THE PRINCIPLES OF RATIONAL MECHANICS

BY

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IN WÜRZBURG

TRANSLATED BY

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I. – CONCEPT AND PROBLEM OF MECHANICS.

1. Introduction. – The fact that the results of a mathematical doctrine of fundamental importance have often accelerated progress for a long time now due to their rigorous scientific foundations has manifested itself repeatedly to a great degree in *mechanics*, just as it has in arithmetic or the infinitesimal calculus. One can compare the standpoint that the systematic development of mechanics assumes in its present form with, say, **Cauchy**'s view of the infinitesimal calculus, which one can apply almost verbatim to the remarks of **Hertz** ([1]) in his introduction to mechanics. The following presentation, which endeavors to explain *the principles of mechanics, as they were developed in the course of the Nineteenth Century*, makes no claim to the *elimination* of the logical difficulties that exist everywhere, but rather, it wishes only to *contribute* to them by asserting that a satisfactory union of those principles (which is an inescapable requirement) can be gradually achieved ([2]).

2. Principle and principles of mechanics. – The expression "principle or principles" is applied in many different ways in mechanics. In any science (and mechanics, in particular, here),

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([1]) See the remarks of **Hertz**, *Mechanik*, pp. 8, on the frequently-emphasized ambition in the exposition of the foundations of mechanics to get away from the complications and inconveniences as soon as possible and deal with concrete examples.

The fact that we possess various excellent representations of mechanics in more recent times shall not by any means be underestimated here. Above all, this entire treatise shall go to great lengths to avoid any polemic tendencies. Meanwhile, a glimpse into the present state of the works on mechanics, to the extent that they are not purely-mathematical treatments of it but relate to the development of the actual mechanical models, should show that a great variety exists among them in regard to the principles as long as one is not dealing with the repetition of certain stereotyped phrases. In order to arrive at an examination of them that is as objective as possible in this treatise, an overview of the, in part, completely-differing opinions shall also be given in the footnotes.

([2]) The requirement of a *single* general binding philosophical doctrine shall therefore not be raised unconditionally. Whether it can be satisfied at all might seem all the more doubtful since presently, even in pure mathematics, there exist various theoretical approaches to the fundamental questions.

The vast *literature on the critical analysis of mechanics* that has arisen in the last thirty years has still been relatively unnoticed up to now. **Dühring**'s critical history of mechanics include many interesting remarks about the older periods, some of which one can also find presented already in a clear and appropriate fashion in **Whewell**'s *History of the Inductive Sciences*. Meanwhile, **Dühring** did not actually aspire to a critique in that book. The author also completely failed to understand the development of mechanics that began with the work of **Lagrange**. **Maxwell**'s *Matter and Motion* does not seem to be free from contradictions that are hard to reconcile and prefers to deal with the usual topics of physical mechanics. One can probably regard **Mach**'s *Mechanik*, which attempts to achieve a unified basic picture by means of a subtle understanding of the creation of mechanical principles, as fundamental in many respects. However, here as well (generally when one considers the whole plan of the book), the construction of mechanics in the Nineteenth Century shall be touched upon only in a few places, especially its connection with the simultaneous advancement of the mathematical methods, while the present article attempts only to shift that epoch to the foreground and bring one closer to a general historical and critical understanding of it with the help of the accumulated literature. However, **Hertz**'s critical introduction (even when one overlooks the isolated misunderstandings) seems, on the whole, too one-sided in its intention of deducing everything from a single basic principle, whose detailed implementation remains unfortunately incomplete. **M. Cantor** considered the development of mechanics only up to the time of **Galilei** in his history of mathematics. Almost all of the topics in mechanics and mathematical physics that are discussed by **C. Neumann** are distinguished by their clarity and analytical presentation. The comprehensive viewpoint of **Duhem** in his *Commentaire sur les principes de la thermodynamique* seems to be quite definitive for the recent development of physical mechanics. The entirely-modern spirit of **Th. Young**'s *Lectures on Natural Philosophy* is also especially noteworthy. We believe that we shall not go into a deeper discussion of the *preliminaries* for any critical examination of the principles of mechanics.

one understands *principles* to mean, *first of all*, statements that do not reduce to other statements from the same scientific domain but can be regarded as the implications of other results of what one knows ([3]), e.g., *axioms or postulates* ([4]), and which can be partly logical or methodological in character and partly metaphysical or physical. *Secondly, they are generally laws that are obtained from the basic concepts of mechanics* that, despite the fact that they were previously deduced in some simpler cases, they no longer actually seem to be completely provable in their broadest scope (e.g., the principle of virtual velocities, **d'Alembert**'s or **Gauss**'s principle). *Thirdly* ([5]), the generally *purely-mathematical methods for the treatment of mechanical problems* that are initially provable completely on the basis of principles of the second type and are sufficient for a purely-deductive treatment of an extensive part of mechanics will generally again take on a heuristic character in the broadest extension (e.g., **Hamilton**'s principle, principle of least action). *Finally*, one can arrive at integral equations for the differential equations of dynamics using **C. G. J. Jacobi**'s ([6]) *analytical methods*. In regard to principles of the latter type, we refer to IV 11.a. An overview of the *influence of mathematical methodology* on the current representation and treatment of mechanical problems generally belongs to the study of the principles of mechanics, in essence. Meanwhile, we shall not go into such a discussion *here*, since the specialized mathematical methods will find a thorough presentation in the following articles in the current volume of this *Encyklopädie*, for the most part.

3. Concept and problem of mechanics. – Mechanics ([7]) is the *foundation for all of the physical sciences*, i.e., the sciences that describe the processes of nature *by numerical values that are associated with well-defined laws* ([8]) whose dependency is represented by the mathematical picture of a function ([9]). Since the time of **Aristotle** ([10]), based upon the metaphysical concept of identity, the opinion has become more or less definitive that such an explanation can only be the result of reducing all phenomena to processes of motion for spatially-unvarying substances ([11]).

([3]) **Hertz**, *Mechanik*, pp. 4.

([4]) For the difference between axioms and postulates in the context of physics, cf., **P. Volkmann**, *Theor. Physik*, pp. 11.

([5]) The methodological relationship of the principles of the second and third kind to each other can be assessed in very ways. Even when one regards both of them (as seems to happen many times) as something like incantations in which a lengthy process of inductive reasoning finds its expression, nonetheless, a very essential difference exists in the degree of abstraction that enters in the two cases. Above all, the distinction between the principles of different kinds that one encounters in this treatise can only be a general one. The point at which each individual statement of the multiplicity of "principles" will be chosen in the course of time will depend upon the oft-fluctuating representations that will be given to the expression "principle."

([6]) **Jacobi**, *Dynamik*, 1842, ed., by **A. Clebsch**, pp. 2. Included in it is the analytical use of the law of *vis viva*, the center-of-mass integral, the principle of areas, the last multiplier, **Hamilton**'s principle of varied action, **Poisson-Jacobi**'s principle, and many transformations and equivalence principles, etc.

([7]) The work "mechanics" was initially attributed to **Aristotle** in *μηχανικά προβλήματα* (while explaining a series of useful discoveries), transl. by **F. Th. Poselger**, Berlin, Abhandlungen der Akademie (1829), pp. 56.

([8]) Cf., **J. C. Maxwell**, "On **Faraday**'s lines of forces," 1855, *Scientific Papaers* 1, pp. 155; also **Ostwald**, K. B., no. 69.

([9]) **E. du Bois-Reymond**, *Reden*, second edition, 1848, pp. 6, Leipzig, 1887.

([10]) Cf., **W. Wundt**, *Physikalische Axiome*, pp. 6, *et seq.*; **K. Lasswitz**, *Atomistik* 1, pp. 89; *ibid.* 2, pp. 1.

([11]) **Wundt** that in his *Logik*, 2, pp. 225, *et seq.*; likewise, **H. Petrini**, "Kritische Studien über die grundlegenden Prinzipien der Mechanik," Archiv für system. Philosophie **1** (1895), pp. 204.

However, that opinion can possibly be regarded as too narrow-minded ([12]). Despite the perhaps opposing philosophical misgivings in regard to the concept of a substance, the modern physical intuition also finds no difficulty in speaking of various *states of a substance* ([13]), which is not explained further, whose functional relationship, which also varies in time, is determined merely from a problem in mathematical representation. Here, let us recall only the whole direction of research that **Maxwell** initiated into the construction of a theory of chemical statics and dynamics, which determined, e.g., the rates of solution and reaction with the help of formulas.

In the purely-physical sciences, if one overlooks this very-essential *extension* of the viewpoint then one deals *first and foremost* with the consideration of those processes that are made understandable by the representation of *motion*. To that extent, mechanics is then the study of motion, and its basic problem is to derive all of the consequences that follow from a given state of motion.

Since one is merely dealing with numerical values, mechanics is applied *mathematics*, just like analytic geometry. Like geometry, it is coupled with certain assumptions. However, whereas in geometry they were *previously* sought in the conceptual formulation of *certain a priori statements*, in mechanics, they have a completely-different character from the outset. The naïve opinion of earlier times wished to grasp and *explain* the *true evolution* of phenomena by the *action of things on each other:* Thus, the purely-mathematical elements of the study of motion were combined with metaphysical speculations from the outset. The action of things then consists of the *forces* that they exert upon each other, i.e., the *sources of acceleration* with respect to each other ([14]).

Certainly, many still cleave to that *concept of causality* even now that wishes to "explain" all processes by the *relationship of cause to effect*. However, the concept of the *necessity and uniqueness* ([15]) *of natural phenomena* is increasingly being validated: The single truth that one can actually ascertain is derived from the fact that certain classes of phenomena that can be quantitatively determined completely are coupled with each other in a unique and indissoluble bond such that one of them *appears* to be a temporal consequence of the other ([16]).

([12]) **E. Mach**, *Geschichte der Satzes von der Erhaltung der Arbeit*, Prague, 1872, remarked on pp. 23 that there is no necessity for representing everything that one imagines in a merely-spatial way, so, e.g., the conception of five-atom isomeric molecules that differ only by their relationship to *two* points is impossible in three-dimensional space, *ibid*., pp. 29; likewise, **Mach**, *Mechanik*, pp. 486: "We regard the idea that all physical processes can be explained mechanically to be a prejudice." Cf., also **P. Beck**, "Die Substanzbegriff in der Naturwissenschaften" Diss., Leipzig, 1896, pp. 59.

([13]) **P. Drude**, *Physik der Äthers*, Stuttgart, 1894, pp. 10; for **W. Thomson**, those ideas already appeared in 1847 (*Papers* 1, pp. 76); cf., also **P. Duhem**, *Traité élementaire de mécanique chimique*, 4 vols., Paris, 1897/99, t., pp. 29.

([14]) That definition of mechanics was known already to **P. Varignon**, *Nouvelle mécanique* (1687), 1725, t. 1, pp. 1: "Mechanics, in general, is the science of motion, its causes, and its effects."

([15]) **J. Petzoldt**, "Das Gesetz der Eindeutigkeit," Vierteljahrsschrift für wiss. Philosophie **19** (1895), pp. 257, in agreement with **H. von Helmholtz**, *Wiss. Abh.*, pps. 13 and 68: "It was only later that I first clarified that the principle of causality is nothing but the *assumption of legitimacy*." Also, **P. Volkmann**, *Erkenntnistheor. Grundzüge der Naturwiss.*, Leipzig, 1896, likewise, *Theoretische Physik*, pp. 39, borrowed from the idea of causality at a fundamental level. According to **E. Mach**, *Die Prinzipien der Wärmelehre*, Leipzig, 1896, pp. 433, the concept of causality is nothing but *fetishism*; cf., also **Mach**, *Das Prinzip der Vergleichung in der Physik*, Leipzig, 1894, pp. 12: "I hope that the future natural sciences will abandon the concepts of cause and effect, due to their formal ambiguity."

([16]) **H. Weber**, *Über Kausalität in den Naturwissenschaften*, Leipzig, 1881, generally wished to establish the philosophical concept of causality, but combined with the idea that the necessary evolution of phenomena is no different from the actual one.

If one drops the assumption of a metaphysical *causality* ([17]) then what will remain as the problem of *pure* mechanics is solely the problem of determining (*describing*, resp.) the legitimate evolution of motion ([18]). That concept, which was becoming definitive to a larger community due to **Kirchhoff**'s authority, might require further epistemological explanation, but in any event, it is generally sufficient to also adapt the general conceptual determination of mechanics above to an extended form.

If one ignores the naïve concept that **Kant** abandoned once and for all that one can recognize something intrinsic in the effect of things on each other then it will follow that the metaphysical explanation for natural processes is, above all, not a problem in natural science ([19]), but that mechanics, just like geometry, operates with ideals by posing axioms in regard to the spatial pictures that are defined by the intuition, so it seeks to draw a *picture* of reality ([20]) that is based upon certain facts of experience that are formulated in axioms and postulates, and whose usefulness remains to be confirmed (further tested, resp.) by experiment. Obviously, the problem of drawing such a picture can be solved only to the extent that the mutual relationships between the processes in question can already be assumed to be sufficiently well-known. We cannot investigate the question of whether that already holds true in regard to the phenomena of *organic life* here ([20.a]).

In that way, mechanics *initially* starts with *material things* that make an impression on our senses. However, from time immemorial, even the simplest processes have led us to *go beyond* that purely-sensual realm of experience and speak of atoms, molecules, and material points, and introduce imponderable substances, along with the ponderable ones. However, many people currently see the complete picture of natural science as a study that would like to subsume all processes as *changes of state in the ether.*

([17]) However, just as the advancement of the mathematical sciences is almost independent of the general investigations into basic concept of number in epistemology, the course of history also shows that *the most-successful extensions of our knowledge of natural phenomena start from precisely those men* (e.g., Poncelet, Faraday, R. Mayer, as well as Maxwell and Helmholtz in their initial papers) *that were concerned with the especially living and concrete ideas of causality*. It would seem that the main constraint on them would also rob the research physicist of a large part of the vital intuition that was missing from, e.g., **Kirchhoff**'s abstract presentations. The ideal of science, in its complete form, can probably be summarized then in a way that seems to correspond better to the respective standpoint of epistemology. However, one must always note that our abstract concepts must find an ongoing adaptation and improvement as a result of intuition and observation.

([18]) **G. Kirchhoff**, *Mechanik*, pp. III and 1-5; for the description and explanation, cf., **Wundt**, *Logik*, 2, pp. 282; **Mach**, *Wärmelehre*, pp. 430, *et seq.*; **C. Neumann**, *Prinzipien der Galilei'schen Theorie*, pp. 13 and 22.

([19]) Cf., **H. Burkhardt**, "Mathematisches und naturwissenschaftliches Denken," Beilage Münch. Allg. Zeitung, 1897, no. 264.

([20]) This is essentially the representation that **Maxwell** introduced in "On Faraday's lines of force," *Papers*, 1, pp. 155 and in "On the mathematical classification of physical quantities," Proc. Math. Soc. London **3** (1871) = *Scientific Papers*, 2, pp. 257. Cf., also **Maxwell**, pp. 68 in **Ostwald**'s K. B., no. 69. Cf., also **L. Boltzmann**, "Über die Entwicklung der theoretische Physik in neuerer Zeit," Deutsche Math.-Ver. **8** (1900), pp. 71. Similarly, **C. Neumann**, *Prinzip. d. Galil. Theorie*, 1870. One can, in turn, conclude the existence of **Green**'s, and similar, functions and the uniqueness of certain solutions to other problems from the fact that the state that those functions refer to must exist *in reality*. That metaphysical basis will become untenable when one regards the theories of mechanics as only pictures whose agreement with experiments is in no way established *a priori*. Naturally, that should not diminish the *heuristic* value of such representations.

([20.a]) Cf., **J. Larmor**, *Aether and Matter*, pp. 288, **Hertz**, *Mechanik*, pp. 165. In contrast to the entirely-abstract conception of modern theories, we might refer here to the fact that investigations of **G. G. Stokes** and **Helmholtz**, and **W. Thomson**, in particular, generally started from the assumption of the introduction of general coordinates that are based in a mechanical construction, and that **Maxwell** was the first to strip away the last vestiges of that in his derivation of the motion of electricity from **Lagrange**'s equations.

As a viewpoint from which one could assess the value of the representation of a physical theory, **Hertz** ([20.b]) indicated: "This picture must be arranged such that the necessary consequences of the conceptual picture must always again be images of the necessary consequences of the natural situation that is being depicted." They must be logically acceptable, simple, and convenient, and as comprehensive as possible.

At the same time, that conception of things frees us from the obligation of *developing the psychological origin of that picture in mechanics itself*, no matter how important that might be, e.g., in a pedagogical treatment of it ([21]). That is because that picture does not at all possess an intrinsic truth that could be verified by psychological analysis; its only justification lies in its convenience. *Here*, we then believe that the question of from whence arise those special forms for the basic representations of mechanics should be left to the realm of philosophical analysis, in the same spirit by which in the teaching of elementary arithmetic, we can skip over the psychological justification for the primitive laws of combination of numbers themselves.

4. The different branches of mechanics ([21.a]). – One cares to distinguish *theoretical, pure, rational, general* (*mécanique générale, rationelle*) mechanics from *applied* mechanics and to characterize the latter as *astronomical mechanics*, *mathematical physics,* and *applied mechanics* (*mécanique appliquée*).

One cannot draw a completely-sharp boundary between those disciplines ([21.b]). Since antiquity, the individual parts of astronomical mechanics (e.g., planetary motion) and mathematical physics (e.g, hydrodynamics, the theory of elasticity) have been treated in the purely-theoretical books on mechanics. If one regards the problem of mechanics as being that of producing a perfectly-clear image of phenomena in terms of a mathematical representation then the character of *rational* mechanics, in particular, will be determined by the demand that *this picture should be based exclusively upon the representation of the behavior of pure motions.* Of course, that corresponds to astronomical mechanics to a high degree. However, whereas pure mechanics selects its problems on the basis of whether their mathematical implementation by means of analysis can be *carried out* completely, at least in principle (by which, obtaining the final numerical result often seems to be incidental), one confronts certain problems in astronomy that can only be solved with the help of *approximation processes* whose limits will always *shift* by the refinement of the art of observation. Things are different in *mathematical physics*. There, one prefers to use *specialized mathematical methods* that lend their special character to the investigations, beginning with potential theory, by means of **Green**'s theorem, *as well as completely expressing the renunciation of the explanation for physical phenomena in the older sense*.

([20.b]) **H. Hertz**, *Mechanik,* Einleitung, pp. I; cf., also **H. Kleinpeter**, "Entwicklung des Raum- und Zeitbegriffs in der neueren Mathematik und Mechanik und seine Bedeutung für die Erkenntnistheorie," Archiv für system. Philosophie **4** (1898), pp. 32.

([21]) In regard to this, one might cf., **Mach**, *Beiträge zur Analyse der Empfindungen*, Jena, 1886; *Wärmelehre*, pp. 422, *et seq.*; **Föppl**, *MEchanik*, 1, pp. 21; **Klein** and **Sommerfeld**, *Theorie des Kreisels*, pp. 70, Leipzig, 1897; **Budde**, *Mechanik*, 1, pp. 111; **Lasswitz**, *Atomistik*, 2, pp. 23.

([21.a]) Cf., **Newton**, *Principia*, praefatio ad lectorem: "Mechanicam vero duplicem *veteres* constituerunt: *rationale*, quae per demostrationes accurate procedit, et *practicum*. Quo sensu mechanica rationalis erit scientia motuum qui ex viribus quibusconque resultant." (However, the ancients established a twofold mechanism: the rational, which proceeds accurately through demonstrations, and the practical. In this sense, rational mechanics will be the science of the motions that result from each of the forces.) Meanwhile, **Newton**'s *mechanica practica* is engineering.

([21.b]) The following discussion should also be considered as only an attempt to delimit those domains from each other in an appropriate way.

Finally, the problem of *applied mechanics*, in its broadest sense, is the examination of the statics and dynamics of buildings and machines (see IV 8 and 23), as well as the study of the mechanical theories of those physical processes that come under consideration when performing precision measurements in physics (IV 7 and 25). Since the assumptions that one infers from experiment are already preferably expressed by *mean values* here, which is required by the *variable* nature of materials, the purely-mathematical processes that one applies to them (even when they are possible) no longer have any purpose at all and must be replaced with special mean-value estimates that are always controlled by experiment if it is to even be likewise possible for one to overcome the mathematical difficulties at all. Furthermore, the problems of applied mechanics almost always demand that one must consider phenomena in which changes of energy occur *in the dissipative sense* as a result of the unknown nature of the forces (in particular, those of friction, hardening, and incomplete elasticity), and for them, rational mechanics tends to develop only a general schema (see IV 8). Obviously, the general Ansatz for those problems must also be addressed in spirit of a purely-mathematical treatment.

In consideration of those discussions, in what follows, we shall include in rational mechanics all investigations that strive to reduce natural processes to motions by employing nothing but pictures that are sharply defined mathematically, with no regard to an immediately-practicable application of the solution of the problem, but with all of the precision that the current state of mathematical analysis permits.

5. Historical remarks. – **Galilei**'s investigations in *Discorsi* and *Scienza meccanica* include not only the dynamics of falling bodies, but also the theory of simple machines, the beginnings of solid mechanics, etc. **Varignon** regarded mechanics as *statics* in his *Nouvelle mécanique*, while **Euler** regarded it exclusively as *dynamics* in his *Mechanica sive motus scientia*. One can regard **Newton** as the founder and **Laplace** as the completion of the classical epoch in astronomical mechanics that began with **Clairaut**, **d'Alembert**, **Lagrange**, and others. **Lagrange** gave rational mechanics its characteristic form in *Mécanique analytique*. Along with that, the **Bernoulli**'s also developed engineering mechanics, in the form of mathematical physics, as well as hydrostatics and hydrodynamics. The truly productive era of engineering mechanics began with **Poncelet** and **Coriolis**, and generally with a predominantly-dynamical coloration. Later, it was built up by **Culmann**, in particular, in regard to its static questions, while the development of mathematical physics resulted mainly from the work of **Fourier**, **Cauchy**, **Poisson**, **Green**, **Gauss**, **Lamé**, **B. de Saint-Venant**, **F. E. Neumann**, **Stokes**, **Maxwell**, **W. Thomson**, **Kirchhoff**, **von Helmholtz**, and others.

II. – THE GENERAL PRINCIPLES OF MECHANICS.

A) Philosophical principles.

6. The causality principle and the law of sufficient grounds. – One can subdivide the general (i.e., not arising from the specialized realm of mathematical analysis) principles of mechanics into *philosophical*, *purely-mathematical*, and *physical-mechanical* ones.

Of the philosophical principles, along with the *causality principle* that was already mentioned above, one must emphasize the *law of sufficient grounds*. One concludes from the latter that the motion of a material point that is represented independently of all other things cannot change in *direction* ([22]), while it will first become possible to say the same thing about the *magnitude* of the velocity when one makes a *dialectical distinction* between cause and effect that shifts the cause *outside* of the moving body ([23]). That law also plays a historical role in the development of the concept of a force, the proof of the parallelogram law of forces, the consideration of action-at-a-distance between two material points, etc. ([24]).

The fact that one can make no decision regarding *real* relationships from *mere logical* premises is probably not in doubt nowadays ([25]). However, things will be different when the law of sufficient grounds appears in the form of a logical conclusion *whose premises are assumed to be known completely from experience.* For example, if one assumes that the resultant of the forces that act upon a material point is *determined* uniquely and completely by the position and magnitude of the latter then it will *follow* that the resultant of two equal and opposite forces or three equal forces with equal angles of 120° between them will be equal to zero. Under the assumption that *only* the relative position of the forces is crucial for the equilibrium of a lever, it will follow that the persistence of the rest state is the only one possible for the uniformly-loaded lever with equal arms in its "rest configuration."

7. Teleological principles. – *Teleological* principles have had a very essential influence on the development of mechanics. **Euler** ([25.a]) derived the principle of least action from essentially that viewpoint. **Gauss**'s principle of least constraint, as well as certain principles of the theory of elasticity ([25.b]), are likewise connected with such conceptions. Meanwhile, we do not need to touch upon the question of whether actual situations exist in nature that confirm the idea that the greatest

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([22]) **Euler**, *Mechanica*, § 56, Theoria motus, §83, also "Recherches sur l'origine des forces," Berlin, Mém. de l'Acad. (1750), pp. 419; **Laplace**, *Mém. céleste* (*Œuvres* 1, pp. 15); likewise, **S. D. Poisson**, *Mécanique*, 2nd ed. (transl. by **Stern**), pp. 167. See also footnote 142.

([23]) Thus, e.g., **Wundt**, *Axiome*, pp. 121.

([24]) **Wundt** employed this argument (*Axiome*, pp. 115, *et seq.*) in order to systematically deduce the axioms of physics.

([25]) Quite aptly, **Mach** remarked in *Mechanik*, pp. 135 that the law of "cessante causa cessat effectus" (if the cause ceases then so will the effect) is just as correct as its converse, according to whether one applies it to the concept of velocity or acceleration. Similarly, **Helmholtz**, *Erhaltung der Kraft*, **Ostwald**, K. B., pp. 58: "What many regard as the pinnacle of **Mayer**'s achievements, namely, the metaphysical fallacy of the *a priori* necessity for this law, would seem to be the weakest aspect of his viewpoint to any natural scientist that is accustomed to rigorous scientific methodology."

([25.a]) See footnote 257.

([25.b]) See no. **39** and footnote 224.

effect will be achieved with the smallest expenditure of resources here. In arguments of that type, neither the expended resources nor the effect achieved must usually be based upon a certain *measurement*, *such that the assertion will possess no clear meaning at all*. However, as far as its application to *mechanics* is concerned, that teleological viewpoint is, *in a real sense*, already completely inapplicable ([26]) due to the fact that by no means (neither for the principle of least action nor **Gauss**'s principle) is the *actual* motion distinguished from all other *possible* motions by a minimum principle, but only from certain motions that are purely-fictitious, in general, but also *impossible*. In fact, that teleological viewpoint has generally proved to be very necessary for the advancement of science, and in many respects, it would seem interesting to seek the general basis for that fact ([27]).

