"Ueber kürzeste Integral curven einer Pfaff'schen Gleichung," Math. Ann. **52** (1899), 417-432.

On the shortest integral curves of a Pfaff equation

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In a discussion (*) of my paper "Grundlagen einer Krümmungslehre der Curvenscharen," Sommerfeld pointed out the difference between straightest and shortest paths that Hertz presented (**), which I had not considered in that paper. That prompted me to address the question of the shortest connecting line between two points that is, at the same time, an integral curve of a given Pfaff equation, or in other words, an orthogonal trajectory to a given doubly-infinite family of curves.

In what follows, I shall first derive the characteristic equation for that shortest line in two different ways, the first of which was given by Hertz in terms of the usual calculus of variations, while the second one, which is more geometrically intuitive, is better adapted to the situation.

The integration of the differential equations that were presented will then be discussed, and finally two cases will be given in which the integration is simplified.

I will refer to the aforementioned paper by the symbol "G", although only the last paragraph of what follows will assume any knowledge of that paper.

§ 1.

First derivation of the differential equation of the shortest lines.

Let a doubly-infinite family of curves be established by the equations:

$$dx : dy : dz = \xi : \eta : \zeta, \tag{1}$$

in which ξ, η, ζ are functions of x, y, z that should satisfy the condition:

$$\xi^2 + \eta^2 + \zeta^2 = 1.$$

The rectangular intersection curves of that family of curves are the integral curves of the Pfaff equation:

(*) Göttingische gelehrte Anzeigen (1898), no. 11.
(**) *Die Principien der Mechanik*, pps. 100 and 106.

(2) $$\xi\,dx + \eta\,dy + \zeta\,dz = 0.$$

One now imagines two points P_0 (x_0 , y_0 , z_0) and P_1 (x_1 , y_1 , z_1) being connected by one such integral curve and considers the coordinates x, y, z of its points to be functions of its arc-length s, which might run through the values from 0 to σ when one goes from P_0 (x_0 , y_0 , z_0) to P_1 (x_1 , y_1 , z_1) .

If one now varies the curve in question under the requirement that first of all the varied curve must still go through the points P_0 and P_1 , and furthermore that it must remain an integral curve of equation (2) then the variations δx, δy, δz must vanish for $s = 0$ and $s = \sigma$, and in addition, they must satisfy the varied equation (2); i.e., the relation:

$$\begin{aligned}\xi\,d\,\delta x + \eta\,d\,\delta y + \zeta\,d\,\delta z + dx\left(\frac{\partial\xi}{\partial x}\delta x+\frac{\partial\xi}{\partial y}\delta y+\frac{\partial\xi}{\partial z}\delta z\right)&\\ + dy\left(\frac{\partial\eta}{\partial x}\delta x+\frac{\partial\eta}{\partial y}\delta y+\frac{\partial\eta}{\partial z}\delta z\right)&\\ + dz\left(\frac{\partial\zeta}{\partial x}\delta x+\frac{\partial\zeta}{\partial y}\delta y+\frac{\partial\zeta}{\partial z}\delta z\right)&= 0.\end{aligned}$$

If one applies the relations (G, pp. 90) here:

$$2e_1 = \frac{\partial\zeta}{\partial y}-\frac{\partial\eta}{\partial z}, \qquad 2e_2 = \frac{\partial\xi}{\partial z}-\frac{\partial\zeta}{\partial x}, \qquad 2e_3 = \frac{\partial\eta}{\partial x}-\frac{\partial\xi}{\partial y}$$

then the relation in question will take the form:

$$\begin{aligned}\xi\,d\,\delta x + \eta\,d\,\delta y + \zeta\,d\,\delta z + \delta x\,(d\xi + 2e_3\,dy - 2e_2\,dz)&\\ + \delta y\,(d\eta + 2e_1\,dz - 2e_3\,dx)&\\ + \delta z\,(d\zeta + 2e_2\,dx - 2e_1\,dy) &= 0,\end{aligned}$$

or, more briefly:

(3) $$\xi\,d\,\delta x + \eta\,d\,\delta y + \zeta\,d\,\delta z + A = 0.$$

In order for the curve that is imagined to be the shortest of all of the integral curves of (2) that connect the points P_0 and P_1, the variation of the integral:

$$\int_0^\sigma ds$$

must vanish under the condition (3).

When one introduces a temporarily-undetermined function φ of s, that will imply the equation:

$$\int_0^\sigma\left\{\frac{dx\,d\delta x+dy\,d\delta y+dz\,d\delta z}{ds}+\varphi(\xi\,d\delta x+\eta\,d\delta x+\zeta\,d\delta x+A)\right\} = 0,$$

or, after performing a partial integration:

$$\int_0^\sigma \left\{ \varphi A - \delta x \left(d\frac{dx}{ds} + d(\varphi\xi) \right) - \delta y \left(d\frac{dy}{ds} + d(\varphi\eta) \right) - \delta z \left(d\frac{dz}{ds} + d(\varphi\zeta) \right) \right\} = 0.$$

Here, the coefficients of δx, δy, δz under the integral sign must be set equal to zero. In that way, when one goes from differentials to derivatives, it will follow that (*):

$$(4)\qquad \left\{ \begin{aligned} &\frac{d^2x}{ds^2} + \xi\frac{d\varphi}{ds} + 2\varphi\left(e_2\frac{dz}{ds} - e_3\frac{dy}{ds} \right) = 0, \\ &\frac{d^2y}{ds^2} + \eta\frac{d\varphi}{ds} + 2\varphi\left(e_3\frac{dx}{ds} - e_1\frac{dz}{ds} \right) = 0, \\ &\frac{d^2z}{ds^2} + \zeta\frac{d\varphi}{ds} + 2\varphi\left(e_1\frac{dy}{ds} - e_2\frac{dx}{ds} \right) = 0. \end{aligned} \right.$$

