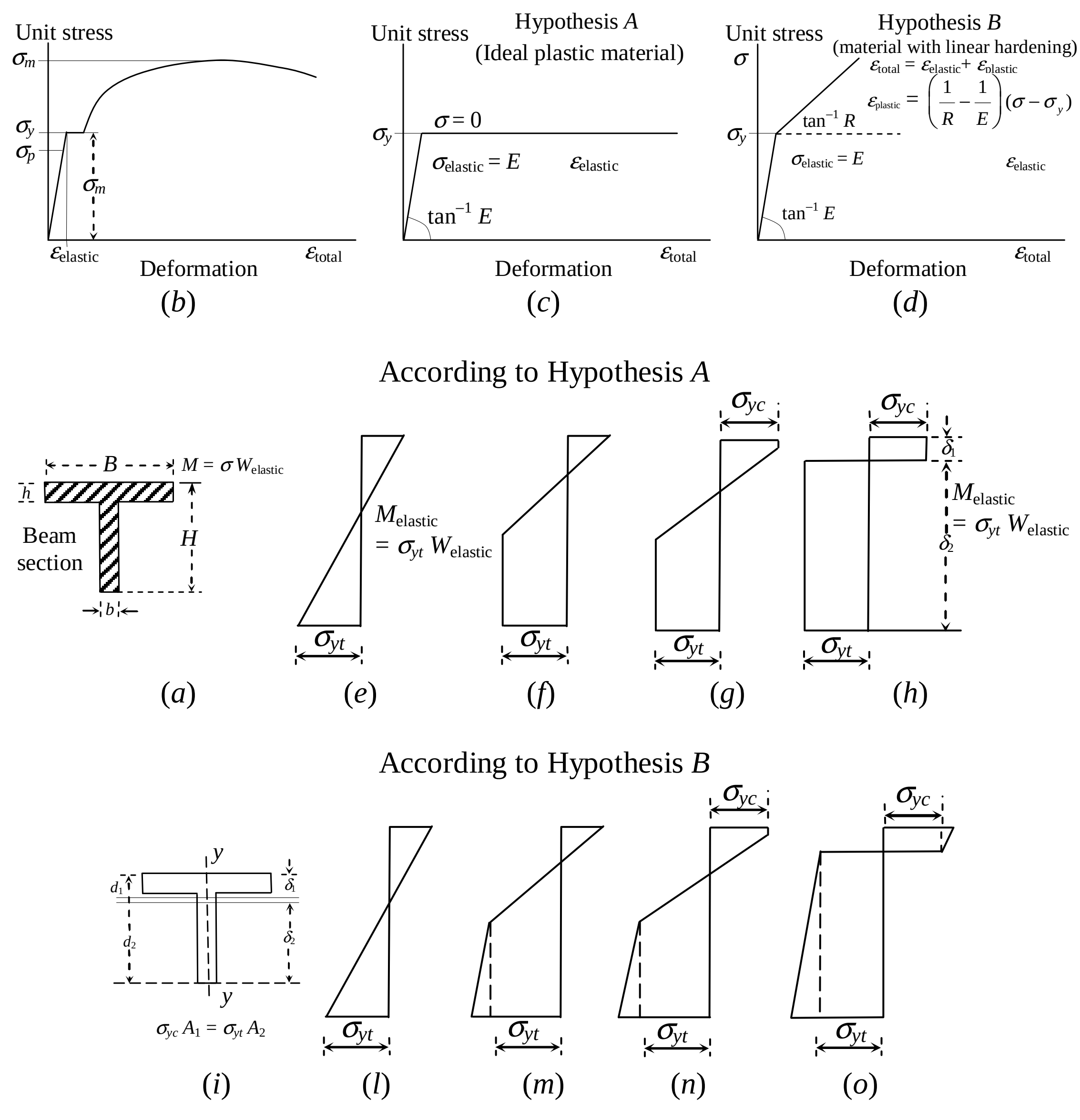


Figure 27.

3. – If a metallic beam with a profile that is similar to the one that is illustrated in Fig. 27.*a* is subjected to a moment of flexure M that increases from zero to a final value at which the beam yields then the extreme fibers of the beam will rapidly become deformed considerably. The section of that beam is referred to its principal axes of inertia (X, Y) and is symmetric with respect to the axis Y, which is found in the plane of the moment of flexure, and in the most general case, it is not symmetric with respect to the X-axis.

The stress-deformation diagram that is obtained when a specimen that is taken from the beam is represented in Fig. 27.*b*. The proportional limit of the material is represented by σ_p , the maximal elastic limit by σ_y , the minimal elastic limit by σ_x , and the final stress by σ_m .

In order to simplify the stress-deformation diagram for the sake of numerical applications, one can suppose that Hooke's law is no longer applicable past the elastic limit and the material will gradually be deformed at constant stress.

However, for some special materials, the phenomena can be described more precisely by taking into account the effect of curing. In that case, it is convenient to assume that above the elastic limit, the increase in the deformations is a linear function of the stress and the material keeps its characteristic property of not returning to its original state when the stress is suppressed.

In Fig. 27.*d*, if one takes R to be the new modulus of proportionality then the total deformation above the elastic limit will be expressed by:

$$\varepsilon_{\text{total}} = \varepsilon_{\text{elastic}} + \varepsilon_{\text{plastic}} = \frac{\sigma}{R} - \sigma_y \left(\frac{1}{R} - \frac{1}{E} \right),$$

the elastic deformation by:

$$\varepsilon_{\text{elastic}} = \frac{\sigma}{E},$$

and the plastic deformation by:

$$\varepsilon_{\text{plastic}} = (\sigma - \sigma_y) \left(\frac{1}{R} - \frac{1}{E} \right).$$

Melan [**69**] called those two hypotheses that of the "ideal plastic material" (A) and the "material with a linear limit of hardness" (B).

