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Remarks on the principles of mechanics

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I. – On the energetic foundation of mechanics.

From the extraordinary importance of the energy principle in all questions of physical mechanics, it is no wonder that one might seek to also derive it from the foundations of theoretical mechanics themselves. In all of those attempts, one deals with the problem of arriving at **d'Alembert**'s principle, or any form of the equations of motion that is equivalent to it, from the energy principle.

For the conception of mechanics that knows of only conservative forces that depend upon the coordinates of points, but are completely devoid of *conditions*, as, e.g., **Boussinesq** ([1]) developed in his lectures, that poses no difficulty. By differentiating the equation:

$$E = T + V = C$$

with respect to time t, in which V is the potential energy, which depends upon only the coordinates x, y, z, and T is the kinetic energy, one will get:

$$\sum m\,(x'\,x'' + y'\,y'' + z'\,z'') + \sum\left(\frac{\partial V}{\partial x}x' + \frac{\partial V}{\partial y}y' + \frac{\partial V}{\partial z}z'\right) = 0. \tag{1}$$

If it were now assumed that the accelerations, multiplied by the masses, are completely independent of the velocities and the constant C then it would follow from (1) that:

$$m\,x_i'' + \frac{\partial V}{\partial x_i} = 0, \qquad m\,y_i'' + \frac{\partial V}{\partial y_i} = 0, \qquad m\,z_i'' + \frac{\partial V}{\partial z_i} = 0.$$

However, the conclusion can no longer be applied when conditions between the coordinates are assumed, since in that case, the $m\ x''$, etc., actually depend upon the

([1]) **J. Boussinesq**, "Recherches sur les principes de la mécanique," J. de Math. (2) **18** (1873), pp. 315; *Leçons synthétiques de mécanique générale*, Paris, 1889, pp. 23.

velocities (1). **Helm** (2) then sought the assistance of variational procedures and gave the basic principle of energetics the form: *The variation of the energy $E = T + V$ is equal to zero in any possible direction*. However, at the same time, one must demand that the concept of variation is introduced *in a consistent way into both types* of energy. Now, if E were varied in a direction then one would have to replace x, y, z with the quantities $x + \varepsilon\,\xi$, $y + \varepsilon\,\eta$, $z + \varepsilon\,\zeta$, in which ξ, η, ζ are arbitrary functions of t and ε is a constant that converges to zero. One understands the variation δA of an expression A to mean the coefficient of ε in the development of A in powers of ε.

In fact, one then has:

$$\delta V = \sum \frac{\partial V}{\partial x}\xi + \frac{\partial V}{\partial y}\eta + \frac{\partial V}{\partial z}\zeta\,,$$

but one finds the following value for δT:

$$\delta T = \frac{d}{dt}\sum m\,(x'\xi + y'\eta + z'\zeta) - \sum m\,(x''\xi + y''\eta + z''\zeta)\,,$$

and that expression is in no way equal to:

$$\sum m\,(x''\xi + y''\eta + z''\zeta)\,,$$

which would be necessary if one were to assert the identity of this principle with that of **d'Alembert**. Since the discussion between **Boltzmann** and **Helm** on the derivation of the equations of motion has not led to any completely-conclusive result (3), it would nonetheless not be superfluous to summarize those simple relationships, and all the more so since **Helm** emphasized his viewpoint with particular vigor in his *Energetik*, and it has also been assumed by others since then (4).

I do not believe that I should go into the principle of the *superposition* of energy that was expressed by **Planck** (5) and **Boltzmann** with a similar purpose. In fact, it is nothing but an arbitrarily-chosen representation that forces the identity with **d'Alembert**'s principle. By contrast, **Schütz** (6) presented a principle of *absolute conservation of energy* in order to avoid **Helm**'s variational process. It will generally achieve the desired

(1) See the remark of **R. Lipschitz** on **Helmholtz**'s conservation of force, **Ostwald**'s Klassiker-Bibliothek, no. 1, pp. 55, and likewise, **L. Boltzmann**, "Ein Wort der Mathematik an die Energetik," Wiedem. Ann. **57** (1896), pp. 39.

