“Über die Bewegung eines starren Körper, der ohne Gleitung auf einer beliebigen Fläche rollt,” Math. Ann. **70** (1911), 410-453.

On the motion of a rigid body that rolls without slipping on an arbitrary surface

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Table of contents

The handful of particular solutions to the problem of the motion of a rigid body that rolls on a given surface that have been investigated up to now refer to mainly two special cases. One treats either the motion of a rigid ball that rolls on an arbitrary surface (*) or the motion of an arbitrary rigid body that rolls on a plane under the action of gravity (**).

In the present work, it shall be shown that most of the results that have been achieved by the treatment of the aforementioned two problems can be easily extended to the more general problem of the motion of a rigid body that rolls on a sphere with almost no restrictions. In that problem, gravity will be replaced with a force that points from the center of mass of the body to the center of the sphere and depends upon only the distance between those points.

The study of that problem will define the contents of Chapter III.

In Chapter IV, the equations of motion of a rigid body that rolls without slipping on an arbitrary surface will be developed and some simple particular solutions of it will be given.

The first two chapters (***) can be regarded as an introduction to the two aforementioned ones.

Chapter I includes some theorems on the kinematics of rolling motion. Those theorems will be employed in order to determine the projections of the instantaneous angular velocity of the rolling body onto axes that are fixed in the body by means of C. Neumann's coordinates (†).

(*) That problem was discussed quite thoroughly in the more celebrated textbooks on dynamics; e.g., in Routh ("The advanced part of a treatise on the dynamics of a system of rigid bodies," Chap. V). Cf., also the interesting Dissertation of Fr. Noether, "Über die rollende Bewegung einer Kugel auf Rotationsflächen," Munich, 1909.

(**) For the literature on that topic, cf., Enc. math. Wiss. IV **6** (P. Stäckel), "Elementare Dynamik der Punktsysteme und starren Körper," no. 38.

(***) Cf., also my Russian treatises: "On the equations of motion of non-holonomic systems," Moscow math. Collection 1902 and "The equations of motion of a rigid body that rolls without slipping on a plane at rest," Kiev Univ. Reports 1903.

(†) C. Neumann, "Grundzüge der analytischen Mechanik," Leipziger Berichte (1899). Cf., also, Vierkandt, "Über gleitende und rollende Bewegung," Monatshefte für Math. und Physik **3** (1892).

In Chapter II, a method will be given for exhibiting the equations of motion of non-holonomic systems (without the Euler-Lagrange multipliers) that is analogous to Hamilton's method for holonomic systems as long as one has to calculate with only first-order differential expressions that are functions of the independent velocities. However, the number of those expressions will be larger for non-holonomic systems. Along with the expressions for the force functions and the *vis viva*, there will also be as many expressions for the impulses as there are non-holonomic condition equations.

CHAPTER I

Kinematical examination of the motion of a rigid body that rolls on a given surface

§ 1. – Introductory remarks.

In the present study of the problem of the motion of a rigid body that rolls without slipping on a given surface S_1, following C. Neumann (*), one chooses the following quantities to be the coordinates of the body: The Gaussian coordinates u and v of the point M on the outer surface S of the body at which the surface S contacts the surface S_1, the Gaussian coordinates u_1 and v_1 of the same point M on the surface S_1, and the angle ϑ that the coordinate line v = const. makes with the coordinate line u_1 = const. at the point M.

We imagine that an orthogonal system of coordinate axes $Oxyz$ is fixed in the body, we let w and ω denote the velocity of the coordinate origin O and the instantaneous angular velocity of the rigid body, resp., and we pose the problem of expressing the projections k, l, m; p, q, r of the vectors w and ω, resp., onto the x, y, z axes, resp., in terms of the Neumann coordinates u, v, ϑ, u_1, v_1, and their differential quotients with respect to time $\dot{u}$, $\dot{v}$, $\dot{\vartheta}$, $\dot{u}_1$, $\dot{v}_1$, resp. In order to do that, we will need to have some simple theorems and formulas from the theory of surfaces that we would like to derive in the following paragraphs using the method of Lord Kelvin (W. Thomson) and P. Tait (**). In it, we shall employ the following notations: We denote the first and second-order fundamental quantities (***) of a surface S, which might be given by the equations:

$$z = z(u, v), \qquad y = y(u, v), \qquad z = z(u, v),$$

by E, F, G; D, D', D''. We think of a system of axes $Muvn$ as being drawn at each point M of the surface S whose u, v, n-axes coincide with the positive directions of the lines u (v = const.) and v (u = const.) and the normal n to S at M. The positive n-axis might be laid in relation to the u and v-axes in the same way that the z-axis lies with respect to the

(*) *Ibid.*

(**) Thomson and Tait, *Treatise on Natural Philosophy*, vol. I, part. I, art. 110, *et seq.* In an extended form, in Sonslov, "On the rolling of one surface on another," Kiev Univ. Reports 1892, and in my treatise, "The rolling motion of a rigid body that rolls without slipping," Chap. IV, *ibid.*, 1903.

(***) Stahl and Kommerell, *The Grundformeln der allgemeiner Flächentheorie*, 1893, form. (4), § 1 and (1), § 2.

x and y-axes (*). We shall denote the nine cosines of the angles between the u, v, n-axes and the x, y, z-axes by α, α', …, γ''. We will then have (**):

$$\alpha = \frac{1}{\sqrt{E}}\frac{\partial x}{\partial u}, \quad \beta = \frac{1}{\sqrt{G}}\frac{\partial x}{\partial v}, \quad \gamma = \frac{1}{H}\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial v}\right),$$

(1) $$(H = +\sqrt{EG - F^2}),$$

……………………………………………………..

§ 2. – Total bending, pure bending, and twisting.

Draw a curve L on the surface S, and let M and M_1 be two infinitely-close points on L. Lay tangent planes T and T_1 through the points M and M_1, resp., and let $\Delta\varepsilon$ denote the infinitely-small angle between T and T_1, and let Δs denote the length MM_1 of the curve L. We carry the length $\Delta\varepsilon$ along the line of intersection of the planes T and T_1, and Δs points in a direction such that that line segment lies in relation to the normals n_1 and n at M_1 and M, resp., in the same way that the z-axis lies in relation to the x and y-axes. If we now let the point M_1 approach the point M along L until they coincide then we will get a vector Ω in that way that we would like to call the *total bending* of the surface S at the point M in the direction L.

It is clear that the rolling motion of the surface S on the contacting plane T along the curve L means that the vector Ω that represents the component of the instantaneous angular velocity of the surface S in the plane, if that angular velocity is referred to a unit length, as G. Darboux did (***).

If we let γ, γ', γ'' and γ_1, γ_1', γ_1'' denote the cosines of the angles that the normals n and n_1 make with the x, y, z-coordinate axes, resp., then the projections of the vector Ω onto the x, y, z-axes will be:

$$\Omega_x = \frac{1}{\Delta s}(\gamma_1'\gamma'' - \gamma_1''\gamma'),$$

……………………….

If we substitute:

$$\gamma_1 = \gamma + \left(\frac{\partial \gamma}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial \gamma}{\partial v}\frac{\partial v}{\partial s}\right)\Delta s + \ldots,$$

……………………………………

in this and let Δs go to zero then we will get:

(*) The positive z-axes should always be drawn in such a way that the positive x-axis will be made to coincide with the positive y-axis by means of a clockwise rotation around the positive z-axis through an angle of $\pi / 2$.

(**) *Ibid.*, form. (22) and (23), § 1.

(***) Darboux, *Leçons sur la théorie générale des surfaces*, v. V.

$$\Omega_x = \left(\frac{\partial\gamma}{\partial u}\gamma'' - \frac{\partial\gamma''}{\partial u}\gamma'\right)\frac{\partial u}{\partial s} + \left(\frac{\partial\gamma}{\partial v}\gamma'' - \frac{\partial\gamma''}{\partial v}\gamma'\right)\frac{\partial v}{\partial s},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

However, as is known (*), one has:

$$H^2\frac{\partial\gamma}{\partial u} = (F\,D' - G\,D)\frac{\partial x}{\partial u} + (F\,D - E\,D')\frac{\partial x}{\partial v},$$

$$H^2\frac{\partial\gamma}{\partial v} = (F\,D'' - G\,D')\frac{\partial x}{\partial u} + (F\,D' - E\,D'')\frac{\partial x}{\partial v},$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots,$$

such that, from (1), one will have:

(2)
$$H\cdot\Omega_x = \left(D\frac{\partial x}{\partial v} - D'\frac{\partial x}{\partial u}\right)\frac{du}{ds} + \left(D'\frac{\partial x}{\partial v} - D''\frac{\partial x}{\partial u}\right)\frac{dv}{ds},$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

From (1), the projections of the vector Ω onto the directions u and v will then be equal to:

(3)
$$\Omega_u = \frac{1}{H\sqrt{E}}\left[(DF - D'E)\frac{du}{ds} + (D'F - D''E)\frac{dv}{ds}\right],$$

$$\Omega_v = \frac{1}{H\sqrt{G}}\left[(DG - D'F)\frac{du}{ds} + (D'G - D''F)\frac{dv}{ds}\right].$$

We now decompose the total bending Ω into two components: Ω_s , which is in the direction of the curve L, and Ω_p , which is the direction p that is perpendicular to s. We would like to call Ω_p the *pure bending* and Ω_s the *twisting* of the surface S at M in the direction of L.

Geometrically, it is clear that the pure bending Ω_p is equal to the curvature of the normal section to the surface S at the point M in the direction L. In order to interpret the twisting Ω_s , we draw a geodetic line through M in the direction L – i.e., a curve whose curvature plane always goes through the normal n to the surface S. The line p is the binormal for that curve, such that Ω_s will be equal to the torsion of the geodetic line that is drawn through M in the direction L.

The tangent to the curve L defines angles with respect to the x, y, z-axes whose cosines are equal to:

(*) Stahl and Kommerell, formula (6), § 2.

$$\frac{dx}{ds} = \frac{\partial x}{\partial u}\frac{du}{ds} + \frac{\partial x}{\partial v}\frac{dv}{ds},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,\tag{4}$$

such that formulas (2) will imply that:

$$H \cdot \Omega_s \cdot ds^2 = (DF - D'E)\, du^2 + (DG - D''E)\, du\, dv + (D'G - D''F)\, dv^2.$$

The lines for which the twisting is equal to zero are the curvature lines of the surface *S*. In general, two such lines will go through any point *M* on the surface *S* then. When the relations:

$$D : E = D' : F = D'' : G$$

are fulfilled at each point of the surface *S*, each line on *S* will be a line of curvature; i.e., the surface *S* will be a sphere. If we exclude that case and choose the lines of curvature to be the *u* and *v*-lines then we will have:

$$F = 0, \quad D' = 0$$

at each point of the surface. The lines of curvature will then define an orthogonal net of coordinate lines (*).

We now go on to the determination of the pure bending Ω_p.

The direction *p* is perpendicular to the tangent *s* to the curve *L* and the normal *n* to the surface *S*, so one will have:

$$\cos(p, x) = \gamma' \frac{dz}{ds} - \gamma'' \frac{dy}{ds} = \left(\gamma' \frac{\partial z}{\partial u} - \gamma'' \frac{\partial y}{\partial u}\right)\frac{du}{ds} + \left(\gamma' \frac{\partial z}{\partial v} - \gamma'' \frac{\partial y}{\partial v}\right)\frac{dv}{ds},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

or, from (1):

$$H\cos(p, x) = \left(E\frac{\partial x}{\partial v} - F\frac{\partial x}{\partial u}\right)\frac{du}{ds} + \left(F\frac{\partial x}{\partial v} - G\frac{\partial x}{\partial u}\right)\frac{dv}{ds},$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\tag{5}$$

We imagine the positive direction of the *p*-axis as being laid with respect to the *s* and *n*-axes in the same way that the *y*-axis lies in relation to the *x* and *z*-axes.

As a result of (5), formulas (2) will imply that:

$$\Omega_p \cdot ds^2 = D\, du^2 + 2D'\, du\, dv + D''\, dv^2.$$

The lines along which the pure bending is equal to zero are the asymptotic lines. In general, two such lines go through each point of the surface.

(*) Stahl and Kommerell, formula (10), § 1.

§ 3. – The spinning of the tangent plane.

We again imagine the tangent planes T and T_1 to the surface S at two infinitely-close points M and M_1, resp., of the curve L, and let Δs denote the length MM_1 of the curve L. We draw the tangents s and s_1 to the curve L in the planes T and T_1 at the points M and M_1. If we now rotate the plane T around the axis of the total bending through the angle $\Delta\varepsilon$ then the plane T will coincide with the plane T_1, but in general, the line s will not point along the line s_1, and in order to make the lines s and s_1 coincide, we must rotate the tangent plane T around the normal n to S at M through an infinitely-small angle $\Delta\eta$. We lay out the line segment $\Delta\eta : \Delta s$ along the normal n and let the point M_1 approach the point M along the curve L until they coincide. We call the vector N that results in this way the *spinning* (*das Kreiseln*) of the tangent plane T at the point M in the direction L.

Obviously, N is equal to the magnitude of the geodetic curvature of the curve L at the point M ; i.e., the projection of the curvature $1 / \rho$ of the curve L onto the tangent plane T:

$$N = \pm \frac{1}{\rho} \cos(\rho, p).$$

If we give p the direction that is defined by formulas (5) and choose the lower sign in the expression for N then the tangent plane will rotate around n in a counterclockwise motion.

For a rolling motion of the surface S along the plane T, which is at rest, along the curve L, N will then be the components of the instantaneous angular velocity of the surface S along the normal n when that angular velocity is referred to unit length.

N shall now be expressed in terms of the quantities E, F, G.

If we denote:

$$E\frac{du}{ds} + F\frac{dv}{ds} = k_1, \qquad F\frac{du}{ds} + G\frac{dv}{ds} = k_2,$$

for brevity, such that we will have (*):

$$k_1\frac{du}{ds} + k_2\frac{dv}{ds} = 1 \tag{5}$$

then (5) will imply that:

$$H\cos(p, x) = k_1\frac{\partial x}{\partial v} - k_2\frac{\partial x}{\partial u},$$
$$\dots\dots\dots\dots\dots\dots$$

In addition, we have, as is known:

$$\cos(\rho, x) = \rho\frac{d^2 x}{ds^2},$$
$$\dots\dots\dots\dots$$

(*) Stahl and Kommerell, formula (3), § 1.

such that:

$$N = -\frac{\cos(\rho, p)}{\rho} = \frac{k_2}{H}\left(\frac{\partial x}{\partial u}\frac{d^2 x}{ds^2} + \cdots\right) - \frac{k_1}{H}\left(\frac{\partial x}{\partial v}\frac{d^2 x}{ds^2} + \cdots\right).$$

However, from (6), one has:

$$k_2\left(\frac{\partial x}{\partial u}\frac{d^2 x}{ds^2} + \cdots\right) = \left[\frac{\partial x}{\partial u}\frac{\partial}{\partial u}\left(\frac{dx}{ds}\right) + \cdots\right] k_2 \frac{du}{ds} + \left[\frac{\partial x}{\partial u}\frac{\partial}{\partial v}\left(\frac{dx}{ds}\right) + \cdots\right]\left(1 - k_1 \frac{du}{ds}\right),$$

……………………………………………………………………………………..

