"Ueber die Principe von Hamilton and Maupertuis," Göttinger Nachr. (1900), 322-327.

On the principles of Hamilton and Maupertuis

By

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The meaning of the *principle of least action*, as **Lagrange** expressed it, is known to have remained doubtful for some time, although **Ostrogradsky** had already exhibited the general variational principle for the isoperimetric problems. **Hölder** deserves the credit for having posed the question with complete clarity, and its renewed treatment of gave rise to the investigations of **Hertz** and **Helmholtz**. If I come back to the work of **Hölder** ([1]) here, who thoroughly explained *the connection between the principles of least action* and *Hamilton's principle* and the nature of variational considerations that one calls upon in so doing, then it is not to add anything essential to main question itself. However, the application of the line of reasoning to the case of *completely-general coordinates* requires some arguments that are not found in his article, in which only coordinates that were *explicitly independent* of time were employed. Following through on that would perhaps be nontrivial. To that end, I will first reproduce **Hölder**'s line of reasoning in a somewhat-different form ([2]) and then explain it based upon completely-general coordinates.

§ 1. - Hölder's formulation of the principles of Hamilton and Maupertuis.

Let a problem in mechanics be given, in which the coordinates x, y, z are subject to conditions that are *independent of time*, some of which can also possess the form of *linear differential equations*. Furthermore, T is the *vis viva*, and:

$$\delta' U = \sum X_h \, \delta x_h + Y_h \, \delta y_h + Z_h \, \delta z_h$$

is the virtual work done by the active forces.

([1]) **A. Hölder**, "Ueber die Principien von Hamilton und Maupertuis," Gött. Nachr., Heft 2 (1896), 1-36.
([2]) The application of the coordinates q was already suggested by **Hölder**, moreover; *loc. cit.*, pp. 14, Rem.

One can then fulfill the finite condition equations by means of the coordinates q_i , $i = 1, 2, \ldots, n$, which will make T become a *homogeneous* function of degree two in the q_i' , and $\delta' U$ will go to:

(1) $$\delta' U = \sum_i Q_i \, \delta q_i ,$$

while the further differential equations:

(2) $$\sum_i p_{ki} \, dq_i = 0 , \qquad k = 1, 2, \ldots, r ; \; i = 1, 2, \ldots, n$$

exist between the q_i .

If one assigns variations to t, q_i such that the displacements P' of the system of points P correspond to the values $t + \delta t$, $q_i + \delta q_i$, so δq_i , δt are the variations at the time t when the original coordinates take on arbitrary *virtual* displacements δx_h , δy_h , δz_h then from known methods ([1]):

$$\delta T = \sum_i \frac{\partial T}{\partial q_i} \delta q_i + \sum \frac{\delta T}{\delta q_i'} \left(\frac{\delta \, dq_i}{dt} - q' \frac{\delta \, dt}{dt} \right) ,$$

so

$$dt \; \delta T + 2 \; T \; \delta \, dt = dt \sum_i \frac{\partial T}{\partial q_i} \delta q_i + \sum \frac{\partial T}{\partial q_i'} \, \delta \, dq_i \, dt + \left(2T - \sum q_i' \frac{\partial T}{\partial q_i'} \right) \delta \, dt ,$$

or in the event that the last term can be dropped by using the theorem on homogeneous functions and the second one is transformed by partial integration under the integration over the time interval $t_1 - t_0$:

$$\int_{t_0}^{t_1} dt \, \delta T + 2T \, \delta \, dt = \left| \sum \frac{\partial T}{\partial q_i'} \delta q_i \right|_{t_0}^{t_1} + \int_{t_0}^{t_1} \sum \left(\frac{\partial T}{\partial q_i} - \frac{d}{dt} \frac{\partial T}{\partial q_i'} \right) \delta q_i \, dt ,$$

i.e., when one adds:

$$\int_{t_0}^{t_1} \delta' U \, dt$$

to both sides and drops the terms on the right that are free of the ∫ sign, from (1), one will have:

$$\int_{t_0}^{t_1} dt \, \delta T + 2T \, \delta \, dt + \delta' U \, dt = \int_{t_0}^{t_1} \sum \left(\frac{\partial T}{\partial q_i} + Q_i - \frac{d}{dt} \frac{\partial T}{\partial q_i'} \right) dt \, \delta q_i .$$