(2) Namely, cf., **G. Helm**, *Die Energetik in ihrer geschichtlichen Entwicklung*, Leipzig, 1898, pp. 220, *et seq.*

(3) Cf., **G. Helm**, "Zur Energetik," Wiedemann's Ann. **57**, pp. 646; **L. Boltzmann**, *ibid.* **58** (1896), pp. 595.

(4) Cf., **P. Gruner**, "Die neueren Ansichten über Materie und Energie," Mitt. d. naturforsch. Ges. zu Bern, 1897.

(5) **M. Planck**, "Das Prinzip der Erhaltung der Energie," Leipzig, 1887, pp. 148; **L. Boltzmann**, Wied. Ann. **57**, pp. 39 *et seq.* Cf., also the note by **C. Neumann** in **Helm**'s *Energetik*, pp. 229.

(6) **J. Schütz**, "Das Princip der absoluten Erhaltung der Energie," Gött. Nachr. (1897), pp. 110. I do not quite understand a derivation of the equations of motion from the law of energy that **E. Padova** carried out ["Sulle equazioni della dinamica," Atti Ist. Veneto (7) **5** (1893), pp. 1641], due to the assumptions that were made in it.

purpose for *one* material point, but it does not admit an extension to a system and might not be compatible, in and of itself, with the representation of the relativity of all states of motion, either.

However, one can avoid the incorrectness that was just pointed out by a *more general* variational process. Namely, if one also varies *time*, along with the coordinates x, y, z such that x, y, z, t go to $x + \varepsilon\,\xi$, $y + \varepsilon\,\eta$, $z + + \varepsilon\,\zeta$, $t + + \varepsilon\,\tau$, in which ξ, η, ζ, τ are arbitrary functions of t then x' will go to:

$$\frac{x' + \varepsilon\,\xi'}{1 + \varepsilon\,\tau'} = x' + \varepsilon\,(x' - \tau' x') + \ldots,$$

and that will imply that:

$$\delta(V + T) = \sum\left(\frac{\partial V}{\partial x} + m\,x''\right)\xi + \left(\frac{\partial V}{\partial y} + m\,y''\right)\eta + \left(\frac{\partial V}{\partial z} + m\,z''\right)\zeta$$

$$+ \frac{d}{dt}\sum m\,(x'\,\xi + y'\,\eta + z'\,\zeta) - 2\sum m\,(\xi\,x'' + \eta\,y'' + \zeta\,z'') - 2\tau' T\,.$$

One is now free to choose τ' in such a way that the right-hand side reduces to **d'Alembert**'s formula, and that is *always possible*, since T does not vanish. The desired result will be achieved *in that way*. However, one can hardly see anything but an abstract formalism in such an arbitrary representation. Since one also has that **Ostwald**'s principle of the maximum of energy exchange can be used only for the case of relative rest, but in general it must be replaced with an entirely different consideration ([1]), it would seem that the attempts that have been made *up to now* do not suggest the possibility of an unforced derivation of **d'Alembert**'s principle, or that of **Gauss**, from the law of energy.

II. – On Hamilton's principle.

It was proved in no. **1** that one can give rise to *any arbitrary relation* for the varied quantities by a suitably-generalized variational process. **Hölder** ([2]) employed such general variations in order to prove that the principles of **Hamilton** and **Maupertuis** are completely equivalent to **d'Alembert**'s principle. However, that viewpoint can be expressed in a much more general form by the following theorem:

Under the assumption of a suitable variational process, the variation of the integral:

$$J = \int_{t_0}^{t_1} (\alpha T + \beta U)\,dt\,,$$

([1]) Cf., **A. Voss**, "Ueber ein energetisches Grundsetz der Mechanik," these Situngsber. (1901), pp. 53.
([2]) **O. Hölder**, "Ueber die Principien von Hamilton und **Maupertuis**," Gött. Nachrichten (1896), issue 2.

in which α, β are two generally completely-arbitrary constants, will be equal to zero because of the differential equations of motion, and conversely, the requirement that δJ should vanish for all allowable displacements will lead to the differential equations of motion ([1]).