If we substitute that in N then, as a result of the obvious formula:

$$\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2 = 1,$$

the coefficients of k_1 and k_2 will be equal to zero, and we will get:

$$N = \frac{1}{H}\frac{\partial}{\partial v}\left(\frac{\partial x}{\partial u}\frac{dx}{ds} + \cdots\right)\frac{du}{ds} - \frac{1}{H}\frac{\partial}{\partial u}\left(\frac{\partial x}{\partial v}\frac{dx}{ds} + \cdots\right),$$

or, from (4):

$$N = \frac{1}{H}\frac{\partial}{\partial v}\left(E\frac{du}{ds} + F\frac{dv}{ds}\right) - \frac{1}{H}\frac{\partial}{\partial u}\left(F\frac{du}{ds} + G\frac{dv}{ds}\right). \tag{7}$$

That is the desired formula.

If the curve L is given by the equation:

$$f(u, v) = 0$$

then we will have:

$$\frac{du}{\frac{\partial f}{\partial v}} = \frac{dv}{-\frac{\partial f}{\partial u}} = \frac{ds}{h}, \qquad h^2 = E\left(\frac{\partial f}{\partial v}\right)^2 - 2F\frac{\partial f}{\partial v}\frac{\partial f}{\partial u} + G\left(\frac{\partial f}{\partial u}\right)^2,$$

and (7) will go to the known formula of Bonnet:

$$N = \frac{1}{H}\left[\frac{\partial}{\partial v}\left(\frac{E\frac{\partial f}{\partial v} - F\frac{\partial f}{\partial u}}{h}\right) - \frac{\partial}{\partial u}\left(\frac{F\frac{\partial f}{\partial v} - G\frac{\partial f}{\partial u}}{h}\right)\right].$$

For geodetic lines, the projection of the curvature $1/\rho$ onto the tangent plane is equal to zero, and therefore, N, as well.

§ 4. – Application to the problem of rolling motion.

We would now like to utilize the formulas of the previous two paragraphs in order to solve the problem that we posed in § **1**. If ω is the instantaneous angular velocity of a rigid body that rolls on a given surface S_1 then we will have to determine the projections p, q, r of ω into the x, y, z-axes, resp., which are fixed in the body, in terms of the coordinates u, v, ϑ, u_1, v_1 of the body and the differential quotients $\dot{u}$, $\dot{v}$, $\dot{\vartheta}$, $\dot{u}_1$, $\dot{v}_1$.

We introduce the following notations: Let E, F, G ; D, D', D'' be the fundamental quantities of the outer surface S of the body, and let E_1, F_1, G_1 ; D_1, D_1', D_1'' be the same quantities for the surface S_1 upon which the body rolls. For the sake of simplicity, we shall assume that the lines u and v on S and the lines u_1 and v_1 on S_1 are the lines of curvature of those surfaces:

$$F = 0,\quad D = 0,\quad F_1 = 0,\quad D_1' = 0.$$

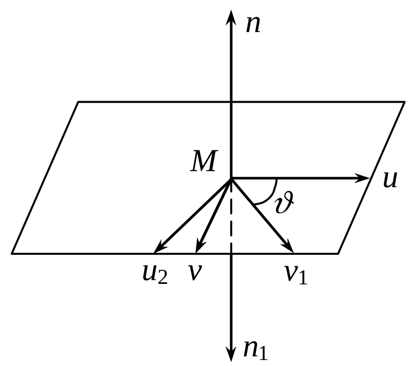

Figure 1.

We imagine two systems of axes $Muvn$ and $Mu_1v_1n_1$ (Fig. 1) being drawn through the point of contact M between the surfaces S and S_1 whose axes coincide with the tangents to the lines u, v; u_1, v_1, resp., through the points M and the common normal to S and S_1, and consider the instantaneous angular velocity ω to be the geometric sum of three vectors ω_1, ω_2, ω_3 :

$$(\omega) = (\omega_1) + (\omega_2) + (\omega_3)\ ,$$

in which ω_1 means the angular velocity of the system of axes $Oxyz$ that is fixed in the body with respect to the system $Muvn$, ω_2 is the angular velocity of the system $Muvn$ with respect to $M_1u_1v_1n_1$, and ω_3 is the angular velocity of the system $M_1u_1v_1n_1$ with respect to the system of axes $O_1x_1y_1z_1$ that is fixed in the surface S_1 .

If the point of contact M on the surface S is shifted along the curve u through the segment $\sqrt{E}\,du$ then, from (3), the component of the angular velocity ω_1 that lies in the tangent plane T will point along the curve v and will be equal to $D : E$ per unit length, while, from (7), the component along the normal n will equal $\frac{1}{2\sqrt{G}}\frac{1}{E}\frac{\partial E}{\partial v}$, also per unit length. If we then shift the contact point M along the curve v through the segment $\sqrt{G}\,dv$ then we will easily find that:

$$
\begin{aligned}
&\omega_1 \cos(\omega_1, u) = \sigma_1 = -\frac{D''}{\sqrt{G}}\dot{v}, \\
(8) \qquad &\omega_1 \cos(\omega_1, v) = \tau_1 = \frac{D}{\sqrt{E}}\dot{u}, \\
&\omega_1 \cos(\omega_1, n) = n_1 = \frac{1}{2\sqrt{EG}}\left(\frac{\partial E}{\partial v}\dot{u} - \frac{\partial G}{\partial u}\dot{v}\right).
\end{aligned}
$$

The angular velocity ω_2 points along the normal n_1 and is equal to $\dot{\vartheta}$, while the angular velocity ω_2 is determined similarly to ω_1. From Fig. 1, we will then get the following formulas:

$$
\begin{aligned}
&\omega \cos(\omega, u) = \sigma = -\frac{D''}{\sqrt{G}}\dot{v} - \frac{D_1''}{\sqrt{G_1}}\dot{v}_1 \sin\vartheta - \frac{D_1}{\sqrt{E_1}}\dot{u}_1 \cos\vartheta, \\
(9) \qquad &\omega \cos(\omega, v) = \tau = \frac{D}{\sqrt{E}}\dot{u} - \frac{D_1}{\sqrt{E_1}}\dot{u}_1 \sin\vartheta + \frac{D_1''}{\sqrt{G_1}}v_1 \cos\vartheta, \\
&\omega \cos(\omega, n) = n = \dot{\vartheta} + \frac{1}{2\sqrt{EG}}\left(\frac{\partial E}{\partial v}\dot{u} - \frac{\partial G}{\partial u}\dot{v}\right) + \frac{1}{2\sqrt{E_1G_1}}\left(\frac{\partial E_1}{\partial v_1}\dot{u}_1 - \frac{\partial G_1}{\partial u_1}\dot{v}_1\right)
\end{aligned}
$$

and that will give the desired expressions for p, q, r :

$$
(10) \qquad p = \sigma\alpha + \tau\beta + n\gamma, \quad q = \sigma\alpha' + \tau\beta' + n\gamma', \quad r = \sigma\alpha'' + \tau\beta'' + n\gamma'',
$$

We move on to the determination of the projections k, l, m of the velocity w of the coordinate origin O onto the x, y, z-axes. To that end, we consider the motion of the contact point M. The absolute velocity $\mathfrak{v}_1$ of the point M has the quantities $\sqrt{E_1}\,\dot{u}_1$ and $\sqrt{G_1}\,\dot{v}_1$ for its components along the curves u_1 and v_1, resp. The components of the relative velocity $\mathfrak{v}$ of the point M along the lines u and v are equal to $\sqrt{E}\,\dot{u}$ and $\sqrt{G}\,\dot{v}$, resp. Finally, the velocity $\mathfrak{w}$ of the point of the rigid body that coincides with the contact point M at the given moment will have $k + qz - ry$, $l + rx - pz$, $m + py - qx$ for its projections onto the x, y, z-axes, resp., where x, y, z are the coordinates of the point M, and are thus given functions of u and v. However, the vector $\mathfrak{v}_1$ is equal to the geometric sum of the vectors $\mathfrak{v}$ and $\mathfrak{w}$:

$$
(\mathfrak{v}_1) = (\mathfrak{v}) + (\mathfrak{w}),
$$

such that, from Fig. 1 and (10), if we project the velocities $\mathfrak{v}_1$, $\mathfrak{v}$, and $\mathfrak{w}$ onto the x, y, z-axes then we will get:

$$
(11)\qquad
\begin{aligned}
k = {} & \left(-\sqrt{E_1}\,\dot{u}_1 \sin\vartheta + \sqrt{G_1}\,\dot{v}_1 \cos\vartheta - \sqrt{E}\,\dot{u}\right)\alpha \\
& + \left(\quad \sqrt{E_1}\dot{u}_1 \cos\vartheta + \sqrt{G_1}\dot{v}_1 \sin\vartheta - \sqrt{G}\,\dot{v}\right)\beta \\
& + y\,(\sigma\,\alpha'' + \tau\,\beta'' + n\,\gamma'') - z\,(\sigma\,\alpha' + \tau\,\beta' + n\,\gamma'), \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
$$

These formulas determine the quantities k, l, m as functions of the coordinates u, v, ϑ, u_1, v_1 of the rigid body and the differential quotients $\dot{u}$, $\dot{v}$, $\dot{\vartheta}$, $\dot{u}_1$, $\dot{v}_1$.

Should the rolling motion of the body proceed without slipping, then the absolute velocity $\mathfrak{v}_1$ of the point M would be geometrically equal to its relative velocity $\mathfrak{v}$, and from Fig. 1, we would get:

$$
(12)\qquad
\begin{aligned}
\sqrt{E_1}\,\dot{u}_1 &= -\sqrt{E}\,\dot{u}\sin\vartheta + \sqrt{G}\,\dot{v}\cos\vartheta, \\
\sqrt{G_1}\,\dot{v}_1 &= \quad \sqrt{E}\,\dot{u}\cos\vartheta + \sqrt{G}\,\dot{v}\sin\vartheta.
\end{aligned}
$$

Formulas (11) simplify to the following ones:

$$
(13)\qquad
\begin{aligned}
& k = (y\;\alpha'' - z\;\alpha')\;\sigma + (y\;\beta'' - z\;\beta') + (y\;\gamma'' - z\;\gamma')\;n, \\
& \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}
$$

in that case, and we can eliminate two of the quantities $\dot{u}$, $\dot{v}$, $\dot{\vartheta}$, $\dot{u}_1$, $\dot{v}_1$ from the expressions (9) for σ, τ, n with the help of (12).

CHAPTER II

On the equations of motion of non-holonomic systems

§ 5. – Eliminating the Lagrange multipliers from the equations of motion.

The motion of a rigid body that rolls without slipping on a given surface serves as an example of the motion of a non-holonomic system.

Before we go on to the special problems of rolling motion, we would like to discuss the equations of motion of non-holonomic systems in the general case.

We let $q_1, q_2, \ldots, q_{n+k}$ denote the coordinates of a material system, let $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_{n+k}$ denote the differential quotients of the coordinates with respect to time t (i.e., the generalized velocities), let $T\,(t, q_1, q_2, \ldots, q_{n+k}, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_{n+k})$ denote the kinetic energy of the system, and let Q_s denote the generalized force that corresponds to the coordinate q_s. The product $Q_s\,\delta q_s$ then determine the work done by the force that is acting under a motion of the system when all coordinates are constant, except for the coordinate q_s, which increases by δq_s.

Let the material system be subject to the condition equations:

(14) $$\dot{q}_{n+\nu} = \sum_{i=1}^{n} a_{\nu i}\,\dot{q}_i + a_\nu \qquad (\nu = 1, 2, \ldots, k),$$

where the coefficients $a_{\nu i}$ and a_ν denote given functions of time and the coordinates.

We will assume that the integration conditions for the differential equations (14) are not fulfilled, such that the quantities:

(15)
$$A_{ij}^{(\nu)} = \left(\frac{\partial a_{\nu i}}{\partial q_j} + \sum_{\mu=1}^{k} a_{\mu j}\frac{\partial a_{\nu i}}{\partial q_{n+\mu}}\right) - \left(\frac{\partial a_{\nu j}}{\partial q_i} + \sum_{\mu=1}^{k} a_{\mu i}\frac{\partial a_{\nu j}}{\partial q_{n+\mu}}\right),$$

$$A_i^{(\nu)} = \left(\frac{\partial a_{\nu i}}{\partial t} + \sum_{\mu=1}^{k} a_{\mu}\frac{\partial a_{\nu i}}{\partial q_{n+\mu}}\right) - \left(\frac{\partial a_{\nu}}{\partial q_i} + \sum_{\mu=1}^{k} a_{\mu i}\frac{\partial a_{\nu}}{\partial q_{n+\mu}}\right),$$

$$(i, j = 1, 2, \ldots, n;\ \nu = 1, 2, \ldots, k)$$

cannot all be simultaneously zero.

When the Lagrange multipliers are denoted by $\lambda_1, \lambda_2, \ldots, \lambda_k$, the equations of motion of material system can be written out thus:

(16)
$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} = \frac{\partial T}{\partial q_i} + Q_i - \sum_{\nu=1}^{k} \lambda_k\, a_{\nu i} \qquad (i = 1, 2, \ldots, n),$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_{n+\nu}} = \frac{\partial T}{\partial q_{n+\nu}} + Q_{n+\nu} + \lambda_\nu \qquad (\nu = 1, 2, \ldots, k).$$

Those equations and the conditions (14) define a system of $n + 2k$ differential equations that determine the $n + k$ coordinates q and the k quantities λ as functions of time t.

The application of the equations of motion in the form (16) to special problems – in particular, to the problem of rolling motion – will encounter some difficulties, first of all, because the multipliers λ cannot be eliminated from (16), and secondly, because the function T is not used in its simplest form with the help of the condition equations (14) (*); i.e., T is a quadratic function of $n + k$ arguments $\dot{q}$, while the formulas (14) yield the possibility of expressing T as a quadratic function of only n arguments $\dot{q}$. For that reason, we would like to seek to obtain the equations of motion of non-holonomic systems in a form that is more convenient for the applications (**).

(*) Cf., e.g., Hölder, "Über die Prinzipien von Hamilton und Maupertuis," Nachrichten der Kgl. Gesellschaft der Wissenschaften zu Göttingen (1896) or Hadamard, "Sur les mouvements de roulement," Mémoires de la Société des science physiques et naturelles de Bordeaux **5** (1895).

(**) Cf., also V. Volterra, "Sopra una classe di equazioni dinamiche," Torino Atti **33** (1898); P. Appel, "Remarques d'order analytique sur une nouvelle forme des équations de la dynamique," J. de math. **7** (1901); L. Boltzmann, "Über die form der Lagrangeschen Gleichungen für nichtholonome, generalisierte

If the dependent velocities $\dot{q}_{n+1}$, $\dot{q}_{n+2}$, …, $\dot{q}_{n+k}$ are eliminated from the function T with the use of (14):

$$T = \Theta\,(t, q_1, q_2, \ldots, q_{n+k}, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n)$$

then:

$$\frac{\partial \Theta}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} + \sum_{\nu=1}^{k} a_{\nu i} \frac{\partial T}{\partial \dot{q}_{n+\nu}} \qquad (i = 1, 2, \ldots, n).$$

If we differentiate these equations with respect to time and apply formulas (16) then we will have:

$$\frac{d}{dt}\frac{\partial \Theta}{\partial \dot{q}_i} = \frac{\partial \Theta}{\partial q_i} + Q_i + \sum_{\nu=1}^{k} a_{\nu i}\left(\frac{\partial T}{\partial q_{n+\nu}} + Q_{n+\nu}\right) + \sum_{\nu=1}^{k} \frac{da_{\nu i}}{dt}\frac{\partial T}{\partial \dot{q}_{n+\nu}} \qquad (i = 1, 2, \ldots, n).$$

These equations no longer contain the multipliers λ.