If time is left unvaried now then it will follow that:

([1]) Cf., **Hölder**, pp. 9.

$$\int_{t_0}^{t_1} (\delta T + \delta' U)\, dt = \int_{t_0}^{t_1} \sum \left(\frac{\partial T}{\partial q_i} + Q_i - \frac{d}{dt} \frac{\partial T}{\partial q_i'} \right) \delta q_i \, dt ,$$

and the demand that **Hamilton**'s integral must vanish will produce the differential equations of mechanics:

$$\frac{\partial T}{\partial q_i} + Q_i - \frac{d}{dt} \frac{\partial T}{\partial q_i'} = \sum_{s=0}^{r} \lambda_s \, p_{si} ,$$

and conversely.

However, if time is varied then, by contrast, the **Hölder** condition:

$$\delta T - \delta' U = 0$$

for the behavior of *energy* must be introduced into the conditions (2), which will then yield:

$$\delta \int_{t_0}^{t_1} 2T \, dt = \int_{t_0}^{t_1} \sum \left(\frac{\partial T}{\partial q_i} + Q_i - \frac{d}{dt} \frac{\partial T}{\partial q_i'} \right) \delta q_i \, dt ,$$

such that *the extended principle of least action* ([1]) *is also completely equivalent to* the differential equations of mechanics.

§ 2. – The case of general coordinates.

Now, let the finite condition equations also depend upon time *explicitly*. When one frees them of the coordinates q_i, $i = 1, 2, \ldots, n$, so one sets the x_h, y_h, z_h equal to functions of the t, q_i now, the differential equations of condition will also have the form:

(1) $$\sum p_{ks} \, dq_s + p_k \, dt = 0 \qquad k = 1, \ldots, \nu,$$

in which the p_{ks}, p_k are now functions of q_i, t, and even when they do not include t explicitly as a differential or an argument, such that this general case, which I already considered in 1884 ([2]), is the one that will be postulated here. At the same time, T will become a function of degree two in the q_i' ([3]).

If one now assigns variations to the x_h, y_h, z_h , in such a way that one varies q_i, t by δq_i, δt, resp., then the x_h will change by:

$$\delta x_h = \sum \frac{\partial x_h}{\partial q_i} \delta q_i + \frac{\partial x_h}{\partial t} \delta t ,$$

([1]) Cf., **Hölder**, pp. 11.
([2]) **A. Voss**, "Ueber die Differentialgleichungen der Mechanik," Math. Ann. **25**, pp. 258.
([3]) This special form of T is entirely irrelevant in what follows.

while one has

$$\frac{dx_h}{dt} = \sum \frac{\partial x_h}{\partial q_i} q_i' + \frac{\partial x_h}{\partial t} .$$

However, in the application of **d'Alembert**'s principle, one merely applies *virtual* displacements $\delta' x_h$, $\delta' y_h$, $\delta' z_h$. Upon considering the equation:

$$\delta x_h - x_h' \, \delta t = \sum \frac{\partial x_h}{\partial q_i} (\delta q_i - q_i' \, \delta t) \;,$$

one now sees from that consideration ([1]) that the virtual displacements δ' will correspond to the values:

(2) $$\delta' q_i = \delta q_i - q_i' \, \delta t \,,$$

such that

$$\delta' x_h = \sum \frac{\partial x_h}{\partial q_i} \, \delta' q_i \;.$$

It then follows from equation (2) that when one increases t by dt and q_i by dq_i , one will have:

(3) $$\delta' dq_i = \delta \, dq_i - q_i' \, \delta \, dt - q_i'' \delta t \, dt \,.$$