Ordinarily, one adds the condition that the variations of the coordinates x, y, z should *vanish* at the limits of the integral. That can generally be in the best interests of a mechanical interpretation, but in itself that further condition is generally superfluous and *inessential*.

Therefore, one might next understand δU to mean the virtual work done by the forces x, y, z under the displacement that corresponds to ξ, η, ζ, so one sets:

$$\delta U = \sum (X\,\xi + Y\,\eta + Z\,\zeta)\,.$$

Now, in order to vary the integral ([2]):

$$I' = \int_{t_0}^{t_1} F(x, x', t)\,dt\,,$$

one can, by the substitution ([3]):

$$t = k\,u + k_0\,,$$

where

$$k = \frac{t_1 - t_0}{1 - t_0}\,, \qquad k_0 = \frac{1 - t_1}{1 - t_0}\,,$$

reduce that to the integral between *constant* limits 0 and 1:

$$I' = \int_0^1 F\left(x, \frac{1}{k}\frac{dx}{du}, ku + k_0\right) k\,dt\,.$$

If one then lets x, y, z, u go to $x + \varepsilon\,\xi$, $y + \varepsilon\,\eta$, $z + \varepsilon\,\zeta$, $u + \varepsilon\,v$, then $k\,v$ will be the arbitrary function that was denoted by τ in no. **1**. At the same time, $\frac{1}{k}\frac{dx}{du}$ will go to ([4]):

$$x' + \varepsilon\,\frac{[(\xi') - (x')(v')]}{k} + \ldots$$

([1]) Obviously, one can also substitute *any arbitrary function* of x, y, z; x', y', z' for the function under the integral sign. However, the linear function of U and T will lead to the forms that are essential from the mechanical standpoint.

([2]) For the sake of brevity, all differential quotients with respect to t are denoted with a prime, such that $x' = \frac{dx}{dt}$, $x'' = \frac{d^2x}{dt^2}$.

([3]) If $t_0 = 1$ then one switches t_1 with t_0 or sets:

$$t = u\,(1 - t_1) + t_1\,.$$

([4]) The symbols x', ξ', v' in brackets mean the differential quotients with respect to u here.

One will then get:

$$\delta I' = \int_0^1 \left\{ \frac{\partial F}{\partial x}\xi + \frac{\partial F}{\partial x'}\left[\frac{(\xi') - (x')(v')}{k}\right] + \frac{\partial F}{\partial t} kv + F(v') \right\} k\,du\,,$$

which, by means of the identities:

$$\frac{(\xi')}{k} = \frac{1}{k}\frac{d\xi}{du} = \frac{d\xi}{dt} = \xi',$$

$$\frac{(x')}{k} = \frac{1}{k}\frac{dx}{du} = \frac{d\xi}{dt} = x',$$

$$(v') = \frac{dv}{du} = \frac{k\,dv}{k\,du} = \frac{d\tau}{dt} = \tau',$$

will once more go to:

$$\text{(A)} \qquad \delta I' = \int_0^1 \left[\frac{\partial F}{\partial x}\xi + \frac{\partial F}{\partial x'}(\xi' - x'\tau') + \frac{\partial F}{\partial t}\tau + F\tau' \right] du\,.$$

Obviously, one can also deduce this formula *immediately* from the *concept of a variation* ([1]). In view of the misunderstanding that arises in presenting the variation by the use of the δ sign, it seems to me that the above consideration, which is also cumbersome, does not seem preferable for entirely elementary purposes. If formula (A) were addressed by the method of partial integration in the well-known way then that would produce the useful formula ([2]):

$$\delta I' = \left| \frac{\partial F}{\partial x'}(\xi' - x'\tau') + F\tau \right|_{t_0}^{t_1} + \int_{t_0}^{t_1} \left(\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial x'} \right)(\xi' - \tau\,x')\;du\,.$$

I shall now consider the integral:

$$J = \int_{t_0}^{t_1} (\alpha T + \beta U)\,dt$$

and set:

$$V = \sum (X\,\xi + Y\,\eta + Z\,\zeta)\,,$$

to abbreviate, which it is equal to the *virtual work done by the given forces*, and:

([1]) See **Hölder**, *loc. cit.*, § **2**, remark.