8. Mach's formal principles. – By contrast, **Mach** ([28]) drew attention to some other principles that should lie at the basis of all natural concepts, namely, *economy* and *simplicity*. For him, the goal of all science is to replace the realm of individual phenomena with a comprehensive description that can encompass the same scope with the least mental labor. Naturally, that is possible only when one seeks the basic *elements* for the individual phenomena and by their legitimate construction, provides an explanation for the processes for whose continuing development purely-formal principles will once more prove to be definitive, such as *continuity* and *constancy* [which are comparable to **Hankel**'s ([29]) principle of the *permanence of formal laws*], but above all, the *principle of analogy* ([30]), i.e., the adaptation of certain trains of thought that have been developed completely in one domain to a *new* domain.

B) Mathematical principles.

9. Mathematical assumptions about the nature of functions. – In particular, the principle of simplicity implies certain *general viewpoints of a purely-mathematical kind.* Such things will

([26]) Cf., **O. Hölder**, "Die Prinzipien von **Hamilton** und **Maupertuis**," Gött. Nachr. (1896). According to **J. Petzoldt**, "Maxima, Minima und Ökonomie" (Diss. Göttingen, Altenburg, 1891), that mechanical maximum-minimum principle *eliminates* exactly that teleological prejudice. Cf., also **R. Henke**, "Über den Zusammenhang der Naturerscheinungen mit der Methode der kleinsten Quadrate, (Dresden 1868, 2nd ed., Leipzig 1894).

([27]) See **Mach**, *Mechanik*, pp. 443, *et seq.* **Petzoldt** (*loc. cit.*, pp. 11) also sought a logical basis for the frequent appearance of invoking max.-min. laws. Cf., the *principle of the distinguished case*, that **W. Ostwald** [Leipziger Ber. **45** (1893), pp. 599; *ibid.* **47** (1895), pp. 37] expressed later, which nonetheless seemed to be expressed unclearly at his level of generality.

([28]) **E. Mach**, Almanach der Wiener Akad. (1882), pp. 293, *Mechanik*, pp. 471, 481, *Wärmelehre*, pp. 372, 494; cf., **Petzold**, "Max., min. u. Ökonomie," pp. 54; likewise, **Mach**, *Populär-wiss. Vorlesungen*, Leipzig, 1896, pp. 203, *et seq.* Obviously, one must draw upon many considerations here, e.g., **Newton**'s regulae philosophandi. One already finds those ideas expressed most clearly by **Galilei**. Cf., **Wundt**, *Axiome*, pp. 38; **P. Natorp**, "Galilei als Philosph," Phil. Monatshefte **18** (1882), pp. 193. Similar viewpoints are also frequently assumed in the spirit of metaphysics, e.g., **P. de Fermat**, *Opera* 1, Paris, 1891, pp. 173 (1662). "Naturam operari per modos faciliores et expeditores…, non ut plerique: naturam per lineas brevissimas operari." (Nature operates in the easiest and most expedient way…unlike most people: Nature takes the shortest path.)

([29]) **H. Hankel**, *Theorie der komplexen Zahlensysteme*, Leipzig, 1867, pp. 11.

([30]) For the principle of analogy, cf., **Mach**, *Mechanik*, pp. 131, **P. Volkmann**, *Theoretische Physik*, pp. 32. An enumeration of the most remarkable analogies is in **L. Boltzmann**, "Über Faraday's Kraftlinien," **Ostwald**, K. B., no. 69; cf., also **W. Dyck**, "Über die wechselseitigen Beziehungen zwischen der reinen und angewandten Mathematik," München, 1897, pp. 25.

come about when we regard the space of motion as **Euclidian**, with its infinite divisibility ([31]), and regard the coordinates of the paths of points as *continuous* functions of time ([32]) that are differentiable arbitrarily often (at least when we ignore isolated singular locations), and when we accordingly speak of the limiting values $\dot{s}$, $\ddot{s}$ ([33]), i.e., the velocities and accelerations ([34]) and when we assume that the ratio of mass to volume for continuously-filled spaces will approach a well-defined limiting value that is once more differentiable, viz., the density, as the latter two quantities decrease continually. Of course, mathematics has presently been developed to such an extent that even when one defines the general concept of a continuous function, certain statements are still possible, but up to now there has been no need to incorporate those abstractions into mechanics, since they lie far from the intuitive form of the processes of motion ([34.a]). One further considers it to be a general principle that a motion that is defined by forces is determined completely by its initial state ([35]). For that to be true, it is sufficient that the forces are assumed to be single-valued and differentiable arbitrarily often, and in particular, as regular functions of the coordinates and velocities.

The modern development of mathematics that touches upon that concept from the critical viewpoint, which **F. Klein** referred to as the arithmeticization of mathematics ([35.a]), insofar as the single foundation for it seems to be the rigorous concept of a number, has influenced mechanics only to a lesser degree up to now. Indeed, one can presently note, here and there, a lesser degree

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([31]) Cf., **H. Hertz**, pp. 53. The non-Euclidian conception of things has already been introduced extensively in numerous works on kinematics, statics, and dynamics. We shall not go into the details of that here, since up to now, the phenomena of non-Euclidian mechanics have given no reason to think that a non-zero curvature is possible. Cf., also the remark of **O. Heaviside** (*Electromagnetic Theory*, 2 vols., London, 1883/99, v. 1, pp. 2): "Now the real object of true naturalists, when they employ mathematics to assist them, is not to make mathematical exercises (though that may be necessary), but to find out the connection of the phenomena."

([32]) That assumption was made in **Helmholtz**, *Dynamik*, pp. 7. In more detail, it is in **Boltzmann**, *Mechanik*, pp. 10.

([33]) For the sake of brevity, and with regard to the historical character of this article, the differential quotients with respect to time shall be denoted as in **Newton**'s [*Tractatus de quadratura curvarum* (1706), *Opuscula*, Lausannae 1744, v. 1, pp. 203], namely, the notation that is used by the English writers (e.g., **Thomson** and **Tait**), $\dot{x} = \frac{dx}{dt}$, $\ddot{x} = \frac{d^2x}{dt^2}$, etc.

([34]) **F. A. Müller**, "Das Problem der Kontinuität in der Mathematik und Mechanik," Diss. Marburg, 1886, attributed the principle of continuity of velocities and accelerations to **Leibniz** (*Mathem. Schriften*, 3rd ed., **Gerhardt**, pp. 538, *et seq.*).

([34.a]) **P. Appell** and **Jannaud**, "Remarques sur l'introduction de fonctions continues n'ayant pas de dérivée dans les éléments de la dynamique," C. R. Acad. Sci. Paris **93** (1881), pp. 1005; also Archiv f. Math. u. Physik **67** (1882), pp. 160. – For the applications of continuous functions to potential theory, cf., **O. Hölder**, *Potentialtheorie*, Diss. Tübingen, 1882.

([35]) Under different assumptions, the motion can very likely become multi-valued, cf., **Poisson**, J. éc. polyt., cah. 13 (1808), pps. 63 and 105. **P. Painlevé**, *Leçons*, pp. 549, *et seq.* **Poisson**'s example $\ddot{x} = k^2 x^{1/3}$, which has the solutions $x = 0$ and $x = ct^3$, for $t = 0$, $x = 0$, $\dot{x} = 0$, can be combined with a whole series of more complicated ones. We would not like to go into the details here of the considerations of **J. Boussinesq** [C. R. Acad. Sci. Paris **74** (1877), pp. 362], who proved the necessity of a *principe directeur* (guiding principle) that would include mechanical indeterminacy in the laws of freedom, and in that way avoid absolute determinism in an otherwise rigorously-mechanistic world-view (cf., **J. Boussinesq**, *Conciliation du véritable déterminisme mécanique avec l'existence de la vie et la liberté morale*, Paris, 1878). For the mathematical investigation of such singular configurations of a system, see **Painlevé**, *Leçons*, pp. 562.

([35.a]) **F. Klein**, "Über Arithmetisierung der Mathematik," Gött. Nachr. (1895), pp. 82.

of rigor and systematics than was previously customary. Nevertheless, one cannot doubt that the arithmeticized mathematics will establish at least a *starting point* for all questions of its application. Initially, the careless use of infinitely-small quantities was *at least mostly* eliminated, as, e.g., **Maggi** ([35.b]) sought to do recently. Here, it would seem meaningful to further turn to the investigations of **Poincaré** ([35.c]) on how to exactly establish the convergence of series in the problems of astronomy, as well as the whole circle of questions that revolve around the existence of solutions of differential equations, namely, in relation to the **Dirichlet**'s principle. However, other questions will appear along with those that relate to their practical use, and especially the *simplification* of the mathematical methods, but whose ultimate goal *here* is not the abstract formulation of numerical quantities, but only demands estimates for them that possess a sufficiently-bounded precision ([35.d]) in regard to the imprecision of a process that can be controlled by experiment. Questions that are important enough to deserve special treatment do not seem to have been discussed thoroughly up to now, given the present inclination of mathematical analysis towards achieving the most abstract generalizations possible.

10. The homogeneity principle. – The *homogeneity principle* also belongs to that. The concepts of mechanics require that one must establish a series of *fundamental units* (e.g., for length, time, mass in pure kinetics), from which, further concepts (such as, e.g., velocity, acceleration, force, etc.) can be derived. Now, it is in the nature of things that for many considerations *relationships between those concepts must be independent of the choice of those basic units*. Such equations will then remain invariant when the fundamental units of measurement are replaced with any other mutually-independent ones. The *principle of homogeneity* ([36]) amounts to that character of invariance. It will attribute a *formal character* to those equations of mechanics that should serve to describe processes that are independent of the choice of units that proves to be useful in testing those laws ([37]); see the article on Units and Measurement in Volume V. The principle is used in a somewhat-different form in the examination of dynamical relationships as the *principle of similarity* (*principe de similitude*) ([37.a]). – Finally, one might recall the *principle of superposition*, which is an immediate consequence of the properties of the solutions of linear homogeneous differential equations.

([35.b]) **G. A. Maggi**, in *Principii der movimento*. In regard to kinematics, one might cf., **J. Tannery**, "Deux leçons de cinématique," Ann. Éc. normal (3) **3** (1886), pp. 43.

([35.c]) **H. Poincaré**, *Méthodes Nouvelles de la mécanique celeste*, 2 vols., Paris, 1892/93; "Sur les équations aux dérivées partielles de la physique mathématique," Amer. J. of Math. **12** (1890), pp. 220.

([35.d]) One might cf., the documents of the philosophical faculty at Göttingen on the attainment of the **Beneke** Prize for work that was submitted in 1898, Gött. Nachr. (1901) and Math. Ann. **55** (1901), pp. 143.

([36]) The study of *dimensions*, which goes back to **J. B. Fourier**, 1822 (*Œuvres*, 1, pp. 137), was first taken up by **Poisson** in his *Traité* (*Mechanik*, 1, pp. 23). One should also see **Maxwell**, *Elektrizität und Magnetismus*, 1, § 3; there is an especially thorough physical discussion of that topic in **W. Voigt's** *Kompendium*.

([37]) There also presently seem to exist some ambiguities in the employment of the principle in a methodological context, cf., e.g., the article of **F. Pietzker** and others, Unterrichtsblätter f. Mathem. u. Naturwiss. **4** (1898), pp. 64, *et seq.*; *ibid.* **5** (1899), pp. 31.

([37.a]) That was first posed by **Newton** in his *Principia*, pp. 294; one might cf., **J. Bertrand**, J. Éc. polyt. cah. **32** (1848), as well as **F. Reech**, *Cours de mécanique*, pp. 265.

C) Physical-mechanical principles.

11. The continuity principle. – The *physical-mechanical principles* are closely related to mathematical principles, and in particular, the *continuity hypothesis for moving substances* that one makes for *matter* ([38]). It seems inappropriate here to discuss the question of whether mechanics should have any interest in preserving the concept of matter, along with the only one that is definitive for it, namely, *mass*. Mechanics initially starts from the concept of a *material point* ([39]), i.e., a geometric point that is nonetheless accessible to observation due to its mass, and then goes on to a system of *n* such points, and one will then be inclined to assign an arbitrarily-large *n* to the process, at least to the extent that it comes into question in the general theorem. However, it is easy to see that one cannot arrive directly at the representation of the motion of a space that filled continuously with mass ([40]).

Moreover, for the mechanical picture, it is inessential to know how one represents the continuity of masses that fill up spaces, as in the naïve picture of a fluid, since it is only necessary to know that *all of the characteristics that determine the motion are continuous functions of position*, whereby empty space can also still remain at arbitrarily-many places ([41]). Continuum mechanics ([42]), as opposed to the mechanics of *n*-component systems, is then based upon *much-narrower* mathematical assumptions, which comes to light, e.g., when one compares the study of the *n*-body problem with that of hydrodynamics. It is expressed essentially by **Euler**'s continuity equation and by the transition from ordinary to partial differential equations.

([38]) According to **K. Pearson**, *Grammar*, pp. 251, this concept, which is entirely-useless for the natural sciences, was introduced by **Descartes** (cf., **P. Beck**, *Substanz*, etc., footnote 12, pp. 25). For various approaches to the concept of matter, cf., e.g., **P. G. Tait**, *Properties of Matter*, Edinburgh, 1885, Ger. transl. by **G. Siebert**, Vienna, 1888, pp. 13 and 288.

([39]) **Lagrange** still did not know of the term "material point," but, like **Euler**, in his *Mechanica*, which was *the first textbook on analytical mechanics*, he introduced the *point* as the *element* of all dynamical considerations, and **d'Alembert** introduced the term *(petit) corps*, which was recalled in the corpuscular theories of the Eighteenth Century (**Hobbes**). **Laplace** began his *Mécanique céleste* with the *équilibre du point matériel*, but without explanation. **É. Bour** (*Mécanique*, **2**, pp. 6) had every reason to remark that the main statements in mechanics take on a clear meaning only for points that are geometrically endowed with mass, which he called its *raison d'être*, and then **Th. Despeyrous**, *Mécanique*, **1**, pp. 5. Naturally, along with that, one would like to establish that small systems (i.e., bodies) can also be regarded as material points when one considers mean values, as long as one does not treat rotational phenomena (such as, e.g., **F. Reech**, *Cours de mécanique*, pp. 39; **A. Föppl**, *Mechanik*, 1, pp. 17). **Boltzmann** understood material points to mean "individual points that are selected from a body" (*Mechanik*, pp. 7). For **Poisson** (*Mechanik*, 1, § 1) and **Kirchhoff** (*Mechanik*, pp. 2), the material point had infinitely-small dimensions [see also **C. Neumann**, Leipz. Ber. **39** (1887), pp. 135], which is a terminology that seems to make sense only when one regards infinitely-small things as actually existing, although that is a purely-mathematical construction. For **H. Resal** (*Mécanique générale*, 2nd ed., 1, pp. 71), that means "One regards a molecule as a geometric point that one calls material. Since matter is indestructible, so it cannot be divided indefinitely, its final state of subdivision will be a molecule." The viewpoint of **Maggi** (*Principii*, pp. 149), which is based upon the concept of *figura materiale*, is free of any objections, and which is also included in **Love** (*Mechanics*, pp. 85).

([40]) **Weierstrass** rigorously developed the concept of a continuum in mathematics; cf., **G. Cantor**, Math. Ann. **21** (1883), pp. 575, cf., also I A 5, pp. 201.

([41]) For the question of whether the representation of the indefinite divisibility of ideal space must necessarily imply the properties of *observable continua*, cf., **Mach**, *Wärmelehre*, pp. 71.

([42]) For the representation of a continuum, cf., **Pearson**, *Grammar*, pp. 171. **H. Poincaré** showed [Am. J. Math. **12** (1890), pp. 283] how ordinary differential equations are converted into partial ones under the transition from a molecular medium to a continuous one: "It is by a true passage to the limit that one, in turn, passes from the molecular hypothesis to that of continuous matter.:

Discontinuous functions then appear only by means of the initial state and boundary conditions; cf., **Cauchy**, "Mémoires sur les fonctions discontinues," C. R. Acad. Sci. Paris **28** (1849); pp. 27 = *Œuvres* (1) **9**, pp. 120.

Those questions of continuity also play an essential role in the development of the theory of elasticity ([43]) and capillarity. The older theory of **Navier** represented the components of pressure by six-fold sums that **Navier** determined by integration ([44]), but in so doing, it was also assumed that the effect of immediately-neighboring particles would vanish, just like an (improper) integral over an infinitely-small region, which is an assumption cannot be justified completely, at least up to now ([45]).

12. Action-at-a-distance and field action. – Starting from **Galilei**'s simpler description of falling motion, **Newton** extended the problem in his *Principia* by means of the accelerating forces that implied the law of gravitation and *explained* the motions of celestial bodies as a necessary consequence of general mechanics. The *mechanics of distant forces*, which, in turn, posed the problem in **Laplace**'s *Mécanique céleste* of reducing *all* natural processes to the action of material points on each other in a manner that would follow from the model of the law of gravitation ([46]), and whose mathematical form took on a unified characteristic as a result of **Lagrange** that hardly seemed to require any further completion, enjoyed its greatest success in the theoretical results that, it would seem, assume the evolution of celestial bodies into the distant past, and indeed their discovery itself, which was unknown up till then.

The mechanics of distant forces took on a new impetus by its successful application to the theories of magnetism and electricity, which began with **Laplace** and **Poisson**, and ultimately culminated in **Weber**'s law ([47]), which desired to include all phenomena in that domain within a *single* fundamental formula.

It is interesting to pursue the evolution that the representation of distant forces has experienced in the course of time. **Galilei**'s viewpoint was based firmly upon the ground of **Maxwell**'s epistemology, which did not at all wish to present physical theories, but only a description of the processes on the basis of the necessary relations ([48]). The theory of distant forces initially provoked

([43]) **Poincaré**, Am. J. of Math. **12** (1896), pp. 290.

([44]) **C. L. Navier**, (1821), Paris, Mém. de l'Acad. **7** (1827), pp. 381.

([45]) See **Cauchy**, *Exerc.*, 1828 = *Œuvres* (2), **8**, pp. 236; **Boltzmann** raised that objection against **Gauss**'s molecular theory of capillarity, Ann. Phys. Chem. **141** (1870), pp. 582.

([46]) **Laplace**, *Méc. cél*, 1, pp. 1: "I propose to present, from a unified viewpoint, some theories that, when taken together, will embrace all of the results of universal gravitation concerning the equilibrium and motion of bodies…that define celestial mechanics. It is extremely important to banish everything that is empirical from them…"

([47]) **W. Weber**, *Elektrodynamische Maassbestimmungen*, Leipzig, 1846. *Werke*, 6 vols., Berlin, 1891/94, Bd. 3, pp. 132.

([48]) **Galilei**, Il saggiatore (*Opere*, 6, pp. 232): "Philosophy is written in that great book that continually reveals itself to our eyes (I call it the universe), but cannot be understood, since one must first learn its language and understand the characters in which it is written. *It is written in the language of mathematics, and its characters are triangles, circles, and other mathematical figures.*"

a lively debate ([49]). It was only the success of astronomical mechanics that made its ideas take on a familiarity that became a dogma that prevailed up to the middle of the Nineteenth Century ([50]).

Meanwhile, that also implies that the theory of point-like distant forces is by no means *always* necessary for the study of the behavior of continuously-extended bodies. The theories of the equilibrium and motion of fluids would hardly be affected by such representations. They show that notions of that kind are entirely irrelevant to them, and that one deals with only differential equations that characterize the changes in position of immediately-neighboring particles. In general, the theory of fluids employs the concept of internal pressure, about which, there has probably been much speculation, but it had already appeared in **Lagrange** in the form of the factor λ that was applied analytically in the variational problem, and which was a completely-abstract element of the description. The phenomena of capillarity, which **Laplace** and **Gauss** ([51]) explained using the principle of the action of distant forces, can also be represented more simply by such differential formulas.

However, it is the theory of elasticity that has a definitive significance. Of course, **Navier** ([51.a]) also obtained the equations of elastic media on the basis of molecular pictures, but **Cauchy**'s work showed that one only dealt with *field actions*, in the **Faraday** sense, by which the processes in the vicinity of each point would be described completely independently of such hypotheses. Thus, **Cauchy** arrived at not only the stress components, which were actually coupled with only the *presence* of a pressure that is directed transverse to the separation surface ([52]), and which is presently referred to as *stress*, by means of the deformation ellipsoid of the components of deformation, namely, the *strain* or elastic deformation. Thus, the mechanics of physical bodies become the *analysis* of *stress* and *strain* ([53]) (the relationship between the two systems of quantities, resp.). In that way, one also succeeds in eliminating the erroneous consequences of the

([49]) **Leibniz**, *Mathem. Schriften*, ed., **Gerhardt**, 3, pp. 964: "Ita quiquid ex naturis rerum inexplicable est, quemadmodum attractio generalis materiae Neutoniana aliaque hujus modi vel miraculosum est vel absurdum." (Thus, whatever is inexplicable from the nature of things, such as the general attraction of Newtonian matter and others of this kind, is either miraculous or absurd.) Moreover, **Newton** himself said in a well-known letter to **Bentley**: "The fact that a body can influence a body… at a distance through a vacuum is, for me, such a great absurdity that I believe that no one that possesses a sufficient capacity for philosophical thinking can ever fall for it." (Fourth letter to **R. Bentley**, 25 Feb. 1693, cf., **F. Rosenberger**, *Newton*, pp. 267). For the historical context of the study of action-at-a-distance, cf., also **J. C. F. Zöllner**, *Wissenschaftliche Abhandlungen*, 4 vols., Leipzig, 1878, esp., Bd. 1, pp. 16; *ibid.*, **2**, pp. 1 and 181.

([50]) **E. du Bois-Reymond**, "Über die Grenzen des Naturerkennens," *Reden*, 1, pp. 105, in 1872, referred to the understanding of nature as the resolution of all processes into atoms that are endowed with central forces, which was first taught systematically in **R. G. Boscovich**, *Theoria philosophiae naturalis*, Venice, 1758 (cf., also **G. Th. Fechner**, *Die physikalische und philosophische Atomlehre*, **Leipzig**, 1864, pp. 153 and 239). We are presently much further from that ideal in the spirit of **Laplace** ("Essai philosophique sur les probabilités," 1814 = *Œuvres*, **7**, pp. VI) than we were before. Cf., **J. Larmor**, *Aether and Matter*, pp. 272.

([51]) **Laplace**, *Théorie de l'action capillaire*, *Méc. celeste*, 4; **Gauss**, *Principia generalia theoriae figurae fluidorum*, 1829 = *Werke*, 5, pp. 29. Cf., **Volkmann**, *Theoret. Physik*, pp. 240; likewise, footnote, 45.

([51.a]) See footnote 44, and then the *Historical Introduction to the Theory of Elasticity* by **Love**, 1, pp. 1-34; **2**, pp. 1-24.

([52]) **Cauchy** had already done this in 1823 in Paris Soc. Philom. Bull. = *Œuvres* (2), **3** (still unpublished); cf., *Exercises*, 1827 = *Œuvres* (2), **7**, pp. 61.

([53]) This terminology goes back to **Rankine** [Cambr. and Dubl. Math. J. (1851) = *Misc. Scientific Papers*, London, 1881, pp. 68; Proc. Roy. Soc. London (1855), pp. 119).

older **Navier-Poisson** theory and arrives at a much more general and suitable insight into the essence of the elastic constants ([54]).

Of course, the mathematical-physical finesse that is obviously required for the implementation of field action must be learned from the guiding principles of molecular theories and seems to take much more advantage of **Maxwell**'s theories, which were based upon **Faraday**'s ideas, and which have, in the last thirty years, conferred upon the study of field action ([55]) the definitive position that it currently assumes. That representation was expressed fruitfully in the mechanics of deformable bodies in the treatise of **Thomson** and **Tait**, as well as in **Hertz**'s ([56]) development of **Maxwell**'s fundamental equations.

The general ideas about the *epistemological value of the atomistic picture*, which assumes that material points are coupled by distant forces, and the *phenomenological conception of the continuum* will also be affected only *indirectly* in that way at present. This is not the place to go into those questions, which depend, in part, upon the idiosyncrasies of the individual researchers ([57]). The definitive ideas will be the ones that prove to have the greatest success. The physicist will have no doubt as to where they can be found. However, it should be pointed out that it is precisely in the classic example of a distant force, namely gravitation, that no *generally* satisfactory derivation of it from a representation of continuously-distributed matter has emerged up to now. None even dare to say whether the phenomenological picture cannot be reshaped into a psychological one that replaces apparently-continuous processes with *mean values of discontinuous processes*. Such representations have already been built up considerably in the mechanical theory of gases and also in other general ideas that presently seem to be widespread. In order to pursue them exactly, investigations into the *theory of probability* take on a preeminent significance that also comes under consideration in the main concepts of rational mechanics. Meanwhile, we shall not go into that theory here, since it has not been addressed systematically in the latter topic so far, and we shall refer to IV 26, as well as Band V.

It is not our intention to actually go into the details of *energetic* phenomenology here. Whereas the older theory of action-at-a-distance sought to study the motion of the smallest particles of

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([54]) **Poisson**, Paris, Mém. de l'Acad. **18** (1842), pp. 3, had, of course, made more general assumptions that were supposed to eliminate that flaw; In general, that was achieved by **W. Voigt**'s assumption of molecular rotational moments, Gött. Abh. **34** (1887), pp. 11.

([55]) That is what **Maxwell** did since 1864 in his *Dynamical Theory of the Electromagnetic Field = Papers*, 1, pp. 256. **W. Thomson**, Phil. Mag. (4) **1** (1851), pp. 179.

([56]) **H. Hertz**, Gött. Nachr. (1890), pp. 106.

([57]) **H. Poincaré**, *Électricité et optique*, *Introd.*, pp. VI: "The old theories of physics gave us complete satisfaction. It seems that we would like to give the same precision to each of the branches of physics that belongs to celestial mechanics." According to **Boltzmann** (Verhandl. deutscher Naturf., Leipzig, 1900, pp. 112), the "old" mechanics is the only one that included clear presentations. He also validated the argument ["Über die Unentbehrlichkeit der Atomistik in der Naturwissenschaft," Ann. Phys. Chem. (2) **60** (1897), pp. 231] that the description of a continuous medium by differential equations would make sense only by means of a passage to the limit of the atomistic picture.

Volkmann said the opposite ["Über die notwendige und nicht notwendige Verwertung der Atomistik in der Naturwiss.," Ann. Phys. Chem. (2) **61** (1897), pp. 195], although (*Theor. Physik*, pp. 242) atomistic pictures still do not seem essential for certain physical processes at present.