If one multiplies the foregoing equations by $\frac{dx}{ds}$, $\frac{dy}{ds}$, $\frac{dz}{ds}$, in turn, and adds them then one will obtain the condition (2) that was required from the outset. In order to find the other two conditions that the system (4) implies in a geometrically-intuitive form, we set:

$$(5)\qquad \left\{ \begin{aligned} 2e_1 &= 2\varepsilon\xi + \frac{1}{P_s}\left(\eta\frac{dz}{ds} - \zeta\frac{dy}{ds} \right) + \frac{1}{P'_s}\frac{dx}{ds}, \\ 2e_2 &= 2\varepsilon\eta + \frac{1}{P_s}\left(\zeta\frac{dx}{ds} - \xi\frac{dz}{ds} \right) + \frac{1}{P'_s}\frac{dy}{ds}, \\ 2e_3 &= 2\varepsilon\zeta + \frac{1}{P_s}\left(\xi\frac{dy}{ds} - \eta\frac{dx}{ds} \right) + \frac{1}{P'_s}\frac{dz}{ds}, \end{aligned} \right.$$

and furthermore:

$$(6)\qquad \left\{ \begin{aligned} \frac{d^2x}{ds^2} &= \frac{1}{R_s}\left(\eta\frac{dz}{ds} - \zeta\frac{dy}{ds} \right) + \frac{\xi}{h_s}, \\ \frac{d^2y}{ds^2} &= \frac{1}{R_s}\left(\zeta\frac{dx}{ds} - \xi\frac{dz}{ds} \right) + \frac{\eta}{h_s}, \\ \frac{d^2z}{ds^2} &= \frac{1}{R_s}\left(\xi\frac{dy}{ds} - \eta\frac{dx}{ds} \right) + \frac{\zeta}{h_s}. \end{aligned} \right.$$

$$(7)\qquad \frac{1}{R_s} + 2\,\varepsilon\,\varphi = 0, \qquad \frac{d\varphi}{ds} - \frac{\varphi}{P_s} + \frac{1}{h_s} = 0.$$

(*) Cf., A. Voss, "Ueber die Differentialgleichungen der Mechanik," Math. Ann., Bd. **25**, pp. 282.

The elimination of φ from those relations will take different forms according to whether ε is continually zero or does not vanish in general. Since:

$$2\varepsilon = \xi\left(\frac{\partial\zeta}{\partial y}-\frac{\partial\eta}{\partial z}\right)+\eta\left(\frac{\partial\xi}{\partial z}-\frac{\partial\zeta}{\partial x}\right)+\zeta\left(\frac{\partial\eta}{\partial x}-\frac{\partial\xi}{\partial y}\right),$$

in the case of $\varepsilon = 0$, the curves that are defined by (1) will be the orthogonal trajectories to a family of curves. Here, we will get simply:

$$\frac{1}{R_s} = 0, \tag{8}$$

which makes the second equation in (7) represent only a condition on the function φ, which is foreign to the problem.

However, when ε is non-zero, in general, eliminating φ will imply the relation (*):

$$\frac{1}{R_s}\left(\frac{d\log\varepsilon}{ds}+\frac{1}{P_s}\right)+\frac{2\varepsilon}{h_s}-\frac{d(1/R_s)}{ds} = 0. \tag{9}$$

We would now like to discover the geometric meaning of the quantities R_s, h_s, P_s that appear in this.

If one regards the coordinates of a spatial curve as functions of its arc-length then the equations for the curvature axis that belongs to the point (x, y, z) will become:

$$\left\{\begin{aligned} u &= x+\rho^2\frac{d^2x}{ds^2}+\rho l\left(\frac{dy}{ds}\frac{d^2z}{ds^2}-\frac{dz}{ds}\frac{d^2y}{ds^2}\right),\\ v &= y+\rho^2\frac{d^2y}{ds^2}+\rho l\left(\frac{dz}{ds}\frac{d^2x}{ds^2}-\frac{dx}{ds}\frac{d^2z}{ds^2}\right),\\ w &= z+\rho^2\frac{d^2z}{ds^2}+\rho l\left(\frac{dx}{ds}\frac{d^2y}{ds^2}-\frac{dy}{ds}\frac{d^2z}{ds^2}\right),\end{aligned}\right. \tag{10}$$

in the event that one denotes the coordinates of its points by u, v, w. Here, ρ means the radius of the first curvature of the curve, which then has the value:

$$\frac{1}{\sqrt{\left(\frac{d^2x}{ds^2}\right)^2+\left(\frac{d^2y}{ds^2}\right)^2+\left(\frac{d^2z}{ds^2}\right)^2}}. \tag{11}$$

(*) In that way, my previous assertion in these Annalen, Bd. **32**, pp. 555, lines 8, 9, is restricted to the case of $\varepsilon = 0$.

The quantity l means the abscissa of the point (u, v, w) relative to the center of the first curvature.