([2]) It was assumed in that form in, e.g., **Routh**, *Dynamik starrer Körper*, transl. by **A. Schepp**, v. 2, pp. 327.

$$S = \sum m\,(x'\,\xi + y'\,\eta + z'\,\zeta)\,,$$

which is equal to the *virtual moment of the quantities of motion*, and:

$$W = \sum m\,(x''\,\xi + y''\,\eta + z''\,\zeta)\,,$$

which is equal to the *virtual moment of the accelerations times the masses.* That will then yield:

$$\delta J = \int_{t_0}^{t_1} [(\beta U - \alpha T)\,\tau' + \alpha\,S' - \alpha\,W + \beta\,V]\,dt$$

or

(I) $$\delta J = \beta \int_{t_0}^{t_1} (V - W)\,dt + \int_{t_0}^{t_1} [(\beta U - \alpha T)\,\tau' + (\beta - \alpha) W + \alpha\,S']\,dt\,,$$

(II) $$\delta J = \alpha \int_{t_0}^{t_1} (V - W)\,dt + \int_{t_0}^{t_1} [(\beta U - \alpha T)\,\tau' + (\beta - \alpha) V + \alpha\,S']\,dt\,.$$

If one then chooses the arbitrary function t in such a way that the second partial integral in formulas (I), (II) vanishes then one will have:

$$\delta J = \beta \int_{t_0}^{t_1} (V - W)\,dt\,,$$

$$\delta J = \alpha \int_{t_0}^{t_1} (V - W)\,dt\,;$$

i.e., *the demand that* $\delta J = 0$ *will be completely equivalent to d'Alembert's principle.* Depending upon the choice of constants α, β, there can be various *special forms for the general variational principle.*

First: If one sets $\alpha = \beta$ then, from (I), that will demand the condition:

$$(U - T)\,\tau' + S' = 0\,;$$

i.e., when the part S' is dropped by integration, as usual, and the variations of the x, y, z are equal to zero at the limits then τ = const. or 0, resp. ([1]) In particular, if $U - T$ = const. = h then one can also set $\tau h + S = 0$. *That is Hamilton's principle.*

Secondly: If one takes $\beta = 0$ and one now sets, from (II):

$$T\,\tau' + V + S' = 0$$

([1]) When one adds the variations at the limits, one will find that the principle is *true without exception;* it is applicable even when $U - T$ vanishes between the limits of integration.

then one will have the *extended form of the principle of least action* ([1]). Since T is not zero, *that way of determining* τ *is always possible*, which should be emphasized here especially.

Third: By contrast, if one takes $\alpha = 0$ then, from (I), one sets:

$$U\,\tau' + W = 0,$$

which means that a possible addition to the variations at the limits would be entirely superfluous to further simplification. However, it must be assumed here that *U does not vanish between the limits of the integral* ([2]). Under those circumstances, the expression:

$$\delta\int_{t_0}^{t_1} U\,dt = 0$$

will also lead to the differential equation of motion.

Fourth: Finally, one will get:

$$\delta\int_{t_0}^{t_1} E\,dt = 0$$

for $\beta = -\alpha$, with the condition $(T + U)\,\tau' + 2V - S' = 0$.

A generally useful form for the principle will arise only in the first two cases. In the last two, as well as in the general case, the appearance of the symbolic expression U will already be a hindrance, even when one overlooks the fact that $\alpha T - \beta U$ cannot vanish insider the limits on the integral, which is generally not possible for arbitrary values of α, β. One can, however, avoid the symbolic expression U completely in a variational concept that is this general.