If the moments of flexure of the section of the metallic beam considered is increased, while preserving the greatest stress in the material above the proportionality limit (i.e., above the elastic limit σ_y), then the distribution of the stress across the section of the beam will be illustrated by a triangular diagram. Those normal stresses will bring equilibrium to an internal moment of flexure whose value is:

$$M = \sigma W_{\text{elastic}} .$$

In that expression, W_{elastic} represents the elastic modulus of the calculated section, which is referred to its neutral axis XX whose position is determined by the well-known method of applied elasticity. The triangular distribution of the stresses remains valid up to the point that the stress in the most-deformed element of the section attains the elastic limit, whether from tension of compression (σ_{yt} , σ_{yc} , resp.). The corresponding moment of flexure will then be:

$$M_{\text{elastic}} = \sigma_y \, W_{\text{elastic}} \; .$$

When starting from that moment, according to hypothesis *A* of the "ideal plastic material," if the moment of external flexure increases then there will be no more increase in the stress in the most elongated element. However, the stress in the elements that are closest to the neutral axis will successively attain the same value, and finally the stress diagram will be represented as in Fig. 27.*h*, with the entire section then being in a plastic condition. Upon supposing that the sections remain planar during the entire deformation, the maximal moment of flexure to which the section is subject in that final state will be:

$$M_{\text{plastic}} = \sigma_y \, W_{\text{plastic}} \; .$$

It must be calculated by referring to a new neutral axis *XX* (Fig. 27.*i*), and in the limiting case, that section will divide the section into two areas A_1 and A_2 in such a fashion that:

$$\sigma_{yc} \, A_1 = \sigma_{yt} \, A_2 \; .$$

In the case for which $\sigma_{yc} = \sigma_{yt}$, it is possible to express the values W_{elastic} , W_{plastic} , and the ratio $w = \dfrac{W_{\text{plastic}}}{W_{\text{elastic}}}$ for the most common sections in a simple form.

The ratio $\dfrac{W_{\text{plastic}}}{W_{\text{elastic}}}$, which is represented by w and is referred to as "the coefficient of plasticity of the section," represents a measure of the capacity of the section to resist a plastic deformation. The values of W_{elastic} , W_{plastic} , and w are given for eight different sections in Fig. 28. [**70**].

4. – In the elasto-plastic state of deformation, it is possible to distinguish an elastic fraction (c_{elastic}) of the total curvature (c_{total}) of a section of a bent beam; i.e., the curvature that the section will take on if it can elastically resist the moment of flexure *M*, and likewise a fraction that will be called the plastic curvature (c_{plastic}), which represents the difference between them.

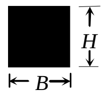
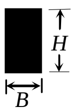
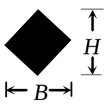
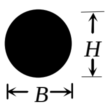
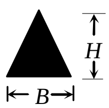
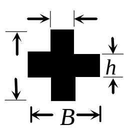
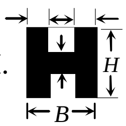
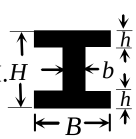
Hence:

$$c_{\text{total}} = c_{\text{elastic}} + c_{\text{plastic}} \; .$$

Therefore:

$$c_{\text{elastic}} = \frac{M}{EI} \; .$$

For a rectangular section (Fig. 29), when one supposes that the material has the same resistance to compression as it has to tension, it is very simple to determine the relation between the total curvature and the moment of flexure [**70**]. In order to do that, it is necessary to express the equilibrium condition for a section that is subjected to the moment of flexure *M*.

Form of the section	Moment of inertia I	Elastic modulus of the section W_{elastic}	Plastic modulus of the section W_{plastic}	$w = \dfrac{W_{\text{plastic}}}{W_{\text{elastic}}}$
I.  $H = B$	$\dfrac{H^4}{12} = 0.0833\,H^4$	$\dfrac{H^3}{6} = 0.1666\,H^3$	$\dfrac{H^3}{4} = 0.25\,H^3$	1.50
II. 	$\dfrac{BH^3}{12} = 0.0833BH^3$	$\dfrac{BH^2}{6} = 0.1666\,BH^2$	$\dfrac{BH^2}{4} = 0.25\,BH^2$	1.50
III.  $H = B$	$\dfrac{H^4}{48} = 0.0208\,H^4$	$\dfrac{H^3}{24} = 0.1666\,H^3$	$\dfrac{H^3}{12} = 0.0833\,H^3$	2.00
IV.  $H = B$	$\dfrac{\pi H^4}{64} = 0.0491H^4$	$\dfrac{\pi H^3}{82} = 0.0982H^3$	$\dfrac{0.2122\pi H^3}{4} = 0.0167H^3$	1.70
V. 	$\dfrac{BH^3}{36} = 0.0278\,BH^3$	$\dfrac{BH^2}{24} = 0.0416\,BH^3$	$\dfrac{BH^2}{10.126} = 0.0987\,BH^2$	2.37
VI. 	$\dfrac{BH^3}{12}[\beta + (1-\beta)\alpha^3]$	$\dfrac{BH^2}{6}[\beta - (1-\beta)\alpha^3]$	$\dfrac{BH^2}{4}[\beta - (1-\beta)\alpha^2]$	$1.5\,\dfrac{\beta+(1-\beta)\alpha^2}{\beta+(1-\beta)\alpha^3}$
VII. 	$\dfrac{BH^3}{12}[2\beta - (1-2\beta)\alpha^3]$	$\dfrac{BH^2}{6}[2\beta - (1-2\beta)\alpha^3]$	$\dfrac{BH^2}{4}[2\beta - (1-2\beta)\alpha^2]$	$1.5\,\dfrac{2\beta+(1-2\beta)\alpha^2}{2\beta+(1-2\beta)\alpha^3}$
VIII. 	$\dfrac{BH^3}{12}[1-(1-\beta)(1-2\alpha)^3]$	$\dfrac{BH^2}{6}[1-(1-\beta)1-2\alpha)^3]$	$\dfrac{BH^2}{4}[1-(1-\beta)(1-2\alpha)^2]$	$1.5\,\dfrac{1-(1-\beta)(1-2\alpha)^2}{1-(1-\beta)(1-2\alpha)^3}$

$$\sigma_y\ (\text{traction}) = \sigma_y\ (\text{compression}) \cdot \alpha = \frac{h}{H}\ \beta = \frac{b}{B}$$

For symmetric sections: $\delta_1 = \delta_2 = d_1 = d_2 = H/2$

For sections with Δ: $d_1 = \dfrac{2}{3}H$, $d_2 = \dfrac{1}{3}H$, $\delta_1 = \dfrac{H}{\sqrt{2}}$, $\delta_2 = H\left(1 - \dfrac{1}{\sqrt{2}}\right)$

Figure 28.

One can deduce the following relation between the two moments of flexure M_{plastic} and M from the diagram of the stresses in the bent section:

$$M_{\text{plastic}} - M = \tfrac{1}{3} B\, d^2 \sigma_y\,, \tag{44}$$

in which d represents the distance from the neutral axis to the plane whose fibers have just reached the elastic limit.