If we then eliminate the derivatives of T with respect to the coordinates q_s with the help of the obvious formulas:

$$\frac{\partial \Theta}{\partial q_s} = \frac{\partial T}{\partial q_s} + \sum_{\nu=1}^{k} \frac{\partial T}{\partial \dot{q}_{n+\nu}}\left(\sum_{i=1}^{n} \frac{\partial a_{\nu i}}{\partial q_n} q_i + \frac{\partial a_\nu}{\partial q_s}\right) \qquad (s = 1, 2, \ldots, n + k)$$

and denote the generalized impulses that correspond to the dependent velocities by K_1, K_2, …, K_k :

$$\frac{\partial T}{\partial \dot{q}_{n+\nu}} = K_\nu\,(t, q_1, q_2, \ldots, q_{n+k}, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n) \qquad (\nu = 1, 2, \ldots, k)$$

then after a brief calculation, we will get the equations of motion of a non-holonomic system in the form:

(17)
$$\frac{d}{dt}\frac{\partial \Theta}{\partial \dot{q}_i} = \frac{\partial \Theta}{\partial q_i} + Q_i + \sum_{\nu=1}^{k} a_{\nu i}\left(\frac{\partial \Theta}{\partial q_{n+\nu}} + Q_{n+\nu}\right) + \sum_{\nu=1}^{k} K_\nu \left(\sum_{j=1}^{n} A_{ij}^{(\nu)} \dot{q}_j + A_i^{(\nu)}\right)$$

$$(i = 1, 2, \ldots, n).$$

The quantities $A_{ij}^{(\nu)}$ and $A_i^{(\nu)}$ in this have the meanings that were given in (15).

The $n + k$ differential equations (17) and (14) determine the $n + k$ coordinates q in terms of time t.

Koordinaten," Sitzungsberichte der Wiener Akademie **111**, Abt. IIa (1902); G. Hamel, "Die Lagrange-Eulerschen Gleichungen der Mechanik," Zeit. Math. Phys. **50** (1903).

§ 6. The equations of motion in special cases.

As is known (*), the motion of a non-holonomic system is still not determined completely when the expression Θ for the kinetic energy of the system and the expressions Q_s for the generalized forces are given. In fact, the equations of motion (17) still include the functions K_ν , and therefore just as many generalized impulses as non-holonomic condition equations. Those functions K_ν are first-order differential expressions, and their calculation presents no complications, at least, in the problem of rolling motion.

As is easy to see, for the case in which the integrability conditions for (14) are fulfilled:

$$A_{ij}^{(\nu)} = 0, \qquad A_i^{(\nu)} = 0 \qquad (i, j = 1, 2, \ldots, n\,;\ \nu = 1, 2, \ldots, k),$$

formulas (17) go over to the Lagrange conditions equations.

We shall refer to the following special cases that frequently occur in the applications. The coordinates q_{n+1} , q_{n+2} , …, q_{n+k} , which correspond to the eliminated velocities, might be "cyclic"; i.e., those coordinates might not be included in either the kinetic energy, the expressions Q_s , or ultimately the condition equations (14). The problem of the determination of the coordinates in terms of time will then split into two autonomous problems that are solved in succession. First of all, we seek the non-cyclic coordinates q_1, q_2, …, q_n . In order to do that, we must integrate the n second-order differential equations (17), in which we have replaced:

$$\frac{\partial\Theta}{\partial q_{n+\nu}} = 0, \qquad A_{ij}^{(\nu)} = \frac{\partial a_{\nu i}}{\partial q_j} - \frac{\partial a_{\nu j}}{\partial q_i}, \qquad A_i^{(\nu)} = \frac{\partial a_{\nu i}}{\partial t} - \frac{\partial a_\nu}{\partial q_i}$$

$$(i, j = 1, 2, \ldots, n\,;\ \nu = 1, 2, \ldots, k).$$

When that problem has been solved, we determine the cyclic coordinates q_{n+1} , q_{n+2} , …, q_{n+k} from the condition equations (14) by quadratures.

If we now assume that a force function U exists:

$$Q_s = \frac{\partial U}{\partial q_s} \qquad (s = 1, 2, \ldots, n + k),$$

which also depends upon the first n coordinates, then one will obviously have the following theorem, which was first presented by Ferrers (**):

If the condition equations:

$$A_{1,j}^{(\nu)} = 0, \qquad A_1^{(\nu)} = 0 \qquad (j = 1, 2, \ldots, n\,;\ \nu = 1, 2, \ldots, k)$$

(*) Cf., e.g., P. Appell, "Sur une forme générale des équations de la dynamique et sur le principe der Gauss," J. f. reine u. angew. Math. **122** (1900), 205-208.

(**) Ferrers, "Extension of Lagrange's equations," Quart. J. of math. **45** (1878).

are fulfilled for one of the coordinates – e.g., for q_1 *– then the corresponding equation of motion will have the Lagrangian form.*

If the time t is not included explicitly in the kinetic energy T, the force function U, or the condition equations (14) then the coefficients a_1, a_2, …, a_n must all be equal to zero in (14), since otherwise the rest position would not belong to the possible positions of the system, which was not assumed. If the coordinates q_{n+1}, q_{n+2}, …, q_{n+k} are cyclic, as well, then the equations of motion (17) will assume the simple form:

$$\frac{d}{dt}\frac{\partial \Theta}{\partial \dot{q}_i} = \frac{\partial \Theta}{\partial q_i} + \frac{\partial U}{\partial q_i} + \sum_{\nu=1}^{k} K_\nu \left(\frac{\partial a_{\nu i}}{\partial q_j} - \frac{\partial a_{\nu j}}{\partial q_i} \right) \dot{q}_j \qquad (i = 1, 2, \ldots, n).$$

Those equations were given by Chaplygin ([*]).

§ 7. – A formula for non-holonomic systems that is analogous to the Hamilton integral.

The equations of motion for a non-holonomic system in the form (17) will be obtained very easily with the help of the following theorem:

Let q_1, q_2, …, q_{n+k} *denote the coordinates of a material system, let* T *be its kinetic energy, and let* Q_s *be the generalized force that corresponds to the coordinate* q_s. *The system might be subject to the condition equations:*

$$\dot{q}_{n+\nu} = \sum_{i=1}^{n} a_{\nu i}\, \dot{q}_i + a_\nu \qquad (\nu = 1, 2, \ldots, k).$$

If we use those equations to express the kinetic energy of the system and the generalized impulses that corresponds to the dependent velocities $\dot{q}_{n+1}$, $\dot{q}_{n+2}$, …, $\dot{q}_{n+k}$ *in terms of time t, the coordinates* q_1, q_2, …, q_{n+k}, *and the independent velocities* $\dot{q}_1$, $\dot{q}_2$, …, $\dot{q}_k$:

$$T = \Theta\,(t, q_1, q_2, \ldots, q_{n+k}, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_k)$$

$$\frac{\partial T}{\partial \dot{q}_{n+\nu}} = K_\nu\,(t, q_1, q_2, \ldots, q_{n+k}, \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_k) \qquad (\nu = 1, 2, \ldots, k)$$

then we will have the formula:

([*]) Chaplygin, "On the motion of a heavy body of revolution on a horizontal plane," Reports of the physical section, no. 9, Moscow, 1897.

(18) $$\int_{t_1}^{t_2}\left[\delta\Theta+\sum_{s=1}^{n+k}Q_s\,\delta q_s+\sum_{\nu=1}^{k}K_\nu\,\delta\left(\dot q_{n+\nu}-\sum_{i=1}^{n}a_{\nu i}\,\dot q_i-a_\nu\right)\right]dt = 0$$

for all variations δq_1, δq_2, …, δq_n *that vanish at the moments* t_1 *and* t_2. δq_{n+1}, δq_{n+2}, …, δq_{n+k} *are defined by the equations:*

(19) $$\delta q_{n+\nu} = \sum_{i=1}^{n} a_{\nu i}\,\delta q_i \qquad (\nu = 1, 2, …, k),$$

and the differences $\delta\dot q_s - \frac{d}{dt}\delta q_s$ *are all set equal to zero.*

In fact, as a result of (19), (18) can be transformed by partial differentiation in such a way that a linear function of the variations δq_1, δq_2, …, δq_n will appear under the integral sign. If we set the coefficients of those variations equal to zero then we will obtain the equations of motion (17).

We would like to make the following remarks in regard to formula (18) (*).

If we consider the conditions (14) for the velocities and the conditions (19) for the variations *simultaneously* and set:

$$\delta\dot q_i - \frac{d}{dt}\delta q_i = 0 \qquad (i = 1, 2, …, n)$$

then the differences $\delta\dot q_{n+\nu} - \frac{d}{dt}\delta q_{n+\nu}$ ($\nu = 1, 2, …, k$) will be non-zero, in general. If we then multiply the known d'Alembert formula:

$$\sum_{s=1}^{n+k}\left[-\frac{d}{dt}\left(\frac{\partial T}{\partial\dot q_s}\right)+\frac{\partial T}{\partial q_s}+Q_s\right]\delta q_s = 0$$

by dt and integrate between t_1 and t_2, in which t_1 and t_2 denote two moments at which all variations vanish, then we will get the formula:

(20) $$\int_{t_1}^{t_2}\left[\delta T+\sum_{s=1}^{n+k}Q_s\,\delta q_s+\sum_{\nu=1}^{k}\frac{\partial T}{\partial\dot q_{n+\nu}}\left(\frac{d}{dt}\delta q_{n+\nu}-\delta\dot q_{n+\nu}\right)\right]dt = 0.$$

If we replace the functions T and $\frac{\partial T}{\partial\dot q_{n+\nu}}$ in this with Θ and K_ν, resp., and calculate the differences $\frac{d}{dt}\delta q_{n+\nu} - \delta\dot q_{n+\nu}$ ($\nu = 1, 2, …, k$) directly from (14) and (19) then we will get

(*) On this, cf., my treatise and the treatise of Souslov in v. 22 of the Moscow Mathematical Collection, 1901.

the equations of motion (17) from (20) in a manner that is similar to how we got them from (18).

The reason that we prefer the formula (20) over the formula (18) is as follows:

When one examines the problem of rolling motion, one does not usually employ the generalized velocities, but one introduces linear functions of them into the corresponding formulas. Hence, e.g., the differential quotients $\dot{\varphi}$, $\dot{\psi}$, $\dot{\theta}$ of the three Euler angles φ, ψ, θ with respect to time t are ordinarily replace with the projections p, q, r of the instantaneous angular velocities of the rolling bodies onto its central principal axes of inertia:

$$p = \dot{\varphi}\sin\theta - \dot{\psi}\sin\varphi\cos\theta,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

If we, with Kirchhoff (*), consider the quantities p', q', r':

$$p' = \delta\varphi \sin\theta - \delta\psi \sin\varphi \cos\theta,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

along with the quantities p, q, r, then we must employ formula (18) to ascribe the meaning to the differences $\delta p - dp' / dt$, etc., that Kirchhoff (**) gave to them:

$$\delta p - \frac{dp'}{dt} = q\,r' - r\,q',$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

However, if we had chosen formula (20) then the aforementioned differences would have other values that would depend upon the form of the non-holonomic condition equations. For that reason, the investigation would take a somewhat more complicated form if we had started from formula (20), instead of (18).

§ 8. – Introducing linear functions of velocity into the equations of motion.

The equations of motion (17) and the formula (18) shall now be generalized in such a way that the velocities $\dot{q}$ will be replaced with arbitrary linear functions of them in the equations of motion. We denote those functions by p_1, p_2, …, p_{n+k} and set:

$$\dot{q}_r = \sum_{s=1}^{n+k} \alpha_{rs}\, p_s + \alpha_r \qquad (r = 1, 2, \ldots, n + k), \tag{21}$$

in which the coefficients α_{rs} and α_r depend upon the time t and the coordinates q.

We will assume that equations (21) can be solved for the variables p.

Along with p, we also consider $n + k$ quantities p' that satisfy the equations:

(*) Kirchhoff, *Vorlesungen über mathematische Physik*, Bd. I, lect. VI.
(**) *Ibid.*, formula (9).

(22) $$\delta q_r = \sum_{s=1}^{n+k} \alpha_{rs}\, p'_s \qquad (r = 1, 2, \ldots, n + k).$$

One can then calculate the differences $\delta q_s - dp'_s / dt$ (s = 1, 2, …, n + k) from the conditions:

$$\delta \dot{q}_r = \frac{d}{dt}\delta q_r \qquad (r = 1, 2, \ldots, n + k)$$

as linear functions of the p':

(23) $$\delta p_s - \frac{dp'_s}{dt} = \sum_{r=1}^{n+k} P_{sr}\, p'_r \qquad (s = 1, 2, \ldots, n + k),$$

in which the coefficients P_{sr} are linear functions of the p.

Let the non-holonomic condition equations be expressed in terms of the quantities p :

(24) $$p_{n+\nu} = \sum_{i=1}^{n} b_{\nu i}\, p_i + b_\nu \qquad (\nu = 1, 2, \ldots, k),$$

in which the $b_{\nu i}$ and b_ν depend upon time t and the coordinates q.

The equations of motion of the material system then have the form (*):

(25) $$\begin{aligned} \frac{d}{dt}\frac{\partial T}{\partial p_i} &= \sum_{r=1}^{n+k} \alpha_{ri}\left(\frac{\partial T}{\partial q_r} + Q_r\right) + \sum_{r=1}^{n+k} \frac{\partial T}{\partial p_r} P_{ri} - \sum_{\nu=1}^{k} \lambda_\nu b_{\nu i} \qquad (i = 1, 2, \ldots, n), \\ \frac{d}{dt}\frac{\partial T}{\partial p_{n+\nu}} &= \sum_{r=1}^{n+k} \alpha_{r,n+\nu}\left(\frac{\partial T}{\partial q_r} + Q_r\right) + \sum_{r=1}^{n+k} \frac{\partial T}{\partial p_r} P_{r,n+\nu} + \lambda_\nu \qquad (\nu = 1, 2, \ldots, k)\,. \end{aligned}$$

Here, T denotes the kinetic energy of the system, Q_s , the generalized force that corresponds to the coordinate q_s , and λ_1 , λ_2 , …, λ_k are Lagrange multipliers. From (21), T is a function of time t, the coordinates q, and the quantities p.

The formulas (25), (24), and (21) define a system of $2n + 3k$ first-order differential equations that determine the $n + k$ coordinates q, the $n + k$ quantities p, and the multipliers λ as functions of time t.

We can say the same thing in regard to formulas (25) that we expressed above in regard to the equations of motion (16). An application of (25) to the problem of rolling motion will encounter difficulties firstly, because the multipliers λ have not been eliminated from (25) and secondly, because the function T is included in (25), and it has not been converted into its simplest form by the use of (24).

(*) Cf., e.g., the method by which G. Kirchhoff (*ibid.*, Lect. VI) derived the differential equations of the motion of a rigid body. See also the treatise of K. Heun, "Die Bedeutung des D'Alembertschen Prinzipes für starre Systeme und Gelenkmechanismen," Arch. Math. Phys. (3) **2** (1902), § 17.