If one takes the spatial curve here to be an orthogonal trajectory of the family of curves that is given by (1) then:

$$\xi \frac{dx}{ds} + \eta \frac{dy}{ds} + \zeta \frac{dz}{ds} = 0.$$

Now, since one also has:

$$(u-x)\frac{dx}{ds} + (v-y)\frac{dy}{ds} + (w-z)\frac{dz}{ds} = 0,$$

the equations:

$$u = x + h\,\xi, \quad v = y + h\,\eta, \quad w = z + h\,\zeta$$

will be compatible with each other and will serve to determine h and l ; the curvature axis that belongs to a point P of an orthogonal trajectory will cut the tangent to the individual curve of the family of curves that goes through P. The quantity h measures the distance from the point of intersection in question to the point P.

If one multiplies the latter equations by $\frac{d^2x}{ds^2}$, $\frac{d^2y}{ds^2}$, $\frac{d^2z}{ds^2}$, in turn, and adds them then one will get:

(12) $$\frac{1}{h} = \xi \frac{d^2x}{ds^2} + \eta \frac{d^2y}{ds^2} + \zeta \frac{d^2z}{ds^2},$$

so from (6), one will have:

$$h = h_s.$$

We let $1 / h_s$ denote the normal curvature of the trajectory considered with respect to the family of curves.

Since, at the same time:

$$\frac{1}{h_s} = -\frac{dx}{ds}\frac{d\xi}{ds} - \frac{dy}{ds}\frac{d\eta}{ds} - \frac{dz}{ds}\frac{d\zeta}{ds},$$

the normal curvatures that we speak of for all trajectories of the family of curves that go through the same point in the same direction will be equal to each other.

As a result of the relation:

$$(u-x)\frac{dx}{ds} + (v-y)\frac{dy}{ds} + (w-z)\frac{dz}{ds} = 0,$$

the equations:

$$u = x + R\left(\eta \frac{dz}{ds} - \zeta \frac{dy}{ds}\right),$$

$$v = y + R\left(\zeta \frac{dx}{ds} - \xi \frac{dz}{ds}\right),$$

$$w = z + R\left(\xi \frac{dy}{ds} - \eta \frac{dx}{ds}\right)$$

are also compatible with each other; i.e., the curvature axis considered will also cut the tangents to the trajectory of the family of curves that go through P and are perpendicular to the trajectory considered.

If one multiplies the foregoing equation by $\frac{d^2x}{ds^2}$, $\frac{d^2y}{ds^2}$, $\frac{d^2z}{ds^2}$, in turn, and adds them then that will yield:

$$\frac{1}{R} = \sum \left(\eta \frac{dz}{ds} - \zeta \frac{dy}{ds} \right) \frac{d^2x}{ds^2}; \tag{13}$$

i.e., from (6):

$$R = R_s .$$

We call $1 / R_s$ the *geodetic curvature* of the trajectory considered relative to the family of curves.

From (6) and (11), one gets:

$$\frac{1}{\rho^2} = \frac{1}{R_s^2} + \frac{1}{h_s^2} .$$

From that, it is clear that of all orthogonal trajectories of the family of curves that go through the same point P in the same direction, the ones that possess the smallest first curvature at the location P will be the ones for which the geodetic curvature vanishes at that location. I have called a trajectory along which that curvature continually vanishes a *geodetic line* of the family of curves (G, pp. 50). Following Hertz's precedent (*Principien der Mechanik*, pp. 101), I will refer to such a line as the *straightest* one; however, that terminology is much less felicitous from a grammatical standpoint.

Secondly, we apply equations (10) to the individual curve of the family that goes through the point P.

Here, we have:

$$\frac{dx}{ds} = \xi, \qquad \frac{dy}{ds} = \eta, \qquad \frac{dz}{ds} = \zeta$$

$$\frac{d^2x}{ds^2} = \frac{\partial \xi}{\partial x}\xi + \frac{\partial \eta}{\partial y}\eta + \frac{\partial \zeta}{\partial z}\zeta ,$$

or since:

$$\frac{\partial \xi}{\partial x}\xi + \frac{\partial \eta}{\partial x}\eta + \frac{\partial \zeta}{\partial x}\zeta = 0,$$

one will have:

$$\frac{d^2x}{ds^2} = 2\,(e_2\,\zeta - e_3\,\eta),$$

and correspondingly:

$$\frac{d^2y}{ds^2} = 2\,(e_3\,\xi - e_1\,\zeta), \qquad \frac{d^2z}{ds^2} = 2\,(e_1\,\eta - e_2\,\xi),$$

such that:

$$\frac{1}{\rho^2} = 4(e_1^2 + e_2^2 + e_3^2 - \varepsilon^2) .$$

One then obtains:

$$\begin{aligned} u &= x + 2\rho^2 (e_2\,\zeta - e_3\,\eta) + 2\,\rho\,l\,(e_1 - \varepsilon\,\xi) , \\ v &= y + 2\rho^2 (e_3\,\xi - e_1\,\zeta) + 2\,\rho\,l\,(e_2 - \varepsilon\,\eta) , \\ w &= z + 2\rho^2 (e_1\,\eta - e_2\,\xi) + 2\,\rho\,l\,(e_3 - \varepsilon\,\zeta) . \end{aligned}$$

If one now forms the equations:

$$u = x + \mathsf{P}\frac{dx}{ds}, \quad v = y + \mathsf{P}\frac{dy}{ds}, \quad w = z + \mathsf{P}\frac{dz}{ds}$$

then, due to the relation:

$$(u - x)\,\xi + (v - y)\,\eta + (w - z)\,\zeta = 0,$$

those equations will be compatible with each other, and P means the distance from the point P to the intersection point of the curvature axis that was spoken of with the tangent to the trajectory under consideration.