However, from the deformation diagram of that same section, one will have:

$$c_{\text{total}} = \frac{\sigma_y}{Ed}\,.$$

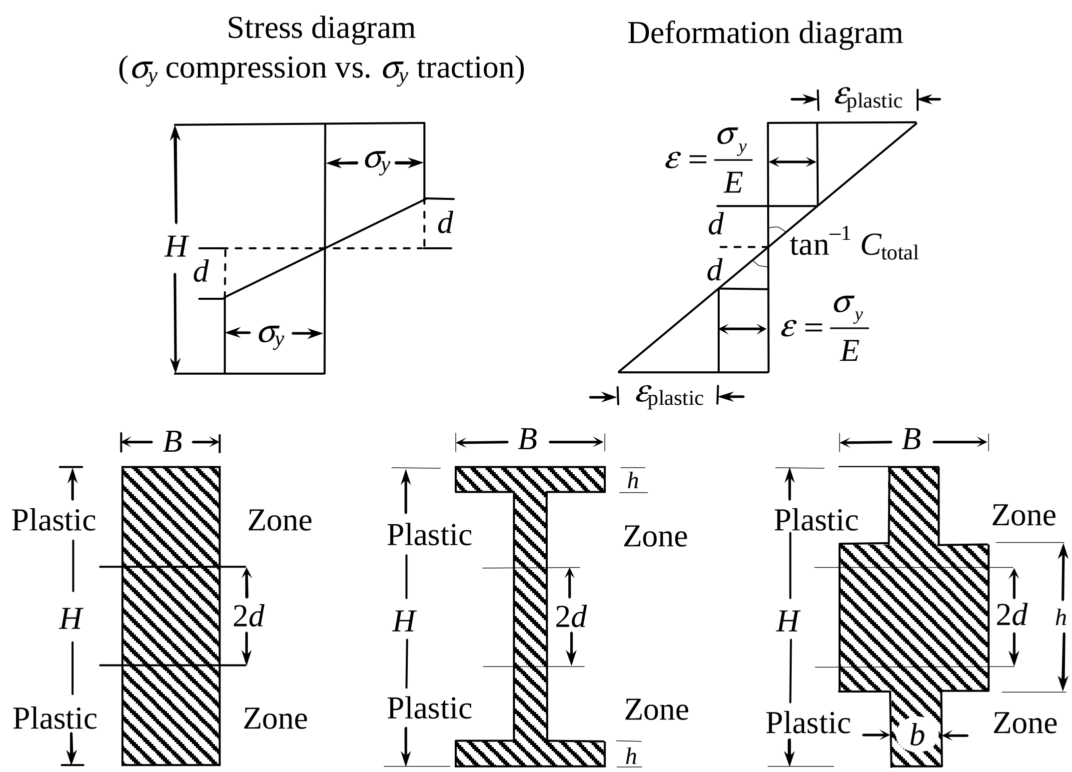


Figure 29.

Hence:

$$d = \frac{\sigma_y}{Ec_{\text{total}}}\,,$$

and formula (44) will become:

$$M_{\text{plastic}} - M = \frac{B\sigma_y^3}{3E^2 c_{\text{total}}^2}\,,$$

from which, one can deduce the following expression for the c_{total} :

(45) $$c_{\text{total}} = \frac{1}{E}\sqrt{\frac{B\sigma_y^3}{3(M_{\text{plastic}} - M)}}\,.$$

Formula (45) can be applied with no change to no particular sort of bent beam that has a symmetric section with respect to the *X*-axis, and for which the elastic zone of stress resides entirely within the rectangular portion of the section, whose size is *B* (Fig. 29).

One can find the relation between the moment of flexure and the total curvature in a more general fashion by a graphical method that Colonetti gave [**71**].

Figures 30 and 31 give the calculated values for the curves that give the total curvature versus the moments of flexure for the two different types of beam profiles (viz., a beam that has a cruciform section and one that has a rectangular section), and the calculated values are compared to the ones that were derived from experiments.

As the diagrams show, the experimental curves are always above the ones that were calculated.

The divergence between the two curves will become more and more pronounced when the moment of flexure increases above the value $M > M_{\text{elastic}}$, and the curing phenomena at the boundaries will become increasingly important [**70**].

5. – As one knows, under elastic conditions, the theory of the bending of beams is based upon the possibility of integrating the deflection curve equation:

(46) $$\frac{d^2y}{dx^2} = -\,c_{\text{elastic}} = -\frac{M}{EI}\,.$$

When one passes beyond the elastic limit for the deflection curve of the beam, one can appeal to equation (46), provided that one substitutes the total curvature for the elastic curvature; hence:

(47) $$\frac{d^2y}{dx^2} = -\,c_{\text{total}} = -\,(c_{\text{elastic}} + c_{\text{plastic}}).$$

If one supposes that the moment of flexure *M* returns to zero after having attained a value $M > M_{\text{elastic}}$ then the elastic curvature will also become zero, but the plastic curvature will not vanish, and equation (47) will become:

(48) $$\frac{d^2y}{dx^2} = -\,c_{\text{plastic}}\,.$$

One can calculate the permanent deflection of a beam that has been bent beyond the elastic limit with the aid of equation (48).

Under elastic conditions, while considering the deflection curve to be a funicular curve for an imaginary load that is represented by the curvature diagram, one can deduce a very simple grapho-analytical method for calculating the deformations of beams.