According to **Mach**, *Wärmelehre*, pp. 428, atomism is the attempt to make the representation of substance in its crudest and most naïve form into a foundation for physics. For the current physical theories that many times fluctuate between atomistic representations and the assumption of a continuum, cf., the **Beneke** prize submissions of the Göttingen philosophy faculty, Gött. Nachr. (1901) and Math. Ann. **55** (1901), pp. 143.

matter ([58]) on the basis of explicit formulas for the forces that were mostly obtained with the help of triple *integrals*, the theory of field action replaced that with relations between *differential expressions* that governed the relationships between neighboring particles without needing to know anything about the other parts of space. It is possible to assume yet a *third* standpoint: Namely, that one can express the well-defined relationships between whole systems with each other with the help of *integral formulas without needing to observe the intermediate states* that each system has gone through in so doing. That is the direction along which the *energetic way of treating mechanics* ([58.a]) has developed, at least in its methodological form.

([58]) **H. Hertz**, *Mechanik*, pp. 15, very clearly described the character of this picture, which is not just mathematically complicated.

([58.a]) See no. **47**.

III. – THE BASIC CONCEPTS OF MECHANICS

A) Basic phoronomic concepts.

13. The concepts of space and time. – The basic concepts of mechanics are initially of a *purely-phoronomic type*, insofar as they are merely concerned with processes of motion ([59]) in space and time. One can regard it as an *axiom* that every change of position occurs *only in time* ([60]). The fact that all motions are *continuous* changes of position that correspond to continuously-changing time values seems to belong to our most fundamental intuitions ([60.a]). In that way, time initially seems to be a continuously-increasing variable, but the consideration of *negative* time values is finding continuing application, especially in the mechanics of reversible processes ([60.b]). Naturally, one can imagine a *theory of pure motion* that moves geometry completely off to one side and operates with abstract space as the substrate for all geometric constructions and abstract time values that will then play the role of a fourth variable ([61]). *However, the main connection between mechanics and reality will then be lacking*, in which, however, the time evolution of the motions should be known ([61.a]).

Newton then saw it as permissible to attribute a general transcendent *reality* to *space*, as well as *time* ([62]). **Newton**'s absolute space is an immaterial medium that is intrinsically immobile and

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([59]) For what follows, cf., in particular, **L. Lange**, *Bewegungsbegriff*, Leipzig, 1886.

([60]) As **Jak. Hermann** did before in his *Phoronomia*, Amstelod., 1716, pp. 1, which addressed the dynamics of solid and fluid bodies in connection with **Newton**'s *Principia*.

([60.a]) **W. K. Clifford** (*Lectures and Essays*, 1, pp. 112) occasionally developed the viewpoint that time consists of discrete moments that likewise corresponds to discrete positions of the "moving" body on the basis of known optical-physiological phenomena.

([60.b]) Moreover, the application of *imaginary* time values, as a special case of projective transformations, can also be useful, cf., e.g., **P. Appell**, "Sur une interprétation des valeurs imaginaires du temps en mécanique," C. R. Acad. Sci. Paris **87** (1878), pp. 1074, and also **L. Lecornu**, *ibid.*, **110**, pp. 1244; **P. Painlevé**, *Leçons sur l'intégrat.*, pp. 226. The introduction of imaginary times will already become necessary from a purely-mathematical standpoint in very-simple problems when one wishes to represent the coordinates as *single-valued* analytical functions, in general. Cf., e.g., **F. Klein**, *The Mathematical Theory of the Top*, Princeton Lectures, New York, 1897, pp. 33, 52.

([61]) The idea of such a theory of motion (phoronomy, kinematics), which one often attributes to **A. M. Ampére**, "Essai sur la classification des sciences," 2 vols., Paris, 1834-43; 1, pp. 50, was already expressed in **Kant**'s metaphysical prolegomena, 1786. From the standpoint of the pure analyst, e.g., **Lagrange** stated in *Théorie des fonctions*, ed. 2, 1813), pp. 311: "Therefore, one can regard mechanics as geometry in four dimensions and mechanical analysis as an extension of geometric analysis." Such opinions are still popular, cf., e.g., **O. Rausenberger**, *Analyt. Mechanik*, pp. 1, Naturally, one can also go down the path of *first* developing a consistent, completely-abstract picture of phenomena by means of fundamental geometric-mechanical concepts and *then* introducing further ways of determining things, while the latter must be subject to the constraint that it must be possible to relate them to reality. With that classification, the theory of pure motion would take on, e.g., a different systematization. That is the view that **Hertz** sharply established in *Mechanik*, pps. 53 and 157.

([61.a]) Naturally, the proofs in the discipline of *kinematics*, cf., IV, 4, can be supported by that abstract concept of time as a methodological tool. It defines an important part of descriptive mechanics, which has developed the graphical representation of velocities and accelerations (in regard to the latter, see **R. Proell**, *Versuch einer graphischer Dynamik*, Leipzig, 1874) since the time of **Poncelet**'s lectures (1836) and then the idea of *graphical dynamics*, whose general objective was, of course, first sketched out recently by **K. Heun** [**K. Heun**, "Die kinetische Probleme der wissenschaftlichen Technik," Deutsche Math.-Ver. **9** (1900), pp. 112]. **A. Schoenflies** developed a synthetic representation of the concept of a pure geometry of motion, Leipzig, 1886, that avoided the concepts of velocity and acceleration in its proofs.

([62]) **Newton**, *Principia*, pp. 5-7; cf., **Lange**, pp. 47-72. **Laplace** began his *Mécanique cél.*, pp. 4, with the words: "One imagines a space with no boundaries that is immobile and penetrable by matter. It is the parts of that space that are *real* or *ideal* that we shall associate with the *thought* of the position of the body."

unknowable ([63]) and that should likewise provide a fixed reference system for all motions, and absolute time is comparable to an ordinary clock that carries along all processes with each swing of its pendulum. The difficulties that lie in this representation were already expressed very clearly by **Euler**, whereas later on, one became accustomed to overlooking them as unavoidable. Thus, with the current figure of speech in mechanics ([64]) "one defines neither time nor space." At best, one regards it as necessary to discuss the concept of equal times.

However, since the time of **Kant** ([64.a]), the idea has become pervasive that space and time are forms in our imagination that possess only a transcendental reality, and from that standpoint, nothing will change in all deeper psychological-physiological understanding of the *process* of the creation and further development of those imaginary forms ([65]). However, at the same time, that also says that one can speak of motion in a mechanical-mathematical sense only when one gives a coordinate system relative to which the position of a moving point is defined ([66]).

Euler ([67]) already expressed the idea very clearly that all motion, i.e., change of position, is only *relative*, but of course without adding that one must nonetheless establish the absolute motion ([68]). **Kant** ([69]) said, more precisely: "Absolute space is, in itself, nothing, and not at all an object…but means only an individual relative space…that I shall highlight in any given one. *To make it into an actual thing…is to misunderstand the reason for the idea*," but at the same time preferred to regard rotational motion as something real. Although from the standpoint of mathematics, the study of relative motion ([70]) has been built up to the highest degree of completeness, nonetheless in mechanics there still exists the assumption of an *absolute fixed*

([63]) **Newton**, *Princ.*, pp. 7: "Verum quoniam hae spatii partes videri nequent." (Because those parts of space cannot be seen.)

([64]) **Newton**, *Princ.*, pp. 5: "Nam tempus spatium locum et motum ut omnibus notissima non definio." (For I do not define time, space, place, and motion as well known to all.) **Poisson**, *Mécanique*, 1, § 112, **Duhamel**, *Mécanique*, 1, pp. 3: "The notion of time is one of those notions that is not susceptible to definition, but what can be defined is the equality of times."

Generally, it is naïve and ignorant of the way that science works for one to wish to define everything [**Boltzmann**, Wien. Ber. **106** (1897), pp. 83]. However, although it is not the problem of the individual sciences to define their concepts by psychological analysis, nonetheless, they must indicate *their particular character* precisely in any event.

([64.a]) Naturally, the opinion that space and time possess no objective reality is much older than **Kant**'s *Critique of Pure Reason* (1781). **Euler** had already polemicized ["Reflexions sur l'espace et le temps," Berlin, Mém. de l'Acad. (1748), pp. 324] against the "metaphysicist," which he felt was an unacceptable viewpoint for mechanics. With the use of the terminology that goes back to **Kant**, one says that *transcendent* facts are ones that lie beyond the limits of experience, and therefore statements that refer to such facts, as well. By contrast, investigations are called *transcendental*, i.e., concerned with the assumptions of knowledge, when they refer to such assumptions, as well as the latter themselves. The study of the existence of an absolute space and an absolute time is therefore transcendent. The study of the ideal nature of time and space is transcendental.

([65]) By contrast, along with eliminating **Kant**'s study of the *a priori* nature of space and time in the sense that **Kant** himself wished to give them, the dogma that the only things in mechanics that can be regarded as objectively true are things that are regarded as temporal and spatial also fell by the wayside; cf., **P. Beck**, Diss., pp. 37.

([66]) Cf., **Pearson**, *Grammar*, pp. 233. We shall not touch upon the question of the transcendental character of motion at all (cf., e.g., **A. Höfler**, *Studien*, pps. 127, 133). It does not belong to mechanics.

([67]) **Euler**, *Theoria motus*, 1765. Likewise, "Reflexions sur l'espace et le temps," Berlin, Mém. de l'Acad. (1748), pp. 324. For **Euler**'s viewpoint, cf., **Lange**, *Bewegungsbegriff*, pp. 87-97.

([68]) *Theoria motus*, § 81.

([69]) **Kant**, *Metaph. Anfangsgründe*, Wiener Ausgabe, pp. 16; in **Lange**, pp. 97-108.

([70]) **G. Coriolis**, J. Éc. polyt. cah. **24** (1835), pp. 142.

coordinate system ([71]) (as well as that of *absolute time*, into which astronomical discussions of the measurement of time get mixed in an ambiguous way).

Duhamel and **C. Neumann** ([72]) have once more emphatically drawn attention to that flaw. According to **Neumann**, the principle of inertia is impossible to understand ([73]) unless nothing further is added to it in regard to which coordinate system the "material point left to itself" moves rectilinearly and what one means by uniform motion. In order to be able to understand that statement, one must assume the existence of a hypostasized immobile space ([74]), namely, "the rigid body A," whose three principle axes of inertia provide the coordinate axes. By contrast, to him, it seemed that one had achieved a new approach to the measurement of time in connection with *this* concept: *Any two* material points that are left to themselves move in such a way that equal path segments on one of them will always correspond to equal path segments on the other ([75]). If one refers to the time intervals during which the equal segments are traversed as *equal* then the law of inertia expresses the *statement* that *any* point that is left to itself will advance uniformly and rectilinearly ([76]).

14. The measurement of time. – **D'Alembert** ([77]) sought to arrive at a metaphysical definition of equal times: Equal times are ones during which identical bodies will exhibit identical processes of motion under identical circumstances. No matter how intuitive that definition might be (one might imagine a mathematical pendulum that swings to equal elongations), one must still *assume that the identity* of the effects *is knowable*.

That difficulty is closely connected with the assumption of *unaffected reference systems*, which is based upon **Neumann**'s viewpoint, as well as later ones (e.g., **Lange**, **Streinz**) on material points, resp. One might perhaps say that a point is unaffected when it experiences no changes in

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([71]) E.g., in **F. Minding**, *Mechanik*, pp. 1: "It is clear that one associates every body with an absolute motion." By contrast, from the remark of **R. Hoppe** [Archiv f. Math. (2) **16** (1898), pp. 8], one can always agree to explain the concept of motion as absolute as a constraint on knowledge, even when one admits to the relativity of motion.

([72]) **J. M. C. Duhamel**, "Sur les principes de la science des forces," C. R. Acad. Sci. Paris **69** (1869), pp. 773, and he said, more thoroughly, in *Méthodes* **4** (1870), pp. 454: "Up to now, absolute motion has generally been assumed to be a *pure chimera* that is based upon another chimera, which is that of a space that is eternal and absolute. We must once more do battle with a concept that is just as chimerical as that of space and makes time into a real entity that is necessarily independent of how it was created." Cf., *loc. cit.*, pp. 224 and XVI in the Foreword. **Duhamel**'s opinions seem to have attracted little attention. **F. Reech**, *Cours de mécanique*, Paris, 1852, had already expressed that idea as follows: "That law of inertia will no longer be a principle nor a fact of experience, but only a pure convention that one would like to base statics upon, since without it, one would never need to invoke either a hypothesis on the rigidity of the body, nor that of an absolute state of rest, nor that of an absolute state of uniform rectilinear motion in space." Moreover, **Th. Young**, thoroughly developed a viewpoint in his *Lectures*, 1, pps. 1 and 2, pp. 27 that was entirely similar to that of **Duhamel**. **C. Neumann**, *Die Prinzipien der Galilei-Newton'schen Theorie*, Leipziz, 1870.

([73]) **Neumann**, *Prinzipien*, pp. 14.

([74]) **Neumann**, *Prinzipien*, pp. 15.

([75]) **Neumann**, *Prinzipien*, pp. 18.

([76]) **Neumann**'s representation of time is also found in **Maxwell**, *Substanz und Bewegung*, pp. 35.

([77]) **D'Alembert**'s *Traité*, 2nd ed., 1758, which is *not* found in the transl. of **A. Korn** in **Ostwald**, K. B., no. 106. Similarly, it is also found in **Poisson**, *Mécanique*, 2, § 111, and that viewpoint was explained satisfactorily by **H. Streinz** (*Grundlagen*, pp. 85). **Euler** said simply in *Theoria motus*, § 18: "Everyone can see what the same times are, even when perhaps equal changes never occur in both from which one can conclude that equality." **D'Alembert**'s idea was already developed much more completely by **J. Locke** (1690).

its state of motion as a result of changes in the position of the bodies that exist outside of it ([78]). However, the latter can be concluded only from the lack of differences between the coordinates and velocities, i.e., it can be known from determinations of times, so it already *assumes* a unit of time.

That is, in fact, how one gets lost in ever more unresolvable contradictions. A way around that seems possible here only when one *either* starts from the entirely-abstract concept of assuming *absolute determinism*, in which all variable quantities that determine the nature of the body depend upon only one *basic variable*, which will now be called t ([79]), or *what seems to correspond more to a return to a natural viewpoint*, by satisfactorily explaining it by means of a *practical reference* that is given by the period of Earth rotation (the best chronometer that can be made, resp.) ([80]).

Thus, it is inessential whether the time unit in a perfect chronometer, which is corrupted by the non-uniformity of the Earth rotation over a long time interval, is replaced with, say, the period of oscillation of light of a certain color – say, one of the sodium lines ([81]) – or by the orbital period of an ideal planet around a central body, which was proposed by **Maxwell**, **Helmholtz**, and others.

15. Philosophical opinions of the current era. – One sees from this that a general *unification* of those questions that belong to the boundary between philosophy and mechanics has still not been achieved to date, despite many attempts. Even the philosophical viewpoints are still diametrically opposing each other. **Volkmann** referred to the existence of an absolute uniformly-flowing time as a necessary postulate ([82]), and the same idea was expressed by **Liebmann** ([83]). Others agreed wholeheartedly with the relativity of time and space. All of those difficulties were previously highlighted in detail by **Locke** ([84]), but they were also resolved in more recent works.

J. Epstein ([85]), in analogy with the **Helmholtz** axioms for geometry, preferred to regard it as an axiom for the measurement of time that the *duration* of a process should be independent of the point in time and the location where it takes place ([86]). The question of whether or not one might attribute a clear meaning to that statement might remain open. In any event, a *difference* exists between the concepts of space and time that makes such an analogous treatment impossible.

The fact that we possess the means in our conception of space to make completely-defined constructions on the basis of axiomatic statements, in particular, the means for defining the equality of two line segments by "superposition," seems beyond any doubt, just like the fact that those constructions can be applied to our spatial picture of reality. However, there is a fundamental

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([78]) **P. Duhem** (*Commentaire*, 1892, pp. 274) spoke of the concept of independent systems in the case where the parameter of the one system is independent of the parameter of the other, which one cannot object to from the abstract standpoint.

([79]) One will then come back to **Lagrange**'s viewpoint (footnote 61) by means of that concept.

([80]) Thus, **Hertz**, *Mechanik*, pp. 158; **Boltzmann**, *Mechanik*, pp. 8; **Love**, *Mechanics*, pp. 3, *et seq.*

([81]) **Thomson** and **Tait**, *Treatise*, 1, part 1, pp. 226. For the unit of length in such investigations, see **A. A. Michelson**, "Les méthodes interférentielles en métrologie," J. de phys. (3) **3** (1894), pp. 1. The determination of time that is based upon the *present* period of Earth rotation does not contradict the picture in which the period that is measured in one and the same ideal chronometer can vary in the course of time.

([82]) **Volkmann**, *Theor. Physik*, pps. 51 and 71.

([83]) **O. Liebmann**, *Zur Analysis der Wirklichkeit*, pp. 70, *et seq.*, 87, 93-95. **Wundt**, *Logik*, 1, pp. 430, seemed to be of the opposite opinion.

([84]) **J. Locke**, *An Essay Concerning Human Understanding*, Book 2, Chap. 14, § 3, 1690. Cf., the Oxford edition, 1894, vol. 1, pp. 249; see footnote 77.

([85]) **J. Epstein**, "Die logischen Prinzipien der Zeitmessung," Diss., Leipzig, 1887 (Berlin, 1887).

([86]) See also **Maxwell**, *Substanz und Bewegung*, pp. 15.

difference between that dogma and the other one that would like to assert similar statements for the representation of time. We initially only encounter the representation of *succession*, and then *duration*, which should not allow us to compare *simultaneous* events, but only events where one of them ends before the other one, although they are based upon the ill-defined sensations of fatigue and relaxation ([87]). However, in no way can we compare the durations of events that take place at *different* times by using our *inner* intuition, which would be necessary if one, with **Epstein**, would like to introduce a measurement of time that is based upon the axioms of one's intuition in a manner that is analogous to the processes of geometry ([88]).

16. The reference system of mechanics. – If **C. Neumann** had indicated the lack of a reference system that is based upon the demand that a body A must exist then **Streintz** ([89]) sought to produce one. Absolute translation was obviously something unknowable to him. He then replaced the statement that the reference system can possess no absolute translational *acceleration* with the requirement that it should be *unaffected*. By contrast, **Streintz** preserved the possibility of an absolute rotational motion ([90]). Naturally, one can see that in apparent proper motion of a *gyroscopic compass*, i.e., a suspension of a rotating body around a freely-moving axis, using **Cardani**'s method, viz., a *gyroscope*, and one can also understand that when one knows the relevant lemmas of mechanics. From a practical standpoint, one must concede that in the gyroscopic compass, one has a means for defining a reference body relative to which the **Galilean** principle ([91]). However, one can still complain about the assumption of absolute rotation, which is hardly compatible with the current fundamental laws of epistemology, as well as methodology, since one has introduced a complicated experiment as a precondition for any advanced knowledge whose understanding can first be achieved only after many dynamical preliminaries have been established.

L. Lange has sought to fill in that gap by giving a system that possesses at least no logical (methodological, resp.) flaws, and at the same time, is not a *transcendent real* one, like **Neumann**'s body A, but only an *ideal* one ([92]). It is the following one:

([87]) By contrast, **O. Lodge** stated in Phil. Mag. (5) **36** (1893), pp. 8: "The conception of uniform motion is based on a simple primary muscular sensation."

([88]) It also remains incomprehensible then that **Hertz** (*Mechanik*, pp. 53) simply defined time to be the time of "our inner intuition"; similarly, when one would like to regard capabilities that are obviously physiologically acquired, such as counting beats, *et al.*, as primary concepts.

([89]) **H. Streintz**, *Die physical. Grundlagen der Mechanik*, Leipzig, 1883. Similar ideas were already expressed, in part, by **Mach** in his *Mechanik* of 1883; see his critique of **Streintz**'s argument in *Mechanik*, pp. 232.

([90]) **Maxwell** also assumed that *directional state of rest* in *Substanz und Bewegung*, pp. 95. **B.** and **J. Friedländer**, *Absolute und relative Bewugung* (Berlin, 1896), in turn, attempted to resolve that question. A general agreement on that question (cf., also **Kant**, metaphys. Anfangsgründe, pp. 96) has not been reached up to now. Cf., the critique of the various viewpoints in **Mach**, *Mechanik*, pp. 221-240. **L. Lange** (*Bewegungsbegriff*, pp. 63) aptly pointed out that one can arrive at no clarification of the absolute nature of motion from forces, which include only statements about perceived motions, as **Newton** wished to do in his *rotation experiment*, *Principia*, pp. 9.

([91]) The gyroscope determines a body that does not rotate relative to a **Streinz**ian *fundamental system*, but by no means a body for which one say that an absolute rotation does not exist. According to **Mach**, *Mechanik*, pp. 218, the latter is an entirely-irrelevant metaphysical picture.

([92]) **L. Lange**, "Über das Beharrungsgesetz," Leipz. Ber. **37** (1885), pp. 353; "Die wissenschaftl. Fassung des Galilei'schen Beharrungsgesetz," Phil. Studium **2**, pp. 266, 539; cf., also the references in **H. Seeliger**, Vierteljahrsschr. d. astronomischen Gesellsch, **22** (1887), pp. 252.

W. Thomson ([93]) had proposed that one might regard a configuration of four material points that are simultaneously thrown from a location as a reference system. According to **Lange**, any coordinate system, say, **Descartes**'s, whose axes are continually represented by three points that are thrown out from a point (in various spatial directions) like rigid wires through that point on which along which the points can slide like smooth balls, is a **Galilean** reference system. It will then be easy to show that every such system will again have the character of such a system relative to another such system. Now, the path of a point left to itself relative to such a system will be *defined to a line*. If one then introduces the concept of time, like **Neumann**, then one can make the further *statement* that every point moves uniformly and rectilinearly relative to an *inertial system* and an *inertial scale of time*.

A general convention regarding those question has still not been achieved up to now, and one should probably conclude from the fact that even recently the old doubts about absolute and relative motion and the meaning of **Newton**'s theory have been discussed thoroughly that those fundamental questions have still not been completely clarified and thought through ([93.a]).

Meanwhile, it should be pointed out that **James Thomson** ([94]) had already referred to the necessity for a "frame of reference" ([95]), in a similar way to **Lange**, such as, say, a **Thomson-Tait** coordinate tetrahedron. **Muirhead** sought to sketch out the viewpoint that lay at the foundation for the **Newton-Galilei** system in a more logical way when he said ([96]):

> "It is possible to choose the masses of the solar system, the axes, the chronometry…, so that the masses shall correspond with those of astronomy and the forces shall be resolvable into such as will be expressed by the law of universal gravitation… Then true time, absolute velocity and mass-measurement being defined from this system, there would be a further law of physics that the forces of the various particles composing the members of the solar system and others are expressed by our various physical laws or theories."

Meanwhile, one might obviously choose a less-abstract standpoint in order to overcome the present difficulties. If one considers the acceleration relative to *any coordinate system that is at our disposal* to be a measure of the force that "produces" that motion then when the latter is zero, it will create a uniform rectilinear motion for the material point in question. Based upon that *assumption*, one will arrive at certain statements that either do or do not agree with reality to a satisfactory degree. In the latter case, one will discard the reference system as unsuitable ([97]).

([93]) **Thomson** and **Tait**, *Treatise*, 1, Part 1, pp. 242**. J. Tilly** had already considered a similar coordinate system that established the representation of absolute motion, moreover [Brux. Bull. de l'Acad. Roy. (3) **14** (1887)], in 1878 in his "Essai sur les principes fondamentaux de la géométrie et de la mécanique," Bordeaux, Mém. (2) **3** (1878), pp. 1.

([93.a]) Cf., the reference to a discussion that took place in various English journals between **Love**, **MacGregor**, **A. Basset**, **E. Dixon**, **McAulay**, **A. Gray**, **O. Lodge** on absolute and relative motion by **E. Lampe**, Forthschritte d. Mathem. **25** (1897), pp. 1318, as well as a series of notes by **E. Goedseels**, **P. Mansion**, **Pasquier**, **É. Vicaire** in in the Bruxelles, Ann. Soc. Scientif. 16-21 (1890-1897).

([94]) **J. Thomson**, "On the law of inertia," Proc. Roy. Soc. Edinburgh **12** (1882/84), pp. 568, 730.

([95]) **J. Thomson**, *loc. cit.*; **P. G. Tait**, Proc. Roy. Soc. Edinburgh **12**, pp. 743.

([96]) **F. Muirhead**, "The laws of motion," Phil. Mag. (5) **23** (1887), pp. 473. Similarly, see **Petrini**, footnote 11, pp. 231, *et seq.*; likewise, **Love**, *Mechanics*, pp. 92, as well as **J. Hadamard**, "Sur les principes fondamentaux de la mécanique," Bordeaux, Ann. Soc. Phys. (1897).

([97]) The choice of "frame of reference" is at our discretion. However, the description of the motions will prove to vary according to that choice, cf., **Love**, *Mechanics*, pp. 8. That also seems to be the standpoint of **Hertz**, *Mechanik*, However, at the same time, the *fact* that it has always been successful up to now (at least in the mechanics of ponderable bodies) implies that another one should be introduced that is suited to the phenomena of reality to a sufficient approximation. Thus, one goes from the phenomena of the case in which one refers everything to a reference system on the Earth to a geocentric system that does not participate in the rotation of the Earth. For the advanced question about a system whose origin is the center of the Sun, whose *Z*-axis is the normal to the invariable plane (to a system whose axes are determined by the directions to the fixed stars, resp.). In that case, there remains some doubt as to whether there actually is a fundamental reference system, but the expectation is justified that upon advancing to deeper questions, one will always succeed in seeing how to test the applicability of such a thing.