If one multiplies the equations in question by 2 $(e_2\,\zeta - e_3\,\eta)$, 2 $(e_3\,\xi - e_1\,\zeta)$, 2 $(e_1\,\eta - e_2\,\xi)$, in turn, and adds them then that will give:

$$\frac{1}{\mathsf{P}} = 2\left[\frac{dx}{ds}(e_2\,\zeta - e_3\,\eta) + \frac{dy}{ds}(e_3\,\xi - e_1\,\zeta) + \frac{dz}{ds}(e_1\,\eta - e_2\,\xi)\right],$$

or also:

$$\frac{1}{\mathsf{P}} = 2\left[e_1\left(\eta\frac{dz}{ds} - \zeta\frac{dy}{ds}\right) + e_2\left(\zeta\frac{dx}{ds} - \xi\frac{dz}{ds}\right) + e_3\left(\xi\frac{dy}{ds} - \eta\frac{dx}{ds}\right)\right]. \tag{14}$$

As a result, from (6), one has:

$$\frac{1}{\mathsf{P}} = \frac{1}{\mathsf{P}_s} .$$

The family of curves that is given by (1) will be a system of rays when the quantities $\frac{d\xi}{ds}$, $\frac{d\eta}{ds}$, $\frac{d\zeta}{ds}$ vanish along every curve of the family; i.e., when:

$$e_1 : e_2 : e_3 = \xi : \eta : \zeta .$$

In that case, the expression $1 / \mathsf{P}_s$ will be continually zero.

Otherwise $1 / \mathsf{P}_s$ will vanish only when the tangent to the trajectory under consideration at a point is, at the same time, the binormal to the individual curve of the family that goes through that point.

We then have the result that: If the expression ε is continually equal to zero then the shortest line will coincide with the geodetic line and will be established by equations (2) and (8). If e is, in general, non-zero then equations (2) and (8) will characterize the geodetic lines, while equations (2) and (9) will characterize the shortest lines.

§ 2.

Second derivation of the differential equations of the shortest lines.

We once more understand x, y, z to mean the coordinates of one of the orthogonal trajectories of the give family of curves that connect the points P_0 and P_1. They will be assumed to be functions of the arc-length s, which might range from the value 0 to σ when one goes from P_0 to P_1. We shall now seek to find other trajectories that likewise connect the points P_0 and P_1 and lie in the neighborhood of the trajectory that is envisioned.

If one next takes:

$$x' = x + s\,(s - \sigma) \sum_{\nu=1}^{\infty} a_\nu \frac{\tau^\nu}{\nu!},$$

$$y' = y + s\,(s - \sigma) \sum_{\nu=1}^{\infty} b_\nu \frac{\tau^\nu}{\nu!},$$

$$z' = z + s\,(s - \sigma) \sum_{\nu=1}^{\infty} c_\nu \frac{\tau^\nu}{\nu!}$$

and considers the quantities a_ν, b_ν, c_ν to be functions of s that make it possible for the sums that appear to converge for sufficiently-small values of τ and all values of s from 0 to σ then x', y', z' will represent the coordinates of a curve that connects the points P_0 and P_1 for such a τ.

One must now find the conditions under which the curves τ = const. will also be orthogonal trajectories of the given family of curves.

To that end, we set:

$$s\,(s - \sigma)\, a_\nu = \alpha_\nu, \qquad s\,(s - \sigma)\, b_\nu = \beta_\nu, \qquad s\,(s - \sigma)\, c_\nu = \gamma_\nu,$$

to abbreviate. The values of the direction cosines ξ, η, ζ at the location $x = x'$, $y = y'$, $z = z'$ are:

$$\xi' = \xi + \tau\left(\frac{\partial \xi}{\partial x}\alpha_1 + \frac{\partial \xi}{\partial y}\beta_1 + \frac{\partial \xi}{\partial z}\gamma_1\right) + \ldots,$$

$$\eta' = \eta + \tau\left(\frac{\partial \eta}{\partial x}\alpha_1 + \frac{\partial \eta}{\partial y}\beta_1 + \frac{\partial \eta}{\partial z}\gamma_1\right) + \ldots,$$

$$\zeta' = \zeta + \tau\left(\frac{\partial \zeta}{\partial x}\alpha_1 + \frac{\partial \zeta}{\partial y}\beta_1 + \frac{\partial \zeta}{\partial z}\gamma_1\right) + \ldots$$

The coefficients of the powers of τ in the development of the expression:

$$\xi' \frac{dx'}{ds} + \eta' \frac{dy'}{ds} + \zeta' \frac{dz'}{ds} = 0$$

must now vanish. If one sets the coefficients of τ equal to zero then that will imply the condition for the quantities α_1 , β_1 , γ_1 in the form:

$$\alpha_1 \left(\frac{\partial \xi}{\partial x} \frac{dx}{ds} + \frac{\partial \eta}{\partial x} \frac{dy}{ds} + \frac{\partial \zeta}{\partial x} \frac{dz}{ds} \right) + \beta_1 \left(\frac{\partial \xi}{\partial y} \frac{dy}{ds} + \frac{\partial \eta}{\partial y} \frac{dy}{ds} + \frac{\partial \zeta}{\partial y} \frac{dz}{ds} \right)$$

$$+ \ \gamma_1 \left(\frac{\partial \xi}{\partial z} \frac{dx}{ds} + \frac{\partial \eta}{\partial z} \frac{dy}{ds} + \frac{\partial \zeta}{\partial z} \frac{dz}{ds} \right) + \xi \frac{d\alpha_1}{ds} + \eta \frac{d\beta_1}{ds} + \zeta \frac{d\gamma_1}{ds} = 0$$

or:

$$\frac{d(\alpha_1 \xi + \beta_1 \eta + \gamma_1 \zeta)}{ds} + 2\alpha_1 \left(e_3 \frac{dy}{ds} - e_2 \frac{dz}{ds} \right) + 2\beta_1 \left(e_1 \frac{dz}{ds} - e_3 \frac{dx}{ds} \right) + 2\gamma_1 \left(e_2 \frac{dx}{ds} - e_1 \frac{dy}{ds} \right) = 0.$$