Namely, if one varies the expression:

$$A = \int_{t_0}^{t_1} \sum (X\,x' + Y\,y' + Z\,z')\,dt,$$

which represents the *total* work that is done by the effective forces from t_0 to the variable time t, so from formula (A), that will yield:

$$\text{(B)} \qquad \delta A = V - V_0 + \int_{t_0}^{t_1} \sum (Z)\,dt,$$

in which:

([1]) Cf., **Hölder**, *loc. cit.*, § **2**.

([2]) Naturally, a similar assumption must always be made when one takes an arbitrary function under the integral sign (cf., remark 1 on pp. 4). It will be fulfilled by itself from the principle of least action and **Hamilton**'s principle.

$$Z = \left[y'\left(\frac{\partial Y}{\partial x} - \frac{\partial X}{\partial y}\right) + z'\left(\frac{\partial Z}{\partial x} - \frac{\partial X}{\partial z}\right) - \frac{\partial X}{\partial t}\right](\xi - \tau\, x')$$

$$+ \left[z'\left(\frac{\partial Z}{\partial y} - \frac{\partial Y}{\partial z}\right) + x'\left(\frac{\partial X}{\partial y} - \frac{\partial Y}{\partial x}\right) - \frac{\partial Y}{\partial t}\right](\eta - \tau\, y')$$

$$+ \left[x'\left(\frac{\partial X}{\partial z} - \frac{\partial Z}{\partial x}\right) + y'\left(\frac{\partial Y}{\partial z} - \frac{\partial Z}{\partial y}\right) - \frac{\partial Z}{\partial t}\right](\zeta - \tau\, z'),$$

and one only has to show that the arbitrary function t is subjected to the conditions that arise when one employs the non-symbolic equation (B) in the variation of the integral:

$$\int_{t_0}^{t_1} (\alpha T + \beta A)\, dt\,,$$

in place of the previous equation $\delta U = V$.

If one considers that **d'Alembert**'s principle can be expressed in the forms:

$$\delta \int (T + A)\, dt = 0, \qquad \delta \int T\, dt = 0, \qquad \delta \int U\, dt = 0, \qquad \delta \int E\, dt = 0,$$

$$\delta \int (\alpha T + \beta A)\, dt = 0$$

then this variational principle, in its general form, will prove to be a *completely conventional rule* that no longer has anything to do with special representations that belong to the actual realm of mechanical intuitions, but are solely conceived for the sake of expressing the differential equations of motion in the most condensed form possible. I do not consider it trivial to once more repeat that remark (which is obvious by itself), which I already made on a previous occasion ([1]), since very differing opinions seem to be circulating at present in the conception of **Hamilton**'s principle, in principle. From the abstract standpoint, one can even see how the special form of the principle that employs the *energy integral* $\int E\, dt$ can have an advantage. However, there seems to be no doubt that the *actual* ***Hamilton*** *integral* is likewise recommended for its simplicity and general validity. Therefore, it was also used by **v. Helmholtz** in all of his investigations (under the name of the principle of least action).

III. – On the principle of least constraint.

If one denotes the coordinates of the points of a material system indifferently by x_i ([2]) then the *vis viva* will be:

([1]) **A. Voss**, "Ueber die Differentialgleichungen der Mechanik," Math. Ann. **25** (1884), pp. 267.
([2]) For the notation, see **H. Hertz**, Ges. Werke, III, pp. 62.

$$T = \tfrac{1}{2}\sum m_i\, x_i^2 \,.$$

Now, if one introduces just as many new variables y_i , which are mutually-independent functions of the x_i that can include time t, as well, in place of the x_i , then, by assumption, the functional determinant:

$$\Delta = \begin{vmatrix} \dfrac{\partial x_1}{\partial y_1} & \cdots & \cdots & \dfrac{\partial x_1}{\partial y_n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \dfrac{\partial x_n}{\partial y_1} & \cdots & \cdots & \dfrac{\partial x_n}{\partial y_n} \end{vmatrix}$$

will be non-zero, so:

$$m_1 \ldots m_n\, \Delta^2 = A$$

will also vanish, in which A is the determinant of the elements ([1]):