By eliminating the λ from (25) using the method that was applied in § **5**, we will come to the equations of motion, which can be derived most easily from the following theorem, which is analogous to the theorem in § **7**:

If we express the kinetic energy T and the derivatives of T with respect to the quantities $p_{n+1}, p_{n+2}, \ldots, p_{n+k}$ *as function of time t, the coordinates q, and the quantities* $p_1, p_2, \ldots, p_n$:

$$T = \Theta\,(t, q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n),$$

$$\frac{\partial T}{\partial p_{n+\nu}} = K\nu\,(t, q_1, q_2, \ldots, q_n, p_1, p_2, \ldots, p_n) \qquad (n = 1, 2, \ldots, k)$$

then the integral expression:

$$\text{(26)} \qquad \int_{t_1}^{t_2}\left[\delta\Theta + \sum_{r=1}^{n+k} Q_r\,\delta q_r + \sum_{\nu=1}^{k} K_\nu\,\delta\left(p_{n+\nu} - \sum_{i=1}^{n} b_{\nu i}\,p_i - b_\nu\right)\right] dt$$

will vanish for all $p_1', p_2', \ldots, p_n'$ *that vanish at the limits. The quantities* $p_{n+1}', p_{n+2}', \ldots, p_{n+k}'$ *are eliminated with the help of the formulas:*

$$\text{(27)} \qquad p_{n+\nu}' = \sum_{i=1}^{n} b_{\nu i}\,p_i' \qquad (\nu = 1, 2, \ldots, k),$$

the variations $\delta q_1, \delta q_2, \ldots, \delta q_{n+k}$ *are eliminated with the help of* (22), *and the differences* $\delta p - dp' / dt$ *are calculated from* (21) *and* (22) *and the conditions:*

$$\delta\dot{q}_r = \frac{d}{dt}\delta q_r \qquad (r = 1, 2, \ldots, n + k).$$

§ 9. – Application to the problem of rolling motion.

We shall now move on to the special problem of the motion of a rigid body that rolls without slipping on a given surface S_1 under the action of given forces.

We imagine two orthogonal coordinate systems $Oxyz$ and $O_1\,x_1\,y_1\,z_1$, the former of which is fixed in the rigid body, and the latter of which is fixed in the surface S_1.

The coordinates of the body are the coordinates a, b, c of its point O relative to the system of axes $O_1\,x_1\,y_1\,z_1$ and the three Euler angles φ, ψ, θ, which determine the position of the x, y, z axes relative to the x_1, y_1, z_1 axes.

In place of the generalized velocities $\dot{a}$, $\dot{b}$, $\dot{c}$, $\dot{\varphi}$, $\dot{\psi}$, $\dot{\theta}$ we introduce linear functions of them by way of the formulas:

$$k = \dot{a}\,\cos(x, x_1) + \dot{b}\,\cos(x, y_1) + \dot{c}\,\cos(x, z_1), \qquad p = \dot{\varphi}\,\sin\theta - \dot{\psi}\,\sin\varphi\cos\theta,$$

$$l = \dot{a}\cos(y, x_1) + \dot{b}\cos(y, y_1) + \dot{c}\cos(y, z_1), \qquad q = \dot{\varphi}\cos\theta + \dot{\psi}\sin\varphi\sin\theta,$$
$$m = \dot{a}\cos(z, x_1) + \dot{b}\cos(z, y_1) + \dot{c}\cos(z, z_1), \qquad r = \dot{\theta} + \dot{\psi}\cos\varphi,$$

in which the nine cosines are known functions of the Euler angles.

As in § **1**, the quantities k, l, m ; p, q, r denote the projections onto the x, y, z axes of the velocity w of the point O and the instantaneous angular velocity ω of the body, resp.

If we introduce the additional quantities k', l', m'; p', q', r' by way of the formulas:

$$k' = \delta a\cos(x, x_1) + \delta b\cos(x, y_1) + \delta c\cos(x, z_1); \qquad p' = \delta\varphi\sin\theta - \delta\psi\sin\varphi\cos\theta,$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

then we will have (*):

$$\delta k - \frac{dk'}{dt} = l\,r' - r\,l' + q\,m' - m\,q'; \qquad \delta p - \frac{dp'}{dt} = q\,r' - r\,q'; \tag{28}$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

If we let x, y, z denote the coordinates of the point M at which the outer surface S of the body contacts the surface S_1 relative to the system of axis $Oxyz$ and express the idea that the point is momentarily at rest then we will get the non-holonomic condition equations to which the body is subject:

$$k = yr - zq, \qquad l = zp - xr, \qquad m = xq - yp. \tag{29}$$

If the point O coincides with the center of mass of the body and the x, y, z axes coincide with the principal axes of inertia through the point O then the kinetic energy T of the body will be equal to:

$$T = \tfrac{1}{2}M\,(k^2 + l^2 + m^2) + \tfrac{1}{2}(A\,p^2 + B\,q^2 + C\,r^2)\,, \tag{30}$$

in which M means the mass of the body, and A, B, C mean the moments of inertia about the x, y, z axes, resp.

In the present case, the functions Θ and K that are included in (26) will then have the values:

$$\begin{aligned} &\Theta = \tfrac{1}{2}M\,[(x^2 + y^2 + z^2)\,(p^2 + q^2 + r^2) - (xp + yq + zr)^2] + \tfrac{1}{2}(A\,p^2 + B\,q^2 + C\,r^2), \\ &K_1 = M\,(y\,r - z\,q), \qquad K_2 = M\,(z\,p - x\,r), \qquad K_3 = M\,(x\,q - y\,p), \end{aligned} \tag{31}$$

such that formula (26) will imply that:

$$\int_{t_1}^{t_2} [\delta\Theta + \delta U + M\,(tr - zq)\,\delta(k - yr + zq) + \cdots]\,dt = 0.$$

(*) *Ibid.*, formulas (8) and (9).

We assume that a force function U exists

From the theorem in § **8**, when one transforms this formula, one must consider only the equations:

$$k' = y\,r' - z\,q', \qquad l' = z\,p' - x\,r', \qquad m' = x\,q' - y\,p', \tag{32}$$

in addition to (28).

We will then have:

$$\delta(k - y\,r + z\,q) = \frac{d}{dt}(y\,r' - z\,q') + l\,r' - r\,l' + q\,m' - m\,q' - \delta(yr - zq)$$

$$= \dot{y}\,r' - \dot{z}\,q' - (r\,dy - q\,dz),$$

such that the integral expression (26) will ultimately assume the form:

$$\int_{t_1}^{t_2} \{\delta\Theta + \delta U + M[\rho\dot{\rho}\,(pp' + qq' + rr') - (xp' + yq' + zr')(\dot{x}\,p + \dot{y}\,q + \dot{z}\,r)] \tag{33}$$
$$- M\,[\rho\;\delta\rho \cdot \omega^2 - (x\,p + y\,q + z\,r)\,(p\;\delta x + q\;\delta y + r\;\delta z)]\}\;dt = 0,$$

in which one has set:

$$x^2 + y^2 + z^2 = \rho^2, \quad p^2 + q^2 + r^2 = \omega^2,$$

for brevity.

That formula allows one to derive the equations of rolling motion of a rigid body in arbitrary coordinates. If we choose the coordinates u, v, ϑ, u_1, v_1 of § **1** then x, y, z will be given functions of u and v. Formulas (10) and (9) determine the p, q, r in terms of the generalized velocities $\dot{u}$, $\dot{v}$, $\dot{\vartheta}$, $\dot{u}_1$, $\dot{v}_1$. Those velocities must satisfy the non-holonomic conditions (12). Two of the quantities $\dot{u}$, $\dot{v}$, $\dot{\vartheta}$, $\dot{u}_1$, $\dot{v}_1$ (e.g., $\dot{u}_1$ and $\dot{v}_1$) can be eliminated from (9) with the help of those formulas:

$$\sigma = -\left(\frac{D''}{G} + \frac{D_1''}{G_1}\right)\sqrt{G}\,\dot{v} + \left(\frac{D_1''}{G_1} + \frac{D_1}{E_1}\right)\left(-\sqrt{E}\,\dot{u}\sin\vartheta + \sqrt{G}\,\dot{v}\cos\vartheta\right)\cos\vartheta,$$
$$\tau = \left(\frac{D}{E} + \frac{D_1}{E_1}\right)\sqrt{E}\,\dot{u} + \left(\frac{D_1''}{G_1} - \frac{D_1}{E_1}\right)\left(\sqrt{E}\,\dot{u}\cos\vartheta + \sqrt{G}\,\dot{v}\sin\vartheta\right)\cos\vartheta. \tag{34}$$

We then see that these formulas will assume an especially simple form when the surface S_1 is a sphere, i.e., when:

$$D_1 : E_1 = D_1'' : G_1 .$$

The examination of that special case shall be prefaced with the general examination.

CHAPTER III

On the motion of a rigid body that rolls on a sphere

§ 10. – The differential equations of motion of a rigid body that rolls on a sphere.

The problem that defines the topic of the present chapter can be formulated thus:

A given rigid body is constrained to roll without slipping on an immobile sphere S_1 *. A force acts upon the body that is applied to the center of mass O of the body and points to the center* O_1 *of the sphere, and it depends upon only the distance between the points O and* O_1 *. Determine the motion of the body.*

If we set:

$$x_1 = R_1 \sin u_1 \cos v_1 , \quad y_1 = R_1 \sin u_1 \sin v_1 , \quad z_1 = R_1 \cos u_1$$

then we will get:

$$E_1 = R_1^2 , \qquad G_1 = \sin^2 u_1 , \quad D_1 = - R_1 , \qquad D_1'' = - R_1 \sin^2 u_1 , \tag{35}$$

and the formulas (34) and (10) will imply:

$$\begin{aligned} &\sigma = \nu \dot{v} , \qquad \tau = \mu \dot{u} , \\ &p = \nu \dot{v} \alpha + \mu \dot{u} \beta + n \gamma , \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \end{aligned} \tag{36}$$

in which:

$$\nu = -\left(\frac{D''}{G} - \frac{1}{R_1}\right)\sqrt{G} , \qquad \mu = \left(\frac{D}{E} - \frac{1}{R_1}\right)\sqrt{E} . \tag{37}$$

We calculate the kinetic energy Θ of the body with the help of formulas (36) according to (31):

$$\begin{aligned} 2\Theta = {} & M \rho^2 (\nu^2 \dot{v}^2 + \mu^2 \dot{u}^2 + n^2) - M \left(\rho \frac{\partial \rho}{\partial u} \frac{\nu \dot{v}}{\sqrt{E}} + \rho \frac{\partial \rho}{\partial v} \frac{\mu \dot{u}}{\sqrt{G}} + \varepsilon n \right)^2 \\ & + A(\nu \dot{v} \alpha + \mu \dot{u} \beta + n \gamma)^2 + B(\nu \dot{v} \alpha' + \mu \dot{u} \beta' + n \gamma')^2 + C(\nu \dot{v} \alpha'' + \mu \dot{u} \beta'' + n \gamma'')^2 . \end{aligned} \tag{38}$$

In this, ρ and ε denote the distances from the center of mass O to the contact point M and from the tangent plane at M to the outer surface S of the body:

$$\rho^2 = x^2 + y^2 + z^2, \qquad \varepsilon = x \gamma + y \gamma' + z \gamma''.$$

The kinetic energy Θ is then a second-degree homogeneous function of the arguments $\dot{u}$, $\dot{v}$, n. The coefficients of those functions depend upon only u and v.

As for the force function U, in the present case, U is a function of the distance between the points O and O_1. However, one has:

$$\overline{O_1O}^2 = \overline{O_1M}^2 + \overline{OM}^2 - 2\overline{O_1M}\cdot\overline{OM}\cos\left(\overline{O_1M},\overline{OM}\right) = R_1^2 + \rho^2 + 2R_1\,\varepsilon,$$

so as a result, U will include only u and v.

If we introduce the quantities, p', q', r', along with the p, q, r in (36):

$$p' = \alpha\,\nu\,\delta v + \beta\,\mu\,\delta u + \gamma n' \tag{39}$$

then the basic formula (33) will yield:

$$\int_{t_1}^{t_2}[\delta\Theta + \delta U + M\rho\dot\rho(\nu^2\dot v\,\delta v + \mu^2\dot u\,\delta u + nn')$$

$$- M\left(\rho\frac{\partial\rho}{\partial u}\frac{\nu\,\delta v}{\sqrt{E}} + \rho\frac{\partial\rho}{\partial v}\frac{\mu\,\delta u}{\sqrt{G}} + \varepsilon\,n'\right)\left(\sqrt{E}\,\nu + \sqrt{G}\,\mu\right) - M\,\omega^2\,\rho\,\delta\rho$$

$$+ M\left(\rho\frac{\partial\rho}{\partial u}\frac{\nu\,\dot v}{\sqrt{E}} + \rho\frac{\partial\rho}{\partial v}\frac{\mu\,\dot u}{\sqrt{G}} + \varepsilon\,n\right)\left(\sqrt{E}\,\nu\,\dot v\,\delta u + \sqrt{G}\,\mu\,\dot u\,\delta v\right)\Big]\,dt = 0,$$

in which we have set:

$$\nu^2\,\dot v^2 + \mu^2\,\dot u^2 + n^2 = \omega^2,$$

for brevity.

In order to get the expression for the equations of motion from this, we must calculate the differences $\delta\dot u - \frac{d}{dt}\delta u$, $\delta\dot v - \frac{d}{dt}\delta v$, $\delta n - \frac{dn'}{dt}$.