Here, it should be emphasized that the quantities α_1 , β_1 , γ_1 refer to the three directions that are given by the curve of the family, the trajectory being considered, and the trajectory that is perpendicular to both of them, resp., and are set to:

$$\alpha_1 = n_1 \frac{dx}{ds} + n_2 \left(\eta \frac{dz}{ds} - \zeta \frac{dy}{ds} \right) + n_0 \, \xi ,$$

$$\beta_1 = n_1 \frac{dy}{ds} + n_2 \left(\zeta \frac{dx}{ds} - \xi \frac{dz}{ds} \right) + n_0 \, \eta ,$$

$$\gamma_1 = n_1 \frac{dz}{ds} + n_2 \left(\xi \frac{dy}{ds} - \eta \frac{dx}{ds} \right) + n_0 \, \zeta .$$

The quantities n_0 , n_1 , n_2 are proportional to the cosines of the angles that the tangent to the curve s = const. that goes through the point (x, y, z) makes with the three aforementioned directions.

The relation in question now assumes the form:

$$\frac{dn_0}{ds} + \frac{n_0}{P_s} - 2\varepsilon \, n_2 = 0. \tag{15}$$

The arc-length of the curve τ = const. between the points P_0 and P_1 will be given by the integral:

$$J = \int_0^\sigma \sqrt{\left(\frac{dx'}{ds} \right)^2 + \left(\frac{dy'}{ds} \right)^2 + \left(\frac{dz'}{ds} \right)^2} \, ds .$$

That is a function of τ. We ask what the condition might be under which it would be a minimum when $\tau = 0$.

One has:

$$\sum \left(\frac{dx'}{ds} \right)^2 = 1 + 2\tau \left(\frac{dx}{ds}\frac{d\alpha_1}{ds} + \frac{dy}{ds}\frac{d\beta_1}{ds} + \frac{dz}{ds}\frac{d\gamma_1}{ds} \right) + \ldots,$$

$$\sqrt{\sum \left(\frac{dx'}{ds} \right)^2} = 1 + \tau \sum \frac{dx}{ds}\frac{d\alpha_1}{ds} + \ldots,$$

and as a result:

$$\left(\frac{\partial J}{\partial \tau} \right)_{\tau=0} = \int_0^\sigma \left(\frac{dx}{ds}\frac{d\alpha_1}{ds} + \frac{dy}{ds}\frac{d\beta_1}{ds} + \frac{dz}{ds}\frac{d\gamma_1}{ds} \right) ds$$

$$= \int_0^\sigma \left\{ \frac{dn_1}{ds} + n_2 \sum \frac{dx}{ds} \left(\eta \frac{d^2 z}{ds^2} - \zeta \frac{d^2 y}{ds^2} \right) + n_0 \sum \frac{dx}{ds}\frac{d\xi}{ds} \right\} ds$$

$$= \int_0^\sigma \left(\frac{dn_1}{ds} - \frac{n_2}{R_s} - \frac{n_0}{h_s} \right) ds .$$

However, since n_1 is zero at the locations $s = 0$ and $s = \sigma$, one has:

$$\left(\frac{\partial J}{\partial \tau} \right)_{\tau=0} = - \int_0^\sigma \left(\frac{n_2}{R_s} + \frac{n_0}{h_s} \right) ds .$$

That integral should vanish, while the condition (15) will persist. The latter can be satisfied by only $n_0 = 0$ when ε is continually zero. The function n_2 under the integral then remains arbitrary, and in that way we come to the result that:

$$\frac{1}{R_s} = 0.$$

However, if ε is non-zero, in general, then we infer the expression for n_2 from the condition (15) and find that:

$$\left(\frac{\partial J}{\partial \tau} \right)_{\tau=0} = - \int_0^\sigma \left\{ \frac{1}{2\varepsilon R_s} \frac{dn_0}{ds} + n_0 \left(\frac{1}{h_s} + \frac{1}{2\varepsilon P_s R_s} \right) \right\} ds ,$$

or after partial integration, since n_0 also vanishes at the locations $s = 0$ and $s = \sigma$:

$$\left(\frac{\partial J}{\partial \tau}\right)_{\tau=0} = -\int_0^\sigma n_0 \left(\frac{d\,\dfrac{1}{2\varepsilon R_s}}{ds} - + \frac{1}{h_s} + \frac{1}{2\varepsilon P_s R_s} \right) ds\,.$$

Due to the arbitrariness of the function n_0 , the factor of n_0 under the integral sign must vanish, with which, one will obtain equation (9) of § **1**.

§ 3.

On the integration of the differential equations that were found.

