"Sulla integrazione di una classe di equazioni dinamiche," Atti R. Accad. sci. Torino **33** (1897/98), 542-558.

On the integration of a class of dynamical equations

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In a previous article that had the title "Sopra una classe di equazione dinamiche" (*), I established the differential equations for *spontaneous motion with independent characteristics*.

We will then take advantage of the results that were found in it in order to examine those motions in more detail. We shall first study the case of second-order motions, i.e., those motions that depend upon only two characteristics, and show how they can be obtained as functions of time by means of exponential functions. We will then take up the examination of a *non-holonomic* system that can be assumed to be the typical second-order system and completely study the course of the motion of that system while observing the peculiarities that are observed in the case of permanent motions and distinguishing the case of stability from that of instability.

In two succeeding sections, we will treat the ν-order motions when there exist $\nu - 2$ linear integrals or $\nu - 3$ linear integrals and one quadratic one. In that case, we will benefit from a result that we established some years ago (**) and show that whenever the determinant equation has unequal roots, the characteristics will be elliptic functions of time.

Finally, in the last section, by applying a brilliant observation of Poincaré that was already employed successfully by Picard (***) and Painlevé (****) in regard to some mechanical questions, we will obtain the expression for the characteristic in the most general case by means of a series of functions of time that is valid for all values of time and whose coefficients can be obtained by means of rational operations on the known constants of the differential equations and the initial values of the characteristics.

The problem of the analytical determination of the characteristics as functions of time will then be solved completely.

§ 1. – Spontaneous second-order motions with independent characteristics.

1. – In the case of second-order systems, the general equations [*loc. cit.*, § **7**, form. (E′)] will become:

(*) Session on 27 February 1898.
(**) Session on 31 March 1895.
(***) PICARD, *Traité d'analyse*, t. III, Chapter X.
(****) PAINLEVÉ, *Leçons sur la théorie analytique des équations différentielles professées à Stockholm*, page 577.

$$p_1' = g_{12}^{(1)}\, p_2\, p_1 + g_{12}^{(2)}\, p_2^2\,,$$
$$p_2' = g_{21}^{(2)}\, p_1\, p_2 + g_{21}^{(1)}\, p_1^2\,,$$

or, when one sets:

$$g_{12}^{(1)} = -\,\alpha\,, \qquad g_{12}^{(2)} = -\,\beta\,,$$

one will have:

(1)
$$\begin{cases} p_1' = -p_2\,(\alpha\, p_1 + \beta\, p_2) \\ p_2' = \quad p_1\,(\alpha\, p_1 + \beta\, p_2). \end{cases}$$

2. – We shall now show the effective existence of second-order systems with independent characteristics that correspond to arbitrary values of the constants α and β. An infinitude of them exist for any system of values of those quantities. In particular, we shall examine a special one that can be considered to be typical, and which will give a concrete idea of the course of the corresponding motion.

Take a system of axes x_1, x_2, x_3 such that the cosines of the angles that they originally form with the coordinates axes ξ_1, ξ_2, ξ_3 are represented by the following table:

(2)
$$\begin{array}{c|ccc} & \xi_1 & \xi_2 & \xi_3 \\ \hline x_1 & \alpha_1 & \alpha_2 & \alpha_3 \\ x_2 & \beta_1 & \beta_2 & \beta_3 \\ x_3 & \gamma_1 & \gamma_2 & \gamma_3 \end{array}$$

Rotate the axes x_1, x_2, x_3 through an angle θ around the parallel to ξ_1 that goes through their origin, so their cosines will become:

$$\begin{array}{c|ccc} & \xi_1 & \xi_2 & \xi_3 \\ \hline x_1 & \alpha_1 & \alpha_2 & \alpha_3 \\ x_2 & \beta_1 & \beta_2 & \beta_3 \\ x_3 & \gamma_1 & \gamma_2 & \gamma_3 \end{array}$$

in which:

$$\xi_{11} = \alpha_1\,, \qquad \xi_{21} = \alpha_2 \cos\theta - \alpha_3 \sin\theta\,, \qquad \xi_{31} = \alpha_3 \cos\theta + \alpha_2 \sin\theta\,,$$
$$\xi_{12} = \beta_1\,, \qquad \xi_{22} = \beta_2 \cos\theta - \beta_3 \sin\theta\,, \qquad \xi_{32} = \beta_3 \cos\theta + \beta_2 \sin\theta\,,$$
$$\xi_{13} = \gamma_1\,, \qquad \xi_{23} = \gamma_2 \cos\theta - \gamma_3 \sin\theta\,, \qquad \xi_{33} = \gamma_3 \cos\theta + \gamma_2 \sin\theta\,.$$

Let ξ_1, ξ_2, ξ_3 denote the coordinates of the origin of the axes x_1, x_2, x_3 and set:

(3)
$$\theta = D\,\xi_1\,,$$

in which D denotes a constant. We will then have that when any point in space is taken to be the origin of the axes x_1, x_2, x_3 will correspond to an orientation of those axes.