One will now get, as before:

$$\int_{t_0}^{t_1} dt \, \delta T + 2T \, \delta \, dt$$

$$= \left| \sum \frac{\partial T}{\partial q_i'} \, \delta q_i \right|_{t_0}^{t_1} + \int_{t_0}^{t_1} dt \, \frac{\partial T}{\partial t} \delta t + \int_{t_0}^{t_1} \sum \left(\frac{\partial T}{\partial q_i} - \frac{d}{dt} \frac{\partial T}{\partial q_i'} \right) \delta t \, dt + \int_{t_0}^{t_1} \left(2T - \sum q_i' \, \frac{\partial T}{\partial q_i'} \right) \, \delta \, dt \,,$$

or when the last integral is dealt with by partial integration:

$$= \left| \sum 2T \, dt + \sum \frac{\partial T}{\partial q_i'} \right|_{t_0}^{t_1} + \int_{t_0}^{t_1} \left[\frac{\partial T}{\partial t} \, \delta t + \sum \left(\frac{\partial T}{\partial q_i} - \frac{d}{dt} \frac{\partial T}{\partial q_i'} \right) \delta q_i - \frac{d}{dt} \left(2T - \sum q_i' \, \frac{\partial T}{\partial q_i'} \right) \delta t \right] dt \;.$$

If one now considers that:

$$\frac{d}{dt} \left(2T - \sum q_i' \, \frac{\partial T}{\partial q_i'} \right) = 2 \frac{\partial T}{\partial t} + 2 \sum \frac{\partial T}{\partial q_i} \, q_i' + \sum \frac{\partial T}{\partial q_i'} \, q_i'' - \sum q_i' \, \frac{d}{dt} \left(\frac{\partial T}{\partial q_i'} \right)$$

([1]) Cf., a remark in regard to that by **J. Routh**, *Dynamik*, v. II, pp. 329.

then when one adds the integral:

$$\int \delta' U\, dt = \int \sum Q_i\, \delta' q_i\, dt$$

to both sides, the part of the right-hand side that is free of the integral will drop out when one suitably disposes of the variations at the beginning and end of time, and that will produce the equation:

$$\int dt\,\delta T + 2T\,\delta\, dt + dT\,\delta t + \delta' U\, dt = \int \sum \left(\frac{\partial T}{\partial q_i} - \frac{d}{dt}\frac{\partial T}{\partial q_i'} \right) \delta' q_i\, dt \,. \tag{4}$$

It will then follow from (1) that:

$$\sum p_{ki}\, q_i' + p_k = 0 \,,$$
$$\sum p_{ki}\, \delta q_i + p_k\, \delta t = 0 \,,$$

or from (2):

$$\sum p_{ki}\, \delta' q_i = 0$$

is the expression for the conditions that the *virtual* displacements are subjected to.

Now, if δt is set equal to zero, then the left-hand side will once more imply that **Hamilton**'s integral condition is equivalent to the equations:

$$\frac{\partial T}{\partial q_i} - \frac{d}{dt}\frac{\partial T}{\partial q_i'} + Q_i = \sum_{k=0}^{r} \lambda_k\, p_{ki} \,.$$

On the other hand, if one imposes the relation that the *energy* is subject to for the varied motion in the form:

$$\delta' U\, dt + dt\,\delta t = \delta T\, dt \tag{5}$$

then the *extended principle of least work will follow*.

One has:

$$\delta t\; dT = \delta t \left[\frac{\partial T}{\partial t} dt + \sum \frac{\partial T}{\partial q_i}\, q_i'\, dt + \sum \frac{\partial T}{\partial q_i'}\, q_i''\, dt \right] ,$$

$$dt\;\delta T = dt \left[\frac{\partial T}{\partial t} \delta t + \sum \frac{\partial T}{\partial q_i}\, \delta q_i + \sum \frac{\partial T}{\partial q_i'}\, \frac{\delta\, dq_i - q_i'\,\delta\, dt}{dt} \right] ,$$

or from (5) and (3):

$$\delta' U = \sum \left(\frac{\partial T}{\partial q_i}\, \delta' q_i + \frac{\partial T}{\partial q_i'}\, \delta' q_i' \right) ,$$

with suitable notations, such that the *total energy* will not be changed by the *virtual* displacements here either since the last equation can also be written in the form:

$$\delta'(T - U) = 0 .$$

However, the investigation in question in also completely resolved in the most general case with that.

Würzburg, in July 1900.