Equations (8) and (9) of § **1** express geometric properties of the geodetic and shortest lines and serve as a means of deciding whether a given orthogonal trajectory of the family of curves is a geodetic line or a shortest line or neither. However, when one addresses the search for geodetic or shortest lines, those equations will mostly be useless, since the arc-length appears as an independent variable. In that case, it is better for one to replace the differentiation with respect to arc-length with differentiation with respect to one of the three coordinates x, y, z.

If we take – e.g., x – to be the independent variable then we will next get the relation:

$$\xi + \eta \frac{dy}{ds} + \zeta \frac{dz}{ds} = 0.$$

When we set:

$$N = \sqrt{1 - \eta^2 + 2\xi\eta \frac{dy}{dx} + (1 - \xi^2)\left(\frac{dy}{dx}\right)^2}$$

and preserve dy / dx, the direction cosines of the tangent to an orthogonal trajectory of the basic family of curves will then follow from:

$$\frac{dx}{ds} = \frac{\zeta}{N}, \qquad \frac{dy}{ds} = \frac{\zeta \dfrac{dy}{dx}}{N}, \quad \frac{dz}{ds} = \frac{-\xi - \eta \dfrac{dy}{dx}}{N}.$$

Geometrically speaking, the direction cosines in question are expressed here by the angle that the tangent to the projection of the trajectory onto the XY-plane makes with the X-axis.

Now, if $\mathfrak{F}$ is a function of x, y, z, and the first ν derivatives of y, namely, of:

$$p_1 = \frac{dy}{dx}, \qquad p_2 = \frac{d^2 y}{dx^2}, \qquad \ldots, \qquad p_\nu = \frac{d^\nu y}{dx^\nu},$$

then one will have:

$$\frac{d\mathfrak{F}}{ds} = \frac{1}{N}\left\{\frac{\partial\mathfrak{F}}{\partial x}\zeta + \frac{\partial\mathfrak{F}}{\partial y}\zeta\frac{dy}{dx} - \frac{\partial\mathfrak{F}}{\partial z}\left(\xi + \eta\frac{dy}{dx}\right) + \zeta\left(\frac{\partial\mathfrak{F}}{\partial p_1}\frac{d^2y}{dx^2} + \frac{\partial\mathfrak{F}}{\partial p_2}\frac{d^3y}{dx^3} + \cdots + \frac{\partial\mathfrak{F}}{\partial p_\nu}\frac{d^{\nu+1}y}{dx^{\nu+1}}\right)\right\}.$$

The expression $1 / R_s$ then includes the unknowns $x, y, z, \frac{dy}{dx}, \frac{d^2y}{dx^2}$, and the geodetic lines will be established by a system of the form:

$$\text{(15)} \qquad \left\{\begin{aligned} &\xi + \eta\frac{dy}{dx} + \zeta\frac{dz}{dx} = 0, \\ &F\left(x, y, z, \frac{dy}{dx}, \frac{d^2y}{dx^2}\right) = 0. \end{aligned}\right.$$

If one eliminates z then one will get a third-order ordinary differential equation in y and x; i.e., y, and therefore z, as well, will be functions of x with three parameters. A doubly-infinite family of curves then possesses a triple infinitude of geodetic lines. The ones that radiate from one and the same point define a surface.

The shortest lines will be established by a system of the form:

$$\text{(16)} \qquad \left\{\begin{aligned} &\xi + \eta\frac{dy}{dx} + \zeta\frac{dz}{dx} = 0, \\ &G\left(x, y, z, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}\right) = 0. \end{aligned}\right.$$

The elimination of z leads to a fourth-order ordinary differential equation in y and z. For a doubly-infinite family of curves that is not a normal family ($\varepsilon \neq 0$), there is then a fourfold infinitude of shortest lines. Any two points will be connected with each other by a shortest trajectory. The shortest lines that radiate from the same point in the same direction then define a surface.

The actual calculation of the expressions F and G would then be very laborious, and hardly allows one to foresee the possibilities under which a simplification of the integration procedure might be achieved.

We shall therefore pursue a different path and replace the system (15) and (16) with a system of first-order differential equations by applying that system to curved coordinate lines.

One can easily find three functions u_x, u_y, u_z of x, y, z that satisfy the two relations:

$$\xi\, u_x + \eta\, u_y + \zeta\, u_z = 0,$$

$$u_x^2 + u_y^2 + u_z^2 = 1.$$

In order to do that, if one assumes that one has three arbitrary functions f_1, f_2, f_3 of x, y, z then one needs only to set:

$$u_x : u_y : u_z = \eta f_3 - \zeta f_2 : \zeta f_1 - \xi f_3 : \xi f_2 - \eta f_1 .$$

In addition, we take:

$$\frac{dy}{dx} = \frac{u_y}{u_x}, \qquad \frac{dz}{dx} = \frac{u_z}{u_x},$$

on the one hand, and:

$$\frac{dy}{dx} = \frac{v_y}{v_x}, \qquad \frac{dz}{dx} = \frac{v_z}{v_x},$$

on the other, that will determine two doubly-infinite families of curves that are perpendicular to each other, as well as to the given family of curves. We consider those three families to be a system of coordinate lines.

The direction cosines of the tangent to an orthogonal trajectory of the given family can be expressed with the help of the angles that they make with the tangents to the those coordinate lines, whose direction cosines we have denoted by u_x, u_y, u_z.