In order to obtain those various orientations, observe that the axes x_1, x_2, x_3, whose origins belong to the same plane parallel to ξ_2, ξ_3, are parallel to them, and if their origins are chosen to be in the coordinate plane ξ_2, ξ_3 then their common orientation will correspond to the one that was originally represented by the table (2). Therefore, imagine that those axes x_1, x_2, x_3 are drawn through any point in that coordinate plane in their original orientation, and then move that plane with a screwing motion around the axis ξ_1 in such a way that the pitch of the screw is $2\pi / D$. If the plane drags along the axes x_1, x_2, x_3, which are supposed to be drawn through any point of it and are assumed to be rigidly coupled with that plane, in the course of its motion they will then take on the orientations that correspond to the various positions that their origins can assume.

9. – From what we have just seen, any point A in space corresponds to a plane $x_1\, x_2$ that passes through A, and which we shall denote by σ_A . Suppose that we have a moving point, such that in any position that it occupies in space, it can move only tangentially to the plane σ_A that corresponds to that point. The system will be *non-holonomic* (*) and the equation of constraint will be:

$$\xi_{13}\, d\xi_1 + \xi_{23}\, d\xi_2 + \xi_{33}\, d\xi_3 = 0 .$$

We can then set:

$$\text{(4)} \qquad \begin{cases} \xi_1' = \xi_{11}\, p_1 + \xi_{12}\, p_2 , \\ \xi_2' = \xi_{21}\, p_1 + \xi_{22}\, p_2 , \\ \xi_3' = \xi_{31}\, p_1 + \xi_{32}\, p_2 , \end{cases}$$

in which p_1 and p_2 are the characteristics of the motion.

If one supposes that the mass of the point is equal to 1 then its *vis viva* will be:

$$T = \tfrac{1}{2}(p_1^2 + p_2^2) .$$

In order to calculate the coefficients $g_{12}^{(1)}$, $g_{12}^{(2)}$. one makes use of the formulas (see *loc. cit*, § **1**):

$$\begin{aligned} g_{12}^{(1)} = {} & \xi_{11}\left(\xi_{11}\frac{\partial \xi_{12}}{\partial \xi_1} + \xi_{21}\frac{\partial \xi_{22}}{\partial \xi_1} + \xi_{31}\frac{\partial \xi_{32}}{\partial \xi_1}\right) \\ & + \xi_{21}\left(\xi_{11}\frac{\partial \xi_{12}}{\partial \xi_2} + \xi_{21}\frac{\partial \xi_{22}}{\partial \xi_2} + \xi_{31}\frac{\partial \xi_{32}}{\partial \xi_2}\right) \\ & + \xi_{31}\left(\xi_{11}\frac{\partial \xi_{12}}{\partial \xi_3} + \xi_{21}\frac{\partial \xi_{22}}{\partial \xi_3} + \xi_{31}\frac{\partial \xi_{32}}{\partial \xi_3}\right) , \end{aligned}$$

(*) Indeed:

$$\xi_{12}\left(\frac{\partial \xi_{23}}{\partial \xi_3} - \frac{\partial \xi_{33}}{\partial \xi_2}\right) + \xi_{23}\left(\frac{\partial \xi_{33}}{\partial \xi_1} - \frac{\partial \xi_{13}}{\partial \xi_3}\right) + \xi_{23}\left(\frac{\partial \xi_{13}}{\partial \xi_2} - \frac{\partial \xi_{23}}{\partial \xi_1}\right) = (1 - \gamma_1^2) D \neq 0 .$$