Moment of flexure vs. curvature diagram for a beam in the elasto-plastic regime

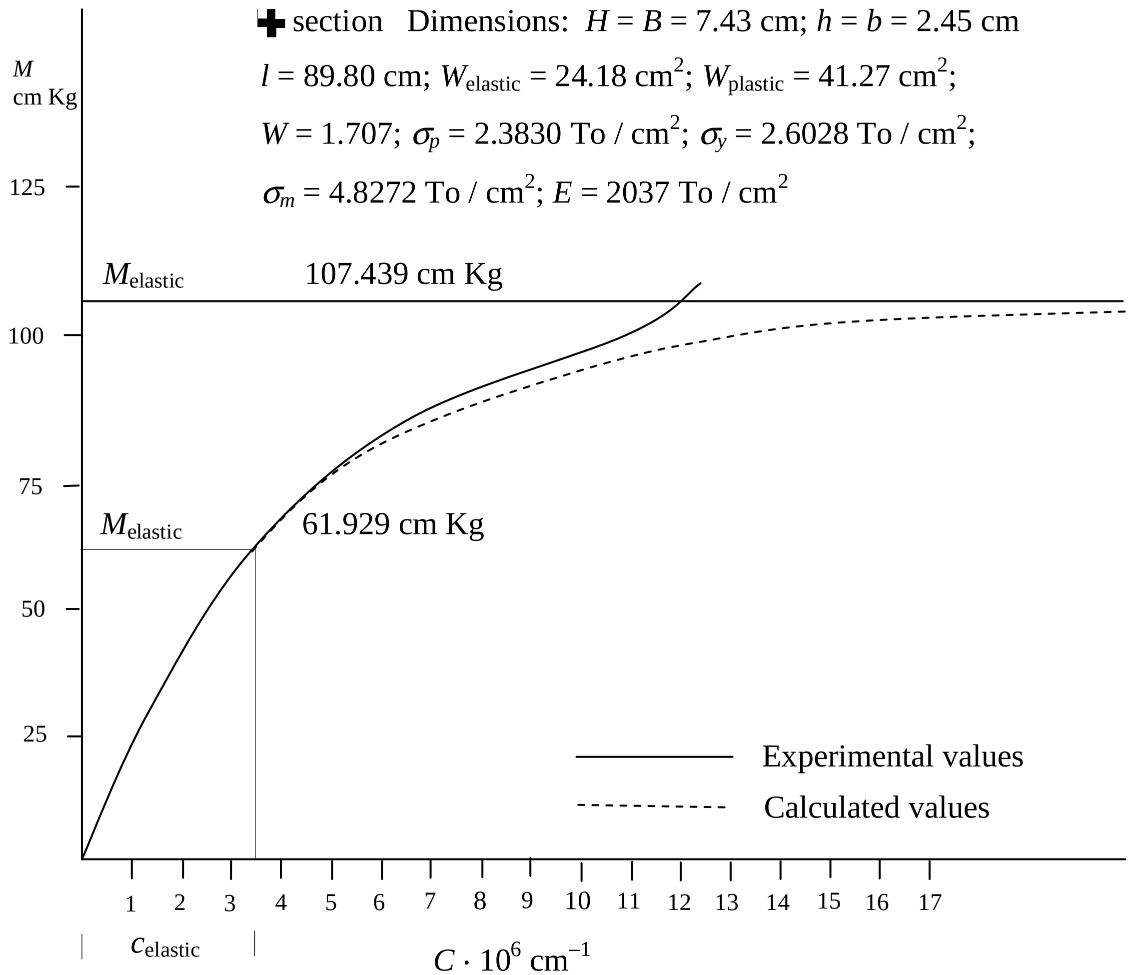


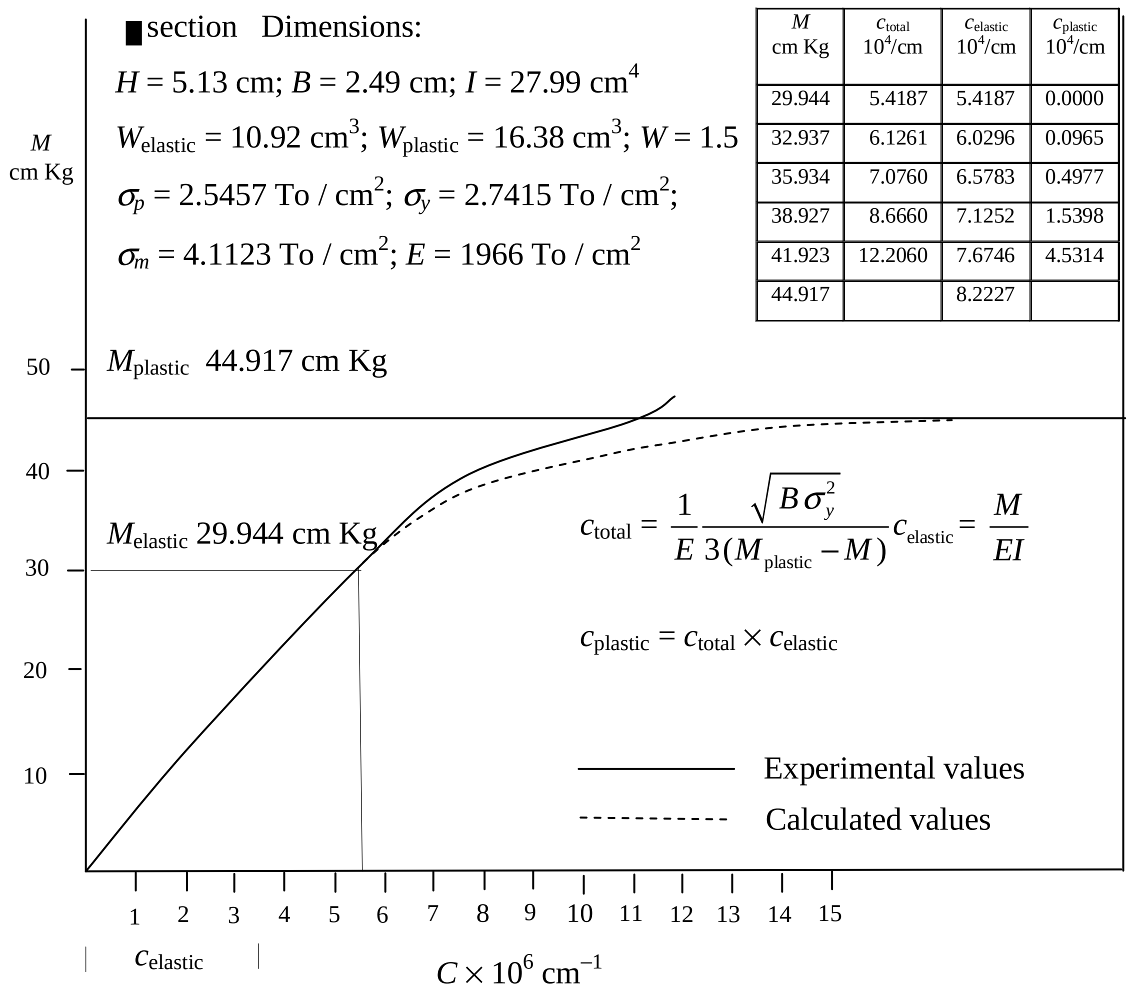
Figure 30.

One can extend the same method to the case of interest to us by considering the total curvature diagram, instead of the elastic curvature diagram, and one can then calculate the deformations of beams under plastic conditions [**70**]. (See § **8** of this chapter)

6. – Some considerations that are similar to the ones that are made for metallic beams that are bent beyond their elastic limit can be extended to beams of reinforced concrete for a limited period of their deformation [**72**], [**73**], [**74**].

Tests involving the compression and traction of concrete prisms have shown that the relation between the stress and the deformation can be represented graphically by a diagram that is similar to the one that is illustrated in Fig. 32 [**75**]. The stresses are represented vertically and the deformations horizontally. Positive values represent tractions, while negative values represent compressions.