17. New theories. – This standpoint, which has been taken many times in recent years, since it is also very popular, to the extent that it concerns the mechanics of ponderable bodies, has been temporarily allowed to offer some advantage over those considerations that would like to introduce **Newton**'s absolute space again (but generally only in a certain sense) by way of the *assumption of a medium that fills up space*, namely, the ether. That alteration is generally impossible in the older conception of things that described the phenomena of light and heat by the *behavior of motions of the ether*. However, things will be different when the ether in the advanced **Maxwell** theory is regarded as an existing medium with certain well-defined properties whose geometric configuration is unvarying ([98]). In so doing, one must only recall that the electric and magnetic processes are no longer regarded as motions, but as *states* of polarization, etc., that one can further represent with the *picture* of a perturbation. As opposed to such an ether, which then defines the absolute reference system [but of course, nothing can or should be said about its state of absolute rest ([99]), and whose states are definable by observable processes in a field], one can also imagine *relative motions* ([100]) of "ponderable" bodies then as long as one still does not succeed in ascribing *all* phenomena to states of the ether, first and foremost. Perhaps, the latter is the tendency of an electric world-view that is presently popular with many physicists. Above all, to the extent that one can decide by experiments, that will depend upon whether the actual phenomena are consistent with the assumption of an ether at rest, for which the aberration of light defines a well-known support. The formal foundations of mechanics would then be able to assume an entirely-different character than before. However, it would seem premature to go further into that.

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pp. 158 and **A. Föppl**, *Mechanik*, 1, pp. 1. Cf., also **P. Duhamel**, *Commentaire*, 1892, pp. 271: "If we regard a hypothesis as exact when the consideration of absolute motion is involved, and if the application of that hypothesis to motions relative to a certain trihedron leads to inexact results then we will declare that the trihedron is not absolutely fixed." Likewise, **Mach**, *Mechanik*, pp. 236: "The most natural standpoint is still the one that considers the law of inertia to initially be a sufficient approximation and refers to the fixed stars spatially and the rotation of the Earth temporally, and which one would expect to be corrected by extended experiments."

([98]) Cf., the presentation in **W. Wien**, "Über die Fragen, welche die translatorische Bewegung des Lichtäthers betreffen," Beibl. Ann. Phys. Chem. (2) **65** (1898).

([99]) **H. A. Lorentz**, *Versuch einer Theorie der elektrischen und optischen Erscheinungen in bewegten Körpern*, Leiden, 1895, pp. 4; likewise, **J. Larmor**, *Aether and Matter, including a discussion of the influence of the Earth's motion on optical phenomena*, Cambridge, 1900. That had already been done before in **J. J. Thomson**, *Anwendungen*, pp. 40.

([100]) Cf., e.g., **Volkmann**, *Theor. Physik*, pp. 54; **E. Budde**, *Mechanik*, 1, pps. 112, 135; likewise, **Mach**, *Mechanik*, pp. 225. **W. Wien** made an attempt to sketch out a theory of mechanics in this sense, "Über die Möglichkeit einer elektromagnetischen Begründung der Mechanik," Livre Jubilaire dedié à **H. A. Lorentz**, La Haye 1901, pp. 96, also Ann. Phys. Chem. (3) **5** (1901), pp. 501.

In connection with the ideas of **F. Reech**, in *Cours de mécanique*, 1852, **Andrade** ([101]) recently made a novel attempt to base a formal theory of mechanics without having to resolve the principle of inertia from the outset. For him, forces were static and measured by stresses. He then distinguished a final acceleration j (*accélération finissante*) for every material particle from its initial acceleration (*acc. commençante*) J ([102]), whose vectorial difference relative to any relative coordinate system is independent of the properties of the latter, from the **Coriolis** formulas ([103]) for relative motion, and assumed that the quantity T in the formula for the relative force:

$$F = m\,(J - j)\,T$$

is a factor that depends upon only the unit of time. Since one will have:

$$(J - j)\,dt^2 = (J' - j')\,dt'^2$$

under a transition to a new unit of time t' that is coupled with the old t by the equation $t' = f(t)$, it will follow that:

$$F = m\,(J' - j')\left(\frac{dt'}{dt}\right)^2 T\,.$$

One can then always introduce an *absolute time-scale*, for which:

$$\left(\frac{dt'}{dt}\right)^2 T = 1\,.$$

In general, either the *accélération finissante* or the course of the natural motion must be known if those formulas are to find any practical employment.

B) The basic concepts of statics.

18. Forces in statics. – We shall now go further into the basic concepts of mechanics. Mechanics developed from the study of the simplest machines and then by the geometric teachings of **Archimedes** about the center of mass. It initially appeared as *statics*, in which forces are regarded as tensions and compressions, as might be due to human hands or weights acting on tensed cables, so in an entirely anthropomorphic way ([104]). It is not in doubt that these *static forces*

([101]) **F. Reech**, *Cours de mécanique*, Paris, 1852, pp. 21; **J. Andrade**, *Leçons de mécanique physique*, Paris, 1898.

([102]) **Andrade**, *Leçons*, pp. 51, *et seq.*

([103]) **G. Coriolis**, "Mémoire sur les équations du movement relatif des systèmes de corps," J. Éc. polyt. cah. **24** (tome 15) (1835), pp. 142.

([104]) E.g., **Galilei** in *Scienza della meccanica* (1592), *Opere* 2, **Varignon**, in *Mécanique nouvelle*, etc. Those manual illustrations gradually disappeared and gave way to the now-customary vectorial pictures by directed line segments (i.e., vectors). **C. Neumann** also spoke of *commands* that are imposed on bodies, *Prinzipien*, pp. 5, and also Math. Ann. **1** (1869), pp. 317. Incisive expressions of that sort are certainly not worthless. The fact that abstract scientific representations then strive to eliminate anthropomorphic pictures is probably self-explanatory. However, even in a general context, it can seem only premature to regard those representations as definitive of all phenomena when they were inferred from an entirely-restricted domain of sensations. In relation to that, one should also note the *are measured in well-defined units of weight* and can be expressed in numbers as *spatial vectors*. In that way, the *statics of rigid bodies* was, in fact, developed under **Varignon**'s influence, and its completion seems to have been achieved in **Poinsot**'s theory of force-pairs (couples), and with complete rigor as soon as one accepts more or less sharply expressed axiomatic foundations that will lead to the equilibrium considerations for rigid bodies.

Two forces will then be equal if they preserve equilibrium when they act in opposite directions. If one further assumes that forces, whether acting in the same or opposite directions, can be combined by algebraic summation then all that will remain is the problem of composing forces with different directions at the same mathematical point by means of the parallelogram rule.

The concept of a *rigid body* that one encounters is a concept that is derived from statics itself, which originally includes only the representation of a substance whose geometric configuration is unvarying, and for whose points the possibility of relocating forces along direction of their line of action is defined as an *axiom, namely, the principle of relocation of the point of application along their own direction*. **E. Budde** (*Mechanik* 2, pp. 537) aptly called such vectors *sliding vectors*, as opposed to *free* ones (e.g., force-couples).

The *historical* character of those investigations, whose further evolution (namely, to the extent that one deals with the action of various rigid bodies on each other by collisions and related questions) also drew upon the representation of a *mass* that is particular to any rigid body [and thus, the entire study of the *geometry of masses* (see IV 3), when it is carried out with the help of infinitesimal methods, which go far beyond the barycentric studies of **Archimedes**], is connected with actual dynamics, in addition. It is juxtaposed with the *dynamical representation of the rigid body*, which regards it as an aggregate of mass-particles from the outset that are constrained into a system by forces that obstruct any change of the configuration ([104.a]). That seems to lead one, in a much more unforced way, to the idea of a general material system, in which the singular cases of static indeterminacy (see IV 5 and 22) do not appear. Moreover, the fundamental value of statics is based, not so much on the reality of a suitable invariability of the configuration (but only to a very restricted degree), but only on the development of the study of the *equivalence of forces*, which also continually emerges from the dynamical pictures later on. Here, we can only refer to the worthwhile methodological content of statics, namely, as it appears in the theory of moments, and the important duality between statics and the study of motion that is shown by the theory of *vectors* and *screws*.

19. The parallelogram of forces. – **Newton** and **Varignon** ([105]) had deduced that fundamental theorem directly from the study of the *composition of motions*, which **Stevins**, according to **Cantor** (*Vorlesungen*, 2, pp. 449) had not referred to in exactly those words. However, for the strictly *static* concept of force that one must have mind here, the recursion to a *motion that does not at all*

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remark of **Th. Young**, *Lectures* 1, pp. 28: "We must not imagine that the idea of force is naturally connected with that of labour or difficulty; this association is only derived from habit."

([104.a]) One ordinarily replaces the abstract conception of things that was chosen in this treatise with the picture in which very large opposing forces appear under small changes in the configuration. Equilibrium then refers to that state of deformation, which is actually *unknown*, however.

([105]) **Newton**, *Principia*, pp. 13: "Corpus viribus compenetis diagonalem parallelogramme eaoden tempore describere quo latera separatis." (The body is composed of forces that describe a diagonal parallelogram whose sides are, at the same time, separated.) **Varignon**, *Nouvelle mécanique*, 1 (1725). One finds the well-known rule for the equilibrium of three forces that act upon a point there.

come about ([106]) does not seem immediately permissible. That viewpoint gave rise to many attempts to prove the *parallelogram of forces* on the basis of axioms that were easier to overlook, i.e., special cases of the theorem. It also does not seem to be worthless at present, although **Mach** remarked ([107]) that many of those proofs are based upon an induction about the essence of the phenomena that one might expect would be difficult to acquire.

Dan. Bernoulli ([108]) was the first, and later **d'Alembert** ([109]), and others, to determine the *resultant* of two forces from the purely-static concept of force. The functional equation that the latter found appeared in a somewhat-different form in **Poisson** ([110]). For its proof, one reverts to the composition of two forces with the same direction by algebraic addition, under the assumption that the uniquely-determined resultant R bisects the *inner* angle $2x$ into two *equal* forces P, and its magnitude is given by the *continuous* function:

$$R = P \cdot \varphi(x) .$$

In that way, one then has $R = 2P$ for $x = 0$, and for $x = 60^{\circ}$, one has $R = P$, by the symmetry principle. If one replaces each of the forces P with two equal forces Q that define the angle z with P and assumes that the resultant will not change in that way then φ must satisfy the equation ([111]):

([106]) **Joh. Bernoulli**, *Opera* 4, pp. 256: "Peccant, qui compositionem virium cum compositione motuum confundunt." (They sin who confuse the composition of forces with the composition of motions.) Likewise, **H. Lambert**, *Beiträge zum Gebrauche der Mathematik*, 2, Berlin 1779, pp. 451. **F. Reech**, *Cours de mécanique*, pp. 61: "We reject absolutely all of the putative proofs of the theorem of the parallelogram of forces by means of the obvious rule of the parallelogram of velocities in geometry." **E. Bour**, *Mécanique*, 2, pp. 16; cf., also **R. Heger**, Schul-Programm Dresden, no. 398 (1887), pp. XVII. **Thomson** and **Tait**, *Treatise*, 1, Part 1, pp. 244, have a different opinion.

([107]) **E. Mach**, *Mechanik*, pp. 45. – It should be emphasized that the way that things are implemented in the book deals with *only* the static conception of force.

([108]) **D. Bernoulli**, Comm. Ac. Sc. imp. Petrop. (1726), 1 (1728), pp. 126.

([109]) **D'Alembert**, *Opusc. math.* 1, Paris, 1761, pp. 269. Cf., also **D. de Foncenex**, *Mélanges de philosophie et mathématiques*, Turin, 1760/61, pp. 305. Moreover, **d'Alembert**, Berlin, Mém. de l'Acad. (1750), pp. 350. It is found in a simplified form in **Aimé**, J. de math. **1** (1836), pp. 335. For the older literature, cf., **C. Jacobi**, "Praeciporum inde a Newtone conatuum compositionem virium demonstrandi recensio," Diss. Gottingae, 1817. The critical discussion of the proof of the parallelogram law by **A. H. C. Westphal** (Diss. Göttingen, 1868) is neither complete nor clear, with its all-too-abstract schematization of it. A thorough presentation of that generally more methodological-mathematical question that extends to the most recent times would probably not be trivial, despite **Mach**'s disparaging judgment of it (*Mechanik*, pp. 48). Proofs of the parallelogram law no longer appear at all in recent presentations of mechanics. **O. Heaviside** (*Electromagnetic Forces*, 1, pp. 147, London, 1883) remarked: "Is it not sufficient to recognize that a quantity is a vector, to know that it follows the laws of the geometrical vector?" **A. E. H. Love** (*Mechanics*, pp. 89) said, more precisely: "We *define* the force exerted upon the particle *m* to be a vector localised at a point." On that subject, one should confer the remark in footnote 61.

([110]) **Poisson**, "Du parallélogramme des forces," Corresp. de l'éc., polyt. **1** (1804/8), pp. 357. *Mécanique*, 1, pp. 6 in the edition by **Stern**.

([111]) That was done before by **d'Alembert**, Paris, Mém. de l'Acad. (1769), pp. 278. Another functional equation is in **d'Alembert**, Opusc. math. **6** (1773), pp. 360. A simpler one is in **O. Schlömilch**, Zeit. Math. Phys. **2** (1857), pp. 85. – **P. S. Laplace**, *Méc. céléste*, pp. 3-8 gave a proof that first determined the *magnitude* of the resultant in rectangular components but found its *direction* by infinitesimal considerations that were not as clear. **Cauchy** did something similar (*Exerc. de math.*, 1826, pp. 29), but he found the direction only by a functional equation. Similar things were also in **W. G. Imschenetzky**, Kharkow Ges. (2) **2**, pp. 108. Other proofs are in **E. Brassienne**, Nouv. ann. de math. (3) **1** (1882), pp. 320. **Delèque**, *ibid.* (2) **12** (1873), pp. 495. The presentation by **Möbius**, *Statik*, pp. 22-39 [cf., also J. f. Math. **42** (1851), pp. 170] is distinguished by the careful implementation of its assumptions, as well as **Poinsot**, *Élémens de statique*, 11th ed., pp. 11-25.

$$\varphi(x + z) + \varphi(x - z) = \varphi(x) \cdot \varphi(z) .$$

Since $\varphi(0) = 2 \cos 0$, $\varphi(\pi / 3) = 2 \cos \pi / 3$, it will then follow that *for all rational value of a*:

$$\varphi\left(\frac{a\,\pi}{3}\right) = 2\cos\left(\frac{a\,\pi}{3}\right) ,$$

so when one extends to the irrational domain, one will have:

$$\varphi(x) = 2 \cos x .$$

One can then go on to the case of two equal forces by means of the transition from rectangular forces to the general case ([112]) by **Bernoulli**'s elementary considerations.

Boltzmann sought to avoid the objection that **Bour** ([113]) and **Mach** ([114]) made that the resultants must be assumed to be *intrinsically* angle bisectors, in addition to the constraints of single-valuedness, continuity, and differentiability, by a somewhat-different arrangement of **Poisson**'s representation ([115]).

A certain resolution of the whole question was achieved by **Darboux** ([116]), who deviated from the considerations of most of the earlier authors ([117]) by drawing upon *spatial* ones, and thus avoided **d'Alembert**'s functional equation. Under the assumptions that the *resultant of n vectors* $P_1, P_2, \ldots, P_n$, is *first of all*, single-valued, *secondly*, it does not change when one replaces arbitrarily many of the P with their resultants, and *thirdly*, it is independent of the position of the vectors relative to the coordinates system, that will imply that according to the parallelogram rule, that resultant is the vector that is defined by:

$$\varphi(P_1),\ \varphi(P_2),\ \ldots,\ \varphi(P_n) ,$$

in the event that φ means an arbitrary function, and every $\varphi(P_i)$ will be carried by the direction of P_i for positive values. Under the assumption that *equally-directed vectors can be added*, it will then follow that:

$$\varphi(P + Q) = \varphi(P) + \varphi(Q) ,$$

from which it will follow that $\varphi(x) = A \cdot x$ when one assumes, *fourthly*, that either $\varphi(x)$ is *continuous* or that $\varphi(x)$ is *positive* ([118]).

([112]) **D. Bernoulli**'s, footnote 108, pp. 134, prop. III.
([113]) **Bour**, *Mécanique*, 2, pp. 45.
([114]) **Mach**, *Mechanik*, pp. 48.
([115]) **Boltzmann**, *Mechanik*, pp. 31.
([116]) **G. Darboux**, Bull. sci. math. **9** (1875), pp. 281; it also appeared as Note 1 in vol. 1 of **Despeyrous**'s *Mécanique*, pp. 371.
([117]) Such a thing is already found, in part, in the proof that **Andrade** attributed to **A. Morin** in *Mécanique*, pp. 357, which **F. Sciacci** once more adopted in Napoli, Rend. Accad. reale (1899).
([118]) More generally, **Darboux** showed in Math. Ann. **17** (1880), pp. 56, that the equation $\varphi(x) = A \cdot x$ will already follow when one replaces assumptions (4) in this treatise with just the assumption that $\varphi(x)$ assumes only positive or negative values whose absolute values lie below a *finite* limit in any finite interval.

C) The basic concepts of dynamics.

20. Galilei and Newton's Principia. – Galilei recognized the possibility of giving a simple description of natural phenomena in the concept of *acceleration* ([119]). It is very remarkable that the concept of velocity will not suffice for that ([120]), but would bring about the greatest complications, and that on the other hand, it has not been necessary up to now to go on to the concept of higher-order accelerations, which would, however, characterize the process of motion in a much more general way. Based upon **Galilei**'s ideas, one erects the edifice of **Newton**'s doctrine, which defines the acceleration to be $\varphi = K : m$ as the *effect of the force K* with the help of the concept of the *mass m*.

Newton placed four *definitiones* and three *axiomata* sive leges motus (or laws of motion) at the pinnacle of his *Principia:*

1) *Definitiones* ([121]).

1) Quantitas materiae est mensura ejusdem orta ex illius densitate et magnitudine conjunctim. (The quantity of matter is the measure of a thing that arises from its density and size, when taken together.)

2) Quantitas motus est mensura ejusdem orta ex velocitate et quantitate materiae conjunctim. (The quantity of motion is the measure of a thing that arises from its velocity and quantity of matter, when taken together.)

3) Materiae vis insita est potential resistendi, qua corpus unum quodque quantum in se est, perseverat in statu suo vel quiescendi vel movendi uniformiter in directum. (The inherent force of matter is its resistive potential, by which each and every quantity that is intrinsic to a body persists in its state of rest or uniform motion in a direction.)

4) Vis impressa est action in corpus exercita, as mutatum ejus statum vel quiescendi vel movendi uniformiter in directum. (Impressed force is the action exerted on a body that changes either its state of rest or of moving uniformly in a direction.)

2) *Axiomata sive leges motus* ([122]).

1) Corpus omne perseverare in statu quo quiescendi vel movendi uniformiter in directum, nisi quaternus a viribus impressis cogitur statum illum mutare. (The whole body must remain in a state of rest or of moving uniformly in a straight direction unless the body is forced to change that state by the forces exerted upon it.)

([119]) **Galilei**, *Opere* 2, cf., in particular, pp. 261, *de motu acceleratio* for the definition of acceleration.

([120]) Naturally, that is understood to mean when one looks back upon the actual development of mechanics, which defined motions by differential equations for the trajectories of points. The energetic pictures, which (perhaps rightfully so) dispense with such detailed pictures, would generally allow a different description of processes to still be conceivable.

([121]) *Principia*, pp. 1.

([122]) *Principia*, pp. 12.

2) Mutationem motus proportionalem esse viri motrici impressae et fieri secundum lineam rectam, qua vis illa imprimatur. (The change of motion is proportional to the motive force impressed upon it, and it takes place along the straight line by which the force is impressed.)

3) Actioni contrariam semper et aequalem esse reactionem, sive corporum duorum actiones in se mutuo semper esse aequales et in partes contrarias dirigi. (There is always an opposite and equal reaction to an action, or the actions of two bodies on each other are always equal and directed in opposite directions.)

21. The dynamical study of motion. – Based upon the foregoing general principles, the system of classical mechanics in the Nineteenth Century gradually developed in conjunction with a well-made phoronomy, but we shall not go further into a discussion of the questions connected with the representation of space and time.

The pure study of motion, which describes the position of a point by its three coordinates x, y, z, which are given as functions of time t, starts from the idea that under a motion along an arbitrary path, the:

$$\lim \frac{\Delta s}{\Delta t} \qquad \text{for} \qquad t = t_0$$

will represent a well-defined limiting value relative to the path length $s - s_0 = \Delta s$ that was traversed and the associated time interval $t - t_0 = \Delta t$, namely, the *speed* v of the point. If the motion is *projected* onto an axis X, say by parallel projection, then the speed of the projected motion will be equal to the projection of v. The motion will be determined completely by its projections onto three coordinate axes x, y, z, which are most conveniently mutually rectangular, and:

$$\dot{x} = \frac{dx}{dt} = f_1(x, y, z, t) ,$$
$$\dot{y} = \frac{dy}{dt} = f_2(x, y, z, t) ,$$
$$\dot{z} = \frac{dz}{dt} = f_3(x, y, z, t)$$

are three first-order differential equations that determine the coordinates x, y, z as single-valued functions of t for a given initial position (x_0, y_0, z_0, t_0).

One calls the autonomous location that each of the velocities possesses, e.g., $\dot{x}$, for an observer that participates in the projected motion in the YZ-plane, its *components*; their resultant is the *vector* v. By that consideration, which associates every motion with its orthogonal or also general parallel projections onto any axes, one will avoid *the irrelevant examination of the composition and decomposition of velocities*, while at the same time, the representation that simultaneously attributes *several* motions to a point, as a purely-logical abstraction that is generally mediated by one's imagination, will not at all correspond to any real process.

For constant $\dot{x}$, $\dot{y}$, $\dot{z}$, the path will be a straight line that is traversed *uniformly*. In every other case, the vector v is combined with an infinitely-small vector dv over the time dt, which is regarded as *a measure of the change in velocity*. Here, as well, we shall consider that *acceleration* to be a limiting value whose direction and magnitude are determined completely. From that standpoint, it is obvious that the accelerations are composed by the parallelogram law in any event. In particular, that is brought to light by **Hamilton**'s concept of the **hodograph** ([122.a]). Conversely, for a given initial state $(x_0, y_0, z_0, \dot{x}_0, \dot{y}_0, \dot{z}_0, t_0)$, the equations:

$$\begin{aligned}\ddot{x} &= f_1(x, y, z, \dot{x}, \dot{y}, \dot{z}, t),\\ \ddot{y} &= f_2(x, y, z, \dot{x}, \dot{y}, \dot{z}, t),\\ \ddot{z} &= f_3(x, y, z, \dot{x}, \dot{y}, \dot{z}, t)\end{aligned}$$

will once more define the motion completely, with suitable restrictions on the functions. The process of phoronomic concept building that is introduced in that way is therefore not excluded, but *dynamics* has no reason to make use of higher-order accelerations ([122.b]).

22. The system of classical dynamics ([123]). – One achieves the transition to the dynamics of material points in the customary representations by the following basic laws ([124]), which can be regarded as abstractions from experience:

1. Whenever a material point does not move uniformly and rectilinearly, other bodies will always be present whose positions and state of motion seem to be the conditions that determine that deviation from the inertial path. One says that *active forces* are present that *assign* the *acceleration* in question ([125]).

2. *Mass* is the property of material points to take on *accelerations* under the same conditions of that type whose scalars have different magnitudes. The *unit of mass* can be chosen arbitrarily.

([122.a]) **W. R. Hamilton**, Dublin, Trans. **3** (1846), pp. 345. *Elements of Quaternions*, London, 1866, pp. 100, 718, 2nd ed., 2 vols., London, 1899/1901. Meanwhile, the concept of a hodograph already appeared in 1843 in **Möbius**, *Mechanik des Himmels* = *Werke* 4, Leipzig, 1887, pp. 35 and 47.

([122.b]) **C. G. J. Jacobi** already referred to the theory of higher-order accelerations in 1825 (*Werke* 3, pp. 44), whose general concept had been employed by **Möbius**, J. f. Math. **36** (1838), pp. 91. **A. Transon**, "Note sur les principes de la mécanique," J. de math. (1) **20** (1843), pp. 320, introduced $m\dddot{x}$, in turn, as the *virtualité motrice*, along with $m\ddot{x}$, while **H. Resal** (*Traité de cinématique pure*, Paris, 1862) produced a systematic theory of first and second order accelerations that was developed much further by others since then.

([123]) We understand this to mean, say, the lessons that seem to have achieved general validity under the influence of the French mathematicians in the first half of the Nineteenth Century.

([124]) These foundations of dynamics mostly seem to be coupled with metaphysical ideas about inertia, causes, etc. (cf., e.g., the presentation by **E. Bour**, *Mechanik*, 2, pp. 6, **Duhamel**, and others). The presentation here intends to adapt them to the newer presentations *as much as possible*. Cf., e.g., **Streinz**, *Grundlagen*, pp. 99, *et seq.* The principle of inertia is then replaced with statement no. 1. The connection between those basis axiomatic laws and experiment will not infrequently be represented by calling upon entirely-uncontrollable observations excessively. For example, **Coriolis** said in *Mécanique*, pp. 5: "If the force *F varies* during the duration of motion then *observation* will show that the ratio φ (acceleration) varies in the same way, i.e., the values of φ will remain proportional to the values of *F*."

([125]) Even **Kirchhoff** and **Mach** spoke of *active* forces. According to **A. Höfler**, *Studien*, pp. 56, action is a special type of relation, namely, a relationship of necessity between realities of a certain type whose representation does not include any anthropomorphic elements at all.

3. *The unit of force* is the one that assigns an acceleration of unity to a unit of mass. *Any mass* that obtains a unit acceleration "by means of" a unit force represents a unit mass.

4. If a unit mass is acted on by several unit forces *at the same time* then each of those forces will act precisely as if the remaining ones were not present at all. One thus arrives at the representation of the *force n* that assigns the acceleration *n* to a unit mass.

5. A material point that takes on a unit acceleration under the influence of force *n* possesses a *mass* of *n*. A material point of mass *m* will then take on the acceleration $\varphi = n$ by the force $F = mn$. Thus:

$$F = m\,\varphi\,.$$

6. The *proportionality of mass and weight*, in its simplest form, can be confirmed by experiments for ponderable material points, e.g., by **Atwood**'s machine ([126]).

23. Critical remarks about the system of dynamics. – The fact that **Newton**'s *definitiones* were, in part, chosen quite carelessly, e.g., *quantitas materiae* was explained in terms of *densitas*, *vis* was explained in terms of *actio*, has often been remarked ([127]). It seems much less appropriate to speak of a *vis inertiae* ([128]). This is not the place to treat those formal questions. However, a brief factual discussion of the axioms would seem imperative.