$$
\begin{aligned}
g_{12}^{(2)} = \;& \xi_{12}\left(\xi_{11}\frac{\partial\xi_{12}}{\partial\xi_1}+\xi_{21}\frac{\partial\xi_{22}}{\partial\xi_1}+\xi_{31}\frac{\partial\xi_{32}}{\partial\xi_1}\right) \\
&+ \xi_{22}\left(\xi_{11}\frac{\partial\xi_{12}}{\partial\xi_2}+\xi_{21}\frac{\partial\xi_{22}}{\partial\xi_2}+\xi_{31}\frac{\partial\xi_{32}}{\partial\xi_2}\right) \\
&+ \xi_{32}\left(\xi_{11}\frac{\partial\xi_{12}}{\partial\xi_3}+\xi_{21}\frac{\partial\xi_{22}}{\partial\xi_3}+\xi_{31}\frac{\partial\xi_{32}}{\partial\xi_3}\right),
\end{aligned}
$$

and obtain:

$$
\begin{aligned}
g_{12}^{(1)} &= \alpha_1\left(\xi_{21}\frac{\partial\xi_{22}}{\partial\theta}+\xi_{31}\frac{\partial\xi_{32}}{\partial\theta}\right)\frac{d\theta}{d\xi_1} = \alpha_1\,(\xi_{31}\,\xi_{22}-\xi_{21}\,\xi_{32}) = -D\,\alpha_1\,\gamma_1\,, \\
g_{12}^{(2)} &= \beta_1\left(\xi_{21}\frac{\partial\xi_{22}}{\partial\theta}+\xi_{31}\frac{\partial\xi_{32}}{\partial\theta}\right)\frac{d\theta}{d\xi_1} = \beta_1\,(\xi_{31}\,\xi_{22}-\xi_{21}\,\xi_{32}) = -D\,\beta_1\,\gamma_1\,,
\end{aligned}
$$

so:

$$
\alpha = D\alpha_1\sqrt{1-\alpha_1^2-\beta_1^2}\,, \qquad \beta = D\beta_1\sqrt{1-\alpha_1^2-\beta_1^2}\,.
$$

Since D, α_1, β_1 are arbitrary (as long as the sum of the squares of the last two quantities is less than 1), it can then happen that α and β have arbitrary values.

In everything that follows, we will suppose that we take D, γ_1, and the radical $\sqrt{1-\gamma_1^2}$ to always be *positive*.

To summarize everything that we have found, we can say that *the motion of a point with coordinates* ξ_1, ξ_2, ξ_3 *that is not subject to any force and whose constraints are representable by means of the equation:*

$$
\xi_{13}\,d\xi_1 + \xi_{23}\,d\xi_2 + \xi_{33}\,d\xi_3 = 0
$$

will constitute the most-general type of spontaneous second-order motion with independent characteristics.

4. – Lay the plane σ_A through an arbitrary point A in space that corresponds to that point and the plane that is parallel to the coordinate plane $\xi_2\,\xi_3$. Their intersection l_A forms angles ξ_1, ξ_2, ξ_3 with the axes whose cosines will be:

$$
(A) \qquad\qquad 0\,, \qquad \frac{\pm\xi_{22}}{\sqrt{1-\gamma_1^2}}\,, \qquad \frac{\mp\xi_{23}}{\sqrt{1-\gamma_1^2}}\,,
$$

respectively.

Take the positive direction of l_A to be the one whose cosines correspond to the upper sign.

Any point A in space corresponds to a line l_A, and obviously all l_A that relate to points that are equidistant from the $\xi_2\,\xi_3$-plane will be parallel and have the same direction.

Take two points A, B whose distances from the $\xi_2\,\xi_3$-plane differ by ε, so the angle between the lines will be:

$$l_A\, l_B = D\,\varepsilon\,.$$

Therefore, if $\varepsilon = (2h + 1)\,\pi / D$ (h is an integer) then the lines l_A and l_B will be parallel and have opposite directions, while if $\varepsilon = 2\,h\,\pi / D$ then they will be parallel have the same direction.

The line l_A, as well as the plane σ_A, has a noteworthy importance in all questions of motion.

§ 2. – Integrating the equations of spontaneous second-order motion with independent characteristics.

1. – Recall the general equations (1).