Moment of flexure vs. curvature diagram for a beam in the elasto-plastic regime



M cm Kg	c_{total} 10^4/cm	c_{elastic} 10^4/cm	c_{plastic} 10^4/cm
29.944	5.4187	5.4187	0.0000
32.937	6.1261	6.0296	0.0965
35.934	7.0760	6.5783	0.4977
38.927	8.6660	7.1252	1.5398
41.923	12.2060	7.6746	4.5314
44.917		8.2227	

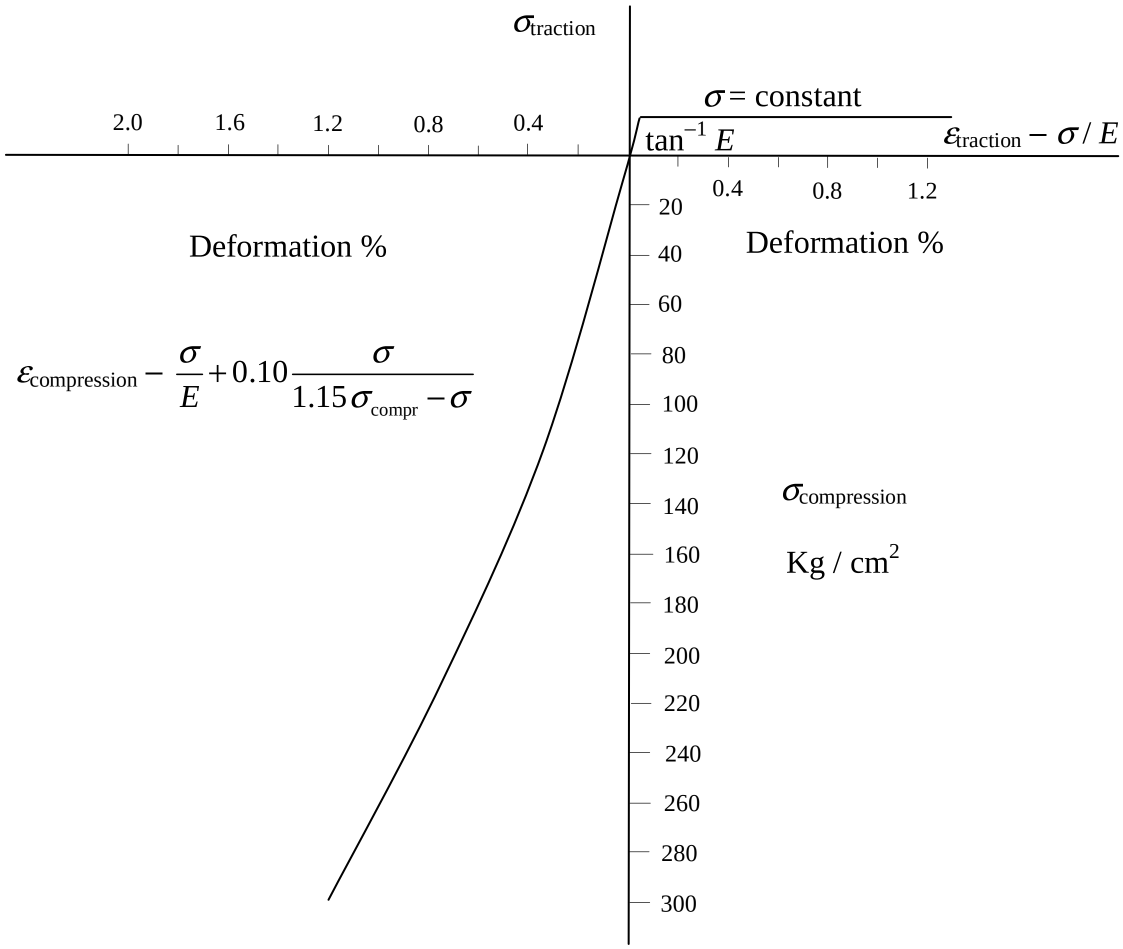
Figure 31.

The deformation due to a compression $\varepsilon_{\text{total}}$ can be expressed to a sufficient degree of approximation in the stress terms by the following law [**76**]:

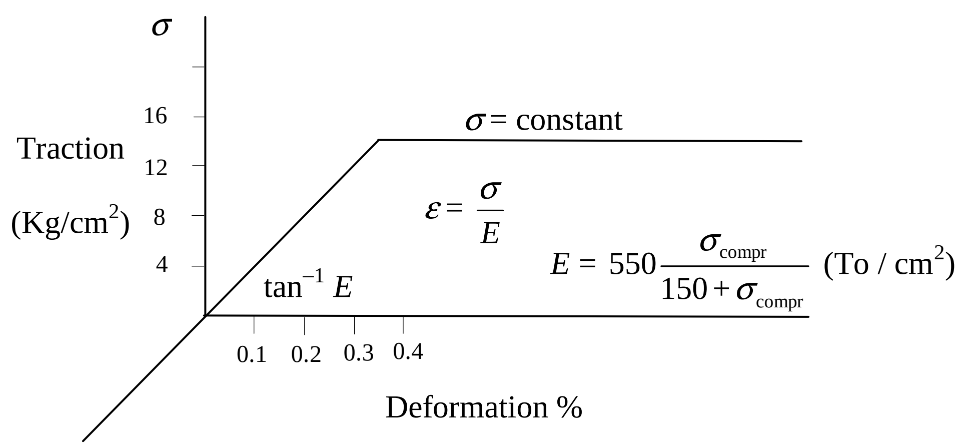
$$\varepsilon_{\text{total}} = \varepsilon_{\text{elastic}} + \varepsilon_{\text{plastic}} = \frac{\sigma}{E} + 0.10\frac{\sigma}{1.15\sigma_{\text{compr}} - \sigma}.$$

E represents the initial value of the modulus of elasticity for concrete; i.e., the inclination of the tangent with respect to the origin in the stress-deformation diagram. σ_{compr} represents the maximum resistance to compression of concrete in prismatic form. One can further express that quantity to a sufficient degree of approximation as a function of the prismatic resistance to compression of concrete by the formula [**77**]:

$$E = 550 \frac{\sigma_{\text{compr}}}{150 + \sigma_{\text{compr}}} \quad (\text{To} / \text{cm}^2).$$



(*a*)



(*b*)

Figure 32. – Relations between the unit tension and the unit deformations for concrete.

The branch that represents traction begins at the origin with the same inclination E as the other one, but for a small value of stress, it will change its direction and become almost parallel to the deformation axis. If the concrete is not reinforced then the last phase of the phenomenon will not exist, so to speak, and the specimen will rapidly break. However, if the specimen is sufficiently reinforced then the concrete can resist more appreciable deformations before it fractures, and one can see that second phase of the phenomenon very clearly then [**78**].

One can represent this phenomenon to a sufficient degree of approximation by two straight lines, the first of which starts at the origin and represents positive stresses that are within certain easily-determined values that correspond to the elastic phase, and the second of which, which is parallel to the deformation axis and consequently represents deformations that increase to infinite at constant stress, represents the plastic phase of the process [**79**].

The summit of the angle represents the elastic limit of concrete under traction.