1. *The independence of the axioms.* The difficulties that oppose one's understanding of **Newton**'s axioms are much greater. They have produced an extensive body of literature, only the most essential of which can be cited here. We shall first mention the questions that relate to the mutual independence of the statements. For example, the fact that the first law is included in the

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([126]) **F. Bessel**'s papers "Versuche über die Kraft, mit welcher die Erde Körper von verschiedener Beschaffenheit anzieht," Berlin, Anhandl. d. Akad. (1830), pp. 41 and Ann. Phys. Chem. **23** (1832), pp. 401 belong to that category. **Bessel**'s investigations were recently taken up again, with more precise tools, by **R. v. Eötvös**, "Über die Anziehung der Erde auf verschiedene Substanzen," Mathematische und naturwissenschaftliche Berichte aus Ungarn **8** (1891), pp. 65; cf., also Ann. Phys. Chem. (2) **50** (1896), pp. 354.

([127]) That opinion was not shared by all. **Thomson** and **Tait** said [*Treatise* (1) 1, pp. 279]: "The introduction to the Principia contains *in a most lucid form* the general foundations of dynamics." **Volkmann** made a similar statement in *Theor. Physik*, pp. 70. It is customary in England, in particular, to read into **Newton**'s brief laws much more far-reaching conceptions, e.g., **d'Alembert**'s principle, the conservation of energy, etc. We shall leave those issues unresolved (which are by no means generally maintained even in England). Meanwhile, if one observes that despite its flawed superficial form, all of dynamics has been developed from **Newton**'s principles without one finding it inadmissible up to recent times to increase the number axiomatic statements then one will be just as inclined to guard against assigning *too little* to **Newton**'s ideas. Moreover, the fact that **Newton**'s *leges*, which are suitable for the demands of *astronomical mechanics*, are *not* sufficient for deducing the mechanics of *constrained material systems*, can hardly be doubted at present.

([128]) The *vis inertiae*, viz., the so-called *power of persistence* (*Beharrungsvermögen*, i.e., inertia) can seem to be an entirely-irrelevant scholastic concept. **Euler** had already rejected it in the strongest terms in his *Reflexions sur l'espace*. On the occasion of his quite-useful *concept of the deviation* of the motion of a material point, **A. M. Ampère** developed, in turn, the picture that the *force d'inertie* was always equal and opposite to the active force [J. de math. **1** (1836), pp. 211]. Discriminating between forces of motion and accelerating forces or *forces d'inertie* (which currently appear in **d'Alembert**'s principle as $-m\ddot{x}$, $-m\ddot{y}$, $-m\ddot{z}$) does not seem to be very fortunate, in any event. That is also the basis for the false conceptions about the *centrifugal force*, in which **Hertz**, in turn, believed he glimpsed an essential flaw in conventional mechanics (*Mechanik*, pp. 8). On this, one might cf., **Boltzmann**, *Mechanik*, pp. 45.

second one is probably generally recognized. Obviously, those questions can be resolved only by a purely *logical-mathematical* examination, which we shall not go into here. It also still seems *doubtful* whether the attempts to do that, which appear in the English literature, in particular ([129]), have already arrived at an actual *conclusive* result. One might recall the difficulties that analogous investigations have indicated in the much-clearer realm of geometry, and which have been advanced decisively by **Hilbert** in recent times ([130]).

2. *The concept of mass*. **Newton** and **Poisson** explained mass as the *quantity of matter* ([131]). **Hertz** ([132]) referred to a mass-particle as a feature by which one can distinguish *one* geometric point from *another* and arriving at a purely-phoronomic concept of mass comes down to enumerating hypothetical atoms in a certain unit of volume, while he made the "ponderable" masses of dynamics proportional to weight. French mathematicians ([133]) define mass by the ratio of force to acceleration. Obviously, one must first define force, which can be defined by weight only in the static sense. For **Kirchhoff** ([134]), mass simply seemed to be a numerical coefficient that is introduced in the differential equations of motion of *systems* that *does not eliminate* the possibility of a unique description of motion, so it remains completely undefined. According to **Andrade** ([135]), the phenomenon of the *collision* of two material points is constructed from a concept of mass *that is entirely independent of the assumption of a certain coordinate system and a previously-chosen unit of time*. Others are content with mere nominal definitions, such as, e.g., when mass is referred to as the acceleration capacity ([136]).

([129]) Cf., in particular, **J. MacGregor**, "On the fundamental hypotheses of abstract dynamics," Trans. Roy. Soc. Canada **3** (1892). Likewise, Phil. Mag. (5) **35** (1892), pp. 134; *ibid.* **36** (1893), pp. 243, as well as Trans. Roy. Soc. Canada **6** (1895), pp. 95.

([130]) **D. Hilbert**, "Über die Grundlagen der Geometrie," Festschrift zur Enthüllung des Gauss-Weber-Denkmals zu Göttingen, Leipzig, 1899; also in French, *Les principes fondamentaux de la géométrie*, transl. **L. Laugel**, Paris, 1900.

([131]) One also finds that in **Kant**'s *Metaphysische Anfangsgründe*, pp. 95. That expression is also presently used by physicists, such as **W. Voigt**, *Kompendium*, 1, pp. 14. It makes sense only when one speaks of an abstract material that is capable of superposition, and assumes states of different density in that way. **L. M. N. Carnot** (*Principes*, 1803) did not define mass at all.

([132]) **Hertz**, *Mechanik*, pp. 54. Similar statements are also in **Schell**, *Mechanik*, 1, pp. 2, 72.

([133]) E.g., **Duhamel**, *Mécanique*, 1, pp. 132.

([134]) **Kirchhoff**, *Mechanik*, pp. 13-21. Similarly, **C. Neumann**, *Grundzüge der analytischen Mechanik*, Leipziger Ber. **39** (1887), pp. 155. The fact that **Kirchhoff**'s introductory considerations are in striking contrast to the careful analytical representation of the later parts of the book, which raise objections only in a few places, is probably not in doubt nowadays.

([135]) **Andrade**, *Mécanique*, pp. 54. Naturally, the phenomena of collisions, in and of themselves, were already called upon much earlier. For experimental tests, one might cf., e.g., **W. M. Hicks**, *Elementary Dynamics of Solids and Fluids*, London, 1890.

([136]) For example, **A. Höfler**, *Studien*, pp. 70. **Ch. Freycinet** (*Essais sur la philosophie des sciences*, Paris, 1896, pp. 177) compared mass with the specific heat, as the *capacité dynamique*. For **Euler**, mass was the quantity of inertia (*Theoria motus*, § 153). **Volkmann**, *Theoret. Physik*, referred to the concept of mass as quantity that prevents change in phenomena, so as a *postulate*. Naturally, one would have to keep that in mind in any discussion of the subject. One can initially regard the mechanics of points with *variable masses* that **Jacobi** (*Dynamik*, ed. by **A. Clebsch**, pp. 57) alluded to as an entirely-abstract extension of the differential equations of dynamics. Meanwhile, it can find application to certain questions (e.g., the gradual changes of mass of a celestial body by the absorption of cosmic mass). However, one is not actually dealing with that abstraction, but with an investigation of collision phenomena. One might cf., **J. Mesterchesky**, "Dynamik des Punktes mit verändlicher Masse," article in the Fortschr. d. Mathematik **28** (1897), pp. 645.

Mach ([137]) treated the concept of mass most thoroughly. According to him, bodies of equal mass are ones that impart equal and opposite accelerations to each other when they act upon each other. Naturally, that statement includes the assumption, which agrees with experiment and was based upon still-further ones by **Mach** ([138]), that two bodies A_1 and A_2 that have equal masses relative to a unit body A will also behave that way relative to any other reference body B. That concept, which is based upon the most-recent presentations ([139]) and comes close to what **Andrade** proposed, seems to recommend itself to the same degree as the latter. A general unification of them has not resulted up to now.

Furthermore, there seems to exist no obstacle to assuming the very-popular standpoint and simply initially making the mass of a ponderable body proportional to its weight ([140]), while calling upon certain facts of experience regarding its dependency upon position, time, and other physical states, and then introducing other masses in order to preserve the analogy that would have to enter into the consideration of hidden masses, as **Hertz** did. Moreover, in that sense, one must always introduce non-gravitating (e.g., electric, magnetic, etc., …, positive and negative) masses ([141]), in which one reduces their units to the unit of force that was obtained already. That will also be necessary if the concept of *work* is to take on a general sense.

3. *The principle of inertia*. **Newton**'s first axiom is known by the name of the **Galilean** principle of inertia: A point that is represented as independent of all other bodies will move uniformly and rectilinearly when left to itself ([142]). Along with the aforementioned difficulties that relate to the representation of unaffected points, this axiom seems entirely *superfluous*. Moreover, it is *delusional* ([143]) for one to believe that one can *verify it by experiment*. one can generally see that the deviations from the inertial part will become smaller and smaller the more one eliminates certain "circumstances" that affect the motion. However, the fact that *when* the point moves uniformly and rectilinearly, such circumstances will *no longer* be present is already assumed and lies beyond the reach of any possible experiment ([144]). The principle of inertia will have an actual

([137]) **Mach**, *Mechanik*, pp. 210.

([138]) **Mach**, *Mechanik*, pp. 213. In conjunction with that conception of mass and the fact of experience that the accelerations that several bodies A_1, A_2, A_3 impart upon a body A are independent of each other (see the fourth indented line in the book), **Mach** (*Mechanik*, pp. 242) arrived at a system of statements that seems suitable to replace the one in no. **22**.

([139]) See **Love**, *Mechanics*, pp. 87. **Maggi**, *Principii*, pp. 150. **Boltzmann**, *Mechanik*, pp. 22.

([140]) The objections to that opinion are well-known, cf., e.g., **G. A. Greenhill**, "On weight," Nature **46** (1892), pp. 247; *ibid*., **51** (1894), pp. 105. However, on closer inspection, they ultimately seem to be no greater than the difficulties that exist in the definition of a physical unit of length.

([141]) **A. Coulomb**, Paris, Mém. de l'Acad. (1745), pp. 569, **Ostwald**, K. B., no. **13**; **C. F. Gauss**, *Intensitas vis magneticae terrestris*, etc., 1833 = *Werke*, 5, pp. 79.

([142]) **Galilei**, *Discorsi*, third and fourth day, **Ostwald**, K. B., no. 24, pp. 57, 81. According to **E. Wohlwill**, Zeit. f. Völkerpsychologie u. Sprachwissenschaft **15** (1883/84), pp. 101, **Galilei**, who did not at all seem to find an *a priori* basis for the principle of inertia in the sense of footnote **22**, restricted his statement to the processes on the surface of the Earth, moreover. Its extension to celestial mechanics goes back to **Newton**.

([143]) On the logical difficulties that are inherent to the law of inertia, and are, for the most part, created by misunderstandings, cf., **F. Poske**, "Der empirischer Ursprung von der Allgemeingültigkeit Beharrungsgesetzes," Vierteljahrsschr. f. wiss. Philosophie **8** (1884), pp. 385, with an addendum by **W. Wundt**, pp. 405; likewise, **L. Weber**, *Über das Galile'sche Prinzip*, Kiel, 1891; **P. Johannesson**, "Über das Beharrungsgesetz," Schul-Programm, Berlin, 1896, no. 98.

([144]) One might probably imagine a theory of mechanics in which the motion of the unaffected points is completely different from this, as e.g., **F. Reech** did in *Cours de mécanique*, cf., footnote 72; then see **J. Andrade**, *Mécanique* sense, but also *only* a metaphysical one, only when one also assumes a real absolute unit of time, along with **Newton**'s real space in which the motion of the isolated points can be considered.

4. *The concept of force*. The purely-dynamical *definition of force as merely an abbreviated way of saying that a mass-particle possesses a certain component of acceleration* generally has no connection with the static concept of force. However, one will be all the more inclined to *abandon* the latter, and thus to abandon the *duality between statics and dynamics* that **Gauss** ([145]) had already criticized as a satisfactory representation of static processes as ones for which the acceleration is zero, as not successful by itself, but at the same time, further assumptions seem to be dispensable. The *principle of independence* ([146]), which is included in **Newton**'s *lex secunda*, in the opinion of many English authors, and is based upon the parallelogram law, was emphasized only later in the French school with especial vigor. It will seem to become irrelevant when one *adopts the study of the composition of accelerations* as simply a requirement for the description of phenomena by the concept of force ([147]). As an expression for the basic representation of dynamics, one now has the *equations of motion of a free material point:*

$$m\ddot{x} = X,$$
$$m\ddot{y} = Y,$$
$$m\ddot{z} = Z,$$

in which X, Y, Z should be regarded as the values of the force components that one obtains by observing the accelerations and masses (the static effects, resp), according the concept of force that one uses as a basis ([148]).

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physique; similar statements were also expressed by **H. Poincaré** and **P. Painlevé** [Revue de métaphys. (8) **5** (1900), pp. 557]; **Jacobi** had already gone much further in his *Vorlesungen*, 1847/48, pp. 1.

([145]) **Gauss**, J. f. Math. **4** (1829), pp. 233; "However, that is what the spirit of the inverse process demands, in which all of statics appears to be a special case of dynamics." **Laplace**, *Méc. cél.*, pp. 16, would still like to *prove* the proportionality of force and acceleration, as well as **Poisson**, *Mécanique*, 2nd ed., **1**, § 116.

A general union of those two conceptions of force has not been achieved up to now. Whereas abstract dynamics, of which statics is only a *limiting case*, considers the concept of $m\,\varphi$ to be sufficient, the representations of mechanics that are directed towards applications initially start from static phenomena (as the apparently *simpler* ones), i.e., from the measurement of forces by *weight* or the deformations of elastic systems (e.g., *spring scale*, *dynamometer*), resp. Cf., the detailed explanation in **Ch. Freycinet**, *Essais sur la philosophie*, Paris, 1896, pp. 153, *et seq.* **A. Ritter**, *Lehrbuch der analytischen Mechanik*, 2nd ed., Leipzig, 1883, pp. 66, whose view was certainly shared by many, indeed defined force by $m\,\varphi$, but be preferred to measure mass itself only by forces. "Rather, one must assume that there is yet another experimentally-accessible criterion for recognizing the equality of two forces," namely, the identity of the deformations in the dynamometer. Similar statements are in **A. Ritter**, *Lehrbuch der technischen Mechanik*, 7th ed., Leipzig, 1896, pp. 42; cf., also, **E. Budde**, *Mechanik*, **1**, pp. 111.

([146]) See the version of it in **Duhamel**, *Mechanik*, **1**, pp. 338; **E. Bour**, *Mécanique*, **2**, pp. 7. **D. Bernoulli** [Comm. Ac. Petrop. (1728), pp. 126] was *the first* to express it. **Poisson** based his mechanics in 1811 upon it (pp. 277). In the 2nd ed., **1**, pp. 172, he sought to abandon it by a consideration whose untenability had already been pointed out by the translator **M. A. Stern**. The arbitrary character of the independence principle, which is still suitable for a convenient description of the processes of motion, has probably been rarely in doubt. **Poincaré** (cf., footnote 144) recently remarked that the *conservation of magnetic masses* is not directly consistent with the law. On that topic, cf., **Painlevé**'s remark "The principles of mechanics are conventions that experiments can never contradict."

([147]) As is, e.g., **Tait**, Encycl. Brit. 9th ed. (1881), article "Mechanics," pp. 701; **MacGregor**, "On the hypotheses…," footnote 129. **F. Muirhead**, "On the laws of motion," (5) **23** (1887), pp. 489.

([148]) There exists no obstacle to regarding those values of X, Y, Z as depending upon the state of motion, i.e., x, y, z, $\dot{x}$, $\dot{y}$, $\dot{z}$. If they also include the $\ddot{x}$, $\ddot{y}$, $\ddot{z}$ then that will imply an analogous problem when one solves for them.

5. The law of *action and reaction*, or **Newton**'s *lex tertia*, which was not taken up in no. **22** in the study of classical dynamics, is not required for the academic example of the motion of a free point, but it contains exactly the most important part of **Newton**'s mechanics for the consideration of material systems. According to **Hertz**, who *proved* that it is a consequence of his basic law for the actions between *subsystems* ([149]), when that law is extended beyond the former case, it will become an unprovable, perhaps even incorrect, statement. More general forms of the law appear in electrodynamics and the theory of elasticity, moreover ([150]). It does not seem incorrect that **Petrini** ([151]) saw in the law a method for establishing suitable limits on the investigations that is useful in the interests of simplicity.

24. Instantaneous forces, impacts, or impulses ([152]). – Since every force K will produce the rectilinear *deflection* or *deviation* ([153]) in direction:

$$\frac{1}{2}\frac{K}{m}\,dt^2$$

in the time interval dt, while it produces the infinitely-small velocity:

$$\frac{K}{m}\,dt\,,$$

abstract mechanics is not initially in a position to explain the apparently sudden changes in the state of motion that one believes to be perceived during collisions. Without going into the earlier treatment of that question by **Huygens** ([154]) and others, let it only be remarked that it does not seem possible to object to the *presentation* ([154.a]), since one treats continuous processes with it that evolve over such a short time interval ([155]) that an apparent discontinuity arises. When X is regarded as very large in the formula:

$$m\,\ddot{x} = X$$

or

However, things will be completely different when the forces depend upon even higher derivatives, since the initial state (in the usual sense) would no longer suffice to determine it then. That fact has not always been emphasized enough.

([149]) **Hertz**, *Mechanik*, pp. 215.

([150]) One extension of it is in **Volkmann**, *Theoretische Physik*, pp. 131. Another in a different direction is in **Voigt**, *Kompendium*, **1**, pp. 79.

([151]) **H. Petrini**, footnote 11, pp. 221.

([152]) Naturally, the choice of terminology makes no difference. The expression *impulse* [which was chosen by **Thomson** and **Tait**, *Treatise* (1) **1**, pp. 282, although it also meant nothing but impact] in **E. Budde**, *Mechanik*, 1, pp. 411, seems to be in more general usage. Cf., **Klein** and **Sommerfeld**, *Theorie des Kreisels*, pp. 69, *et seq.*

([153]) That formula was first used by **Euler**, "Découverte d'un nouveau principe de mécanique," Berlin, Mém. de l'Acad. 1750 (1752), pp. 185; *Theoria motus*, § 169.

([154]) On the subject of **Huygens** and his predecessors, cf., e.g., **Mach**, *Mechanik*, pp. 300-325; likewise, **Galilei**'s investigations of collisions in **Ostwald**, K. B. no. 25, pp. 37.

([154.a]) In the mechanics of **Hertz**, pp. 288, the presentation is *just the opposite*. He dealt with true discontinuities in the motion that would be described by means of the existing time integrals of the accelerations.

([155]) That is the *time of impact* for the English authors, cf., **Thomson** and **Tait**, *Treatise* (1) **1**, pp. 274.

$$m(\dot{x} - \dot{x}_0) = \int_{t_0}^{t} X\,dt = P$$

in an extremely-small time interval $t - t_0$, the right-hand side can even assume a value P that lies below a finite limit with an order of magnitude that is comparable to the usual one that one refers to as the *intensity of the impact or impulse*; on the left-hand side is the increase in the *quantity of motion* (*quantité de movement*, momentum). The instantaneous forces of *impulse* will then measure the quantity of motion that is created in an extremely-small time interval. Since one has, at the same time:

$$x - x_0 = \int_{t_0}^{t} \dot{x}\,dt ,$$

the material point m will experience an increase in velocity of $\dot{x} - \dot{x}_0$ over a change in position of order $t - t_0$ ([156]). In the first approximation, a *sudden change in velocity with an unchanging position of the point* is allowed, and one can speak of the value of the interval as $t - t_0$ converges to zero. Although that conception of things, whose absurdity is concealed only formally by the integral sign that is employed ([157]), is so unsatisfactory in many respects, no contradiction has emerged in that presentation to date since it seems to be in satisfactory agreement with observations and has been subsequently built up by the French school especially.

The equivalence of the representation of the quantity of motion of a material point or mass m by $m\,v$ and an impulse $P = m\,v$ that suddenly communicates a velocity of v to a point at rest is not just incidental, moreover. In particular, it is indicated when one considers the generalized impulse $p_s = \partial T / \partial \dot{q}_s$ (cf., no. **26**) to be one that assigns the velocities $\dot{q}_s$ to the generalized coordinates q_s. It enters, in turn, as something that is *on an equal footing* with the continuous forces from the outset. Moreover, in that way, one does not actually deal with a discontinuous conception of things since the impulse itself is once more regarded as continuously varying, in general. **D'Alembert** seems to have arrived at his principle originally by considering impulses ([157.a]). **Lagrange** also considered *both* representations equally in his *Mécanique*, which emerged later on in **Poinsot**'s synthetic-dynamics theory of motion of rigid bodies especially ([157.b]).

That generalized conception of the notion of impulse, upon which **Maxwell** based his generally-disputed derivation of the differential equations of dynamics, as well as the rich content of the various theorems that **Thomson** and **Tait** developed by pursuing the theorems of **Carnot**

([156]) Naturally, completely-different cases are theoretically conceivable since one is dealing with *limiting values* to which the integral should converge. For example, there is the one where a *finite* displacement results in a time interval that converges to zero *with no* change in velocity. Examples of that kind are in **G. Darboux**, Bull. sciences math. (2) **4** (1880), pp. 128.

([157]) That integral from **Poisson**'s *Mécanique* is found in **Duhamel**, *Mechanik*, **2**, pp. 81. In particular, one might cf., **G. Darboux**, "Étude géométrique sur la percussion et le choc les corps," C. R. Acad. Sci. Paris **78** (1874) and Bull. sciences math. (2) **4** (1880), pp. 126; one will also find instantaneous frictional impulse there.

([157.a]) See **d'Alembert**'s dynamics in **Ostwald**, K. B., no. 106, pp. 138.

([157.b]) **L. Poinsot**, *Théorie nouvelle de la rotation*, Paris, 1834, and then J. de math. (1) **6** (1851), pps. 9 and 289; "Sur la percussion des corps," *ibid.* (2) **2** (1857), pp. 281 and (2) **4** (1859), pp. 421; cf., also **Schell**, *Theorie der Bewegung*, **2**, pp. 352, *et seq.*.

and **Bertrand** ([157.c]), recently came into play in the German literature in a methodically-developed form in **Klein** and **Sommerfeld**'s *Theorie des Kreisels* ([157.d]).

25. Pressures and surface forces. Generalized concept of force. – Under the assumption of a continuous distribution of mass, one will see that it is necessary to introduce surface forces and pressures, along with the forces that are given directly, which are ordinarily denoted by X, Y, Z (per unit mass), so for a mass of $\rho\, dt$ they will be equal to $X\, \rho\, dt$, $Y\, \rho\, dt$, $Z\, \rho\, dt$. Dynamically, the representation of a force of pressure that is spread over a surface creates certain difficulties since those forces are no longer attached to the mass-particles, but appear only as *static resultants* that appear between rigid massless separation surfaces according to the laws of equilibrium. **Cauchy** ([158]), in particular, based his representation of an internal pressure in a continuous mass distribution in that way. Many statements by other authors suggest that he did not develop it completely by any means ([159]).

We shall conclude with a brief reference to *further generalizations of the concept of force*. The fact that mechanics seems to be inclined to treat not only moving forces, but ultimately state-varying ones that are fully general, was already pointed out in no. **3**. However, the laws that are grasped in the sense of the former were also extended in many ways. **Weber**'s law already led to forces that depend upon accelerations, which **C. Neumann** ([159.a]) once more subordinated to the distinction between an *emissive potential* and a *receptive* one, i.e., to a *temporal propagation of the action at a distance* of the ordinary assumptions. The many speculations on the law of distant force that is connected with the study of gravitation suitably delimits it by the demand that a stable state of equilibrium can produce electric masses. However, the extensive investigations that **Koenigsberger** ([159.b]) has carried out since 1896 on the analogies that emerge from the assumption of a kinetic potential of the most general type in the use of **Hamilton**'s principle are of a purely mathematical nature, but still important for the methodology of dynamics.

([157.c]) Cf., **Thomson** and **Tait**, *Treatise* (1) **1**, pp. 284; **E. J. Routh**, *Dynamik*, 1, pps. 335, 350, as well as the literature that is given there. The consideration of initial motions, which is peculiar to the English literature, also belong to this, cf., e.g., **Routh**, *Dynamik*, 1, pp. 420.

([157.d]) **F. Klein** and **A. Sommerfeld**, *Theorie des Kreisels*, Leipzig, 1897, pp. 69, *et seq.*, 93, etc.

([158]) **Cauchy**, *Exerc. de math.*, 1827 = *Œuvres* (2) **7**, pp. 60; 1828; *Œuvres* (2) **8**, pp. 253, *et seq.*; *Œuvres* (2) **9**, pp. 41. Likewise, cf., **Poisson**, "Sur les équations générales de l'équilibre…des corps solides élastiques et fluids," J. Éc. polyt. **20** (1831), pp. 1. The presentation in **Kirchhoff**, *Mechanik*, pp. 110, is entirely abstract, in which he arrived at a determination of the concept of internal pressure by means of **Green**'s partial integration.

([159]) **Duhem**, *Le potentiel thermodynamique*, Ann. Éc. norm. (3) **10** (1893), pps.186, 213. Cf., also, **J. Larmor**, *Aether and Matter*, pp. 270.

([159.a]) **C. Neumann**, *Die Prinzipien der Elektrodynamik*, Tübingen, 1868; The picture of a temporal propagation appeared already in 1845 in **Gauss** (letter to **Weber**), *Werke* **5**, pp. 269. – **C. Neumann**, *Allgemeine Untersuchungen über das Newton'schen Potential*, Leipzig, 1896, pp. 227; cf., **H. Seeliger**, "Über das Newton'schen Gravitationsgesetz," Münch. Ber. **28** (1896), pp. 373.

([159.b]) **L. Koenigsberger**, *Die Prinzipien der Mechanik*, Leipzig, 1901. On pp. 127, the **Neumann** representation that was mentioned in footnote 159.a was once more subordinated to the usual one in an entirely-different way, namely, by contracting hidden motions.

D) The purely-kinetic theories.