We take:

$$\frac{dx}{ds} = u_x \cos \alpha + v_x \sin \alpha,$$
$$\frac{dy}{ds} = u_y \cos \alpha + v_y \sin \alpha,$$
$$\frac{dz}{ds} = u_z \cos \alpha + v_z \sin \alpha,$$

such that for an arbitrary function $\mathfrak{F}$ of x, y, z, and α, that will give:

$$\frac{d\mathfrak{F}}{ds} = \cos \alpha \left(\frac{\partial \mathfrak{F}}{\partial x} u_x + \frac{\partial \mathfrak{F}}{\partial y} u_y + \frac{\partial \mathfrak{F}}{\partial z} u_z \right) + \sin \alpha \left(\frac{\partial \mathfrak{F}}{\partial x} v_x + \frac{\partial \mathfrak{F}}{\partial y} v_y + \frac{\partial \mathfrak{F}}{\partial z} v_z \right) + \frac{\partial \mathfrak{F}}{\partial \alpha} \frac{d\alpha}{ds} .$$

Here, the coefficients of $\cos \alpha$ and $\sin \alpha$ are the derivatives of $\mathfrak{F}$ with respect to the arc-lengths of the coordinate lines under consideration. For that reason, we set:

$$\frac{d\mathfrak{F}}{ds_u} = \frac{\partial \mathfrak{F}}{\partial x} u_x + \frac{\partial \mathfrak{F}}{\partial y} u_y + \frac{\partial \mathfrak{F}}{\partial z} u_z ,$$

$$\frac{d\mathfrak{F}}{ds_v} = \frac{\partial \mathfrak{F}}{\partial x} v_x + \frac{\partial \mathfrak{F}}{\partial y} v_y + \frac{\partial \mathfrak{F}}{\partial z} v_z ,$$

to abbreviate. One must now calculate the quantities $1 / R_s$, $1 / h_s$, $1 / P_s$ with the help of α, and in that way those quantities will become known, to the extent that they are defined for the coordinate lines. Their notations in that case are illuminated by the equations:

$$\frac{d^2 x}{ds_u^2} = \frac{du_x}{ds_u} = \frac{1}{R_u} v_x + \frac{\xi}{h_u}, \qquad \frac{d^2 x}{ds_v^2} = \frac{dv_x}{ds_v} = \frac{1}{R_v} u_x + \frac{\xi}{h_v},$$

$$\frac{1}{P_u} = 2\,(e_1\,v_x + e_2\,v_y + e_3\,v_z), \quad \frac{1}{P_v} = -\,2\,(e_1\,u_x + e_2\,u_y + e_3\,u_z)\,.$$

With the addition of two supplementary expressions l_u and l_v , whose geometric meaning is established in (G, pp. 47), one will further have:

$$\frac{du_x}{ds_v} = -\frac{v_x}{R_v} - \frac{\xi}{l_v}, \qquad \frac{dv_x}{ds_u} = -\frac{u_x}{R_u} - \frac{\xi}{l_u}.$$

It will now follow that:

$$\frac{d^2x}{ds^2} = \cos\,\alpha\left(\frac{du_x}{ds_u}\cos\alpha + \frac{dv_x}{ds_u}\sin\alpha\right) + \sin\,\alpha\left(\frac{du_x}{ds_v}\cos\alpha + \frac{dv_x}{ds_v}\sin\alpha\right)$$
$$+ (v_x\cos\,\alpha - u_x\sin\,\alpha)\,\frac{d\alpha}{ds}$$

$$= (v_x\cos\,\alpha - u_x\sin\,\alpha)\left(\frac{d\alpha}{ds} + \frac{\cos\alpha}{R_u} - \frac{\sin\alpha}{R_v}\right)$$
$$+\xi\left[\frac{\cos^2\alpha}{h_u} - \cos\alpha\sin\alpha\left(\frac{1}{l_u} + \frac{1}{l_v}\right) + \frac{\sin^2\alpha}{h_v}\right].$$

However, since:

$$\eta\frac{dz}{ds} - \zeta\frac{dy}{ds} = v_x\cos\,\alpha - u_x\sin\,\alpha\,,$$

a comparison of this with the formulas (6) in § **1** will show that:

$$\frac{1}{R_s} = \frac{d\alpha}{ds} + \frac{\cos\alpha}{R_u} - \frac{\sin\alpha}{R_v},$$

$$\frac{1}{h_s} = \frac{\cos^2\alpha}{h_u} - \cos\alpha\sin\alpha\left(\frac{1}{l_u} + \frac{1}{l_v}\right) + \frac{\sin^2\alpha}{h_v}.$$

Finally, it follows that:

$$\frac{1}{P_s} = \frac{\cos\alpha}{P_u} + \frac{\sin\alpha}{P_v}.$$

We now get the system:

$$\frac{dx}{u_x\cos\alpha + v_x\sin\alpha} = \frac{dy}{u_y\cos\alpha + v_y\sin\alpha} = \frac{dz}{u_z\cos\alpha + v_z\sin\alpha} = \frac{d\alpha}{\dfrac{\sin\alpha}{R_v} - \dfrac{\cos\alpha}{R_u}}$$

for the geodetic lines. The equations of the shortest lines take a more complicated form. One must then introduce the variable $\beta = d\alpha / ds$, and when one takes:

$$M = \beta\left\{\frac{d\log\varepsilon}{ds}+\frac{1}{P_s}+\frac{\sin\alpha}{R_u}+\frac{\cos\alpha}{R_v}\right\}+\frac{2\varepsilon}{h_s}-\cos\alpha\frac{d(1/R_u)}{ds}+\sin\alpha\frac{d(1/R_v)}{ds}$$

$$+\left(\frac{\cos\alpha}{R_u}-\frac{\sin\alpha}{R_v}\right)\left(\frac{d\log\varepsilon}{ds}+\frac{1}{P_s}\right),$$

one will get the system:

$$\frac{dx}{u_x\cos\alpha+v_x\sin\alpha}=\frac{dy}{u_y\cos\alpha+v_y\sin\alpha}=\frac{dz}{u_z\cos\alpha+v_z\sin\alpha}=\frac{d\alpha}{\beta}=\frac{d\beta}{M}.$$

§ 4.