If a reinforced concrete beam is subject to a moment of flexure that increases from zero up to the value at which the beam ruptures then one can distinguish three different phases of the deformation of the beam [**75**].

a. An elastic phase. – During this phase, the concrete is subject to a compression and a traction at the same time and behaves like an elastic body.

Consequently, if the entire section is considered to be homogeneous then it will resist the external load.

b. An elasto-plastic phase. – When the concrete attains the elastic limit for traction, a second phase will commence. By analogy with the metallic beams that are bent beyond the elastic limit, we shall call that second phase the *elasto-plastic phase.* That second phase has a very short duration in the case of an ordinary reinforced concrete beam. Indeed, some cracks will be produced in concrete when the elongation reaches 0.3 %.

c. A third and final phase. – This is the most important phase in the deformation of a reinforced concrete beam. During that phase, the concrete, which is now broken, offers no resistance to traction, which is, nonetheless, completely absorbed by the steel frame.

When beginning a project involving reinforced concrete beams, one ordinarily neglects the first two phases in order to consider only the third one. For that reason, we shall completely neglect even the very limited resistance to traction that is offered by concrete and calculate the reinforcement as if it were resistant to all traction ([1]).

7. – The fundamental theorem upon which the theory of elasto-plastic equilibrium is based, as it was presented by Colonetti, is the following one:

([1]) In the considerations above, we have not taken "pre-stressing" into account. A complete treatment of that important problem was given by Colonetti in his recent book *L'équilibre des corps déformables*, Paris, Dunod.

The internal stresses (with components τ_{11} , τ_{22} , τ_{33} , τ_{12} , τ_{13} , τ_{23}) *that characterize the equilibrium state are the ones that make the expression:*

$$I = \Phi + \int_V [\tau_{11}\,\bar{\gamma}_{11} + \tau_{22}\,\bar{\gamma}_{22} + \tau_{33}\,\bar{\gamma}_{33} + \tau_{12}\,\bar{\gamma}_{12} + \tau_{13}\,\bar{\gamma}_{13} + \tau_{23}\,\bar{\gamma}_{23}]\,dV$$

minimal with respect to all of the values that the expression can take that are compatible with plastic deformations (with components $\bar{\gamma}_{11}$, $\bar{\gamma}_{22}$, $\bar{\gamma}_{33}$, $\bar{\gamma}_{12}$, $\bar{\gamma}_{13}$, $\bar{\gamma}_{23}$) *and the given external forces.*

Φ represents the elastic potential energy:

$$\Phi = \int_V \varphi\, dV ,$$

in which φ is an essentially-positive homogeneous quadratic form of the six special stress components that were recalled above.

The k equations that give rise to such minimum conditions are linear and inhomogeneous, not only with respect to the k parameters of the stress state (viz., hyperstatic unknowns), but also with respect to the characteristics of the state of coaction (viz., parameters that allow one to express the impressed deformations linearly in terms of them in some particular concrete cases).

8. – Consider a continuous beam on four supports under the load conditions that are illustrated in Fig. 33, cases *A* and *B*.

When the central span is bent beyond the elastic limit, plastic deformations will be produced. It will then be easy to extend the three-moments theorem to those cases, which expresses the continuity conditions at each support with the aid of the grapho-analytic method [**70**].

The angles of rotation at the central supports will be:

$$\theta_1 = \frac{1}{3}\frac{\mathcal{M}_A\, l_1}{EI},$$

$$\theta_2 = \frac{P\,l_2^2}{16EI} + \frac{\mathcal{M}_A\, l_2}{2EI} + \frac{1}{2}\int c_{\text{plastic}}\, ds .$$

The continuity condition is this:

$$\theta_1 = -\,\theta_2 .$$

Hence:

$$\frac{\mathcal{M}_A}{EI}\left(\tfrac{2}{3}l_1 + l_2\right) + \frac{P\,l_2^2}{8EI} + \int c_{\text{plastic}}\, ds = 0, \tag{49}$$

but the plastic curvature (c_{plastic}) is a function of the bending moment at the center of the beam $\mathcal{M}_m$ and also the unknown hyperstatic moment $\mathcal{M}_A$, upon which that bending moment depends. One can always define the relation between the hyperstatic bending moment and the external force P by using formula (49).