26. The elimination of force in the kinetics of W. Thomson (Lord Kelvin). – Finally, we shall move on to some of the recent advances in mechanics. From the **Lagrange** differential equations of motion ([160]) in general coordinates (cf., no. **37**):

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_s} - \frac{\partial T}{\partial q_s} = Q_s \,,$$

one has the right to refer to the Q_s as a *generalized force*, since it is the force component with respect to the q_s, which also finds its basis in the expression for the *work:*

$$\sum Q_s \, dq_s \,,$$

which is *invariant* under arbitrary transformations of q_s into other independent variables r_s . A similar generalization will appear in relation to the generalized *impulse coordinates:*

$$\frac{\partial T}{\partial \dot{q}_s} \,,$$

which takes on an especially intuitive meaning in many problems ([161]).

However, what seems more important is the train of thought that aims to *completely eliminate the concept of force* as the only one that is hard to get rid of since it comes from our accustomed way of expressing metaphysical ideas about the action of things on each other.

In particular, **W. Thomson** always preferred to maintain the opinion that *it is possible for one to develop a purely-kinetic theory of dynamics* by developing detailed pictures and models whose consequences are already known in the specialized domain of the kinetic theory of gases.

A first attempt at that was made by **Thomson** in 1876 with the help of the *properties of vortex motion* ([162]) that **Helmholtz** discovered in 1858 (in regard to that, we might refer to volume IV 16, **3**.b here), which was based upon the behavior of *vortex rings*, as structures in an ideal fluid that are indestructible ([163]) in a certain sense, and the apparent forces that they exert upon each other. **Thomson**, in turn, referred to them as the atoms of the "beings" (*Seienden*) ([164]). Meanwhile, he gave just as little of an actual general implementation of that daring *aperçu* ([165]) as he did of the

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([160]) Here, we must assume some prior knowledge of the **Lagrange** equations of dynamics that will be derived in no. **37**.

([161]) Cf., in particular, **F. Klein** and **A. Sommerfeld**, *Theorie der Kreisels*.

([162]) **Helmholtz**, "Integrale der hydrodynamischen Gleichungen," J. f. Math. **55** (1858), pp. 25; **Cauchy** already discovered this in 1815 [*Œuvres* (1), **1**, pp. 39].

([163]) In particular, the *connection* between closed vortex rings, in the sense of *analysis situs*, remains invariant.

([164]) **W. Thomson**, "On vortex atoms," Phil. Mag. (4) **34** (1867), pp. 15; Proc. Roy. Soc. Edinburgh **6** (1869), pp. 44; Trans. Roy. Soc. Edinburgh **25** (1869), pp. 217. The latter began with: "This work was undertaken in order to show that *all material phenomena* can be explained by the hypothesis that space is filled with an incompressible fluid that is acted on by no external forces." Moreover, Proc. Roy. Soc. Edinburgh **7** (1872), pp. 576, cf., also, **A. E. H. Love**, "On recent English researches in Vortex motion," Math. Ann. **30** (1887), pp. 326.

([165]) "The possibility of forming a theory of elastic solids and liquids may be anticipated," Phil. Mag. (4) **34** (1867), pp. 15. Meanwhile, **Maxwell** remarked in regard to that theory (Encycl. Brit., 9th ed., pp. 45): "The difficulties *theory of gyrostats* that he founded somewhat later, and was further addressed by his students, which essentially intended to explain the effects of force in elastic matter. Such a *gyrostat*, in its simplest form, consists of a rotating body that rotates around an axis (i.e., a flywheel). Once **Thomson** had first examined the equilibrium of a system of that type that consisted of an arrangement of massless, rigidly-coupled gyrostats ([166]), he also extended that representation to the elastic oscillations of a system about its equilibrium configuration ([167]).

27. The kinetic theory of force by J. J. Thomson. – It is only one more step from this *kinetic theory of force* that is based upon the theory of gyrostats to the entirely-abstract and completely-new twist that **J. J. Thomson** had proposed ([168]). Namely, when the equations of a force-free material system (no. **26**):

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right) - \frac{\partial T}{\partial q_s} = 0 \,, \tag{1}$$

in which the *vis viva* T includes certain generalized coordinates q_j only by way of their $\dot{q}_j$, also include no products, in the event that the remaining coordinates are denoted by q_i, one will have from (1) that:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = 0 \tag{2}$$

for $s = i$ and:

$$\frac{\partial T}{\partial \dot{q}_j} = c_j \tag{3}$$

for $s = j$.

If one now goes from T to (T) by eliminating $\dot{q}_j$ ([169]) using (3) then one will have:

$$\frac{d}{dt}\left(\frac{\partial (T)}{\partial \dot{q}_i}\right) - \frac{\partial (T)}{\partial q_i} = \frac{\partial}{\partial q_j} \sum (c_j \, \dot{q}_j) \ ,$$

in which everything on the right-hand side can now be represented as functions of q_i using (3). If one regards the right-hand side as a force function U then one will have the possibility that **J. J.**

of this method are enormous, but the glory of surmounting them would be unique." Cf., also: **W. Thomson**, "On vortex statics," Phil. Mag. (5) **10** (1880), pp. 97; **J. J. Thomson**, *On the motion of vortex rings*, London, 1883.

([166]) "On oscillations and waves in an adynamic gyrostatic system," Proc. Roy. Soc. Edinburgh **12** (1883).

([167]) "Steps towards a kinetic theory of matter," Brit. Assoc. Rep. (Montreal, 1884), pp. 613, London, 1885; likewise, "On a gyrostatic adynamic constitution for ether," (1889); *Math. and Phys. Papers*, **3**, pp. 366; **J. Larmor**, "On the propagation of a disturbance in a gyrostatically loaded medium" Proc. London Math. Soc. **23** (1891), pp. 127.

([168]) **J. J. Thomson**, Trans. London Phil. Soc. **176** (1885), pp. 307; *On some applications of dynamical principles to physical phenomena*, London, 1888; also in German: *Anwendungen der Dynamik auf Physik und Chemie*, Leipzig, 1890.

([169]) That important step was first taken by **E. J. Routh**, *Essay on the stability of motion*, (1877), pp. 61. **J. J. Thomson** called those coordinates q_j *kinetosthenic*, while **Thomson** and **Tait** [*Treatise* (1), **1**, pp. 318] called them *ignored coordinates*. Those processes were later referred to by **Helmholtz** as *hidden* motions [J. f. Math. **100** (1887), pp. 147].

Thomson ([170]), in particular, developed in the sense of **Maxwell**'s general theory of dynamics with the help of the kinosthenic coordinates q_i of interpreting the appearance of force functions in a purely-kinetic way, and thus reducing the potential energy to the kinetic energy of "ignored" masses.

28. The mechanics of H. Hertz. – This conception of things was built up by **Hertz** in his *Mechanik* into the ideal of a *completely-forceless dynamics*. **Hertz** knew of only systems of material points that are bound by constraints, and whose motion is governed by **Gauss**'s principle of least constraint, which he referred to by an expression that recalls **Newton**'s *lex prima*, namely, the *fundamental law of motion along the straightest path* ([171]). With this way of looking at things, the *material point, as an isolated object is an irrelevant fiction*, a certain sense, such that at the same time, the *purely*-mathematical example of the motion of isolated points that analytical mechanics had preferred in previous times, and also could not do without as a means of developing and extending the analytical theory, will be distinguished from the actual mechanical ones.

In general, **Hertz** also spoke of forces that actually consist of *only* values of acceleration that each *part* of a system exerts upon every other one. *One* main service that **Hertz**'s mechanics has performed is to implement *that systematic construction of forces on the basis of a purely-kinetic theory* ([172]) in complete detail ([173]), which was sketched only in very general strokes by **J. J. Thomson**. That is because the introduction of the fundamental law to which the development of his theory of dynamics reduces was already expressed by **J. J. Thomson** ([174]) in a form that was entirely similar to the representation of *brachistichrone* motion in an n-fold manifold. In general, **Hertz** was then required to regard every system as a subsystem of another, i.e., to assume that along with the *visible* masses, there were *hidden* ones that were coupled with the latter by constraints. A further implementation of those general ideas to the treatment of specific questions

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([170]) **J. J. Thomson**, *Anwendungen* (footnote 169), pp. 16, 23-97.

([171]) **Hertz**, *Mechanik*, pp. 162; "Systema omne liberum perseverate in statu sue quiesceni vel movendi uniformiter in directissimam." (Every free system remains in its state of rest or of moving uniformly in the straightest direction.) On the subject of **Hertz**, cf., **Mach**, *Mechanik*, 4th ed., pp. 269; **J. Larmor**, Report of the Brit. Assoc., London, 1900, pp. 620.

([172]) **Hertz**, *Mechanik*, pp. 207-235.

([173]) *Another one* consists of the exceptionally intuitive form in which **Hertz** had interpreted the geometry of n-dimensional manifolds for his own special purposes, as well as the system of concepts that he consistently introduced. A characteristic of it is the concept of the *quantity s of the shift* of the mass-particle m_i from the position with the coordinates x_i to the position x_i':

$$M s^2 = \sum m(x_i - x_i')^2 , \qquad 3 M = \sum m_i ,$$

as well as the *angle* ($s\ \sigma$) *between two shifts of a system:*

$$M s\, \sigma \cos (s\ \sigma) = \sum m(x_i - x_i')(y_i - y_i')$$

that correspond to the coordinates that are denoted by x_i, x_i'; y_i, y_i'. Moreover, **Duhem** had already defined force and work in precisely the same sense as **Hertz** in his *Commentaire*, 1892, pp. 269.

([174]) **J. J. Thomson**, *Anwendungen*, pp. 17; **Jacobi** had already considered the dynamical problem to be that of the brachistochrone in 1847. **R. Liouville**, "Sur les équations de la dynamique," C. R. Acad. Sci. Paris **114** (1892), pp. 1171, showed, in a different way, how *every* dynamical problem, in the older sense, can be reduced to that of geodetic lines.

still does not exist, and the scattered remarks about various aspects of them that have been added along those lines ([175]) do not lead one to expect that **Hertz**'s line of reasoning would find an essential completion along *those lines* in the following years.

A special difficulty arises here due to the fact that no sort of prescriptions exist for a suitably-limited treatment of the question, such as adding other hidden masses to a given system, but rather there can be infinitely-many solutions to the problem ([176]).

Thus, mechanics, starting from the constraint-less theory of action-at-a-distance, gradually developed into the force-less dynamics of Lord **Kelvin**, **J. J. Thomson**, and **Hertz**. The system of classical mechanics always stood at the center of them, which operated with forces *and* constraints, and which was also probably one that was initially given the top priority in the pedagogical literature. In Germany, one can presently regard **Boltzmann** and **C. Neumann** as proponents of constraint-less mechanics, while in France, that would be **J. Boussinesq**. However, the mounting demands of mathematical physics and the peculiarities of its problems seem to have gradually compelled one to believe that the picture of purely distant forces that depend upon distance must also be replaced with more general pictures in theoretical mechanics for being two narrow in scope.

([175]) According to **Mach**, *Mechanik*, pp. 253, the uniform motion of a material point on a circle r can be replaced by coupling it with a hidden mass at a distance of $2r$. See also **A. Brill**, "Über die Mechanik von **Hertz**," Mitth. d. math. Vereins in Württemberg, Stuttgart, 1899; likewise: "Über ein Beispiel des Herrn **Boltzmann** zur Mechanik von **Hertz**," Deutsche Math. **8** (1900), pp. 200.

([176]) This, **Poincaré** remarked (*Electricité et optique, préface*, pp. XII) that, according to **Maxwell**, all physical explanations are based upon two things T and V, which express the energy by a system of $4n$ quantities to be satisfied in any parameters q for a dynamical problem [with P. Stäckel's terminology, J. f. Math. **107** (1891), pp. 319] that consists of n masses and $3n$ coordinates, where n can be arbitrarily large.

IV. – THE SPECIAL PRINCIPLES OF RATIONAL MECHANICS.

A) Elementary variational or differential principles.

α. Statics.

29. The concept of equilibrium. – Very early on, for the case of *simple machines*, elementary statics sought to determine when a material system at rest under the influence of given forces would produce no motion, which is the case of *equilibrium*. However, **Varignon** ([177]) was the first to *reduce all equilibrium problems* to the combination of forces *at individual material points.* Nevertheless, in so doing, one does not deal with the question of the *rest relative* to a coordinate system, but only the case in which the action of forces produces no *change* in the state of motion, i.e., no *acceleration*. In general, it is *only in statics* that it suffices to assume just the case of rest relative to the reference system. However, for the applications to *dynamics*, it is necessary to extend the concept of equilibrium in the suggested way.

In light of that consideration, one can also define the concept of equilibrium for a *material system*, i.e., a union of arbitrarily-many material points that are coupled with each other, in part by geometric *constraints on the configuration*, and in part by the action of *internal* forces: Under the axiomatic assumption ([178]) that the constraints can likewise be *replaced* by forces, equilibrium will exist when the acceleration of any point, which is now considered to be completely free, is zero. Of course, with **Boltzmann** ([179]), one can also extend that definition to the case in which equilibrium exists in a system that is found to be in accelerated motion under the action of any group of forces when that group produces no change in that state of motion.

Meanwhile, if one wishes to extend the concept of equilibrium from the outset, and in particular, in order to apply **d'Alembert**'s principle, then one can pose the following axiom (or something similar):

If a material system A is found in any state of motion at time t then one can always combine it with a second system B that coincides with it at the moment t whose points all move with the same velocities as those of A during the time interval from t to $t + dt$ under the same geometric constraints, and then combine that with a third system C at the moment t that is similarly arranged and remains in a state of rest relative to the coordinate system during that time interval ([179.a]). The system A will be in equilibrium under the action of the applied force if and only if that is true for

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([177]) **P. Varignon**, Preface to volume 1 of his *Nouvelle mécanique:* "At last, I shall apply myself to the search for the source of equilibrium itself, or better yet, how it is generated."

([178]) If one assumes that every acceleration can also be obstructed by suitable forces then that assumption can also be pursued further. Cf., **A. L. Cauchy**, *Exerc. de math.*, 1826 [*Œuvres* (2), 7, pp. 11]. Of course, one must appeal to a further axiom in order to do that.

([179]) **Boltzmann**, *Mechanik,* pp. 233; cf., also Deutsche Math.-V.6 (1898), pp. 142.

([179.a]) The system C, which coincides with B at the moment t and whose points are at rest during the time interval from t to $t + dt$ (which is not the case for B) might seem somewhat superfluous. The introduction of such an axiom (whose conception in this book might be regarded as a first attempt) is nonetheless necessary, since a clear development of the equilibrium conditions seems to succeed only when the system in question is found to be in a state of (relative) rest with respect to the coordinate system or "frame of reference" (which is now the case for C). As soon as that is no longer relevant, because the points of the system are found to be in an arbitrary state of motion, the question of when a *new* group of forces that are introduced at the moment in question will produce no change in that state can no longer be reduced to the previous footnote directly; cf., footnote 208.

the system C under the influence of the reactions that exist in A as a result of the constraints and the applied forces.

30. The principle of virtual velocities. – The fact that equilibrium does not just relate to *special* cases of applicable constraints, but to any of them, was already recognized for simple machines by **Stevin**, **Galilei**, and others ([180]). It appeared without proof in **Joh. Bernoulli** ([181]) by means of a brilliant induction on the general rule that is known by the name of the *principle of virtual velocities or virtual displacements*. However, it was **Lagrange** who first deduced the fundamental analytical principle for mechanics from that, which he then raised to a principle of *analysis*.

We shall initially consider only systems whose material points are coupled to each other by constraint equations and are subject to any sort of given forces P.

If P_i is the force that acts upon the point with the coordinates x_i, y_i, z_i, whose infinitely-small displacements δs_i have the projections δx_i, δy_i, δz_i, then in order to have equilibrium, it is necessary and sufficient that one has:

$$\sum P_i \cos(P_i, \delta s_i)\, \delta s_i = \sum (X_i\, \delta x_i + Y_i\, \delta y_i + Z_i\, \delta z_i) = 0\,,$$

in the event that δx_i, δy_i, δz_i mean *admissible, virtual, but arbitrary* ([182]), displacements, i.e., ones that satisfy the mobility constraints on the system:

$$(1) \qquad \sum (a_{ik}\, \delta x_i + b_{ik}\, \delta y_i + c_{ik}\, \delta z_i) = 0 \qquad (k = 1, 2, \ldots, r).$$

Briefly: The *sum of the virtual works* ([183]) *must vanish for all admissible displacements*.

([180]) As **S. Stevin** (*Hypomnemata mathematica* **4**, lib. 3, pp. 172, Leiden 1608) said: "Ut spatium agentis ad spatium patientis, sic potentia patientis ad potentiam patientis." (As the space of the agent is to the space of the patient, so is the power of the patient to the power of the patient.) **Galilei** (*Opere* **2**, pp. 183, *et seq*.) said: "Quanto si guadagna di forza, tanto perdersi in velocità" (pp. 172) ["What you gain from force, you lose from velocity"]. The first beginnings of that rule were already recognized by **G. Ubaldo** (**Cantor**, *Geschichte der Mathem.*, 2, pp. 524). Meanwhile, according to **Cantor**, such ideas could already be seen in **Aristotle**'s mechanics (*ibidem* 1, pp. 219). **Whewell** (*Hist. of induct. sciences*, **2**, pp. 31) attributed the first definitive conception of it to the *Tractatus de motu* by **Varro** (1584). **Galilei** had already applied the principle to the equilibrium of fluids in *Discorso intorno alle cose he stanno in acqua* (*Opere* **4**, pp. 3), and then in *Scienza meccanica* (ed., **Mersenne**, Leiden, 1634, *Opere* **2**, pp. 152). It was also in **B. Pascal**; cf., **Mach**, *Mechanik*, pps. 52, 86, 96, as well as **F. Montucla**, *Histoire* **3**, pp. 609.

([181]) **Varignon**, *Nouv. méc.* 2, pp. 174, letter by **Joh. Bernoulli**, v. 26, Jan. 1717: "For every arbitrary equilibrium of forces, no matter how they are applied, whether directly or indirectly, the sum of the positive energies will be equal to the negative energies, when taken positively." That energy is $P\, p \cos(p, P)$, and $p \cos(p, P)$ is call the *virtual velocity* there.

([182]) As **Gauss** said to **Möbius** in a letter in 1837 [**C. Neumann**, Leipz. Ber. **31** (1879), pp. 61].

([183]) For this expression, which was introduced by **G. Coriolis**, cf.: "Mém. sur la manière d'établir les différents principes de la mécanique," J. éc. polyt. **15** (1834), pp. 34. Nowadays, one almost always speaks of infinitely-small displacements in mechanics as if they can refer to very-small *well-defined* line segments. The older conception of them sought to avoid that by means of the concept of virtual *velocities*, which is derived from **Newton**'s fluxions. One can establish the concept of differential calculus rigorously in *mechanics* (as in *geometry*) when one starts from virtual *finite* motions and introduce their velocities. Naturally, the virtual work must be regarded as a change in intensity, so not as $\sum (X\, \delta x + Y\, \delta y + Z\, \delta z)$, but as:

31. The proof of the principle of virtual velocities. – Here, we now raise the question of how to prove that principle, and from the discussion in no. **29**, it will suffice to carry out that proof under the assumption of rest relative to the coordinate system. If one agrees that any system of constraints [(1), no. **30**] can be replaced with suitable *reaction forces* with the components Ξ_i, H_i, Z_i then the following equations:

$$\begin{aligned} X_i + \Xi_i &= 0 , \\ Y_i + \mathrm{H}_i &= 0 , \\ Z_i + \mathrm{Z}_i &= 0 \end{aligned} \tag{1}$$

must be true for every point. If one multiplies the by the *arbitrary* variations δx_i, δy_i, δz_i then it will follow that:

$$\sum (X\,\delta x + Y\,\delta y + Z\,\delta z) + \sum (\Xi\,\delta x + \mathrm{H}\,\delta y + \mathrm{Z}\,\delta z) = 0 ,$$

or

$$\delta A + \delta \mathrm{A} = 0 . \tag{2}$$

If (2) vanishes for *all* variations then one will get (1) again: Equilibrium is *certainly* present then. However, instead of that, it is also already necessary and sufficient that (2) should *vanish for all admissible displacements*. If that were not the case then one would be able to counteract the acceleration φ_i that arises at any point, which is considered to be completely free, by a suitable force $- m_i\,\varphi_i$. Thus, from (2), one will *now* have:

$$\delta A + \delta \mathrm{A} - \sum m_i\,\varphi_i\,\delta s_i \cos(\delta s_i, \varphi_i) = 0 , \tag{3}$$

which is an equation that requires that all $\varphi_i = 0$, from (2), since that part of it includes only positive terms in the sum for a δs_i that points in the direction of the φ_i ([184]).

Up to now, the proof has been based upon a *purely-logical basis*, but the principle will first become useful when one shows that $\delta \mathrm{A}$ *vanishes for all admissible displacements*. That essential part of the proof, which **Laplace** ([185]) seems to have regarded as unnecessary, requires a more detailed look into the nature of the reactions (stresses) that are produced by the conditions.

If one considers *rigid* systems to be ones that are composed of points that are separated by unchanging distances and are acted on by equal and opposite forces that act along the connecting lines between each pair of points, so that work will obviously be zero, then the same thing will also be true when points of the systems are, in addition, constrained to remain on completely-flat

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$$\Sigma\left(X\frac{\delta x}{dt} + Y\frac{\delta y}{dt} + Z\frac{\delta z}{dt} \right) .$$

In this book, it already seems necessary to preserve the terminology that is still customary today, for the sake of brevity (but also on other grounds).

([184]) That conclusion was already found in **Laplace**, *Méc. cél.* **1**, pp. 46; see also **Fourier,** footnote 186; cf., **Poisson**, *Mécanique*, **§ 336**.

([185]) *Méc. cél.* **1**, pp. 43; **L. Poinsot**, "Sur une certaine demonstration du principe des vitesses virtuelles," J. de math. **3** (1838), pp. 244, drew attention to the fact that with $\delta \mathrm{A} = 0$, one can also prove that $\delta A = 0$, *eo ipso*. For a theory of mechanics that *excludes* constraints, the further argument does not come under considerations, such as, e.g., **Boltzmann**'s mechanics.

curves or surfaces ([185.a]) or parts of such systems with completely-flat boundaries in contact, etc. Without a doubt, one can go further into the description of such relationships. The principle can then be proved in all such cases then. Naturally, a *more general* proof cannot be produced in the way that **Fourier** had first proposed ([186]), and one will then be required to regard the principle as a rule whose consequences actually agree with experiments for the case of constraint equations that are left completely undetermined.

32. The proofs of Lagrange, Poinsot, and others. – Other proofs seek to partially eliminate that indeterminacy. **Lagrange** himself produced two proofs of the *principe des poulies* or *pulley*, both of which were based upon an axiomatic presentation of *inextensible strings* ([187]) in which the same *tension* prevailed everywhere.

In **Lagrange**'s *first* proof ([188]), the forces were assumed to be commensurable, i.e., whole-number multiples $m\ \varphi$ of a force $2p$ ([189]).

If one associates every point A_i of the system with a fixed point B_i by drawing a string from B_1 to A_1 and back again with m_1 repetitions ([190]) and then proceeding in the same way with A_2, B_2, …, A_n, B_n then one can represent the entire system of forces by a single force that acts at the free end of the string, say, a *weight p*. *In the case of equilibrium, that weight will certainly not drop, and for a virtual displacement of the points of the system, the virtual work done by all forces will be equal to the total elongation of the string, multiplied by p*. However, it in no way follows from this that the work done must be zero for every admissible displacement of the system, since ([191]) "If that work done were not zero then that would produce a corresponding elongation of the string by the weight, which always has a tendency to drop." *That is because in equilibrium, the weight will not drop at all, and the virtual displacement, which exists only in the representation, has no connection with that fact.* Therefore, one cannot assume that the weight drops either: **Jacobi** ([192]) had already criticized that formally-inadmissible argument, which is nonetheless based upon the celebrated evidence of **Lagrange**'s *pulley proof* in recent times. By contrast, others ([193]) have

([185.a]) A curve (surface) is called completely flat when the mobility *within* it is restricted just as little as its coordinates are restricted by the equations that represent it. The reactions are always normal to the curve (surface) then.

([186]) **J. B. Fourier**, J. éc. polyt., cah. **5** (1798), pp. 20 (*Œuvres* **2**, pp. 477, esp., pp. 489). The same thing was also proved for the pulley by **Lagrange** (pp. 115), as well as in a proof by **R. Prony** (pp. 191).

([187]) Simultaneously, but *before* **Lagrange**, a similar viewpoint was developed by **J. B. Fourier** (*Œuvres* **2**, pp. 500), which **L. N. M. Carnot** referred to [*Essai sur les machines en générale* (1783)] with the remark: "It is natural to think that **Jean Bernoulli** knew of an analogous construction."

([188]) See footnote 187; then in his *Mécanique analytique* of 1811 (*Œuvres complètes* **11**, pp. 23) with the remark: "As for the nature of the principle of virtual velocities (which was used as an axiom in the 1788 edition), one must agree that it is not sufficiently obvious in its own right to be raised to the status of a first principle." A similar statement was also made by **É. Mathieu**, *Dynamique analytique* (1878), pp. 2.

([189]) Dropping that assumption should then result from the *Euclidian* methods in the theory of ratios. However, strictly speaking, that process can only carried out with only ideal geometric constructions, so with an ideal mechanical system, only when one ascribes the properties of such a thing to it, in every sense.

([190]) In order to do that, one has to imagine a completely-smooth ring (a pulley, resp.) of arbitrarily-small dimensions at each of the points *A*, *B* that the strings is led through.

([191]) *Méc. anal.* = *Œuvres complètes* **11**, pp. 24.

([192]) **Jacobi**, in a booklet by **Scheibner**, pp. 17, *et seq.*, esp. pp. 21. Cf., the following footnote about **Fourier**.

([193]) See the note by **J. Bertrand**, *Méc. anal.* **1**; pp. 24. **Jacobi**, footnote **192**, pp. 21. **Mach**, *Mechanik*, pp. 64. **Boltzmann**, *Mechanik*, pp. 133. **E. J. Routh**, *Treatise on analytical statics*, v. 1, 2nd ed., Cambridge 1896, pp. 182.

pointed out that the flaw in that method of proof is that the weight *cannot* therefore drop by only an infinitely-small quantity of *first* order, as well as the representation that is based upon the introduction of tensed strings and pulleys.