They admit the *vis viva* integral:

$$p_1^2 + p_2^2 = \text{const.} = C^2\,.$$

Set:

$$p_1 = C\cos\varphi\,, \qquad p_2 = C\sin\varphi\,, \qquad C > 0\,,$$

$$\alpha = -A\cos\varphi_0\,, \qquad \beta = A\sin\varphi_0\,, \qquad A = \left|\sqrt{\alpha^2+\beta^2}\right| = D\,\gamma_1\sqrt{1-\gamma_1^2}\,.$$

(1) reduces to the single equation:

$$\varphi' = A\,C\sin(\varphi - \varphi_0)\,,$$

and when that is integrated, it will give:

$$A\,C\,(t - t_0) = \log\tan\tfrac{1}{2}\,(\varphi - \varphi_0)\,,$$

in which t_0 is an arbitrary constant. Therefore:

$$\tan\tfrac{1}{2}\,(\varphi - \varphi_0) = e^{A\,C\,(t-t_0)}\,,$$

$$\cos\varphi = -\,\frac{\cos\varphi_0\sinh AC\,(t-t_0) + \sin\varphi_0}{\cosh AC\,(t-t_0)}\,,$$

$$\sin\varphi = \frac{-\sin\varphi_0\sinh AC\,(t-t_0) + \cos\varphi_0}{\cosh AC\,(t-t_0)}\,,$$

and finally:

$$(5)\qquad \left\{\begin{aligned} p_1 &= -\frac{C}{A}\,\frac{\beta\sinh AC\,(t-t_0) - \alpha}{\cosh AC\,(t-t_0)}\,,\\ p_2 &= \frac{C}{A}\,\frac{\alpha\sinh AC\,(t-t_0) + \beta}{\cosh AC\,(t-t_0)}\,.\end{aligned}\right.$$

We thus have the solved formulas for all second-order motions.

2. – Let us move on to the *permanent* second-order motions.

We obtain the equations (cf., *loc. cit.*, § **7**):

$$p_2\,(\alpha\,p_1 + \beta\,p_2) = 0\,,$$
$$p_1\,(\alpha\,p_1 + \beta\,p_2) = 0\,,$$

so:

$$\alpha\,p_1 + \beta\,p_2 = 0\,,$$

or when we let λ denote a constant quantity:

$$p_1 = -\,\lambda\,\beta\,, \qquad p_2 = \lambda\,\alpha\,. \tag{6}$$

We will then have all permanent second-order motions.

In order to get those formulas from the general formulas (5), it is enough to set $t_0 = \pm\,\infty$ in them.

If we set $t_0 = +\,\infty$ then we will have:

$$\lim_{t_0=\infty} \frac{C}{A}\frac{\sinh AC\,(t-t_0)}{\cosh AC\,(t-t_0)} = -\,\frac{C}{A}\,,$$

but if we set $t_0 = -\,\infty$ then we will have:

$$\lim_{t_0=-\infty} \frac{C}{A}\frac{\sinh AC\,(t-t_0)}{\cosh AC\,(t-t_0)} = \frac{C}{A}\,.$$

Therefore, in the first case, one will find that:

$$p_1 = -\,\lambda\,\beta\,, \qquad p_2 = \lambda\,\alpha\,, \qquad \lambda = -\,\frac{C}{A} < 0\,,$$

and in the second case:

$$p_1 = -\,\lambda\,\beta\,, \qquad p_2 = \lambda\,\alpha\,, \qquad \lambda = \frac{C}{A} > 0\,.$$

We shall now show that the formulas (6) correspond to the stable permanent motions when one has $\lambda > 0$, and to unstable permanent motions when $\lambda < 0$.

3. – To that end, we will prove the following theorem:

Any spontaneous second-order motion with independent characteristics will indefinitely tend towards becoming a permanent motion that is specified by the formulas:

$$p_1 = -\frac{C}{A}\beta, \quad p_2 = \frac{C}{A}\alpha.$$

Indeed, it follows from (5) that:

$$\lim_{t=\infty} p_1 = -\frac{C}{A}\beta, \qquad \lim_{t=\infty} p_2 = \frac{C}{A}\alpha.$$

It is easy to deduce from this that if one is given a stationary motion that corresponds to positive λ then it will always be possible to perturb it in such a way that p_1 and p_2 will differ from the constant values that those quantities had in their stationary motion during all of their motion by numbers that are as small as one desires. In order for that to be true, it is sufficient that the alterations that the values of p_1 and p_2 are initially subjected to should be less than a given limit. On the contrary, if one considers a stationary motion that corresponds to negative λ then the stated property will not be verified, because no matter how little one alters the values of p_1 and p_2 at an arbitrary instant, as long as one does not keep them proportional to $-\beta$ and α, resp., the motion will cease to be stationary, and the values of p_1 and p_2 will tend to $-\dfrac{C}{A}\alpha$ and $\dfrac{C}{A}\beta$, resp., indefinitely.