(1)
$$a_{s\sigma} = \sum m_i \frac{\partial x_i}{\partial y_s}\frac{\partial x_i}{\partial y_\sigma}$$

of the positive-definite quadratic form:

$$\sum a_{s\sigma}\, u_s\, u_\sigma = \sum m_i \left(\frac{\partial x_i}{\partial y_s} u_s\right)^2 .$$

If one denotes the sub-determinant of the elements (1), divided by A, but $A_{s\sigma}$ then:

$$\sum A_{s\tau}\, a_{s\sigma} = (\sigma\,\tau),$$

in which $(\sigma\,\tau)$ means the known sign. However, since one also has:

$$\sum \frac{\partial y_\tau}{\partial x_i}\frac{\partial x_i}{\partial y_\sigma} = (\sigma\,\tau),$$

it will follow that:

$$\sum\left(A_{s\tau}\frac{\partial x_i}{\partial y_s} m_i - \frac{\partial y_\tau}{\partial x_i}\right)\frac{\partial x_i}{\partial y_\sigma} = 0,$$

or, since $\Delta \neq 0$ ([2]):

([1]) In all cases in which nothing further is said about a summation, it will be extended over all indices s, σ, τ, … that appear *more than once* from 1 to 3.

([2]) In formula (2), the summation over i is obviously *not* performed.

(2) $$\sum A_{s\tau} \frac{\partial x_i}{\partial y_s} m_i = \frac{\partial y_\tau}{\partial x_i}.$$

If one now introduces the equations:

$$x_i' = \sum \frac{\partial x_i}{\partial y_s} y_s' + \frac{\partial x_i}{\partial t}$$

into the expression T then it will follow that:

$$T = \tfrac{1}{2} \sum a_{s\sigma}\, y_s'\, y_\sigma' + \sum a_s\, y_s' + a,$$

if one sets:

(3) $$\begin{cases} a_s = \sum m_i \dfrac{\partial x_i}{\partial y_s} \dfrac{\partial x_i}{\partial t}, \\[2ex] a = \tfrac{1}{2} \sum m_i \left(\dfrac{\partial x_i}{\partial t} \right)^2. \end{cases}$$

T is then a function of second order in the y_s' that is not generally homogeneous. At the same time, one will have:

(4) $$x_i'' = \sum \frac{\partial^2 x_i}{\partial y_s \,\partial y_\sigma} y_s'\, y_\sigma' + 2 \sum \frac{\partial^2 x_i}{\partial y_s \,\partial t} y_s' + \sum \frac{\partial x_i}{\partial y_s} y_s' + \frac{\partial^2 x_i}{\partial t^2}.$$

We employ the value (4) in order to calculate the *constraint Z:*

(5) $$Z = \sum m_i \left(x_i'' - \frac{X_i}{m_i} \right)^2,$$

in which we understand the X_i to mean the components of the effective forces. From a very simple calculation, we find from (4) that:

$$Z = \sum m_i\, \Xi_i^2 + \sum A_{s\sigma} \left[\frac{d}{dt}\left(\frac{\partial T}{\partial y_s'} \right) - \frac{\partial T}{\partial y_s} - Y_s \right] \left[\frac{d}{dt}\left(\frac{\partial T}{\partial y_\sigma'} \right) - \frac{\partial T}{\partial y_\sigma} - Y_\sigma \right] - \sum A_{s\sigma} Q_s\, Q_\sigma ,$$

in which we have set:

$$Y_s = \sum X_i \frac{\partial x_i}{\partial y_s},$$

$$\Xi_s = \sum\left(\frac{\partial^2 x_i}{\partial y_s\,\partial y_\sigma}\,y'_s\,y'_\sigma + 2\frac{\partial^2 x_i}{\partial y_s\,\partial t}\,y'_s + \frac{\partial^2 x_i}{\partial t^2} - \frac{X_i}{m_i}\right),$$

$$Q_s = \sum m_i\,\frac{\partial x_i}{\partial y_s}\frac{\partial^2 x_i}{\partial y_r\,\partial y_\sigma}\,y'_r\,y'_\sigma + 2\sum\frac{\partial x_i}{\partial y_s}\frac{\partial^2 x_i}{\partial t\,\partial y_\sigma}\,m_i\,y'_\sigma + \sum\frac{\partial x_i}{\partial y_s}\frac{\partial^2 x_i}{\partial t^2}\,m_i - Y_s\,,$$

to abbreviate, while we have:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial y'_s}\right) - \frac{\partial T}{\partial y_s} = \sum a_{sr}\,y''_r + \sum m_i\,\frac{\partial x_i}{\partial y_s}\frac{\partial^2 x_i}{\partial y_r\,\partial y_\sigma}\,y'_r\,y'_\sigma + 2\sum m_i\,\frac{\partial x_i}{\partial y_s}\frac{\partial^2 x_i}{\partial t\,\partial y_\sigma}\,y'_\sigma + \sum m_i\,\frac{\partial x_i}{\partial y_s}\frac{\partial^2 x_i}{\partial t^2}\,.$$

One now sees immediately that the first sum in Z will cancel the last one. In order to do that, one needs only to replace the Q_s with their values again in:

$$W = \sum A_{s\sigma}\,Q_s\,Q_\sigma\,.$$

If one also expresses Y_s in terms of the X_i again then that will imply that:

$$W = \sum \frac{\partial x_i}{\partial y_s}\frac{\partial x_j}{\partial y_{s'}}\,m_i\,m_j\,A_{ss'}\,\Xi_i\,\Xi_j\,,$$

which will go to:

$$W = \sum m_j\,(i\;j)\,\Xi_i\,\Xi_j = \sum m_j\,\Xi_i^2\,,$$

from (2).

It follows further by differentiation that:

$$2\sum m_i\,\frac{\partial x_i}{\partial y_s}\,\frac{\partial^2 x_j}{\partial y_r \partial y_\sigma} = 2\begin{bmatrix} r\;\sigma \\ s \end{bmatrix} = \frac{\partial a_{s\sigma}}{\partial y_r} + \frac{\partial a_{sr}}{\partial y_\sigma} - \frac{\partial a_{r\sigma}}{\partial y_s}\,,$$

$$2\sum m_i\,\frac{\partial x_i}{\partial y_s}\,\frac{\partial^2 x_i}{\partial t\,\partial y_\sigma} = [s\;\sigma] = \frac{\partial a_{s\sigma}}{\partial t} + \frac{\partial a_s}{\partial y_\sigma} - \frac{\partial a_\sigma}{\partial y_s}\,,$$

$$\sum m_i\,\frac{\partial x_i}{\partial y_s}\,\frac{\partial^2 x_i}{\partial t^2} = [s] = \frac{\partial a_s}{\partial t} - \frac{\partial a}{\partial y_s}\,,$$

such that:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial y'_s}\right) - \frac{\partial T}{\partial y_s} = \sum a_{s\sigma}\,y''_\sigma + \sum\begin{bmatrix} r\;\sigma \\ s \end{bmatrix} y'_r\,y'_\sigma + [s\;\sigma]\,y'_\sigma + [s]\,.$$

With that, the following theorem is proved:

If one replaces the variables x with just as many new variables y by means of the equations:

$$(6) \qquad x_i = f_i\,(y_1, y_2, \dots, y_{3n}, t), \qquad y_i = \varphi_i\,(x_1, x_2, \dots, x_{3n}, t),$$

which are mutually-independent relative to the y, then the ***Gauss****ian constraint Z will be expressed by the function:*

$$Z = \sum A_{s\sigma}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial y'_s}\right) - \frac{\partial T}{\partial y_s} - Y_s\right]\left[\frac{d}{dt}\left(\frac{\partial T}{\partial y'_\sigma}\right) - \frac{\partial T}{\partial y_\sigma} - Y_\sigma\right],$$

which is covariant in the vis viva T.