Lagrange's second ([194]), much-less-known, proof is free of those flaws, which replaced the forces that arise from the constraints with a similar string construction. It will suffice to explain that idea for points $P_1, \ldots, P_n$, between which *one* constraint equation exists ([195]):

$$f(x_1, y_1, z_1\,;\, x_2, y_2, z_2\,;\, \ldots\,;\, x_n, y_n, z_n) = 0\,. \tag{1}$$

If one assumes that:

$$\sqrt{\left(\frac{\partial f}{\partial x_i}\right)^2 + \left(\frac{\partial f}{\partial y_i}\right)^2 + \left(\frac{\partial f}{\partial z_i}\right)^2} = 2m_i\, p \qquad (i = 1, \ldots, n),$$

in which m_i are once more positive numbers, then one can set:

$$\frac{\partial f}{\partial x_i} = 2m_i\, p \cos \alpha_i\,,$$
$$\frac{\partial f}{\partial y_i} = 2m_i\, p \cos \beta_i\,,$$
$$\frac{\partial f}{\partial z_i} = 2m_i\, p \cos \gamma_i\,,$$

and associate the point P_i with the direction $\cos \alpha_i$, $\cos \beta_i$, $\cos \gamma_i$ that goes through it, which includes a *fixed* point Q_i whose coordinates are a_i, b_i, c_i and is at a distance of l_i from P_i. If one now leads a string from Q_1 to P_1 *and back* m_i times, as before, and so forth, until one arrives at the last point Q_n, where the string is once more attached, then the only possible motions that will still exist are the ones for which:

$$2m_1\, l_1 + 2m_2\, l_2 + \ldots + 2m_n\, l_n = \text{const.}$$

or

$$\sum 2\,m_i (\cos \alpha_i\, dx_i + \cos \beta_i\, dy_i + \cos \gamma_i\, dz_i) = 0\,,$$

i.e.:

$$\frac{1}{p}\, df = 0\,.$$

If one makes the assumption (which was not generally stated explicitly by **Lagrange**) that the influence of the constraint equations depends upon only the first differential quotients of the constraint equation (1), *or that constraint equations that admit the same admissible displacements*

In **Fourier**'s "Mémoire sur la statique" (*Œuvres* **2**, pp. 500), it was quite correctly stated that: When the sum of the moments is zero for a displacement, that displacement cannot arise from the applied forces, regardless of whether equilibrium is or is not present.

([194]) **Lagrange**, *Théorie des fonctions*, 2nd ed., Paris 1813, pp. 350.

([195]) This equation can be replaced with the more general one [no. **30**, (1)] with no essential changes; cf., **A. Voss**, "Über die Differentialgleichungen der Mechanik," Math. Ann. **25** (1885), pp. 258.

are mechanically equivalent, then one can replace them here with the constraint of the invariability of the length of the string, for which the tension that arises in the string will, in fact, do no work under any admissible displacement. That proof likewise shows that the influence of every arbitrary constraint equation can be replaced with the system of forces:

$$\sqrt{\left(\frac{\partial f}{\partial x_i}\right)^2+\left(\frac{\partial f}{\partial y_i}\right)^2+\left(\frac{\partial f}{\partial z_i}\right)^2}$$

with the direction cosines cos α_i, cos β_i, cos γ_i.

Other proofs sought to replace the **Lagrange** assumptions by partially less-abstract ones. In any event, equilibrium in a system that is subject to constraints is based upon the fact that accelerations can be provoked by them in all possible directions of motion. *One will then arrive at the axiomatic assumption that equilibrium will not be perturbed when one adds arbitrarily-many new constraints to the existing ones that do not contradict the old ones* ([196]). One can now add enough of them that the displacement of one individual point determines the displacements of all others uniquely, as in an ideal machine, or that the system becomes *deterministic*. That displacement will then represent an entirely-arbitrary virtual displacement of the original system. **Duhamel** employed an especially lucid proof of the validity of that path, which **Ampère** was the first to go down in that degree of generality ([197]). Moreover, similar presentations are also found in **Poinsot** ([198]), whose proof reproduces that of **Minding** ([199]) almost unaltered.

In any event, under the given assumptions, one can arrive at a proof of the principle for a system of discrete points. That proof will still be applicable in the case of strings with continuous mass distributions, since the representation of the internal stresses that preserve equilibrium with the external forces still remain completely determinate. By contrast, the application of the principle that starts with that assumption that **Lagrange** made to two or three-dimensional systems with continuous mass distributions (surfaces and elastic systems) obviously no longer supports the possibility of establishing the geometric construction of the pulley or any other. Here as well, one will be compelled to make the demand that one must regard the *sufficient validity of the principle* in the previously-mentioned cases is an *axiom* whose consequences agree with experiments. In particular, that viewpoint will become necessary by the argument that all proofs must still start from the assumption of discrete systems with a finite number of degrees of freedom, while the theorem itself can also be applied to systems with infinitely-many degrees of mobility, and **Lagrange** already employed it in that form in his derivation of the equilibrium conditions for fluids.

([196]) As **Poinsot** did: "De l'équilibre et du movement des systèmes," J. éc. polyt., cah. 12, an 12, pp. 206. **Poinsot** nonetheless applied a special principle: "One of the primary elements of the general theory of equilibrium is the *axiom* that if the forces are presently in equilibrium in an arbitrarily-variable system then equilibrium will not cease when one supposes that the system is made invariable all at once." Cf., also **C. Neumann**, "Über eine einfache Methode zur Begründung des Prinzipes der virtuellen Geschwindigkeiten," Leipz. Ber. **38** (1886), pp. 70.

([197]) **A. M. Ampère**, "Démonstration générale du principe des vitesses virtuelles," J. éc. polyt., cah. 13 (1806), pp. 247. **Ch. Duhamel**, *Mécanique,* 3rd ed., 1862, German translation by **H. Eggers**, v. 1, Leipzig, 1853), pps. 114, 119. That proof is also in **Th. Despeyrons**, *Mécanique*, 2, pp. 305. Cf., also **F. Moigno**, *Leçons de mécanique*, Paris, 1868, pp. 281, *et seq.*

([198]) **Poinsot**, "Théorie générale de l'équilibre," J. éc. polyt., cah. 13 (1806), pp. 208.

([199]) **F. Minding**, *Handbuch* **2** (1838), pp. 165.

33. Summary. – The detailed proofs do not seem entirely worthless. From whence does one's confidence in the unrestricted validity of that principle derive then? In part, it comes from the innumerable tests of its agreement with experiments and the possibility of also being able to resolve the question under more general assumptions by further detailing of the constructive method of proof, as well as the uniformity of the results under completely-different Ansätze ([200]). However, on the other hand, that confidence is probably based upon the *energetic concepts* that are employed in conjunction with the principle of the preservation of equilibrium under the introduction of new constraints, namely, that when the forces provoke a well-defined displacement with a certain velocity, that will correspond to an increase in the energy *without* any work being done in the event that the virtual moment of the forces is zero. **Thomson** and **Tait** also based their presentation of the principle of virtual velocities upon that standpoint ([201]).

34. Fourier's principle. Material systems of a more general type. – The foregoing consideration always started from the entirely-abstract assumption that the constraints are expressed by *equations* (whether explicit equations between the coordinates or total differential equations between their differentials). Such things would satisfy the requirement that along with any virtual displacement, the opposite one would also be admissible. Meanwhile, **Fourier** ([202]) had also appealed to completely-general *one-sided* constraints, which are expressed by *inequalities* between those elements, in his investigation, and gave the principle of virtual velocities the form that *the necessary and sufficient condition for equilibrium is:*

$$\sum(X\,\delta x+Y\,\delta y+Z\,\delta z)\leq 0\,.$$

First **Gauss** ([203]), and then **Ostrogradsky** ([204]) had referred to that case, which **Lagrange** had not considered, and independently of **Fourier**.

The lever construction that **Fourier** envisioned in his "Mémoire sur la statique" ([205]) was employed by **C. Neumann** ([206]) in a very intuitive way for this case. An inextensible string that does not intersect itself, which can be displaced along itself in *one* (positive) direction, is obviously in equilibrium when the force components X_i that act in the direction of the string satisfy the condition that for all virtual displacements δs, one must have:

([200]) A critical overview of the proofs of the principle of virtual velocities *still fails to be complete up to now*.

([201]) **Thomson** and **Tait**, *Treatise* (1) 1, pp. 265. In that book, they likewise recalled **Stevin**'s famous remark about the equilibrium of a homogeneous chain on the skew plane: "If it does not exist then that motion will have no end, which is absurd." *Werke* (ed., **A. Giraud**, Leiden, 1634), 2, pp. 448.

([202]) **Fourier**, *Œuvres* **2**, pp. 488. Meanwhile, **Fourier** defined the moment of the virtual work as a *fluxion*, i.e., with the *opposite* sign to the one that is now customary, pp. 479. The constraints that were the only kind that **Lagrange** considered (more generally, the ones in no. **30**, 1) are also called double-sided or *conditions bilatérales,* as **P. Duhem** called them in his *Commentaire*.

([203]) **Gauss**, 1829, *Werke* 5, pp. 27.

([204]) **M. Ostrogradsky**, "Considérations générales sur les momens," 1834, Petersb. Mém. de l'Acad. (6) **1** (1838), pp. 129. **Fourier**'s principle did not remain quite so unnoticed in France. **A. A. Cournot** also developed **Ostrogradsky**'s equations in 1827. See his "Extension du principle des vitesses virtuelles au cas où les conditions de liaison du systèmes sont exprimées par des *inégalités*," Bull. sciences math. de Ferussac **8** (1827), pp. 165.

([205]) **Fourier**, *Œuvres* **2**, pp. 495.

([206]) **C. Neumann**, "Über das Prinzip der virtuellen oder fakultativen Verrückungen," Leipz. Ber. **31** (1879), pp. 53.

$$\delta s \sum X_i \leq 0 ,$$

because $\sum X$ will then be either zero or negative. If one now imagines that the system is made one-sided determinate then one can arrange, by means of a system of levers, that in place of each actual point P_k of the system whose virtual displacement is δs_k, another one P_k' enters, in such a way that all P_k' will possess *the same* displacement magnitude $\delta' s$ with an unchanged direction. If one denotes the components of P_k that acts in the direction of the determinate displacement by X_k and the corresponding component of P_k' that is equivalent to it under the lever rule by X_k' then the necessary and sufficient condition will be:

$$\frac{X_k}{X_k'} = \frac{\delta' s}{\delta s_k} ,$$

so

$$\sum X_k' \, \delta' s = \sum X_k \, \delta s_k \leq 0 .$$

Previously, only the traditionally-introduced cases of constraints were treated. In and of itself, there are no grounds for not assuming that homogeneous quadratic (higher-degree, resp.) equations will also exist for virtual displacements. We shall not go into that, since except for special simple cases, constraints of that *singular* type have hardly been examined in general up to now.

The following extension is much more essential: The constraints on the mobility of a system cannot be restricted to the assumption that relations exist between the virtual displacements, in general. That will already be an issue when *frictional* processes must be considered. In a still-broader context (although one can also start from other viewpoints here, as well), the system might be considered to be in a state that is a *deformation* (*strain*) of its original state in which it is found to be in equilibrium under the action of given (external) forces. In all of those cases, one must obviously add the forces that account for the influence of friction, *stress*, etc., to the forces P that were the only ones that were considered in no. **30** if one would like to apply the principle of virtual velocities.

With **Painlevé**, one can characterize material systems with finite degrees of freedom, i.e., ones whose virtual displacements are determined by a finite number of independent parameters, as being, above all, systems with and without friction (*frottement*). That is because no matter how one might represent the nature of the reaction forces R that act upon the points of the system, one will always be able to decompose the group R into two groups R_1, R_2 in a *single* way such that the virtual work done by the reactions R is equal to the work done by the group R_2 under all admissible displacements, and likewise the vector system in R_2 will correspond to a virtual displacement ([205.a]). The group R_1 then represents the "reactions that arise from the constraint equations," while the group R_2 represents the *frictional resistance*.

([205.a]) **P. Painlevé**, *Leçons sur l'intégration*, pp. 54, *et seq.* For similar decompositions, cf., also **J. König**: "Über eine neue Interpretation der Fundamentalgleichungen der Dynamik," Math. Ann. **31** (1888), pp. 1.

35. Equilibrium conditions. – Since equations (1) in no. **30** must be mutually independent, so not all r-rowed determinants of the a_{ik}, b_{ik}, c_{ik} can vanish, one will obtain the equilibrium conditions in the form ([207]):

$$(1) \qquad \begin{aligned} X_i &= \sum \lambda_k \, a_{ik} \,, \\ Y_i &= \sum \lambda_k \, b_{ik} \,, \\ Z_i &= \sum \lambda_k \, c_{ik} \end{aligned}$$

by **Lagrange**'s method of multipliers. By contrast, when one is given inequalities ([208]), such as:

$$\sum (a_{ik} \, \delta x_i + b_{ik} \, \delta y_i + c_{ik} \, \delta z_i) = \varepsilon_k \qquad (k = 1, 2, \ldots, r),$$

in which:

$$\varepsilon_k \leq 0 \,,$$

one will again get equations (1) in any event when the constraints *include* the case of $\varepsilon_k = 0$. Now, since it will follow from them that:

$$\sum (X_i \, \delta x_i + Y_i \, \delta y_i + Z_i \, \delta z_i) = \sum \lambda_k \, \varepsilon_k \,,$$

that will imply that all of the coefficients λ_k must be *positive* when the moment can never become positive for negative values of any arbitrarily-small ε_k. Its sign will then remain arbitrary only when the associated ε_k is restricted to the value *zero* exclusively. Obviously, constraints for which the case of $\varepsilon_k = 0$ is not included at all are then dropped.

The right-hand sides of equations (1) represent the reactions that arise from the constraints. One now sees how every constraint corresponds to a certain component of that kind that is associated with λ_k. One will get the constraints for the given forces X_i, Y_i, Z_i when one substitutes the values of λ that are calculated from r suitably-chosen equations in (1) in the remaining equations.

β. Dynamics.

36. D'Alembert's principle ([207.a]). – Once one has agreed upon the concept of the principle of virtual velocities, there will be no further obstacle to arriving at **d'Alembert**'s fundamental consideration of the *general equations of dynamics* ([209]).

([207]) Which was first done by **Lagrange** in his *Méc. anal.* of 1788.

([208]) As **Cournot** and **Ostrogradsky** did, footnote 203. There are also examples in the latter (e.g., funicular polygon, incompressible fluids, etc.). In addition to the usual textbooks, cf., also **L. Henneberg**, J. f. Math. **113** (1894), pp. 179.

([207.a]) According to **F. Montucla**, *Histoire* 3, pp. 44 and 627, **A. Fontaine** had already expressed a similar principle in 1739.

([209]) In the opinion of many people, **d'Alembert**'s principle is based upon a *new* axiom, insofar as the equations of equilibrium are adapted to the case of a system that is already found to be *in motion* (e.g., **Jacobi,** *Dynamik*, ed., **Clebsch**, pp. 63, *et seq.*). **C. Neumann**, [Leipz. Ber. **31** (1879), pp. 61]. I can see only a narrow conception of equilibrium in it, such that the difficulty that is present here (cf., no. **29**) *already occurs in the principle of virtual velocities itself.*

A system of material points with masses m_i is found to be in an arbitrary state of motion at time t under the influence of the force components X_i, Y_i, Z_i that are applied to m_i. Let its coordinates be x_i, y_i, z_i . If they are subject to arbitrary constraint equations:

$$f_1 = 0 , f_2 = 0 , \ldots, f_k = 0 ,$$

in addition, which might not include t initially, then the accelerations $\ddot{x}_i$, $\ddot{y}_i$, $\ddot{z}_i$ that result from those constraints will be arranged, in general, such that:

$$X_i - m_i \ddot{x}_i , \qquad Y_i - m_i \ddot{y}_i , \qquad Z_i - m_i \ddot{z}_i$$

do not vanish. Those forces will then be produced by the constraints, so they will be in equilibrium "relative to the latter." If that were not the case then they would communicate accelerations to the system, in addition to the assumed ones $\ddot{x}_i$, $\ddot{y}_i$, $\ddot{z}_i$, which would *contradict* the assumption. In that form, **d'Alembert**'s principle ([210]) is a purely-logical argument that one must couple with only the principle of virtual velocities, according to **Lagrange**, in order to arrive at the basic formula of dynamics:

$$\sum [(X - m\ddot{x})\delta x + (Y - m\ddot{y})\delta y + (Z - m\ddot{z})\delta z] = 0 ,$$

which one can also express more concisely as: The virtual work done by the *lost forces* must vanish.

Of course, the phrase "relative to the constraints" that is used almost everywhere must be made more precise. An entirely-clear understanding of it that is supported by the foundations of mechanics can come about only when one again converts every point of the system into a completely-free one by adding the reactions Ξ_i, H_i, Z_i . *The latter* are the ones that keep the system

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([210]) **J. d'Alembert**'s original argument (*Traité de dynamique*, Paris, 1743 is not essentially different from the one here (cf., **Poisson**, *Mécanique*, § **350**). In **d'Alembert**'s own words: "Let A, B, C, … be the bodies that comprise the system, and suppose that one has compelled the motions a, b, c, … that they are forced to exhibit to change into the motions α, β, γ, … due to their mutual action. It is clear that one can regard the motion a that is imposed upon the body A as composed of the motion α that it takes on and another motion α' . One can likewise regard the motions b, c, … as composed of the motions β, β' ; γ, γ' ; …, so it will follow that the motion of the bodies A, B, C, … between themselves will have been the same if, instead of giving them impulses a, b, c, one gives them the double impulses α, α' ; β, β' , … at once. Now, by hypothesis, the bodies A, B, C have taken on the motions α, β, γ, … by themselves. Thus, the motions α' , β' , γ' , … must be such that they do not disturb anything in the motions α, β, γ, …, i.e., that if the bodies received only the motions α' , β' , γ' , … then those motions would have to mutually cancel, and the system would remain at rest. That implies the following principle: Decompose each of the motions a, b, c, … into two other ones α, α' ; β, β' , … that are such that if one imposes only the motions α, β, γ, … upon the bodies then they can preserve the motions without harming them reciprocally, and if one imposes only the motions α' , β' , γ' , … upon them then the system must remain at rest. It is clear that α, β, γ, … are the motions that the bodies will take on by virtue of their action. That is what we were looking for."

Other authors have not made any essential change to the expression of the principle. The terminologies of *lost forces*, *forces d'inertie*, *effets dynamiques* (**Ostrogradsky**) seem somewhat redundant. The terminology of **G. B. Airy** is by no means clearer (**E. J. Routh**, *Dynamics*, 1, pp. 52). Just as in many other places, mechanics shows an inclination towards stereotypical expressions in the literature, here as well.

in equilibrium, according to the argument that was just presented. In so doing, one can also introduce constraints that vary in time in place of equations that are independent of t, in the event that those changes happen *continuously*. By means of the axioms in no. **29** regarding the extension of the equilibrium state to moving systems, that will, in fact, imply immediately that the equilibrium will now apply to the limiting form of the constraint equations at time t, or as one ordinarily says, to the *virtual displacements independently of time t*. Naturally, for the more general material systems that are considered in no. **34**, the forces X, Y, Z must be added to the reactions that arise from the other kinds of constraints.

37. The Lagrange equations. – The introduction of *differential equations:*

$$\sum (a_{ik}\, dx_i + b_{ik}\, dy_i + c_{ik}\, dz_i) = 0 , \tag{1}$$

or more generally:

$$\sum (a_{ik}\, dx_i + b_{ik}\, dy_i + c_{ik}\, dz_i) + c_k\, dt = 0 , \tag{2}$$

in which the coefficients can be functions of x, y, z, t, in place of *finite equations of constraint* was first presented thoroughly in connection with the principles of mechanics by **Voss** ([211]). However, such cases had already appeared occasionally in problems of rolling motion much earlier and were also assumed in general by others ([212]). **Hertz**, who regarded the assumption of non-integrable differential relations as essentially different from the case of explicit equations of constraint, as a result of a special conception of **Hamilton**'s principle ([213]), had then distinguished between *non-holonomic* and *holonomic* constraints ([214]).

One obtains the equations of motion from **d'Alembert**'s principle directly by **Lagrange**'s method of multipliers in the form ([215]):

$$\begin{aligned} m_i\, \ddot{x}_i &= X_i + \sum \lambda_k\, a_{ik} , \\ m_i\, \ddot{y}_i &= Y_i + \sum \lambda_k\, b_{ik} , \\ m_i\, \ddot{z}_i &= Z_i + \sum \lambda_k\, c_{ik} , \end{aligned} \tag{3}$$

which **Lagrange** first gave them, since **d'Alembert** only employed his principle synthetically ([216]) in order to solve some individual problems. Moreover, **d'Alembert** did not consider it to be

([211]) **A. Voss**, Math. Ann. **25** (1884), pp. 258.

([212]) Such as **M. Ostrogradsky**, Petersb. Mém. de l'Acad. (6) **1** (1858), pp. 565; **N. M. Ferrers**, Quart. J. of math. **12** (1873), pp. 1; also **F. Minding**, Dorpater Gratulationsschr. 1864 [cf., **A. Kneser**, Zeit. math. Phys. **45** (1900), *literar. history.*, Abt., pp. 118]

([213]) **Hertz**, *Mechanik*, pp. 23. On that topic, cf., **O. Hölder**, Gött. Nachr. (1896), pp. 122.

([214]) **Hertz**, *Mechanik*, pp. 91. The exact distinction between holonomic and non-holonomic constraints is made precise on pp. 96.

([215]) Naturally, for explicit equations of constraint $f_k = 0$, the a_{ik}, b_{ik}, c_{ik} are replaced by the differential quotients of the f_k with respect to x_i, y_i, z_i .

([216]) The equations of dynamics relative to three rectangular axes in the form that is customary today were first introduced by **C. Maclaurin** (*A complete treatise on fluxions*, Edinburgh, 1742).

necessary to prove that the equations that **d'Alembert**'s principle produced are also *sufficient* for the complete determination of x, y, z. That proof, which is based upon the independence of the equations of constraint, was given by **Jacobi** ([217]) in regard to equations (3). For **Lagrange**, it first seemed to be a consequence of the introduction of independent coordinates. Furthermore, the determination of the components of the reaction [the sum of the quantities in equations (3)] results from incorporating the expressions for the accelerations in (3) in equations (2) after differentiating them with respect to the independent variable t and calculating their values that arise from the linear relations for λ, which will then be functions of degree two in the velocity components. One will always get equations of constraint that express the second differential quotients as functions of degree two in the first. One will succeed in exhibiting those equations in a much-clearer way by ***Lagrange****'s introduction of independent coordinates*, which we shall now go into.

If only k equations of constraint:

$$f_1 = 0, \quad f_2 = 0, \ldots, f_k = 0$$

between the $3n$ coordinates are now given in explicit form then one can regard the latter as functions of t and $3n - k$ *independent parameters:*

$$q_1, q_2, \ldots, q_r \qquad (r = 3n - k)$$

[the **Lagrange**, or more generally, *generalized* coordinates ([218])], and in infinitely-many ways. From the identities that now exist:

$$\text{(a)} \qquad \sum \left(\frac{\partial f_k}{\partial x_i} \frac{\partial x_i}{\partial q_s} + \frac{\partial f_k}{\partial y_i} \frac{\partial y_i}{\partial q_s} + \frac{\partial f_k}{\partial z_i} \frac{\partial z_i}{\partial q_s} \right) = 0 ,$$

one will get from (3) that:

$$\sum m_i \left(\ddot{x}_i \frac{\partial x_i}{\partial q_s} + \ddot{y}_i \frac{\partial y_i}{\partial q_s} + \ddot{z}_i \frac{\partial z_i}{\partial q_s} \right) = Q_s ,$$

as long as one sets:

$$\sum \left(X_i \frac{\partial x_i}{\partial q_s} + Y_i \frac{\partial y_i}{\partial q_s} + Z_i \frac{\partial z_i}{\partial q_s} \right) = Q_s .$$

If one further sets:

$$\text{(b)} \qquad \begin{aligned} \dot{x}_i &= \sum \frac{\partial x_i}{\partial q_s} \dot{q}_s + \frac{\partial x_i}{\partial t} , \\ \dot{y}_i &= \sum \frac{\partial y_i}{\partial q_s} \dot{q}_s + \frac{\partial y_i}{\partial t} , \\ \dot{z}_i &= \sum \frac{\partial z_i}{\partial q_s} \dot{q}_s + \frac{\partial z_i}{\partial t} , \end{aligned}$$

and

([217]) **Jacobi**, *Dynamik*, ed., **Clebsch**, pp. 133.
([218]) According to **Thomson** and **Tait**, *Treatise* (1) **1**, pp. 286, they are generalized co-ordinates.

(c) $$T = \tfrac{1}{2}\sum m_i(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2)$$

then from (b), one will have:

$$\frac{\partial T}{\partial \dot{q}_s} = \sum m_i \left(\dot{x}_i \frac{\partial x_i}{\partial q_s} + \dot{y}_i \frac{\partial y_i}{\partial q_s} + \dot{z}_i \frac{\partial z_i}{\partial q_s} \right),$$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right) = Q_s + \sum m_i \left(\dot{x}_i \frac{\partial^2 x_i}{\partial q_s\,\partial t} + \dot{y}_i \frac{\partial^2 y_i}{\partial q_s\,\partial t} + \dot{z}_i \frac{\partial^2 z_i}{\partial q_s\,\partial t} \right),$$

so since:

$$\frac{\partial T}{\partial q_s} = \sum m_i \left(\dot{x}_i \frac{\partial^2 x_i}{\partial q_s\,\partial t} + \dot{y}_i \frac{\partial^2 y_i}{\partial q_s\,\partial t} + \dot{z}_i \frac{\partial^2 z_i}{\partial q_s\,\partial t} \right),$$

one will have ([219]):

(4) $$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right) - \frac{\partial T}{\partial q_s} = Q_s .$$

In these *general fundamental equations of dynamics*, *T means the vis viva or kinetic energy* ([218.a]) *of the system, which is a constantly-positive entire rational function of degree two of the generalized velocities* $\dot{q}_s$, which will be a *homogeneous positive-definite form of degree two* in the $\dot{q}_s$ for the case in which the constraint equations are *independent of t*, which is the case that **Lagrange** treated exclusively ([220]).