Those differing properties that one verifies for stationary motions according to whether λ is positive or negative constitute precisely what one assumes to be the *characteristic property of their stability and instability* (*).

4. – We can easily study the course of a motion in a neighborhood where it reduces to a stable motion.

Therefore, take:

$$p_1 = -\lambda\beta + \omega_1, \qquad p_2 = \lambda\beta + \omega_2 \qquad\qquad (\lambda > 0)$$

in (1) and consider ω_1 and ω_2 to be vey small in such a way that one can ignore their powers higher than the first in comparison to those quantities.

(1) will then become:

$$\omega_1' = -\lambda\,\alpha\,(\alpha\,\omega_1 + \beta\,\omega_2),$$
$$\omega_2' = -\lambda\,\beta\,(\alpha\,\omega_1 + \beta\,\omega_2),$$

and set:

(*) In the note that follows the present one, the definitions will be given of stability and instability of stationary motion in the general case, and one will become acquainted with those concepts. (Meanwhile, cf., "Sulle rotazioni permanenti stabili di una sistema in cui sussistono moti interni stazionarii," Annali di Mat., vol. XXIII.)

$$\omega_1 = \psi_1 e^{\rho t}, \quad \omega_2 = \psi_2 e^{\rho t},$$

we will have:

$$\psi_1(\lambda \alpha^2 + \rho) + \psi_2(\lambda \alpha \beta) = 0 ,$$
$$\psi_1(\lambda \alpha \beta) + \psi_2(\lambda \beta^2 + \rho) = 0 ,$$

so

$$\begin{vmatrix} \lambda \alpha^2 + \rho & \lambda \alpha \beta \\ \lambda \alpha \beta & \lambda \beta^2 + \rho \end{vmatrix} = 0 ,$$

or

$$\rho^2 + \lambda \rho (\alpha^2 + \beta^2) = 0 ,$$

from which it will follow that:

$$\rho = \begin{cases} 0 \\ -\lambda(\alpha^2 + \beta^2) = -\lambda A^2. \end{cases}$$

Therefore, if one ignores the zero root then one will have:

$$\omega_1 = K \alpha e^{-\lambda A^2 t}, \qquad \omega_2 = K \beta e^{-\lambda A^2 t},$$

§ 3. – Typical case of spontaneous second-order motions with independent characteristics.

1. – Recall the example in **§ 1** of a system whose motion we assumed to have the type of a spontaneous second-order motion with independent characteristics. We shall show how one can perform the integration and obtain a picture of the course of its motion.

2. – It follows from (3) and (4) that:

(7) $$\theta' = D \xi_1' = D(\alpha_1 p_1 + \beta_1 p_2) .$$

We begin by supposing that the motion is permanent. We deduce from (6) that:

$$\theta' = 0 ,$$

so const. θ = const. (4) will then become:

$$\xi_1' = 0 ,$$
$$\xi_2' = \lambda D \gamma_1 \xi_{33} ,$$
$$\xi_3' = - \lambda D \gamma_1 \xi_{23} ,$$

and when they are integrated:

$$\xi_1 - \xi_1^0 = 0\,, \quad \xi_2 - \xi_2^0 = \lambda\, D\, \gamma_1\, \xi_{33}\, t\,, \quad \xi_3 - \xi_3^0 = -\,\lambda\, D\, \gamma_1\, \xi_{23}\, t\,,$$

in which ξ_1^0, ξ_2^0, ξ_3^0 represent the coordinates of the position A that is occupied by the moving body at the time $t = 0$. The motion is therefore uniform and takes place along a straight line l_A.

It will take place in the positive or negative direction according to whether λ is positive or negative, resp.

We can then conclude that:

The stable permanent motions are uniform motions in the positive direction of l_A and the unstable motions are the ones in the negative direction of that line.

3. – Suppose that the motion is not permanent. If one takes (5) into account then one will deduce from (7) that:

$$\theta' = \frac{AC}{\gamma_1 \cosh AC\,(t - t_0)}\,.$$

One will then get θ as a function of time by a quadrature. Therefore, with three new quadratures, one can also obtain ξ_1 ξ_2, ξ_3 as functions of time by means of (4).