(36) and (39) imply that:

$$\nu\,\dot v = p\,\alpha + q\,\alpha' + r\,\alpha'', \qquad \nu\,\delta v = p'\,\alpha + q'\,\alpha' + r'\,\alpha'',$$

such that, from (28), one will have:

$$\nu\left(\delta\dot v - \frac{d}{dt}\delta v\right) + \frac{\partial\nu}{\partial u}(\dot v\,\delta u - \dot u\,\delta v) = (q\,r' - r\,q')\,\alpha + \ldots + p\,\delta\alpha + \ldots - p'\,\dot\alpha - \ldots$$

If we substitute (36) and (39) in this and remark that as a result of the well-known formulas of kinematics:

$$\beta\,\dot\gamma + \beta'\,\dot\gamma' + \beta''\,\dot\gamma'' = \sigma_1, \qquad \gamma\,\dot\alpha + \ldots = \tau_1, \qquad \alpha\,\dot\beta + \ldots = n_1,$$

in which σ_1, τ_1, n_1 have their meaning in (8), then we will get:

$$\nu\left(\delta\dot{v}-\frac{d}{dt}\delta v\right)+\left(\frac{\partial\nu}{\partial u}+\frac{\mu}{2\sqrt{EG}}\frac{\partial G}{\partial u}\right)(\dot{v}\,\delta u-\dot{u}\,\delta v)=\frac{\sqrt{E}}{R_1}(n\,\delta u-\dot{u}\,n').$$

However, the expression:

$$\frac{\partial\nu}{\partial u}+\frac{\mu}{2\sqrt{EG}}\frac{\partial G}{\partial u}$$

is equal to zero, from (37) and the Mainardi-Codazzi formula (*):

$$2H^2\left(\frac{\partial D''}{\partial u}-\frac{\partial D'}{\partial v}\right)=\left(2\frac{\partial F}{\partial v}-\frac{\partial G}{\partial u}\right)(F\,D'-G\,D)+\frac{\partial G}{\partial v}(F\,D-E\,D')$$

$$-\frac{\partial E}{\partial v}(F\,D''-G\,D')-\frac{\partial G}{\partial u}(F\,D'-E\,D'')$$

such that we will ultimately have:

$$\nu\left(\delta\dot{v}-\frac{d}{dt}\delta v\right)=\frac{\sqrt{E}}{R_1}(n\,\delta u-\dot{u}\,n').$$

In a similar way, we will find that:

$$\mu\left(\delta\dot{u}-\frac{d}{dt}\delta u\right)=\frac{\sqrt{G}}{R_1}(n\,\delta v-\dot{v}\,n'),$$

$$\delta n-\frac{dn'}{dt}=\sqrt{EG}\left(\frac{DD''}{EG}-\frac{1}{R_1^2}\right)(\dot{v}\,\delta u-\dot{u}\,\delta v).$$

If we now transform the basic formula with the use of these equations in such a way that a linear function of the quantities δu, δv, n' will appear under the integral sign and we set the coefficient of those quantities equal to zero then we will get the desired equations of motion in the form:

$$\frac{d}{dt}\frac{\partial\Theta}{\partial\dot{u}}-\frac{\partial(\Theta+U)}{\partial u}=\sqrt{EG}\left(\frac{DD''}{EG}-\frac{1}{R_1^2}\right)\frac{\partial\Theta}{\partial n}\dot{v}+\frac{\sqrt{E}}{R_1}\frac{1}{\nu}\frac{\partial\Theta}{\partial\dot{v}}n-M\rho\frac{\partial\rho}{\partial u}n^2+M\varepsilon\sqrt{E}\,\nu\,\dot{v}\,n,$$

(40) $$\frac{d}{dt}\frac{\partial\Theta}{\partial\dot{v}}-\frac{\partial(\Theta+U)}{\partial v}=-\sqrt{EG}\left(\frac{DD''}{EG}-\frac{1}{R_1^2}\right)\frac{\partial\Theta}{\partial n}\dot{u}+\frac{\sqrt{G}}{R_1}\frac{1}{\mu}\frac{\partial\Theta}{\partial\dot{u}}n-M\rho\frac{\partial\rho}{\partial v}n^2+M\varepsilon\sqrt{G}\,\mu\,\dot{u}\,n,$$

(*) Stahl and Kommerell, formula (6), § **7**.

$$\frac{d}{dt}\frac{\partial\Theta}{\partial n} = -\frac{\sqrt{E}}{R_1}\frac{1}{\nu}\frac{\partial\Theta}{\partial\dot{v}}\dot{u} - \frac{\sqrt{G}}{R_1}\frac{1}{\mu}\frac{\partial\Theta}{\partial\dot{u}}\dot{v} + M\left(\rho\frac{\partial\rho}{\partial u}\dot{u} + \rho\frac{\partial\rho}{\partial v}\dot{v}\right)n - M\varepsilon\left(\sqrt{E}\,\nu + \sqrt{G}\,\mu\right)\dot{u}\,\dot{v}\,.$$

These formulas determine the non-cyclic coordinates u and v and the quantity n as functions of time t.

Before we go further into the study of the equations of motion that we have presented, we would like to test their validity on the basis of more general laws of dynamics.

§ 11. – Developing the equations of motion from the law of the moments of the quantity of motion.

From (36), the non-holonomic condition equations to which the rigid body is subject will take the form:

(41) $$k = (y\,\alpha'' - z\,\alpha')\,\nu\,\dot{v} + (y\,\beta'' - z\,\beta')\,\mu\,\dot{u} + (y\,\gamma'' - z\,\gamma')\,n,$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

for the problem that we spoke of, such that we will have:

(42) $$\frac{1}{\nu}\frac{\partial\Theta}{\partial\dot{v}} = \frac{\partial T}{\partial k}(y\,\alpha'' - z\,\alpha') + \ldots + \frac{\partial T}{\partial p}\alpha + \ldots,$$

from (30), using (36). Now, since $\frac{\partial T}{\partial k}, \ldots, \frac{\partial T}{\partial p}, \ldots$ are equal to the projections onto the x, y, z axes of the resulting vector and the resultant moment about the center of mass O of the quantity of motion of the material points from which we imagine our rigid body is composed, we will see from (42) that $\frac{1}{\nu}\frac{\partial\Theta}{\partial\dot{v}}$, $\frac{1}{\mu}\frac{\partial\Theta}{\partial\dot{u}}$, $\frac{\partial\Theta}{\partial n}$ mean the projections of the aforementioned moments about the contact point M onto the u, v, n axes, resp.

Now, it is known (*) that the geometric derivative $\dot{\Pi}_1$ of the system of vectors Π_1 that consists of the quantities of motion of the material points is equivalent to the system of vectors Π_2 of the applied forces on the bodies and their reactions. If the pole M (x, y, z) relative to which we calculate the resultant moment Γ_1 of the system Π_1 is an immobile point then the resultant moment Γ_1' of the system $\dot{\Pi}_1$ will be equal to the geometric derivative $\dot{\Gamma}_1$ of Γ_1:

$$(\Gamma_1') = (\Gamma_1)\,.$$

However, if the pole M (x, y, z) is mobile, as in the present case, then:

(43) $$(\Gamma_1') = (\dot{\Gamma}_1) + (K),$$

(*) Sousloff, *Foundations of analytical mechanics*, v. 1, § 192, Kiev, 1900 (in Russian).

in which K denotes the moment of the resultant vector of the system Γ_1 when that vector acts upon the "derived" pole $M'(\dot{x}, \dot{y}, \dot{z})$ about the coordinate origin.

If we further remark that the u-axis rotates with an angular velocity whose components along the x, y, z axes are equal to the quantities $\dot{\alpha} + q\alpha'' - r\alpha'$, etc, then when we project the vector along the direction u, we will get:

$$\frac{d}{dt}\left(\frac{1}{\nu}\frac{\partial \Theta}{\partial \dot{v}}\right) - \left(\frac{\partial T}{\partial p} + z\frac{\partial T}{\partial l} - y\frac{\partial T}{\partial m}\right)(\dot{\alpha} + q\alpha'' - r\alpha') - \ldots + \left(\dot{y}\frac{\partial T}{\partial m} - \dot{z}\frac{\partial T}{\partial l}\right)\alpha + \ldots$$

$$= \Gamma_2 \cos(\Gamma_2, u),$$

in which Γ_2 is the resultant moment of the system Π_2 about the contact point M.

If one neglects the moments of the rolling and twisting friction then the only moments of the applied forces will enter into Γ_2, since neither the force of friction nor the normal reaction moment will produce moments about the contact point M. If a force function U exists and we imagine that the differential quotient $\dot{U}$ of U with respect to time t is expressed as a linear function of the quantities $k, \ldots, p, \ldots$ then:

$$\Gamma_2 \cos(\Gamma_2, u) = \left(\frac{\partial \dot{U}}{\partial p} + z\frac{\partial \dot{U}}{\partial l} - y\frac{\partial \dot{U}}{\partial m}\right)\alpha + \ldots$$

If $\dot{U}$ goes to $(\dot{U})$ when we eliminate the quantities $k, \ldots, p, \ldots$ using (36) and (41) then:

$$\Gamma_2 \cos(\Gamma_2, u) = \frac{1}{\nu}\frac{\partial (\dot{U})}{\partial \dot{v}}.$$

If U is a function of u and v then we will have:

$$\Gamma_2 \cos(\Gamma_2, u) = \frac{1}{\nu}\frac{\partial U}{\partial v}.$$

We ultimately get:

$$\begin{aligned} \frac{d}{dt}\frac{\partial \Theta}{\partial \dot{v}} &= \frac{\partial T}{\partial k}\left[\frac{d}{dt}(y\nu\alpha'' - z\nu\alpha') + p\nu(x\alpha + y\alpha' + z\alpha'') - \nu\alpha(xp + yq + zr)\right] + \cdots \\ &+ \frac{\partial T}{\partial p}\left[\frac{d}{dt}(\nu\alpha) + q\nu\alpha'' - r\,y\alpha'\right] + \cdots + \frac{\partial U}{\partial v}. \end{aligned} \tag{44}$$

It can be established by some truly-complicated calculations that this equation is identical with the second of the formulas (40).

If we project the vector Γ_1' onto the directions v and n then we will get the other two equations of motion (40) in a similar way.

Formula (44) can also be derived directly from the general equations of motion of a rigid body that is not subject to the non-holonomic conditions (29):

$$\frac{d}{dt}\frac{\partial T}{\partial k} = r\frac{\partial T}{\partial l} - q\frac{\partial T}{\partial m} + \frac{\partial \dot{U}}{\partial k} + \lambda_1 ,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\frac{d}{dt}\frac{\partial T}{\partial p} = r\frac{\partial T}{\partial q} - q\frac{\partial T}{\partial r} + \frac{\partial \dot{U}}{\partial p} + y\,\lambda_3 - z\,\lambda_2 ,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

by eliminating the multipliers λ_1, λ_2, λ_3.

In order to do that, we must use that formula to differentiate formula (42) with respect to time *t*, after we have multiplied it by ν:

$$\frac{d}{dt}\frac{\partial \Theta}{\partial \dot{v}} = \frac{\partial T}{\partial k}\frac{d}{dt}(y\nu\alpha'' - z\nu\alpha') + \cdots + (y\nu\alpha'' - z\nu\alpha')\left(r\frac{\partial T}{\partial l} - q\frac{\partial T}{\partial m} + \frac{\partial \dot{U}}{\partial k} + \lambda_1\right)$$
$$+ \cdots + \frac{\partial T}{\partial p}\frac{d}{dt}(\nu\alpha) + \cdots + \nu\alpha\left(r\frac{\partial T}{\partial q} - q\frac{\partial T}{\partial r} + \frac{\partial \dot{U}}{\partial p} + y\lambda_3 - z\lambda_2\right) + \cdots$$

or

$$\frac{d}{dt}\frac{\partial \Theta}{\partial \dot{v}} = \frac{\partial T}{\partial k}\left[\frac{d}{dt}(y\nu\alpha'' - z\nu\alpha') + x\nu\,(p\alpha + q\alpha' + r\alpha'') - \nu\,\alpha\,(xp + yq + zr)\right] + \cdots$$
$$\cdots + \frac{\partial T}{\partial p}\left[\frac{d}{dt}(\nu\alpha) + q\nu\alpha'' - r\,y\alpha'\right] + \cdots + \frac{\partial U}{\partial v}.$$

However, from (29) and (30):

$$p\frac{\partial T}{\partial k} + q\frac{\partial T}{\partial l} + r\frac{\partial T}{\partial m} = 0, \qquad x\frac{\partial T}{\partial k} + y\frac{\partial T}{\partial l} + z\frac{\partial T}{\partial m} = 0,$$

so that equation will be identical with formula (44).

§ 12. – Testing the equations of motion with Poinsot's interpretation of the motion of a force-free rigid body.

The validity of formulas (40) can be confirmed (at least, in one special case) with the use of the known interpretation of the equations of a rigid body that has no forces acting upon it that Poinsot gave. As a result of that interpretation, the center of mass of such a body will move uniformly along a line, and the central ellipsoid of inertia of the body:

$$Ax^2 + By^2 + Cz^2 = 1$$

will roll without slipping on a plane that has a translational motion such that its distance from the center of mass of the body remains constant. The contact point of the ellipsoid and the plane will describe a polhode on the outer surface of the ellipsoid and a herpolhode on the plane. The projection of the momentary angular velocity of the body onto the normal at the contact point to the inertia ellipsoid will remain constant during the motion.

If the center of mass is at rest at the initial moment then it will remain at rest during the motion of the body, and the inertia ellipsoid will roll on a plane at rest.

We now transform the inertia ellipsoid and the plane by reciprocal radii (*). The ellipsoid will go over to the fourth-degree surface:

$$Ax^2 + By^2 + Cz^2 = (x^2 + y^2 + z^2)^2,$$

and the plane will go to a sphere.

As a result of what was said, when no forces act upon a body and the body is bounded by the aforementioned fourth-degree surface, the equations of motion (40) must admit a particular solution for which the quantity n remains constant and the contact point M describes a curve on the outer surface of the body that is obtained from the polhode by transformation through reciprocal radii.

In fact, if we set:

$$x = \frac{a}{k-u-v}\frac{\sqrt{a^2-u}\sqrt{a^2-v}}{\sqrt{a^2-b^2}\sqrt{a^2-c^2}}, \ldots$$

$$A = \frac{1}{a^2}, \qquad B = \frac{1}{b^2}, \qquad C = \frac{1}{c^2}, \qquad k = a^2 + b^2 + c^2$$

then we can show that in the case of:

$$U = 0$$

equations (40) will admit the particular solution:

$$n = \text{const.}, \qquad u \cdot v = \text{const.}$$

§ 13. – Solving the equations of motion for the differential quotients of the unknown functions. Integrals of motion. Determining the cyclic coordinates.

We now go on to a detailed examination of the equations of motion (40) that determine the variables u, v, n as functions of time t. The equations for u and v are of second order, while the one for n is of first order. If we add the equations:

$$\frac{du}{dt} = \dot{u}, \qquad \frac{dv}{dt} = \dot{v}$$

(*) Routh, *The advanced part of a treatise on the dynamics, etc.*, art. 147, ex. 4, 1884.

to formulas (40) then we will get five first-order differential equations that determine the five functions u, v, n, $\dot{u}$, $\dot{v}$ in terms of t. Should those equations be solved for the differential quotients of the unknown functions, then we would have to introduce new unknown functions p_1, p_2, p_3 in place of the $\dot{u}$, $\dot{v}$, n:

$$\frac{\partial \Theta}{\partial \dot{u}} = p_1, \qquad \frac{\partial \Theta}{\partial \dot{v}} = p_2, \qquad \frac{\partial \Theta}{\partial n} = p_3 .$$

From a theorem of Donkin (*), we will then have:

$$\dot{u} = \frac{\partial \bar{\Theta}}{\partial p_1}, \qquad \dot{v} = \frac{\partial \bar{\Theta}}{\partial p_2}, \qquad n = \frac{\partial \bar{\Theta}}{\partial p_3}, \qquad \frac{\partial \Theta}{\partial u} = -\frac{\partial \bar{\Theta}}{\partial u}, \quad \frac{\partial \Theta}{\partial v} = -\frac{\partial \bar{\Theta}}{\partial v},$$

in which $\bar{\Theta}$ denotes the kinetic energy of the body when it is expressed as a homogeneous function of degree two of the arguments p_1, p_2, p_3. If we introduce the notation:

$$\bar{\Theta} - U = H$$

then (40) will yield the following five first-order differential equations:

$$\frac{dp_1}{dt} =$$
$$-\frac{\partial H}{\partial u} + \sqrt{EG}\left(\frac{DD''}{EG} - \frac{1}{R_1^2}\right)\frac{\partial H}{\partial p_2}p_2 + \frac{\sqrt{E}}{R_1}\frac{1}{\nu}\frac{\partial H}{\partial p_3}p_2 - M\rho\frac{\partial \rho}{\partial u}\left(\frac{\partial H}{\partial p_3}\right)^2 + M\varepsilon\sqrt{E}\,\nu\frac{\partial H}{\partial p_2}\frac{\partial H}{\partial p_3},$$

$$\frac{dp_2}{dt} =$$
$$-\frac{\partial H}{\partial v} + \sqrt{EG}\left(\frac{DD''}{EG} - \frac{1}{R_1^2}\right)\frac{\partial H}{\partial p_1}p_3 + \frac{\sqrt{G}}{R_1}\frac{1}{\mu}\frac{\partial H}{\partial p_3}p_1 - M\rho\frac{\partial \rho}{\partial v}\left(\frac{\partial H}{\partial p_3}\right)^2 + M\varepsilon\sqrt{G}\,\nu\frac{\partial H}{\partial p_1}\frac{\partial H}{\partial p_3},$$

(45)

$$\frac{dp_3}{dt} =$$
$$-\frac{\sqrt{G}}{R_1}\frac{1}{\mu}\frac{\partial H}{\partial p_2}p_1 - \frac{\sqrt{E}}{R_1}\frac{1}{\nu}\frac{\partial H}{\partial p_1}p_2 - M\left(\rho\frac{\partial \rho}{\partial u}\frac{\partial H}{\partial p_1} + \rho\frac{\partial \rho}{\partial v}\frac{\partial H}{\partial p_2}\right)\frac{\partial H}{\partial p_3} - M\varepsilon\left(\sqrt{E}\,\nu + \sqrt{G}\,\mu\right)\frac{\partial H}{\partial p_1}\frac{\partial H}{\partial p_2},$$

$$\frac{du}{dt} = \frac{\partial H}{\partial p_1}, \qquad \frac{dv}{dt} = \frac{\partial H}{\partial p_2},$$

(*) Phil. Trans. 1854.

which determine p_1, p_2, p_3, u, v in terms of t.