Special cases.

1. The equations of the geodetic lines will possess the integral α = const. when one succeeds in choosing the coordinate lines in such a way that $1 / R_u$ and $1 / R_v$ vanish. In that case, those curves will be isogonal trajectories to the coordinate lines that are likewise geodetic lines, in their own right. The condition for that situation is discussed in (G, pp. 70) in such a way that the differential form:

$$\frac{\mathfrak{S}_1}{R_1}-\frac{\mathfrak{S}_2}{R_2}+\left(\frac{1}{2\rho_1\rho_2\varepsilon}+\vartheta\right)T_0$$

must be a differential.

If one follows [G, (9), pp. 56] and presents the conditions that are necessary for that and considers the seventh and eighth equation in (11) there then one will find that:

$$g_1\left(\frac{1}{2\rho_1\rho_2\varepsilon}\right)=-\frac{1}{h_1P_2}+\frac{1}{P_1}\left(\frac{1}{2\rho_1\rho_2\varepsilon}-\varepsilon\right),$$

$$g_2\left(\frac{1}{2\rho_1\rho_2\varepsilon}\right)=\frac{1}{h_2P_1}+\frac{1}{P_2}\left(\frac{1}{2\rho_1\rho_2\varepsilon}-\varepsilon\right).$$

That raises the question of whether there is a ray system with the property in question, since $1 / P_1$ and $1 / P_2$ will vanish for one, and therefore the right-hand sides of the foregoing equations, as well. Now, from (G, pp. 30), one has, in general:

$$\frac{1}{2\rho_1\rho_2\varepsilon} = \frac{\sqrt{\mathrm{H}\Psi - \Phi^2}}{f - f'},$$

and for the ray system, from (G, pp. 34), the difference $f - f'$ is equal to $f_0 - f_0'$, such that expression in question depends upon only p and q. The foregoing two conditions then condense here into the requirement that the expression:

$$\frac{\sqrt{\mathrm{H}\Psi - \Phi^2}}{f - f'}$$

must be const. In order to resolve the possibility that this requirement can be satisfied, we take our starting surface to the rays of the system in the XY-plane and therefore set:

$$x_0 = p, \; y_0 = q, \; z_0 = 0.$$

In that way, we will have:

$$f_0 - f_0' = \frac{\partial\eta}{\partial p} - \frac{\partial\xi}{\partial q}.$$

Furthermore:

$$\sqrt{\mathrm{H}\Psi - \Phi^2} = \left(\frac{\partial\xi}{\partial p}\frac{\partial\eta}{\partial p} - \frac{\partial\xi}{\partial q}\frac{\partial\eta}{\partial p}\right),$$

so the expression:

$$A = \frac{\dfrac{\partial\xi}{\partial p}\dfrac{\partial\eta}{\partial p} - \dfrac{\partial\xi}{\partial q}\dfrac{\partial\eta}{\partial p}}{\sqrt{1-\xi^2-\eta^2}\left(\dfrac{\partial\eta}{\partial p} - \dfrac{\partial\xi}{\partial q}\right)}$$

must be constant. The problem of finding all ray systems with the property that was spoken of then depends upon the integration of the two partial differential equations:

$$\frac{\partial A}{\partial p} = 0, \qquad \frac{\partial A}{\partial q} = 0$$

in the two unknown functions ξ and η.

2. If one takes the curvature lines of the first kind to be the coordinate lines then it will follow from the first two equations in [G, (11), pp. 56], when one multiplies the first one by – cos α and the second one by sin α and adds them, that:

$$\frac{2\varepsilon}{P_s} = -g_2\left(\frac{1}{h_1}\right)\cos\alpha + g_1\left(\frac{1}{h_2}\right)\sin\alpha + \frac{d\varepsilon}{ds} - \left(\frac{\cos\alpha}{R_1} - \frac{\sin\alpha}{R_2}\right)\left(\frac{1}{h_1} - \frac{1}{h_2}\right).$$

For an isotropic family of curves (G, pp. 24, 96), the quantity $1 / h_s$ is independent of α, and every orthogonal trajectory to the family of curves can be considered to be a line of curvature of the first kind. If we denote the common values of $1 / h_1$ and $1 / h_2$ by $1 / h$ here then we will have:

$$\frac{2\varepsilon}{P_s} = \frac{d\varepsilon}{ds} - \cos\alpha \frac{d(1/h)}{ds_v} + \sin\alpha \frac{d(1/h)}{ds_u}.$$

The known isotropic family of curves, whose normals define a non-special linear complex, has the property that $1 / h$ is continually zero. (G, pp. 24, 25). In that case:

$$\frac{1}{P_s} = \frac{1}{2}\frac{d\log\varepsilon}{ds},$$

and equation (9) of § **1** will then simplify to:

$$\frac{3}{2}\frac{d\log\varepsilon}{ds} - \frac{d\log(1/R_s)}{ds} = 0,$$

so it possesses the integral:

$$\frac{1}{R_s} = c\, e^{3/2}.$$

The further integration must then be done with a system of three first-order differential equations.

Münster i. W., 15 January 1899.