4. – However, let us forego those operations and recognize the course of motion in the following manner:

Let g denote the angle that the direction of motion of the point at any instant forms with the positive direction of the line l_A that passes through the point A that the moving point occupies at that instant. We will have [see (A)]:

$$\cos g = \frac{\xi_2'}{\sqrt{\xi_1'^2 + \xi_2'^2 + \xi_3'^2}} \frac{\xi_{33}}{\sqrt{1-\gamma_1^2}} - \frac{\xi_3'}{\sqrt{\xi_1'^2 + \xi_2'^2 + \xi_3'^2}} \frac{\xi_{23}}{\sqrt{1-\gamma_1^2}}$$

$$= \frac{(\xi_{21}\,\xi_{23} - \xi_{31}\,\xi_{23})\; p_1 + (\xi_{22}\,\xi_{23} - \xi_{32}\,\xi_{23})\; p_2}{\sqrt{p_1^2 + p_2^2}\sqrt{1-\gamma_1^2}} = \frac{\alpha_1\, p_2 - \beta_1\, p_1}{\sqrt{p_1^2 + p_2^2}\sqrt{1-\gamma_1^2}}\,,$$

and upon applying (5), that will give:

$$\cos g = \frac{\sinh AC\,(t - t_0)}{\cosh AC\,(t - t_0)}\,,$$

and therefore:

(8) $$\sin g = \frac{1}{\cosh AC\,(t - t_0)}\,.$$

One deduces from those formulas that:

$$dg = -\frac{AC\,dt}{\cosh AC(t-t_0)} = -\gamma_1\,d\theta,$$

so when that is integrated:

$$\theta - \theta_0 = -\frac{1}{\gamma_1}(g - g_0),$$

in which θ_0 and g_0 denote the values of θ and g, resp., at time $t = 0$.

It will then follow that:

$$\xi_1 - \xi_1^0 = \frac{1}{D\gamma_1}(g_0 - g) \,. \tag{9}$$

5. – We now appeal to the two formulas (8) and (9) that we found. The first one shows that *the angle g will decrease indefinitely*, while the second one gives the projection onto the ξ_1-axis of the path that is traverse by the moving point. *The limit to which that projection will tend as time increases indefinitely will be:*

$$\frac{g_0}{D\gamma_1}.$$

6. – The minimal distance between the z-axis and the line l_A along which the moving point is found at time t will be [see (A)]:

$$r = \frac{\xi_{23}\,\xi_2 + \xi_{33}\,\xi_3}{\sqrt{1-\gamma_1^2}}.$$

Therefore:

$$r' = \frac{\xi_{23}\,\xi_2' + \xi_{33}\,\xi_3' + \xi_2\,\xi_{23}' + \xi_3\,\xi_{33}'}{\sqrt{1-\gamma_1^2}},$$

and with some easy calculations:

$$r' = -\frac{\gamma_1^2}{A}\frac{d\theta}{dt} + \frac{1}{\sqrt{1-\gamma_1^2}}(\xi_{23}\,\xi_3 - \xi_{33}\,\xi_2)\frac{d\theta}{dt},$$

so upon integration:

$$r - r_0 = -\frac{\gamma_1^2}{A}(\theta - \theta_0) + \int_0^t \frac{\xi_{23}\,\xi_3 - \xi_{33}\,\xi_2}{\sqrt{1-\gamma_1^2}} \cdot \frac{AC}{\gamma_1}\frac{1}{\cosh AC(t-t_0)}\,dt,$$

in which r_0 is the value of r at time $t = 0$.

Now observe that ξ_{23} and ξ_{33} are less than 1, and that ξ_2' and ξ_3' are quantities that are always less than a finite value. One can then take:

$$\frac{\xi_{23}\,\xi_3 - \xi_{33}\,\xi_2}{\sqrt{1-\gamma_1^2}} \cdot \frac{AC}{\gamma_1} = M\,t\,,$$

in which M is a quantity that is less than a finite number.

It will then follow that:

(10) $$\int_0^t M\,t\,\frac{1}{\cosh AC\,(t-t_0)}\,dt$$

will always be less than a finite number, no matter what the value of t might be.

We can then conclude that *the moving point can never approach the line l_A when it is at a distance from the z-axis beyond a certain limit.* Obviously, as t_0 increases indefinitely, the integral (10), and therefore $r - r_0$, will decrease indefinitely.

7. – One can easily deduce some other properties from the preceding considerations that relate to the perturbations of stable and unstable permanent motions besides the ones that were established before (**§ 2**). Thus, in the case of stable motions, as long as the initial perturbation is less than a conveniently-chosen limit, for all of time, the perturbed motion will submit to a deviation in the direction of the permanent motion that is less than a number that is as small as one pleases, and the motion will take plane along a trajectory that is indefinitely close to a line l_A whose distance from the trajectory of unperturbed motion is less than an arbitrary small number.