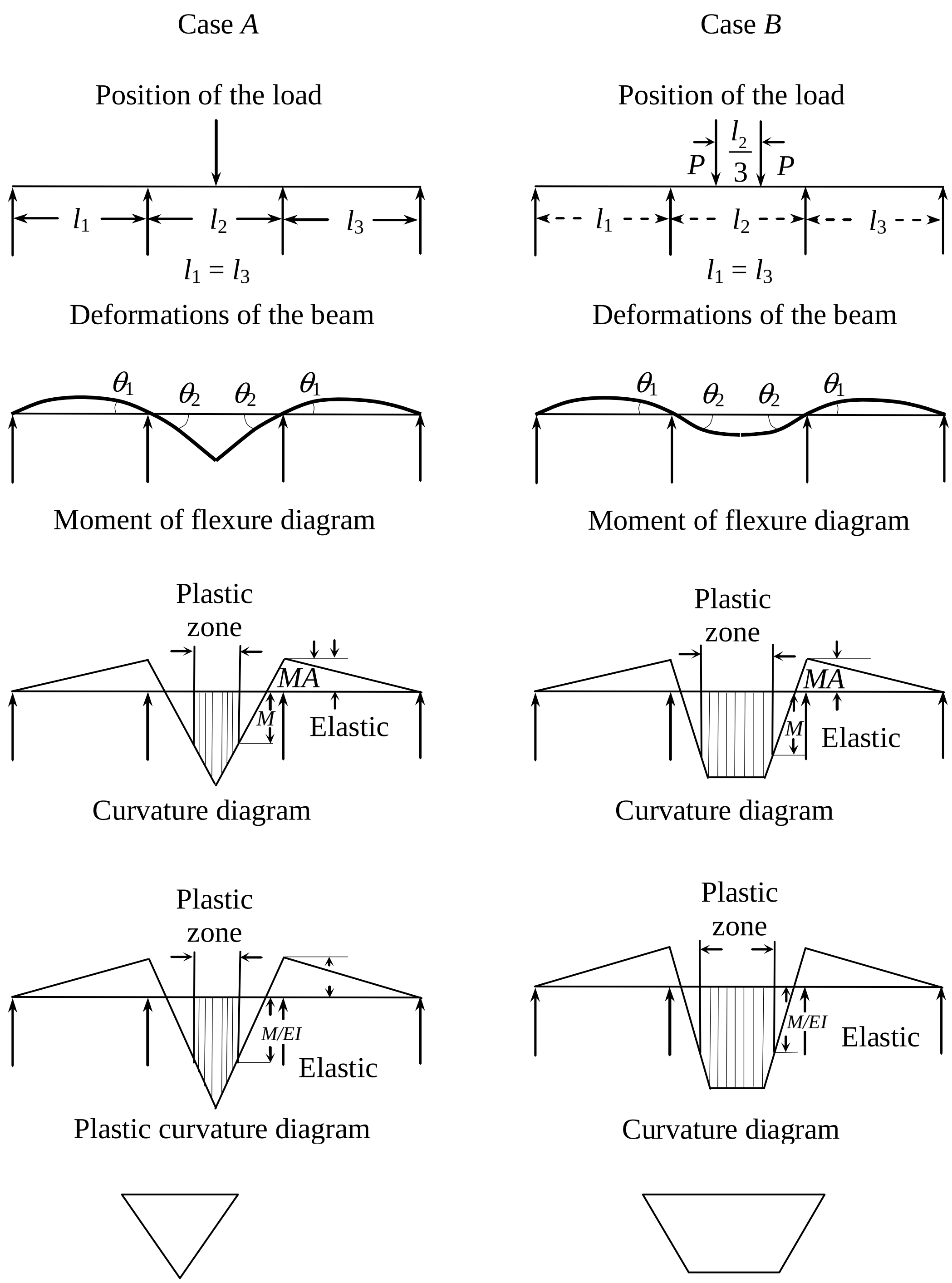


Figure 33.

Colonetti obtained the same result as the one that is given by formula (49) by utilizing his minimum-energy theorem.

For case *B* in Fig. 33, we get, in its place:

$$\theta_1 = \frac{1}{3}\frac{\mathcal{M}_A\, l_1}{EI},$$

$$\theta_2 = \frac{P\, l_2^2}{9EI} + \frac{\mathcal{M}_A\, l_2}{2EI} + \frac{1}{2}\int c_{\text{plastic}}\, ds,$$

but here again:

$$\theta_1 = -\,\theta_2 .$$

Hence:

(50) $$\frac{\mathcal{M}_A}{EI}\left(\tfrac{2}{3} l_1 + l_2\right) + \frac{2P\, l_2^2}{9EI} + \int c_{\text{plastic}}\, ds = 0.$$

If the bending moment M_m in the central beam is less than the limiting value of the elastic bending moment (i.e., $M_m < \mathcal{M}_{\text{elastic}}$) for the two cases *A* and *B* then $c_{\text{plastic}} = 0$, and that will imply the well-known Clapeyron equation in each case:

(49′) $$\frac{\mathcal{M}_A}{EI}\left(\tfrac{2}{3} l_1 + l_2\right) + \frac{P\, l_2^2}{8EI} = 0,$$

(50′) $$\frac{\mathcal{M}_A}{EI}\left(\tfrac{2}{3} l_1 + l_2\right) + \frac{2P\, l_2^2}{9EI} = 0.$$

9. – What are particularly important and interesting from the engineering viewpoint are the conclusions that one can deduce from applying these methods to depressed arches of reinforced concrete. For those arches, the stress curve that is calculated by ordinary methods that are based in the theory of elasticity will be longer than the central curve of the arch at the summit and at the two abutments. That amounts to saying that the stresses in the concrete at the summit and at the two abutments will attain very high values that are often inadmissible, whereas experience shows a more satisfactory state of affairs. Colonetti [**80**] showed that one can explain the apparent contradiction between the theoretical and experimental results by introducing the plastic factor into the calculation and applying his general minimum-energy theorem. The plastic adaptation of the material will make the stress curve almost coincide with the central curve of the arch, and it will reduce the stresses in the sections that are greatly deformed to some normal values.

In that manner, one can plainly justify the static behavior of some works that were constructed already, such as for example, the arch bridge over the Astico in Calvene, which was established in 1909 by Danusso and is a bridge that was found to support loads that were greater than much of the ones for which it was constructed in the first World War.

If one calculates the forces to which that structure was subjected according to the usual theory that takes into account only elastic deformation then they would be so intense that the bridge would not have been capable of supporting them. One can account for the effective behavior of the work by applying the new theory, as the engineer Oberti did [**81**]; i.e., by taking the plastic adaptations of the material into account.

XII. – Systematic deformations.

1. – We have shown the behavior of certain hyperstatic structures in the case in which the stress in certain sections reaches the elastic limit of the material when those structures are subjected to external loads. A distortion will then follow whose effect is to diminish the stresses in those sections. Consequently, the stresses will increase in the sections of the structure that is not also strongly-loaded. That reaction of the material can be compared to a natural defense that is organized by the structure itself. Without that defense, the structure would be destroyed by the external forces. The idea will occur to us naturally that we must imitate and supplement the automatic reaction of the structure by artificially creating certain distortions. They will tend to increase the resistance of the section in question to the external conditions under which the latter would be destroyed, which are conditions such as transient or permanent loads, the effect of temperature variations, the contraction of concrete, or the bending of the internal or external links.

The problem in any case is to perform a very delicate surgical operation on the structure itself when it has just been constructed, since the distortions that one must produce and which must remain impressed permanently in the structure are generally very small, and one must adjust them very precisely if one would like to avoid, on the one hand, an insufficient effect, and on the other, a cure that is worse than the initial defect [**82**], [**83**], [**84**].

2. – Those methods have found their most important applications in the context of reinforced concrete bridges and arches.

One hundred and fifty years ago, Perronet sought "the means that one might employ in order to construct large stone arches of 200, 300, 400, and up to 500 feet long (i.e., 65, 97, 130, and 162 m, resp.) that are destined to span wide valleys bounded by steep rocks." Perronet's predictions have been largely surpassed with the present means.

One can consider the most remarkable advance in the technique of modern construction to be the process that has allowed for extraordinary achievements in the construction of bridges, above all, in France, by the engineer Freyssinet [**85**], [**86**], [**87**].

The first application of the method was made by Freyssinet to the bridge in Voudre over the Allier in 1913 (three spans of 74, 79, and 74 m). In 1914-1919, Freyssinet applied that method to the bridge at Villeneuve-sur-Lot (span: 98 m) [**88**], in 1923, to the bridge at Saint-Pierre-du-Vauvray (span: 131 m) [**89**], and in 1927, to the bridge of the Caille on the torrent of the Usse in Haute-Savoie. The bridge of Plougastel on the Élorn near Brest was completed in 1930 with the aid of Freyssinet, and it has three arches of 185 m, which constituted a record for reinforced concrete bridge constructions [**90**], [**91**].

In the United States, the bridge over the Rogue River in Oregon was built using the Freyssinet method during the years 1931-1932. That structure consists of a series of seven symmetric reinforced concrete arches that each have lengths of 76 m [**92**], [**93**].