Time t can be eliminated from these equations or equations (40) with the use of the *vis viva* integral:

$$H = \Theta - U = h,$$

in which h denotes an arbitrary constant. If we choose, e.g., u to be the independent variable then we will get three first-order differential equations that determine the quantities p_2, p_3, v (v, $\dot{v}$, n, resp.) in terms of u.

In order to obtain a second integral of motion, in addition to the *vis viva*, we make the following special assumptions: The relative velocity $\mathfrak{v}(\dot{x}, \dot{y}, \dot{z})$ of the contact point M at any moment of the motion might lie in the same plane as the velocity w (k, l, m) of the center of mass O of the body. The vector K that is included in (43) will then vanish. When the resultant moment Γ_2 about M of the force that acts upon the body is equal to zero, in addition, from the theorem in § **11**, the resultant moment Γ_1 about M of the quantity of motion of the material point of the body will also be constant. In that special case, we will then have the integral:

$$\frac{1}{\mu^2}\left(\frac{\partial\Theta}{\partial\dot{u}}\right)^2 + \frac{1}{\nu^2}\left(\frac{\partial\Theta}{\partial\dot{v}}\right)^2 + \left(\frac{\partial\Theta}{\partial n}\right)^2 = \text{const.} \tag{46}$$

The conditions that were cited above are obviously fulfilled when the rigid body is either partially or completely bounded by a spherical surface whose center coincide with the center of mass of the body (*).

We shall now return to the general case.

If the equations of motion (40) or (45) have been integrated then we will have to determine the cyclic coordinates ϑ, u_1, v_1. In order to do that, we appeal to the last of formulas (9) and the non-holonomic condition equations (12). For the present case, from (35), those formulas simplify to the following ones:

$$\begin{aligned} R_1\,\dot{u}_1 &= -\sqrt{E}\,\dot{u}\sin\vartheta + \sqrt{G}\,\dot{v}\cos\vartheta, \\ R_1\,\dot{v}_1\ \sin u_1 &= \sqrt{E}\,\dot{u}\cos\vartheta + \sqrt{G}\,\dot{v}\sin\vartheta, \\ \dot{\vartheta} &= -n + \frac{1}{2\sqrt{EG}}\left(\frac{\partial E}{\partial v}\dot{u} - \frac{\partial G}{\partial u}\dot{v}\right) - \dot{v}_1\cos u_1 . \end{aligned} \tag{47}$$

If u, v, n are expressed in terms of time t then, from (36) and (41), the quantities k, …, p, … will be known functions of time, and as a result (**), the determination of all remaining coordinates must come down to the integration of a Riccati equation.

In fact, if we set:

(*) Chaplygin, "On a possible generalization of the surface theorem, with an application to the problem of the rolling of the sphere," Moscow Math. Coll. **20** (1897).

(**) Cf., e.g., Darboux, *Leçons sur la théorie générale des surfaces*, v. I, chap. II.

$$-\frac{\sqrt{E}}{R_1}\dot{u} = f_1 \cos \vartheta_1 , \quad \frac{\sqrt{G}}{R_1}\dot{v} = f_1 \sin \vartheta_1 , \qquad \frac{1}{2\sqrt{EG}}\left(\frac{\partial E}{\partial v}\dot{u} - \frac{\partial G}{\partial u}\dot{v}\right) - n + \dot{\vartheta}_1 = f_2$$

then f_1, f_2, ϑ_1 will be known functions of t. Formulas (47) imply that:

$$\dot{u}_1 = f_1 \sin (\vartheta + \vartheta_1), \quad \dot{v}_1 \sin u_1 = -f_1 \cos (\vartheta + \vartheta_1), \quad \dot{\vartheta} + \dot{\vartheta}_1 = f_2 + f_1 \cot u_1 (\vartheta + \vartheta_1) .$$

If we now introduce new variables ζ_1 and ζ_2, in place of u_1 and ϑ, by way of the formulas:

$$\zeta_1 = -i \cot \frac{u_1}{2} \cdot e^{-i(\vartheta+\vartheta_1)}, \qquad \zeta_2 = i \tan \frac{u_1}{2} \cdot e^{-i(\vartheta+\vartheta_1)},$$

then a brief calculation will show that ζ_1 and ζ_2 are integrals of the Riccati equation:

$$\frac{d\zeta}{dt} = (1 + \zeta^2)\, f_1 (t) - i\, \zeta f_2 (t) .$$

When ζ_1 and ζ_2 (and therefore u_1 and ϑ, as well) are determined in terms of time t, the last coordinate v_1 will be obtained from (47) by quadrature.

§ 14. – Motion of a body of revolution that rolls on a sphere. Reducing the problem to the integration of two Riccati equations. Motion of a cylindrical rod on a sphere.

The equations of motion (40) will simplify significantly when the rigid body is a body of revolution in the dynamical sense, so:

$$A = B,$$

and the outer surface of the body is a surface of revolution around the symmetry axis z.

If we set:

$$x = u \cos v, \qquad y = u \sin v, \qquad z = z (u),$$

and denote the differential quotients of z with respect to u by z', z'', … then we will get:

$$\begin{gathered} E = 1 + z'^2, \quad G = u^2, \qquad \alpha'' = \frac{z'}{\sqrt{1+z'^2}}, \qquad \beta = 0, \qquad \gamma = \frac{1}{\sqrt{1+z'^2}}, \\ D = \frac{z''}{\sqrt{1+z'^2}}, \qquad D'' = \frac{u\, z'}{\sqrt{1+z'^2}}, \qquad \rho^2 = u^2 + z^2, \qquad \varepsilon = \frac{z - u\, z'}{\sqrt{1+z'^2}}, \end{gathered} \tag{48}$$

such that, from (38), we will have:

$$2\Theta = (M\,\rho^2 + A)\ (\nu^2\,\dot{v}^2 + \mu^2\dot{u}^2 + n^2) - M\left(\rho\frac{\partial\rho}{\partial u}\frac{\nu\,\dot{v}}{\sqrt{E}} + \varepsilon n\right)^2 + (C - A)(\nu\,\dot{v}\,\alpha'' + n\gamma'')^2$$

(49)

$$= P\,\dot{u}^2 + Q\,\dot{v}^2 + 2L\,\dot{v}\,n + J\,n^2\,,$$

in which the coefficients P, Q, L, J are functions of the argument u. In addition, we will obviously have:

$$U = \text{funct}\ (u).$$

We then see that in the present case, the coordinate v is also cyclic, so when we eliminate time t from the equations of motion (45) or (40) with the help of the *vis viva* integral, that will yield two first-order differential equations that determine the variables p_2 and p_3 (v and n, resp.) as functions of u.

If the integral (46) exists, or if any other integral of the aforementioned equations is known, then those equations can be solved by quadratures. In fact, it is not difficult to prove that in the present case the form of the last Jacobi multiplier M can be given in advance.

As is known, the multiplier M of the system:

$$\frac{dx_1}{X_1} = \frac{dx_2}{X_2} = \ldots = \frac{dx_n}{X_n}\,, \qquad X_1\,,\,X_2\,,\,\ldots,\,X_n = \text{funct.}\ (x_1\,,\,x_2\,,\,\ldots,\,x_n)$$

satisfies the formula:

$$\frac{d\ln\mathsf{M}}{dx_1} + \frac{1}{X_1}\left(\frac{\partial X_1}{\partial x_1} + \frac{\partial X_2}{\partial x_2} + \cdots + \frac{\partial X_n}{\partial x_n}\right) = 0.$$

If we apply that formula to the system (45) then we will get:

$$\begin{aligned}\frac{d\ln\mathsf{M}}{dt} &+ M\,\rho\frac{\partial\rho}{\partial u}\left(\frac{\partial H}{\partial p_1}\frac{\partial^2 H}{\partial p_3^2} - \frac{\partial H}{\partial p_3}\frac{\partial^2 H}{\partial p_1\,\partial p_3}\right)\\ &+ M\,\rho\frac{\partial\rho}{\partial v}\left(\frac{\partial H}{\partial p_2}\frac{\partial^2 H}{\partial p_3^2} - \frac{\partial H}{\partial p_3}\frac{\partial^2 H}{\partial p_2\,\partial p_3}\right)\\ &+ M\,\varepsilon\sqrt{E}\,\nu\left(\frac{\partial H}{\partial p_3}\frac{\partial^2 H}{\partial p_1\,\partial p_2} - \frac{\partial H}{\partial p_1}\frac{\partial^2 H}{\partial p_2\,\partial p_3}\right)\\ &+ M\,\varepsilon\sqrt{G}\,\mu\left(\frac{\partial H}{\partial p_3}\frac{\partial^2 H}{\partial p_1\,\partial p_2} - \frac{\partial H}{\partial p_2}\frac{\partial^2 H}{\partial p_1\,\partial p_3}\right) = 0.\end{aligned}$$

However, from (48) and (49), ρ is a function of only u, and $\dot{u}$ will be included in Θ, and as a result, p_1 will be included in H only in the second power.

From (45), we then have:

$$\frac{d\ln \mathsf{M}}{dt} + M\left(\rho\frac{\partial\rho}{\partial u}\frac{\partial^2 H}{\partial p_3^2} - \varepsilon\sqrt{E}\,\nu\frac{\partial^2 H}{\partial p_2\,\partial p_3}\right)\frac{du}{dt} = 0,$$

such that:

$$\mathsf{M} = e^{-\int\Phi(u)\,du}, \qquad \Phi(u) = M\left(\rho\frac{\partial\rho}{\partial u}\frac{\partial^2 H}{\partial p_3^2} - \varepsilon\sqrt{E}\,\nu\frac{\partial^2 H}{\partial p_2\,\partial p_3}\right).$$

One then solves, e.g., the problem of the motion of a body of revolution that is partially or completely bounded by a spherical surface and which rolls without slipping on an immobile sphere under the action of forces of the kind that were mentioned at the beginning of the chapter by quadratures (*).

We would now like to actually exhibit the two first-order differential equations by whose integration the variables u, v, n can be obtained by quadratures.

From (48) and (49), the last two equations of motion (40) imply that:

$$\frac{d}{dt}\frac{\partial\Theta}{\partial\dot v} = -\sqrt{EG}\left(\frac{DD''}{EG} - \frac{1}{R_1^2}\right)\frac{\partial\Theta}{\partial n}\dot u + \frac{\sqrt{G}}{R_1}\frac{P}{\mu}n\,\dot u + M\varepsilon\sqrt{G}\,\mu\,n\,\dot u\,,$$

$$\frac{d}{dt}\frac{\partial\Theta}{\partial n} = -\frac{\sqrt{G}}{R_1}\frac{P}{\mu}\dot v\,\dot u - \frac{\sqrt{E}}{R_1}\frac{1}{\nu}\frac{\partial\Theta}{\partial\nu}\dot u + M\rho\frac{\partial\rho}{\partial u}n\,\dot u - M\varepsilon\left(\sqrt{E}\,\nu + \sqrt{G}\,\mu\right)\dot v\,\dot u\,.$$

The equations admit the particular solution:

$$\dot u = 0, \qquad u = u_0\,, \qquad n = n_0\,, \qquad \dot v = \dot v_0\,, \qquad v = \dot v_0\,t + v_0\,,$$

in which u_0, n_0, $\dot v_0$, v_0 denote constants, and the first of the equations of motion (40) gives the condition that those constants must satisfy in order for such a motion of the body to be possible. The determination of the remaining coordinates ϑ, u_1, v_1 by means of formulas (47) presents no difficulties in this case, and we easily convince ourselves that the contact point M will describe a great circle or a minor circle on the outer surface of the sphere according to whether the expression $n_0 + \frac{\dot v_0}{\sqrt{E_0}}$ is equal to zero or non-zero, resp.

If we exclude that particular solution then we will get the equations to be exhibited when we choose u to be the independent variable in them:

$$\frac{d}{dt}\frac{\partial\Theta}{\partial\dot v} = -\sqrt{EG}\left(\frac{DD''}{EG} - \frac{1}{R_1^2}\right)\frac{\partial\Theta}{\partial n} + \frac{\sqrt{G}}{R_1}\frac{P}{\mu}n + M\varepsilon\sqrt{G}\,\mu\,n\,,$$

(*) Cf., the treatise of Chaplygin that was cited above.

$$\frac{d}{dt}\frac{\partial\Theta}{\partial n} = -\frac{\sqrt{G}}{R_1}\frac{P}{\mu}\dot{v} - \frac{\sqrt{E}}{R_1}\frac{1}{\nu}\frac{\partial\Theta}{\partial\nu} + M\rho\frac{\partial\rho}{\partial u}n - M\varepsilon\left(\sqrt{E}\,\nu + \sqrt{G}\,\mu\right)\dot{v}\,.$$

If $\dot{v}$ and n are determined as functions of u by those equations then we will get the *vis viva* integral u as a function of time t by quadrature. A second quadrature will then determine v in terms of t.

The equations that were presented define a system of two first-order linear differential equations. If we eliminate, e.g., n, from them then we will get a second-order linear differential equation that will determine $\dot{v}$:

$$\frac{d^2\dot{v}}{du^2} + \frac{d\dot{v}}{du}f_1(u) + \dot{v}\,f_2(u) = 0.$$

Such an equation will be brought into Riccati form by the substitution:

$$\dot{v} = e^{\int\xi\,du},$$

in which ξ denotes a new variable.

As we saw above, the determination of the cyclic coordinates ϑ, u_1, v_1 will also lead to the integration of a Riccati equation. We might then state the following theorem:

The problem of the motion of a rigid body of revolution that rolls on a sphere under the action of a force that points from the center of mass O of the body to the center O_1 of the sphere and depends upon only the mutual distance between the points O and O_1 can be reduced to the integration of two Riccati equations and quadratures when the motion of the body is rolling without slipping.