Lastly, in the case of an unstable motion, the stated peculiarity cannot be verified for the perturbed motion.

§ 4. – Spontaneous motions of order ν with independent characteristics that have $\nu - 2$ linear integrals.

1. – If the motion has order ν and one knows $\nu - 2$ independent linear integrals then the differential equations will come down to the form [see *loc. cit.*, **§ 8**, eq. (H′)]:

(11) $$\begin{cases} z_1' = z_2\,(M_{12}^{(1)}\,z_1 + M_{12}^{(2)}\,z_2 + N_{12}), \\ z_2' = z_1\,(M_{21}^{(1)}\,z_1 + M_{21}^{(2)}\,z_2 + N_{21}), \end{cases}$$

so when one takes:

$$M_{12}^{(1)} = -\,\alpha\,, \quad M_{12}^{(2)} = -\,\beta\,, \quad N_{12} = -\,\gamma\,,$$

one will have:

$$z_1' = -\,z_2\,(\alpha\,z_1 + \beta\,z_2 + \gamma)\,,$$

$$z_2' = z_1\,(\alpha z_1 + \beta z_2 + \gamma)\,.$$

2. – We shall confine ourselves to showing how the system reduces to a quadrature, although the calculation that must be performed presents no analytical difficulties.

It follows from the *vis viva* integral that:

$$z_1^2 + z_2^2 = \text{const.} = C^2\,,$$

so one can set:

$$z_1 = C\cos\varphi\,,\quad z_2 = C\sin\varphi\,,$$

so (11) will reduce to the single equation:

$$\varphi' = C\,(\alpha\cos\varphi + \beta\sin\varphi) + \gamma\,,$$

and if one sets:

$$\alpha = -A\sin\varphi_0\,,\qquad \beta = A\cos\varphi_0\,,\qquad A = \sqrt{\alpha^2+\beta^2}$$

then one will have:

$$\theta' = C\,A\sin(\varphi - \varphi_0) + \gamma\,,$$

so

$$t - t_0 = \int \frac{d\varphi}{C\,A\sin(\varphi-\varphi_0)+\gamma}\,,$$

in which t_0 denotes an arbitrary constant.

One can then conclude: *If one knows* $\nu - 2$ *linear integrals then the* ν *characteristics can be expressed by means of trigonometric or exponential functions of time.*

It is important to observe that exhibiting one or the other function will depend upon whether $C^2A^2 - \gamma^2$ is less than or greater than zero. *The course of the motion that results will then have an entirely different character according to the sign of the binomial* $C^2A^2 - \gamma^2$.

§ 5. – Spontaneous motion of order ν with independent characteristics that have $\nu - 3$ linear integrals and one quadratic integral.

1. – If one knows $\nu - 3$ independent integrals of degree one and one of degree two for a system of order ν, and if the characteristic equation has simple roots then one can write the equations of motion in the form (see *loc. cit.*, **§ 10**):

$$p_1' = e_{123}\,\frac{d\,(T, F)}{d\,(p_2, p_3)}\,,$$

$$p_2' = e_{231} \frac{d(T, F)}{d(p_2, p_1)},$$

$$p_3' = e_{312} \frac{d(T, F)}{d(p_1, p_2)},$$

in which:

$$T = \tfrac{1}{2} \sum_{r=1}^{3} \sum_{s=1}^{3} E_{rs}\, p_r\, p_s + \text{const.},$$

$$F = \tfrac{1}{2} \sum_{r=1}^{3} \sum_{s=1}^{3} \Lambda_{rs}\, p_r\, p_s + \Sigma_h \Lambda_h\, p_h,$$

which can also be written:

$$f_1 = \tfrac{1}{2} \sum_{r=1}^{3} \sum_{s=1}^{3} E_{rs}\, p_r\, p_s, \qquad f_2 = e_{123}\, F,$$

$$\frac{dp_1}{dt} = \frac{d(f_1, f_2)}{d(p_2, p_3)},$$

$$\frac{dp_2}{dt} = \frac{d(f_1, f_2)}{d(p_3, p_1)},$$

$$\frac{dp_3}{dt} = \frac{d(f_1, f_2)}{d(p_1, p_2)}.$$

2. – If we take into account a result that we established in the article "Sopra un sistema di differenziale equazioni" (*) then we can conclude that the integrals of the preceding equations are elliptic functions of t, so since the ν characteristics are linear functions of p_1, p_2, p_3, we will then have the theorem:

When one knows $\nu - 3$ linear integrals and one quadratic integral whose characteristic equation has unequal roots, the ν characteristics can be expressed as elliptic functions of time.