That is a generalization of result that **Lipschitz** ([1]) derived in the case where the functions f do not include time as a result of his general investigations into the transformation of *homogeneous* differential expressions. However, it is in the nature of things that it cannot be restricted to the case of a homogeneous form T. In that case, it would probably be simpler to derive the transformation result directly.

An *essential* condition for that is, however, that the number of variables y must be just as large as that of the x, because only under that assumption the identity (2), upon which the entire calculation is based, can be applied ([2]).

Now one can choose the variables y in such a way ([3]) that for a mechanical problem with k condition equations:

$$\varphi_l\,(x_1, \dots, x_n, t) = 0, \qquad l = 1, 2, \dots, k,$$

when the first k functions y are set equal to zero, they will represent just those conditions, i.e.:

$$y_l = \varphi_l\,,$$

while the last $h = 3n - k$ of them can be regarded as general coordinates q_m, $m = 1, \dots, h$. Under that assumption, one will then have:

([1]) **R. Lipschitz**, "Bemerkungen zu dem Prinzip des kleinsten Zwamges," J. f. Math. **82** (1877), pp. 328.

([2]) **A. Wassmuth** has ["Ueber die Anwendung des Princips des kleinsten Zwanges auf die Elektrodynamik," these Sitzungsber. (1894), pp. 219] taken advantage of **Lipschitz**'s formula for Z for the case in which the number of variables y is also smaller than that of the x. The fact that this is not permissible could already be seen from the fact that under those circumstances, the constraint that he also denoted by Z would be equal to zero, which only happens for free motions of a system, while condition equations were nonetheless assumed on pp. 220. The formulas that are developed in the further course of the paper must also be replaced with the ones that are derived later in the text, insofar as they do not refer to free motions.

Incidentally, in **Lipschitz**, the fact that the number of variables cannot change is made an assumption expressly (*loc. cit.*, pp. 316 and 328).

([3]) Obviously, one can also drop some of the conditions just as simply by introducing general coordinates.

$$y_l' = 0, \qquad y_l'' = 0 \qquad \text{for } l = 1, 2, \ldots, k.$$

Now, should the constraint Z be a minimum, one would get the equations:

$$\sum A_{s\sigma}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial y_s'}\right)-\frac{\partial T}{\partial y_s}-Y_s\right]a_{sl} = \lambda_s\,,$$

$$\sum A_{s\sigma}\left[\frac{d}{dt}\left(\frac{\partial T}{\partial y_s'}\right)-\frac{\partial T}{\partial y_s}-Y_s\right]a_{sm+k} = 0$$

in the known way by means of the method of **Lagrange** multipliers, or:

(a) $$\frac{d}{dt}\left(\frac{\partial T}{\partial y_s'}\right)-\frac{\partial T}{\partial y_s}-Y_s = \lambda_l\,, \qquad l = 1, \ldots, k,$$

(b) $$\frac{d}{dt}\left(\frac{\partial T}{\partial y_{m+k}'}\right)-\frac{\partial T}{\partial y_{m+k}}-Y_{m+k} = 0, \qquad m = 1, \ldots, h\,.$$

One can drop equations (a) entirely, since they only serve to determine the multipliers λ. Equations (b) imply the equations of motion, as long as one sets:

$$y_l = 0 \qquad \text{for} \qquad l = 1, \ldots, k,$$

$$y_{m+k} = q_m \qquad \text{for} \qquad m = 1, \ldots, h$$

in them, and that will imply the value:

$$Z = \sum A_{ij}\,\lambda_i\,\lambda_j\,, \qquad i, j = 1, \ldots, k$$

for the constraint Z.