Let the rigid body be, e.g., a cylindrical rod:

$$x = R\cos v,\qquad y = R\sin v,\qquad z = u.$$

We then have:

$$E = 1,\qquad G = R^2,\qquad \alpha'' = 1,\qquad \beta'' = 0,\qquad \gamma'' = 0,\qquad \mu'' = -\frac{1}{R_1},$$

$$D = 0,\qquad D'' = R,\qquad \rho^2 = R^2 + u^2,\qquad \varepsilon = -R,\qquad n = \frac{R}{R_1} - 1,$$

such that, from (49), the kinetic energy of the rod will be given by the formula:

$$2\Theta = \frac{1}{R_1^2}(A + MR^2 + Mu^2)\,\dot{u}^2 + (C + MR^2)\left(\frac{R}{R_1} - 1\right)^2\dot{v}^2 + 2MR\left(\frac{R}{R_1} - 1\right)u\,n\,\dot{v} + (A + Mu^2)\,n^2\,.$$

The two linear differential equations that determine $\dot{v}$ and n in terms of u have the form:

$$\frac{d}{du}\left[MR\,u\,n+(C+MR^2)\left(\frac{R}{R_1}-1\right)\dot{v}\right]=M\,\frac{R^2}{R_1^2}(u\,\dot{v}-R_1\,n),$$

$$\frac{d}{du}\left[(A+Mu^2)\,n+MR\left(\frac{R}{R_1}-1\right)u\,\dot{v}\right]$$

$$=\left[\frac{R}{R_1^2}(A+MR^2+Mu^2)-\frac{1}{R_1}\left(\frac{R}{R_1}-1\right)(C+MR^2)-MR\right]\dot{v}-M\left(\frac{R}{R_1}-1\right)un.$$

If we eliminate n from those equations then we will have:

$$\left(A\frac{C+MR^2}{MC}+u^2\right)\frac{d^2\dot{v}}{du^2}+3u\frac{d\dot{v}}{du}+\left[\frac{R^2}{R_1^2}\frac{R+R_1}{R-R_1}\frac{A}{C}-\frac{R}{R_1}\left(\frac{R}{R_1}+2\right)\right]\dot{v}\;=0.$$

We then see that $\dot{v}$ is a hypergeometric function of u.

§ 15. – Equations of motion for a rigid body in whose interior one finds a gyroscope. Motion of a gyroscopic ball that rolls on a sphere.

Most of the results of the foregoing paragraphs can be easily generalized to the case in which the interior of the rigid body (which we would like to think of as hollow) contains a gyroscope that rotates about its axis. We will assume that the center of mass of the gyroscope coincides with the center of mass of the rigid body and that the symmetry axis of the gyroscope coincides with the z-axis.

The friction in the points of the inside of the surface of the body that support the axis of the gyroscope shall be neglected.

If the quantities M, A, and B in the equations of motion (40) mean the mass and the moments of inertia about the x and y axes, resp., of the total system that consists of the rigid body and the gyroscope then we will have to add only one more force-couple (*) to the forces that act upon the body in the case of a gyroscopic body. The moment of that couple is equal to $\bar{C}\omega\varpi\sin(\omega\varpi)$, where $\bar{C}$ denotes the moment of inertia of the gyroscope about the z-axis, ϖ denotes its constant angular velocity, and ω denotes the instantaneous angular velocity of the rigid body. The axes of that moment is normal to the rotational axis of the rigid body and the z-axis and lies in relation to those two axes in the same way that the z-axis lies in relation to the x and y axes.

If we introduce the notation:

(*) The so-called "deviation resistance." F. Klein and A. Sommerfeld, *Über die Theorie des Kreisels*, vol. I, Chap. III.

$$\bar{C}\varpi = \kappa$$

and recall the method that was applied in § **11** in order to present the equations of motion (40) then it will be clear that we have to add the expressions:

$$\mu\kappa(\nu\dot{v}\gamma'' - n\alpha''), \qquad \nu\kappa(n\beta'' - \mu\dot{u}\gamma''), \qquad \kappa(\mu\dot{u}\alpha'' - \nu\dot{v}\beta'')$$

to the right-hand sides of equations (40) in the case of a gyroscopic body.

If we make the same assumptions about the mass distribution of the body and its form that we did in the foregoing paragraphs then, from (48), $\dot{u}$ will once more be included as a common factor in the right-hand sides of the last two equations of motion (40). We will then come once more to two first-order linear differential equations that determine the variables $\dot{v}$ and n as functions of u, except that those linear equations will no longer be homogeneous now. As a result, when we eliminate n from those equations, we will get a second-order differential equation of the form:

$$\frac{d^2\dot{v}}{du^2} + \frac{d\dot{v}}{du} f_1(u) + \dot{v} f_2(u) = f_3(u).$$

If one more integral is known, in addition to the *vis viva* integral, then the equations of motion can be solved by quadratures.

Let the rigid body be, e.g., a hollow sphere in whose interior one finds a gyroscope. The integral (46) will then be valid, and as a result, the variables u, v, n in the problem of the motion of a gyroscopic sphere that rolls on a spherical surface can be determined by quadratures. As was shown in § **13**, the remaining coordinates ϑ, u_1, v_1 will be obtained by integrating a Riccati equation.

The problem that was posed here can be regarded as a generalization of the problem that D. K. Bobylev (*) solved of the motion of a heavy gyroscopic sphere on a horizontal plane. N. E. Joukovski (**) proposed to bring a material ring whose symmetry plane coincided with the *xy*-plane into the interior of a sphere in his own geometric examination of the Bobylev problem. The dimensions of the ring are chosen in such a way that the ellipsoid of inertia of the entire material system is a sphere relative to the center of mass. The motion of the sphere on the horizontal plane can be determined much more simply in that case.

If we also assume that the hollow sphere is provided with such a ring in our general problem then we can prove that the cyclic coordinates ϑ, u_1, v_1 can also be obtained by quadratures. We shall not go further into that problem here, which should probably define the subject of its own treatise. Let us only remark that the quadratures that appear in the problem have elliptic form, such that a complete discussion of the motion of such a sphere on a spherical surface will not encounter any difficulties in its own right.

(*) Bobylev, "On the gyroscopic sphere that rolls without slipping on a horizontal plane," Moscow Math. Coll. 1891.

(**) Joukovski, "On D. K. Bobylev's gyroscopic sphere," Reports of the physical section, 1893.

CHAPTER IV

Differential equations for the motion of a rigid body that rolls on an arbitrary surface.

§ 16. – Exhibiting the equations of motion.

In the foregoing chapter, the equations of motion of a rigid body that rolls without slipping on a sphere were derived from formula (33). In order to apply the other basic formula of Chapter II – namely, formula (18) – the equations of the motion of a rigid body that rolls on an arbitrary surface shall now be developed with the use of that formula.

We shall base that upon the notations and assumptions of §§ **4** and **9** and choose the dependent velocities to be the quantities $\dot{u}_1$ and $\dot{v}_1$, as in (12). We must then determine the generalized impulses K_1 and K_2 that correspond to the velocities $\dot{u}_1$ and $\dot{v}_1$, resp., by using (18).

It we substitute the expressions (11) and (10) for k, l, m ; p, q, r in the kinetic energy T (30) of the body then, from (9), we will get:

$$2T = M(k^2 + l^2 + m^2) + Ap^2 + Bq^2 + Cr^2 = 2T\,(u, v, \vartheta, u_1, v_1, \dot{u}, \dot{v}, \dot{\vartheta}, \dot{u}_1, \dot{v}_1),$$

and as a result, from (11), (10), and (9), we will have:

$$\begin{aligned} K_1 = \frac{\partial T}{\partial \dot{u}_1} &= M\sqrt{E}\,[-(k\alpha + \dots)\sin\vartheta + (k\beta + \dots)\cos\vartheta] \\ &+ \frac{1}{2\sqrt{E_1G_1}}\frac{\partial E_1}{\partial v_1}\left[\frac{\partial T}{\partial k}(y\gamma'' - z\gamma') + \cdots + \frac{\partial T}{\partial k}\gamma + \cdots\right] \\ &- \frac{D_1}{2\sqrt{E_1}}\left\{\left[\frac{\partial T}{\partial k}(y\alpha'' - z\alpha') + \cdots + \frac{\partial T}{\partial p}\alpha + \cdots\right]\cos\vartheta \right. \\ &\qquad \left. + \left[\frac{\partial T}{\partial k}(y\beta'' - z\beta') + \cdots + \frac{\partial T}{\partial p}\beta + \cdots\right]\sin\vartheta\right\}, \end{aligned}$$

and a similar expression for K_2, which is the derivative of T with respect to $\dot{v}_1$.

Should the body roll without slipping on the surface S_1 then k, l, m would have the values (13), and we would get from (10) and (11) that:

$$\begin{aligned} 2T &= M\rho^2(\sigma^2 + \tau^2 + n^2) - M\left(\rho\frac{\partial\rho}{\partial u}\frac{\sigma}{\sqrt{E}} + \rho\frac{\partial\rho}{\partial v}\frac{\tau}{\sqrt{G}} + \varepsilon n\right)^2 \\ &+ A(\sigma\alpha + \tau\beta + n\gamma)^2 + B(\sigma\alpha' + \tau\beta' + n\gamma')^2 + C(\sigma\alpha'' + \tau\beta'' + n\gamma'')^2 \end{aligned} \tag{50}$$

$$= 2\bar{\Theta}(u, v, \sigma, \tau, n),$$

in which we have once more set:

$$x^2 + y^2 + z^2 = \rho^2, \qquad x\,\gamma + y\,\gamma' + z\,\gamma'' = \varepsilon.$$

With the help of this expression $\bar{\Theta}$ for the kinetic energy of the body and formulas (13), which imply that:

$$k\alpha + l\alpha' + m\alpha'' = -\,\varepsilon\tau + \rho\frac{\partial\rho}{\partial v}\frac{n}{\sqrt{G}},$$

$$k\beta + l\beta' + m\beta'' = \quad \varepsilon\sigma - \rho\frac{\partial\rho}{\partial u}\frac{n}{\sqrt{E}},$$

from (1), the quantities K_1 and K_2 can be put into the simpler form:

(51)
$$\begin{aligned} K_1 &= M\sqrt{E_1}\left[\left(\varepsilon\sigma - \rho\frac{\partial\rho}{\partial u}\frac{n}{\sqrt{E}}\right)\cos\vartheta + \left(\varepsilon\tau - \rho\frac{\partial\rho}{\partial v}\frac{n}{\sqrt{G}}\right)\sin\vartheta\right] \\ &\quad + \frac{1}{2\sqrt{E_1 G_1}}\frac{\partial E_1}{\partial v_1}\frac{\partial\bar{\Theta}}{\partial n} - \frac{D_1}{\sqrt{E_1}}\left(\frac{\partial\bar{\Theta}}{\partial\sigma}\cos\vartheta + \frac{\partial\bar{\Theta}}{\partial\tau}\sin\vartheta\right), \\ &\quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}$$

From (12), the coefficients of the quantities K_1 and K_2 in formula (18) have the values:

$$\delta\left(\dot{u}_1 - \frac{-\sqrt{E}\,\dot{u}\sin\vartheta + \sqrt{G}\,\dot{v}\cos\vartheta}{\sqrt{E_1}}\right), \quad \delta\left(\dot{v}_1 - \frac{\sqrt{E}\,\dot{u}\cos\vartheta + \sqrt{G}\,\dot{v}\sin\vartheta}{\sqrt{G_1}}\right),$$

which are calculated with the use of the formulas:

(52)
$$\begin{aligned} \sqrt{E_1}\,\delta u_1 &= -\sqrt{E}\,\delta u\sin\vartheta + \sqrt{G}\,\delta v\cos\vartheta \\ \sqrt{G_1}\,\delta v_1 &= \quad \sqrt{E}\,\delta u\cos\vartheta + \sqrt{G}\,\delta v\sin\vartheta, \end{aligned}$$

as a result of § **7**.

If we once more introduce the quantity:

(53)
$$n' = -\,\delta\vartheta + \frac{1}{2\sqrt{EG}}\left(\frac{\partial E}{\partial v}\delta u - \frac{\partial G}{\partial u}\delta v\right) + \frac{1}{2\sqrt{E_1 G_1}}\left(\frac{\partial E_1}{\partial v_1}\delta u_1 - \frac{\partial G_1}{\partial u_1}\delta v_1\right),$$

along with the quantity n (9), for brevity, then we will easily get the following expressions for the desired coefficients:

$$\frac{1}{\sqrt{E_1}}\left[\sqrt{E}\,(n\,\delta u - \dot u\, n')\cos\vartheta + \sqrt{G}\,(n\,\delta v - \dot v\, n')\sin\vartheta\right],$$

$$\frac{1}{\sqrt{G_1}}\left[\sqrt{E}\,(n\,\delta u - \dot u\, n')\sin\vartheta - \sqrt{G}\,(n\,\delta v - \dot v\, n')\cos\vartheta\right],$$

such that formula (18) (18) will assume the form:

(54) $$\int_{t_1}^{t_2}[\delta\Theta + \delta U + K_1'\sqrt{E}\,(n\,\delta u - \dot u\, n') + K_2'\sqrt{G}\,(n\,\delta v - \dot v\, n')]\,dt = 0$$

for the problem in question, in which we have set:

(55) $$K_1' = \frac{K_1}{\sqrt{E_1}}\cos\vartheta + \frac{K_2}{\sqrt{G_1}}\sin\vartheta, \qquad K_2' = \frac{K_1}{\sqrt{E_1}}\sin\vartheta - \frac{K_2}{\sqrt{G_1}}\cos\vartheta.$$

We assume that a force function U exists. We express the kinetic energy Θ of the body as a second-degree homogeneous function of the independent velocities u, $\dot v$, $\dot\vartheta$. In order to do that, eliminate the dependent velocities $\dot u_1$ and $\dot v_1$ from the functions σ, τ, n (9) with the use of the condition equations (12):

$$\sigma = -\,\Delta'\sqrt{E}\,\dot u - \Delta''\sqrt{G}\,\dot v, \qquad \tau = \Delta\sqrt{E}\,\dot u + \Delta'\sqrt{G}\,\dot v,$$

$$n = -\,\dot\vartheta + \Delta_1\sqrt{E}\,\dot u - \Delta_2\sqrt{G}\,\dot v,$$

(56)

$$\Delta = \frac{D}{E} + \frac{D_1}{E_1}\sin^2\vartheta + \frac{D_1''}{G_1}\cos^2\vartheta, \qquad \Delta' = \left(\frac{D_1''}{G_1} - \frac{D_1}{E_1}\right)\sin\vartheta\cos\vartheta,$$

$$\Delta'' = \frac{D''}{G} + \frac{D_1''}{G_1}\sin^2\vartheta + \frac{D_1}{E_1}\cos^2\vartheta,$$

$$2\Delta_1 = \frac{1}{\sqrt{G}}\frac{\partial\ln E}{\partial v} - \frac{\sin\vartheta}{\sqrt{G_1}}\frac{\partial\ln E_1}{\partial v_1} - \frac{\cos\vartheta}{\sqrt{E_1}}\frac{\partial\ln G_1}{\partial u_1},$$

$$2\Delta_2 = \frac{1}{\sqrt{E}}\frac{\partial\ln G}{\partial u} + \frac{\sin\vartheta}{\sqrt{E_1}}\frac{\partial\ln G_1}{\partial u_1} - \frac{\cos\vartheta}{\sqrt{G_1}}\frac{\partial\ln E_1}{\partial v_1},$$

and substitute the values that are obtained for σ, τ, n in the function $\bar\Theta$ (50). We will then have:

$$\bar\Theta = \Theta\,(u, v, \vartheta, u_1, v_1, \dot u, \dot v, \dot\vartheta).$$