For the actual determination of the unknown functions, we refer to the article that was just cited.

§ 6. – General theorem on the integration by series of the equations of spontaneous motion of a system with independent characteristics.

1. – We begin by establishing the following:

(*) Atti della R. Accademia di Torino, 1895.

Lemma I:

If:

$$\left| a_{sk}^{(r)} \right| < A \,,$$

and the values p_s^0 *of the* p_s *for* $t = t_0$ *are such that:*

$$p_s^0 \leq P$$

then the integrals of the differential equations:

$$p'_s = \sum_{r=1}^{\nu} \sum_{k=1}^{\nu} a_{sk}^{(r)} \, p_r \, p_k \qquad (s = 1, 2, \ldots, \nu) \tag{12}$$

will be holomorphic analytic functions of the plane in the complex variable t inside of the circle of radius:

$$r = \frac{1}{4\nu^2 P}$$

that has its center at the point $t = t_0$.

Indeed, observe that the right-hand sides of (12) are holomorphic functions for all values of the p_s such that:

$$| p_s - p_s^0 | < b \,,$$

in which b is an arbitrary number. The values that the moduli on the right-hand side assume when the p_s satisfy the preceding inequality will obviously be less that:

$$M = \nu^2 A (P + b)^2 \,.$$

If one takes into account the fact that the right-hand sides of (12) are independent of t then from a well-known theorem (*), one will have that the integrals p_s are holomorphic functions of the complex variable t inside of a circle of radius:

$$\frac{b}{M} = \frac{b}{\nu^2 A (P + b)^2} \,.$$

The maximum value of that ratio will occur when $b = P$, so one can assume that the radius of the circle inside of which the p_s are holomorphic is:

(*) PICARD, *Traité d'analyse*, t. II, page 312.

$$r = \frac{1}{4\nu^2 AP}.$$

2. – Lemma II:

If the real numbers $p_1^0, p_2^0, \ldots, p_\nu^0$ *are the values of* $p_1, p_2, \ldots, p_\nu$ *for the real value* $t = t_0$ *then the integrals of* (12) *will be holomorphic functions in all of the infinite strip in the complex plane for* t *found between the two parallels to the real axis that are separated from it by a distance of:*

$$\frac{1}{4\nu^2 A\sqrt{(p_1^0)^2 + (p_2^0)^2 + \cdots + (p_\nu^0)^2}}.$$

Indeed, since (12) admits the integral:

$$p_1^2 + p_2^2 + \cdots + p_\nu^2 = \text{const.},$$

for any real value of t, one will have:

$$|p_i| \leq \sqrt{(p_1^0)^2 + (p_2^0)^2 + \cdots + (p_\nu^0)^2}.$$

Thus, upon applying the preceding lemma, when one makes t traverse the entire real axis, one will find that p_i will remain holomorphic inside of all of the circles that have their centers on the real axis and have a radius equal to:

$$r = \frac{1}{4\nu^2 A\sqrt{(p_1^0)^2 + (p_2^0)^2 + \cdots + (p_\nu^0)^2}}.$$

That proves the proposition.

3. – From the two lemmas that were established, one deduces from an observation of Poincaré that the p_i can be developed in series that are ordered in powers of:

$$z = \frac{e^{\pi t/2r} - 1}{e^{\pi t/2r} + 1},$$

and the development will be valid for all values of t between $-\infty$ and $+\infty$. That implies the theorem:

If the initial vis viva for the spontaneous motion of a system of order ν *with independent characteristics is* $\frac{1}{2}T_0$ *then the characteristics can be expressed as functions of time by means of series in powers of:*

$$z = \frac{e^{2\pi\nu^2 A\sqrt{T_0}\,t} - 1}{e^{2\pi\nu^2 A\sqrt{T_0}\,t} + 1} \;,$$

in which A is a quantity that is greater than $\left| a_{sk}^{(r)} \right|$.

The coefficients can be calculated using rational operations that are performed on the initial values of p_i *and the coefficients* $a_{sk}^{(r)}$.

That theorem shows that the question of effectively determining the characteristics for any value of time has been resolved completely.