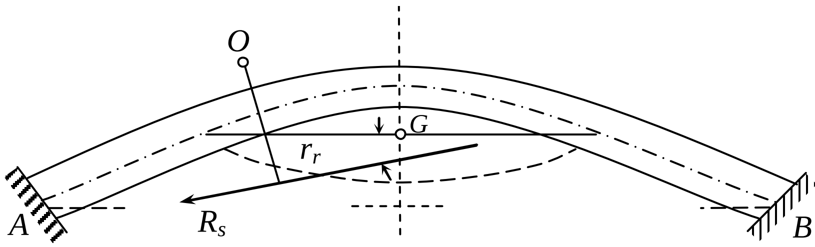


Figure 34.*a*

For the structures of the type above, the predominant effect that one must combat is the contraction of the concrete, which happens gradually and concludes around eighteen months after the material has solidified.

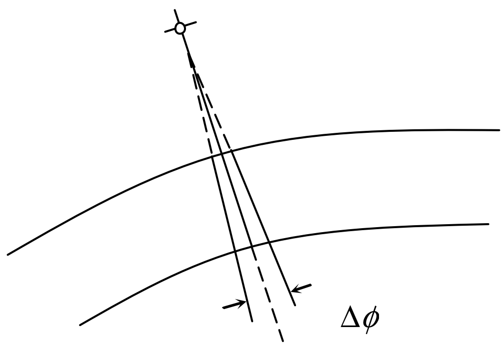


Figure 34.*b*

A single operation will not be sufficient. That compensation process must be performed at the same time that the contraction is produced, and if it is to be truly useful, it must not stop the traffic on the bridge.

That process utilizes a series of Volterra distortions that are produced by hydraulic jacks and maintained by rigid metallic wedges that are introduced into the structure of the bridge.

In order to avoid traffic congestion, and at the same time to permit repeated operations, the following procedure must be followed frequently: A concrete pediment is built on one of the abutments of the bridge during its construction. One leaves special recesses in that pediment into which one introduces hydraulic jacks, and they will produce the desired distortions at regular interval. The deformations thus-produced are measured by delicate instruments that one can easily read.

3. – In practice, the problem is discussed in the simplest manner by means of the theory of Culmann's elasticity ellipse of terminal displacements [**94**], [**95**].

We can reduce the problem to a two-dimensional one by referring it to the symmetry plane of the arch. For any section of the arch, the resultant of the internal stresses will be a certain force R_s whose intensity and direction can be determined directly. If one cuts the arch along an arbitrary section Σ and then displaces the faces of the cut with respect to each other such that the relative motion is a rotation whose value $\Delta\Phi$ is related to the elastic weight G of the arch and to the distance d from the elastic barycenter to the line of action r of the resultant by the relation:

$$\Delta\Phi = R_s\, G\, d,$$

which is valid around the antipole O of r with respect to the elasticity ellipse of the arch (Fig. 34.*a*).

In practice, one can always take the section to be the one that contains the point O. When the arch is cut along that section, one will introduce a force by means of a rigid wedge of opening $\Delta\Phi$ that thus annuls the self-stress R_s . If that wedge is itself elastic then one must take its own deformations into account (Fig. 34.*b*).

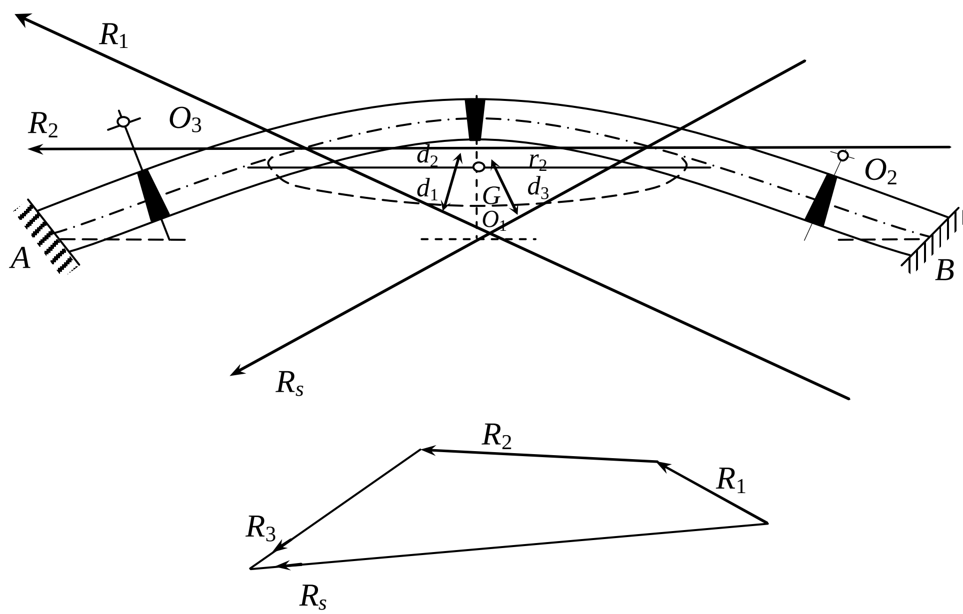


Figure 35.*a*

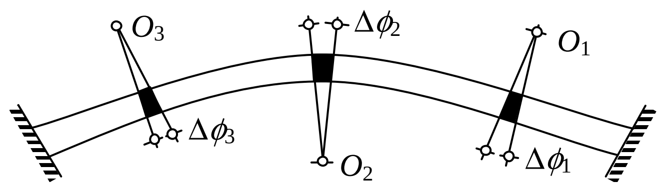


Figure 35.*b*

4. – Instead of correcting the effect of self-stress with just one distortion, one might sometimes find it preferable to utilize several of them that collectively annul the effect of the self-stress R. The process that one agrees to apply in the case of three distortions is indicated, by way of example, in Figs. 35.*a* and 35.*b*: One of them is at the key, while the other two are symmetric to the waist. If three unaligned points O_1, O_2, O_3 are chosen then one can determine the antipolars r_1, r_2, r_3, resp., with respect to the elasticity ellipse of the arch, and one then decomposes R along those three directions.

Hence, one will have three forces O_1, O_2, O_3, resp., that correspond to the rotations:

$$\Delta\Phi_1 = R_1\, \mathcal{G}\, d_1\,,$$
$$\Delta\Phi_2 = R_2\, \mathcal{G}\, d_2\,,$$
$$\Delta\Phi_3 = R_3\, \mathcal{G}\, d_3\,,$$

with respect to O_1, O_2, O_3, respectively. One introduces wedges into the joints that have openings that are equal to the calculated rotations, and one will then find that the preceding state of elastic stresses has been destroyed.

The considerations that we just developed for reinforced concrete arched bridges can obviously be applied to more varied and complex types of constructions.

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