If we now eliminate the quantity n from formula (54) with the help of (53) and the dependent variations δu_1 and δv_1 with the help of (52) and set the coefficients of the independent variations δu, δv, $\delta\vartheta$ equal to zero then we will get the equations for the motion of a rigid body that rolls on an arbitrary surface:

$$\frac{d}{dt}\frac{\partial\Theta}{\partial\dot{u}}-\frac{\partial(\Theta+U)}{\partial u} = \sqrt{E}\left[-\frac{\partial(\Theta+U)}{\partial u_1}\frac{\sin\vartheta}{\sqrt{E_1}}+\frac{\partial(\Theta+U)}{\partial v_1}\frac{\cos\vartheta}{\sqrt{G_1}}-K_1'\dot{\vartheta}\right] - (\Delta_2 K_1'+\Delta_1 K_2')\sqrt{EG}\,\dot{v},$$

(57)

$$\frac{d}{dt}\frac{\partial\Theta}{\partial\dot{v}}-\frac{\partial(\Theta+U)}{\partial v} = \sqrt{G}\left[\frac{\partial(\Theta+U)}{\partial u_1}\frac{\cos\vartheta}{\sqrt{E_1}}+\frac{\partial(\Theta+U)}{\partial v_1}\frac{\sin\vartheta}{\sqrt{G_1}}-K_2'\dot{\vartheta}\right] + (\Delta_2 K_1'+\Delta_1 K_2')\sqrt{EG}\,\dot{u},$$

$$\frac{d}{dt}\frac{\partial\Theta}{\partial\dot{\vartheta}}-\frac{\partial(\Theta+U)}{\partial\vartheta} = K_1'\sqrt{E}\,\dot{u}+K'\sqrt{G}\,\dot{v},$$

in which, from (55) and (51), we have:

$$K_1' = M\left(\varepsilon\sigma-\rho\frac{\partial\rho}{\partial u}\frac{n}{\sqrt{E}}\right)-\left(\Delta''-\frac{D''}{G}\right)\frac{\partial\bar{\Theta}}{\partial\sigma}+\Delta'\frac{\partial\bar{\Theta}}{\partial\tau}-\left(\Delta_2-\frac{1}{2\sqrt{E}}\frac{\partial\ln G}{\partial u}\right)\frac{\partial\bar{\Theta}}{\partial n},$$

(58)

$$K_2' = M\left(\varepsilon\tau-\rho\frac{\partial\rho}{\partial u}\frac{n}{\sqrt{G}}\right)+\Delta'\frac{\partial\bar{\Theta}}{\partial\sigma}-\left(\Delta-\frac{D}{E}\right)\frac{\partial\bar{\Theta}}{\partial\tau}-\left(\Delta_1-\frac{1}{2\sqrt{G}}\frac{\partial\ln E}{\partial v}\right)\frac{\partial\bar{\Theta}}{\partial n}.$$

Since only the expression Θ for the kinetic energy of the body is included in the equations of motion (57), it might be preferable to also express the functions K_1' and K_2' in terms of the derivatives Θ. In order to do that, we have to determine the derivatives of $\bar{\Theta}$ with respect to σ, τ, and n from the equations:

$$\frac{\partial\Theta}{\partial\dot{\vartheta}} = -\frac{\partial\bar{\Theta}}{\partial n}, \qquad \frac{1}{\sqrt{E}}\frac{\partial\Theta}{\partial\dot{u}} = -\Delta'\frac{\partial\bar{\Theta}}{\partial\sigma}+\Delta\frac{\partial\bar{\Theta}}{\partial\tau}+\Delta_1\frac{\partial\bar{\Theta}}{\partial n},$$

$$\frac{1}{\sqrt{G}}\frac{\partial\Theta}{\partial\dot{v}} = -\Delta''\frac{\partial\bar{\Theta}}{\partial\sigma}+\Delta'\frac{\partial\bar{\Theta}}{\partial\tau}-\Delta_2\frac{\partial\bar{\Theta}}{\partial n},$$

which are obvious, from (56), and substitute them in (58):

$$K_1' = M\left(\varepsilon\sigma-\rho\frac{\partial\rho}{\partial u}\frac{n}{\sqrt{G}}\right)+\frac{1}{\sqrt{G}}\frac{\partial\Theta}{\partial\dot{v}}+\frac{1}{R}\frac{D''}{G}\left(\frac{\Delta'}{\sqrt{E}}\frac{\partial\Theta}{\partial\dot{u}}-\frac{\Delta}{\sqrt{G}}\frac{\partial\Theta}{\partial\dot{v}}\right)$$

$$+\left(\frac{D''}{G}\frac{\Delta'\Delta_1+\Delta\Delta_2}{R}-\frac{1}{2\sqrt{E}}\frac{\partial \ln G}{\partial u}\right)\frac{\partial\Theta}{\partial\dot{\vartheta}},$$

$$K_2' = M\left(\varepsilon\sigma-\rho\frac{\partial\rho}{\partial v}\frac{n}{\sqrt{G}}\right)-\frac{1}{\sqrt{E}}\frac{\partial\Theta}{\partial\dot{u}}+\frac{1}{R}\frac{D}{E}\left(\frac{\Delta''}{\sqrt{E}}\frac{\partial\Theta}{\partial\dot{u}}-\frac{\Delta'}{\sqrt{G}}\frac{\partial\Theta}{\partial\dot{v}}\right)$$
$$+\left(\frac{D}{E}\frac{\Delta''\Delta_1+\Delta'\Delta_2}{R}-\frac{1}{2\sqrt{E}}\frac{\partial \ln E}{\partial v}\right)\frac{\partial\Theta}{\partial\dot{\vartheta}},$$

in which:

$$R = \Delta\Delta''-\Delta'^{\,2} = \frac{DD''}{EG}+\frac{D_1D_1''}{E_1G_1}+\left(\frac{D}{E}\frac{D_1}{E_1}+\frac{D''}{G}\frac{D_1''}{G_1}\right)\cos^2\vartheta+\left(\frac{D}{E}\frac{D_1''}{G_1}+\frac{D''}{G}\frac{D_1}{E_1}\right)\sin^2\vartheta$$

is assumed to be non-zero.

The equations of motion (57) *and the non-holonomic condition equations* (12):

(59)
$$\sqrt{E_1}\,\dot{u}_1 = -\sqrt{E}\,\dot{u}\sin\vartheta+\sqrt{G}\,\dot{v}\cos\vartheta,$$
$$\sqrt{G_1}\,\dot{v}_1 = \sqrt{E}\,\dot{u}\cos\vartheta+\sqrt{G}\,\dot{v}\sin\vartheta$$

define a system of five differential equations that determine the coordinates u, v, ϑ, u_1, v_1 of the rigid body as functions of time t.

If the force function U does not include the time t explicitly then equations (57) will admit the *vis viva* integral:

$$\Theta = U + h,$$

in which h denotes an arbitrary constant.

If we eliminate time t from the equations of motion (57) and the conditions (59) with the use of that integral (in which we choose, e.g., the coordinate ϑ to be the independent variable) then we will get four differential equations for the determination of u, v, u_1, v_1 in terms of ϑ that will be of order two relative to u and v and of order one relative to u_1, v_1. If those equations are integrated then the *vis viva* integral will yield the time t as a function of ϑ by quadratures.

§ 17. – The equations of motion in special cases. Particular solutions of the equations of motion.

If the surface S_1 upon which the body rolls without slipping is a sphere [formula (35)] and the force function U depends upon only the variables u and v then the coordinates ϑ,

u_1, v_1 will be cyclic when we introduce the quantity n (9) into the equations of motion (57) in place of $\dot{\vartheta}$, and equations (57) will coincide with formulas (40).

Another simpler case will present itself when the rigid body is a body of revolution:

$$A = B, \qquad x = u \cos v, \quad y = u \sin v, \quad z = \text{funct.}\,(u)$$

and the surface S_1 is a surface of revolution:

$$x_1 = u_1 \cos v_1\,, \qquad y_1 = u_1 \sin v_1\,, \qquad z_1 = \text{funct.}\,(u_1)\,.$$

If the force function U includes only the variables u, ϑ, u_1, in addition, then the coordinates v and v_1 will be cyclic. That condition for the force function will be fulfilled, e.g., when the applied forces have a resultant that acts at the center of mass O, is parallel to the z_1-axis, and has a constant magnitude, or when that resultant points from the center of mass O to a point O_1 on the symmetry axis of the surface S_1 and depends upon only the distance $\overline{OO_1}$.

In that case, it will be preferable to introduce the velocity $\dot{u}_1$ into the equations of motion (57), in place of $\dot{v}$, with the help of the first of the condition equations (59). If we then eliminate t from the equations that arise in that way by making use of the *vis viva* integral:

$$\Theta\,(u,\, \vartheta,\, u_1,\, \dot{u}\,,\, \dot{\vartheta},\, \dot{u}_1) = U\,(u,\, \vartheta,\, u_1) + \text{const.}$$

then we will get two second-order differential equations that determine two of the coordinates u, ϑ, u_1 as a function of the third one. If those equations are integrated then we can get the cyclic coordinates v and v_1 and time t by quadratures.

We would not like to go further into the development of those equations, which is very complicated, and present only two particular solutions of the equations of motion (57) for the special case that we speak of.

If we set:

$$\dot{u} = 0, \qquad u = u_0\,, \qquad \dot{\vartheta} = 0, \qquad \vartheta = \frac{\pi}{2}, \qquad \dot{v} = \dot{v}_0\,, \qquad v = \dot{v}_0\, t + v_0\,,$$

$$\dot{u}_1 = 0, \qquad u_1 = u_{10}\,, \qquad \dot{v}_1 = \dot{v}_{10}\,, \qquad v_1 = \dot{v}_{10}\, t + v_{10}\,,$$

in which u_0, $\dot{v}_0$, … denote constants, then the first of the condition equations (59) and the last two of the equations of motion (57) will be fulfilled when:

$$\frac{\partial U}{\partial \vartheta} = 0 \qquad \left(\vartheta = \frac{\pi}{2}\right). \tag{60}$$

In the examples of the applied forces that were cited above, as was easy to see, U included the angle ϑ only in the form $\sin \vartheta$, such that U actually satisfied condition (60).

The remaining formulas (57) and (59) imply two relations:

$$\sqrt{G_1}\,\dot{v}_1 = \sqrt{G}\,\dot{v}\,, \qquad \frac{1}{\sqrt{E}}\frac{\partial(\Theta+U)}{\partial u} - \frac{1}{\sqrt{E_1}}\frac{\partial(\Theta+U)}{\partial u_1} = \Delta_2\,K_1'\sqrt{G}\,\dot{v}$$

that the constants u_0, $\dot{v}_0$, … must fulfill in order for the motion of the body that is thus defined to be possible.

As a result of the formulas that were presented, the contact point M will describe parallel circles with constant velocity on the outer surface of the body and the surface S_1 . The motion will be stationary.

The other particular solution will be given by the formulas:

$$\dot{v} = 0, \quad v = v_0\,, \quad \dot{\vartheta} = 0, \quad \vartheta = \frac{\pi}{2}, \quad \dot{v}_1 = 0, \quad u = u_0\,, \quad v_1 = v_{10}\,,$$

in which v_0 and v_{10} mean arbitrary constants.

When U satisfies the condition (60), the last two of the equations of motion (57) and the second of the condition equations (59) will be fulfilled. The remaining two formulas (57) and (59) serve to determine the coordinates u and u_1 in terms of time t. If we replace formula (57) with the *vis viva* integral in that way then we will get the two equations:

$$\sqrt{E_1}\,\dot{u}_1 = \sqrt{E}\,\dot{u}\,, \qquad (M\,\rho^2 + A)\;\tau^2 = 2U + \text{const.},$$

from which the variables u and u_1 can be determined by quadratures.

The contact point M describes meridians on the surfaces S and S_1 .

In order to get a particular solution of equations (57) when none of the coordinates of the body is cyclic, we consider the motion of a rigid body that is bounded by the outer surface of an ellipsoid S:

$$x = a\frac{\sqrt{a^2-u}\,\sqrt{a^2-v}}{\sqrt{a^2-b^2}\,\sqrt{a^2-c^2}}\,, \qquad \dots \qquad (a^2 \le u \le b^2 \le v \le c^2)$$

that rolls on an immobile ellipsoid S with the same semi-axes:

$$x_1 = a\frac{\sqrt{a^2-u_1}\,\sqrt{a^2-v_1}}{\sqrt{a^2-b^2}\,\sqrt{a^2-c^2}}\,, \qquad \dots \qquad (a^2 \le u_1 \le b^2 \le v_1 \le c^2)$$

under the action of a force that points from the center of mass O of the body to the center O_1 of the immobile ellipsoid and depends upon only the distance $\overline{OO_1}$ between the two points O and O_1 . The force function U is a function of $\overline{OO_1}$ then, in which:

$$\overline{OO_1}^{\,2} = 2\,(a^2 + b^2 + c^2) - u - v - u_1 - v_1$$

$$+\ 2\frac{a^2b^2c^2}{\sqrt{uv}\sqrt{u_1v_1}}+\frac{1}{2\sqrt{E}}\left(\frac{\sin\vartheta}{\sqrt{E_1}}-\frac{\cos\vartheta}{\sqrt{G_1}}\right)-\frac{1}{2\sqrt{G}}\left(\frac{\cos\vartheta}{\sqrt{E_1}}-\frac{\sin\vartheta}{\sqrt{G_1}}\right).$$

If we assume that the mass distribution in the body is such that the moments of inertia A and C about the largest and smallest of the axes of the ellipsoid S, resp., are equal to each other, such that, from (50):

$$2\bar{\Theta} = (M\rho^2 + A)(\sigma^2 + \tau^2 + n^2) - M\left(\rho\frac{\partial\rho}{\partial u}\frac{\sigma}{\sqrt{E}}+\rho\frac{\partial\rho}{\partial v}\frac{\tau}{\sqrt{G}}+\varepsilon n\right)^2$$
$$+ (B - A)(\sigma\alpha' + \tau\beta' + n\gamma')^2,$$

then it will be clear that a motion of the body on the ellipsoid S_1 must be possible for which the contact point M describes central circular segments on the surfaces S and S_1 :

$$u_1 = u, \qquad v_1 = v, \qquad u + v = a^2 + c^2.$$

Since the projections of the instantaneous angular velocity ω of the body onto the middle axis of the ellipsoid S and the direction OM will be equal to zero for that motion, we can obviously start by replacing the simpler form of $\bar{\Theta}$:

$$2\bar{\Theta} = (M\rho^2 + A)(\sigma^2 + \tau^2 + n^2)$$

in (57). It will not be difficult then to convince ourselves that equations (57) and (59) actually admit the cited particular solution, in which the angle ϑ and the time t are given by the formulas:

$$\tan\vartheta = \frac{1}{2}\frac{b^2(a^2+c^2)-2uv}{\sqrt{uv}\sqrt{u-b^2}\sqrt{b^2-v}}, \qquad t = \frac{1}{\omega_0}\int\frac{(u-v)\,du}{\sqrt{b^2-v}\sqrt{u-b^2}\sqrt{a^2-u}\sqrt{u-c^2}} + \text{const.}$$

$$v = a^2 + c^2 - u,$$

in which ω_0 denotes the constant angular velocity of the body.

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