The importance of those equations is based upon the fact that: *The only quantities that appear upon which the dynamical problem now depends are T and* Q_s, in which, at the same time, the number of variables has been reduced to the smallest number (a smaller number that is suited to the form of the problem, resp.) ([221]).

([219]) In more detail for *fixed* constraints, i.e., ones that do not depend upon *t* explicitly, those equations read:

$$\sum a_{is}\,\ddot{q}_i + \sum \alpha_{irs}\,\dot{q}_i\,\dot{q}_r = Q_s ,$$

in which:

$$2\,\alpha_{irs} = \frac{\partial a_{is}}{\partial q_r} + \frac{\partial a_{rs}}{\partial q_i} - \frac{\partial a_{ir}}{\partial q_s}$$

denotes the **Christoffel** symbol.

([218.a]) The conventional terms in Germany and France *lebendige Kraft, force vive* (cf., footnote 295) will be preserved here, no matter unsuitable they might also be.

([220]) The formulas were first derived by **J. Vielle**, J. de math. **14** (1849), pp. 201, under the extended assumption. They are not considered in the German textbooks, so they are presented in detail in this treatise.

([221]) In *Germany*, and also more recently in *Italy*, it has become customary to distinguish between equations (3) and (4) as **Lagrange**'s equations of the *first* and *second* kind. **Lagrange** himself *did not* make that distinction, and that left him free, moreover, to also prefer a *partial* introduction of independent parameters (*Mécanique*, *Œuvres* **11**, pp. 325 and 336), which is an idea that **Routh** developed further sometime later ("Stability of Motion," *Dynamics*, v.1., pp. 375). **Jacobi** (issue by **Scheibner**, pp. 166) generally spoke of a *first* form of the **Lagrange** equations in his 1847 lecture. That term seems to have first been printed in a problem of dynamics that **Clebsch** addressed (*Dynamik*, pp. 63 and 141, also in the Table of Contents), which had perhaps been popularized by **Jacobi** and his school before. In and of itself, the distinction is not inappropriate.

The set of equations (4) is invariant under arbitrary transformation of the q into just as many new variables k by means of the equation:

$$\sum\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right)-\frac{\partial T}{\partial q_s}\right]dq_s = \sum\left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{k}_s}\right)-\frac{\partial T}{\partial k_s}\right]dk_s .$$

When generalized coordinates are introduced (mostly in purely-theoretical works), the determination of the reactions is often no longer considered any further. Moreover, that would be simplest if one were to again revert to rectangular coordinates. In some situations, the determination of those forces is just as important as determining the motion itself for the applications. Moreover, that is already the case when the constraints have a *one-sided* character, so the quantities λ in equations (3) (cf., no. **35**) must have *well-defined signs*, and the whole Ansatz that there is a location at which the λ change sign when they go through zero *loses its meaning*. One might confer the known examples of the motion of massive points in vertical circles, on the surface of a sphere, the motion of a massive rod whose ends remain on given surfaces or curves, etc.

38. Non-holonomic systems. – By contrast, the transformation is restricted to the case of *holonomic* constraints. In order to be able to perform a formal transformation for non-holonomic ones, **Appell** ([222]) considered the quantity:

$$S = \tfrac{1}{2}\sum m_i(\ddot{x}_i^2+\ddot{y}_i^2+\ddot{z}_i^2),$$

instead of T, which will imply the equations of mechanics in the form:

$$\frac{\partial S}{\partial \ddot{q}_s} = Q_s$$

when one differentiates it with respect to the quantities $\ddot{q}$. However, it should be remarked that it is precisely the advantage that one gains by introducing T, which depends upon only the *first* differential quotients, is lost in that way for the most part. Furthermore, the form of the dynamical equations in the non-holonomic case is the following:

If one has introduced any new parameters $q_1, \ldots, q_k$ in place of the variables x, y, z, between which the l constraints exist:

$$\sum \alpha_{ki}\,\delta q_i = 0$$

then one can make all δx, δy, δz depend upon $k - l$ *independent* variations δq_s ($s = 1, 2, \ldots, k - l$) such that:

$$\delta x_i = \sum a_{is}\,\delta q_s ,$$

([222]) **P. Appell**, C. R. Acad. Sci. Paris **129** (1899), pp. 317, 423, 549; J. f. Math. **121** (1900), pp. 1; J. de math. (5) **6** (1900), pp. 5; *ibid.* (5) **7** (1901), pp. 5.

$$\delta y_i = \sum b_{is}\,\delta q_s\,,$$
$$\delta z_i = \sum c_{is}\,\delta q_s\,.$$

When one sets:

$$\sum X_i\,a_{is} + Y_i\,b_{is} + Z_i\,c_{is} = Q_s\,,$$

the equations of motion will then become:

$$Q_s = \sum (a_{is}\,\ddot{x}_i + b_{is}\,\ddot{y}_i + c_{is}\,\ddot{z}_i)\,m_i$$
$$= \frac{d}{dt}\sum (a_{is}\,\dot{x}_i + b_{is}\,\dot{y}_i + c_{is}\,\dot{z}_i)\,m_i - R_s\,,$$

in which:

$$R_s = \sum m_i\left(\dot{x}_i\,\frac{da_{is}}{dx} + \dot{y}_i\,\frac{db_{is}}{dx} + \dot{z}_i\,\frac{dc_{is}}{dx}\right).$$

Now, it will follow from the equations:

$$\frac{\partial \dot{x}_i}{\partial \dot{q}_s} = a_{is}\,, \qquad \frac{\partial \dot{y}_i}{\partial \dot{q}_s} = b_{is}\,, \qquad \frac{\partial \dot{z}_i}{\partial \dot{q}_s} = c_{is}\,,$$

that:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_s}\right) - R_s = Q_s\,,$$

but R_s itself is still not equal to $\partial T / \partial q_s$ when the coefficients a_{is}, b_{is}, c_{is}, which generally depend upon *all* variables $q_1, \ldots, q_k$, depend upon only the variables $q_1, \ldots, q_{k-l}$, which is the case for many questions, namely, for the simple problems of bodies in rolling motion. That fact is not always noticed, rather it is precisely in the latter case that one repeatedly frees T of the seemingly-superfluous coordinates q with the help of the expressions for $\dot{x}_i$, $\dot{y}_i$, $\dot{z}_i$, and apply that to the expression for T that arises in that way, which **C. Neumann** called the *illegitimate* form of the *vis viva*, which will naturally lead to incorrect results ([221.a]).

39. Gauss's principle of least constraint. – In **Gauss**'s own words, it reads ([223]): The motion of a system of material points, which are always coupled to each other in some way, and whose motions are, at the same time, always coupled by external constraints, will occur at each moment with the greatest-possible agreement with the free motion, or with the smallest-possible constraint, when one considers the measure of the constraint that the entire system experiences at each time-

([221.a]) Cf., on this, **A. Vierkandt**, "Über gleitende und rollende Bewegung," Monatschr. f. Math. Phys. **3** (1892), pp. 31; **J. Hadamard**, "Sur les mouvements de roulement," Bordeaux Mém. (4) **5** (1895); **O. Hölder**, "Die Prinzipien von Hamilton und Maupertuis," Gött. Nachr. (1896), **§ 11**; **D. J. Kortweg**, "Über eine ziemlich verbreitete unrichtige Behandlung eines Problem der rollenden Bewegung," Nieuw Achief voor Wiskunde (2) **4** (1899); **P. Appell**, "Les mouvements de roullement en dynamique," Sammlung Scientia, Phys. math., no. 4, Paris, 1899.

([223]) J. f. Math. **4** (1829) = *Werke* **5**, pp. 23.

point to be the sum of the products of the squares of the "deviation of each point from its free motion" ([224]).

If the position of the point m_i at time t + 2 dt is denoted by:

$$x_i + 2\dot{x}_i\,dt + \tfrac{1}{2}\ddot{x}_i\,dt^2 ,$$

while the position that it would assume as a result of the applied forces if it were completely free is denoted by:

$$x_i + 2\dot{x}_i\,dt + \frac{X_i}{2m_i}dt^2$$

(and corresponding expressions for the remaining coordinates) then the *constraint Z* will be given by ([225]):

$$Z = \sum \frac{1}{m_i}\left[(X_i - m_i\,\ddot{x}_i)^2 + (Y_i - m_i\,\ddot{y}_i)^2 + (Z_i - m_i\,\ddot{z}_i)^2\right].$$

Now, if any varied position is expressed by:

$$x_i + 2\dot{x}_i\,dt + \tfrac{1}{2}\,(\ddot{x}_i + \delta\ddot{x}_i)\,dt^2$$

as **Gauss** did, then if one recalls the constraints, which might be holonomic or non-holonomic, one must have:

$$\sum (a_{ik}\,\delta\ddot{x}_i + b_{ik}\,\delta\ddot{y}_i + c_{ik}\,\delta\ddot{z}_i) = 0 .$$

If one then multiplies the **Lagrange** equations (3) by $\delta\ddot{x}_i$, $\delta\ddot{y}_i$, $\delta\ddot{z}_i$, and adds them then that will give:

(1) $$\sum\left[(X_i - m_i\,\ddot{x}_i)\,\delta\ddot{x}_i + (Y_i - m_i\,\ddot{y}_i)\,\delta\ddot{y}_i + (Z_i - m_i\,\ddot{z}_i)\,\delta\ddot{z}_i\right] = 0 ,$$

([224]) According to **E. Schering**, Gött. Nachr. 18 (1873), pp. 3, 11, that expression is interpreted to mean that one can treat a completely-arbitrary free motion in so doing. By contrast, **R. Lipschitz** [J. f. Math. **82** (1877), pp. 321] drew attention to the fact that neither x nor $\dot{x}$, *but only the acceleration* $\ddot{x}$, can be varied in the deviation in **Gauss** principle.

([225]) For **Lipschitz** [J. f. Math. **82** (1877), pp. 316], Z seems to be *covariant* under arbitrary transformations of the variables *x, y, z,* which are considered to be independent. Cf., also **A. Wassmuth**, Ann. Phys. Chem. (2) **54** (1895), pp. 164. Moreover, see **A. Voss**, "Bemerkungen über die Prinzipien der Mechanik," Münch. Ber. (1901), pp. 167. **Gauss**'s principle can also be formulated as the principle of least work done by lost forces: cf., **Rachmaninoff**, Zeit. Math. Phys. **25** (1879), pp. 206. That is also connected with **Moseley**'s *principle of least resistance* (**Rankine**, *A manual of applied mechanics*, 3rd ed., London,1863, pp. 215), as well as the **Ménabréa-Castigliano** minimum principle, **L. F. Ménabréa**, Rom. Rend. dell'Acc. dei Lincei (2) **2** (1869), pp. 210. See also **A. Castigliano**, *Théorie de l'équilibre des systèmes élastiques et ses applications*, Turin, 1879, German trans. by **E. Hauff**, Vienna 1886. **A. F. B. Müller-Breslau**, *Die neueren Methoden der Festigkeitslehre*, Leipzig, 1886, as well as "Über die Elasticität der Deformationarbeit," Civilingenieur (2) **32** (1886), pp. 553, and the citation by **F. Kötter**, Forthschritte d. Math. **18** (1886), pp. 950. However, among others, **O. Mohr**, Civilingenieur (2) **32** (1886), pp. 395 disputed that way of expressing the principle. Finally, cf., **C. Neumann**, "Das Ostwald'sche Axiom des Energieumsatzes," Leipz. Ber. **44** (1892), pp. 185.

and that is the condition for Z to be a *minimum* with respect to all varied positions.

In this form, **Gauss**'s principle is completely *equivalent* to **d'Alembert**'s, which can also be written in the form (1) now ([226]), which was first given to it by **Gibbs**. At the same time, he remarked that for the case of *equations of motion*, that form of d'Alembert's principle allows one to make an immediate decision about the actual course of motion ([227]).

The fact that, conversely, one can derive everything within the entire scope of mechanics, in particular, the study of statics (theorem of the parallelogram of forces, etc.), from **Gauss**'s or **d'Alembert**'s principle was first shown by **Ritter** ([228]) in his dissertation, which was supervised by **Gauss** himself. **Hertz** had expressed **Gauss**'s principle as a *fundamental law* in his force-less dynamics ([229]).

40. The differential equations of motion for inequality constraints. – If one considers the variations that appear in the principle of virtual velocities to be equivalent to the variations of the accelerations for unchanged x_i and $\dot{x}_i$ then one can also extend **Fourier**'s extension of the aforementioned principle to the case of accelerated motion, and in that way get the most-general form of **Gauss**'s principle ([230]):

$$\sum \left[(X_i - m_i \ddot{x}_i)\,\delta\ddot{x}_i + (Y_i - m_i \ddot{y}_i)\,\delta\ddot{y}_i + (Z_i - m_i \ddot{z}_i)\,\delta\ddot{z}_i \right] \leq 0 \,. \tag{1}$$

As **Gauss** had already remarked, his principle also finds application in statics. One succeeds in that ambition by assuming that the $\ddot{x}_i$, $\ddot{y}_i$, $\ddot{z}_i$ are zero. **Möbius** ([231]) had a different way of expressing that notion. If one denotes the coordinates of the coordinates of each point that is at a distance from the point of application x_i, y_i, z_i of a force that equals the intensity of the force in the direction of the force by a_i, b_i, c_i then if:

$$X_i = a_i - x_i \,, \qquad Y_i = b_i - y_i \,, \qquad Z_i = c_i - z_i$$

then

$$\sum (X_i\,\delta x_i + Y_i\,\delta y_i + Z_i\,\delta z_i) = \sum \left[(a_i - x_i)\,\delta x_i + (b_i - y_i)\,\delta y_i + (c_i - z_i)\,\delta z_i \right] \leq 0$$

([226]) It then comes down to the same thing as solving specific problems of that kind by **Gauss**'s or **d'Alembert**'s principle. For applications of the former, in that sense, to examples in statics and dynamics, cg., **K. Hollefreund**, Schul-Programm Berlin, 1897, no. 97.

([227]) **J. W. Gibbs**, "On the fundamental formulae of dynamics," Am. J. Math. **2** (1897), pp. 49; cf., also **Boltzmann**, *Mechanik*, pp. 230 and 233.

([228]) **A. Ritter**, "Über das Prinzip des kleinsten Zwanges," Diss. Göttingen 1853. **C. G. Reuschle**, "Über das Prinzip des kleinsten Zwanges," Archiv f. Math. Phys. **6** (1845), pp. 238. **H. Scheffler**, "Über das Gauss'sche Grundgesetz d. Mechanik," Zeit. Math. Phys. **3** (1858), pp. 197; **A. Buckendahl**, "Über das Prinzip des kleinsten Zwanges," Diss. Göttingen 1873.

([229]) **Hertz**, *Mechanik*, pp. 185.

([230]) **W. Schell** (*Mechanik*, v. 2, pp. 502) derived the theorem at that level of generality, but in his proof one should notice that one cannot infer the signs of summands from the sign of a sum. The concept of virtual displacements was also given a different meaning from the original one in **Boltzmann**'s presentation (*Mechanik,* pp. 217). In regard to those concerns, it would seem appropriate to regard **Gauss**'s principle, in its *extended* form, *as a fundamental principle that cannot be rigorously proved from prior principles.* That is also true for the case (which was not treated in this book) where the constraint equations include the x, y, z ; $\dot{x}$, $\dot{y}$, $\dot{z}$ in a completely-arbitrary way.

([231]) **Möbius**, *Statik*, v. 1, pp. 330, *et seq.*, cf., however, **Euler**, Mém. de l'Acad. (1752), pp. 246.

will be the condition for equilibrium, which one now expresses by saying that:

$$\sum \left[(a_i - x_i)^2 + (b_i - y_i)^2 + (c_i - z_i)^2 \right]$$

is a minimum with respect to all admissible displacements.

In regard to the dynamical problems, here we must raise the question that **Ostrogradsky** ([232]) first posed (but did not answer completely) of the extent to which equation (1) determines the motion at all. **A. Mayer** has recently taken up that question again ([233]) and showed a simple (but not generally direct) way of resolving it by applying **Gauss**'s principle that is suitable for *excluding* all *unusable* solutions with certainty. However, **Mayer** proved that *one* well-defined solution exists at all only in the case of one and two degrees of freedom. Nonetheless, **Zermelo** ([234]) corrected that flaw completely in the precisely the way that **Jacobi** ([235]) had already suggested in his lectures, namely, by appealing to the special nature of the minimum that exists here, which excludes the existence of several minima.

41. D'Alembert's principle for impulse. – If one integrates the equation:

$$\sum (m_i\, \ddot{x}_i - X_i)\, \delta x_i + \cdots = 0 ,$$

after multiplying it by dt, over an arbitrarily-small time-interval from 0 to τ, while $\ddot{x}_i$, …, X_i, … always have the same sign ([236]), then if one recalls that the impulse is set to:

$$\int_0^\tau X_i\, dt = P_i , \quad \int_0^\tau Y_i\, dt = Q_i , \quad \int_0^\tau Z_i\, dt = R_i ,$$

one can set:

$$\sum \left(|\, m_i\, \dot{x}_i\, |_0^\tau\, \delta \overline{x}_i + |\, m_i\, \dot{y}_i\, |_0^\tau\, \delta \overline{y}_i + |\, m_i\, \dot{z}_i\, |_0^\tau\, \delta \overline{z}_i \right) = \sum \left(P_i\, \delta \overline{x}_i + Q_i\, \delta \overline{y}_i + R_i\, \delta \overline{z}_i \right) ,$$

by the mean-value theorem in the differential calculus, in which one understands $\delta \overline{x}_i$, … to mean the mean values of the virtual displacements. Under the assumption that τ converges to zero, which will likewise make all of those mean values of the displacements go to zero, one will then get the equation:

$$\sum \left[m_i\, (\dot{x}_i - (\dot{x}_i)_0) \right] \delta x_i + \cdots = 0$$

([232]) **M. Ostrogradsky**, "Sur les déplacements instantanées," Petersb. Mém. de l'Acad. (6) **1** (1838), pp. 565.

([233]) **A. Mayer**, "Über die Aufstellung der Differentialgleichungen der Bewegung reibungsloser Punktsysteme," Leipz. Ber. **51** (1899), pp. 224 and 245.

([234]) **E. Zermelo**, Gött. Nachr. (1899), pp. 306 employed an argument that probably goes back to **D. Hilbert** at a crucial step, moreover.

([235]) **Jacobi**, edition by **Scheibner**, pp. 83, *et seq.*

([236]) **A. Ritter** had already treated that case in his Diss. (1853) with the methods of the study of manifolds, but his presentation was not generally rigorous.

for determining the sudden changes in the velocities as a result of the impulse vectors P, Q, R. One can also obtain them directly by applying the **d'Alembert** argument to the velocities that are produced by the impulses, instead of to the accelerations that are produced by continuous forces ([237]).

That consideration can also be applied to the case in which the *constraints on the system are suddenly replaced with other ones*. If the equations that pertain to the time-point 0 are not the ones that were used up to now:

$$f_1 = 0 , \qquad f_2 = 0 , \qquad \ldots, \qquad f_k = 0 ,$$

but new ones:

$$\varphi_1 = 0 , \qquad \ldots, \qquad \varphi_l = 0 ,$$

then one will get:

$$\sum \left[m_i \, (\dot{x}_i - (\dot{x}_i)_0) \right] \delta x_i + \cdots = 0 ,$$

in a similar way, in which the displacements have to make all $\varphi = 0$ ([238]). Obviously, those formulas can also be developed for generalized coordinates q ([239]). Namely, under the assumptions that were made before, it will follow from equations (4) in no. **37** that:

$$\left(\frac{\partial T}{\partial \dot{q}_s} \right) - \left(\frac{\partial T}{\partial \dot{q}_s} \right)_0 = \int_0^\tau Q_s \, dt = P_s .$$

Now if T contains no terms that are *linear* in the $\dot{q}_s$ then one will have:

$$\frac{\partial T}{\partial \dot{q}_s} = P_s ,$$

when all $\dot{q}$ vanish at time 0, as was pointed out before in no. **24**.

B) True variational (isoperimetric) principles.

42. Hamilton's principle. – One can refer to the principle that was treated in the foregoing as an elementary *variational* or *differential* principle in its useful form. Formally, one is dealing with a variational expression in it. Its direct connection to the presentation of forces and accelerations defines the proper foundation for the mechanics of material system. We distinguish that from the

([237]) That is introduced here as an *assumption*, although a continuous function from 0 to τ that has a well-defined number of derivatives in that interval will always have a constant sign for a sufficiently-small positive t, even when it vanishes for $t = 0$, in the event that not all of those derivatives vanish.

([238]) Cf., **Ch. Sturm**, C. R. Acad. Sci. Paris **13** (1841), pp. 1046, and also *Mécanique*, pp. 353, as well as the summary presentation in **Routh** (*Dynamik*, v. 1, pp. 335).

([239]) As **Lagrange** did before in *Mécanique*, t. 2, *Œuvres*, t. 12, pp. 173, and then **W. D. Niven**, Mess. of Math. **4** (1867), **J. Routh**, *Dynamik*, v. 1, pp. 361; **P. Appell**, J. de math. **12** (1896), pp. 5. For the **Lagrange** equations in the case of *friction*, cf., **P. Appell**, C. R. Acad. Sci. Paris **114** (1892), pp. 331.

true variational or *isoperimetric principles* ([240]), whose evidence is no longer based upon an immediate application of mechanical concepts, but only upon the *verification* that one can deduce the equations of dynamics with its help *in any case* ([241]). Whereas the expressions that appear in the differential principles possess only the property of *covariance* (see no. **37**, conclusion), true variational principles will yield *invariant* forms that are exceptionally useful in regard to the transformation of coordinates since they include only the *first* differential quotients (under the usual assumptions about the nature of forces). By contrast, that represents a restriction compared to what the *analytical* treatment of constraint *equations* would assume. A third class is defined by the *true integral principles*, which we will first speak of later on in no. **45**.

Under the assumption that X_i, Y_i, Z_i are partial differential quotients of a function A with respect to the coordinates x_i, y_i, z_i that can also include t, we shall now consider the quantity:

$$A = \int_{t_0}^{t} \sum (X\,\dot{x} + Y\,\dot{y} + Z\,\dot{z})\,\delta t$$

and form its *variation* for an unchanging t. We will then have:

$$\delta A = \Big| \sum (X\,\delta x + Y\,\delta y + Z\,\delta z) \Big|_{t_0}^{t},$$

so when all $\delta x, \ldots$ vanish at t_0 ([242]):

$$\delta A = \sum (X\,\delta x + Y\,\delta y + Z\,\delta z)\,.$$

For an arbitrary X, Y, Z, one then *defines:*

$$\delta A = \sum (X\,\delta x + Y\,\delta y + Z\,\delta z)$$

instead of that, i.e., the *virtual work* ([242.a]). If one now assumes that the variations of the $x, \ldots$ at t_1 are also zero then that will give:

$$\delta H = \delta \int_{t_0}^{t_1} T\,dt + \int_{t_0}^{t_1} \delta A\,dt$$

the following form:

$$\delta H = \int_{t_0}^{t_1} dt \left[\sum \left(X_i + \frac{\partial T}{\partial x_i} - \frac{d}{dt}\frac{\partial T}{\partial \dot{x}_i} \right) \delta x_i + \cdots \right].$$

The statement:

([240]) Moreover, that principle, taken in its broader sense, is no longer truly isoperimetric, since one is dealing with an entirely-different variational concept; cf., **A. Voss**, “Über die Differentialgleichungen der Mechanik,” Math. Ann. **25** (1885), pp. 264.

([241]) In so doing, one does not exclude the possibility that this principle can once more be regarded as a first principle from a different standpoint.

([242]) That is the *only* case in which the *variation of the work* also represents the *virtual work*, under the given assumptions.

([242.a]) For the concept of work, see no. **46**.

$$\delta H = 0$$

is then completely equivalent to the differential equations of mechanics; it is called Hamilton's principle. It is completely independent of the special form that the coordinates and constraints might take, and in particular, for holonomic systems with the independent generalized coordinates q_s, one can set:

$$\delta H = \int_{t_0}^{t_1} (\delta T + \sum Q_s \, \delta q_s) \, dt = 0 \, .$$

If X, Y, Z are once more partial differential quotients of a *force function U*, as **Hamilton** assumed ([243]), then one can introduce the so-called ***Hamiltonian** integral:*

$$H = \int_{t_0}^{t_1} (T + U) \, dt \, .$$

As before, the principle demands that $\delta H = 0$. At the same time, one has:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}_i}\right) = \frac{\partial (T+U)}{\partial x_i} + \sum \lambda_s \, a_{si} \, ,$$

or for independent generalized coordinates and holonomic systems:

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) = \frac{\partial (T+U)}{\partial q_i} \, ,$$

from which it will further follow that:

$$\left[\sum \dot{q}_s \, \frac{\partial T}{\partial \dot{q}_s} - (T+U)\right] + \frac{\partial}{\partial t}(T+U) \, dt = 0 \, .$$

If $T + U$ is explicitly independent of t then:

([243]) **W. R. Hamilton**, Lond. Phil. Trans. (1834) first started from the principle of least action, which is discussed thoroughly in the following section of this book. It was only on pp. 307 of his investigations in the Lond. Phil. Trans. (1835), pp. 95, that he introduced the fundamental integral *H*, without especially emphasizing the variational process that is peculiar to him, moreover, which one can actually find a complete description of in **Lagrange**'s *Mécanique* before him. **Jacobi** (*Dynamik*, *Werke*, Suppl., pp. 58) had referred to it as **Hamilton**'s integral and the variational principle as **Hamilton**'s principle. That terminology did not seem to be used in England. It would also seem much more appropriate to refer to *Hamilton's actual discovery as the principle of varied action* [Lond. Phil. Trans. (1835), pp. 99]. In it, *H* is referred to as the *principal function*. **Routh** (*Dynamik*, v. 1, pp. 375) called $U + T$ the **Lagrangian** function, while **Helmholtz** called it the *kinetic potential*. For the expression "force function," cf., footnote 295.

$$\sum \dot{q}_s \frac{\partial T}{\partial \dot{q}_s} - (T + U) = \text{const.},$$

so when T is a homogeneous function of degree two in the $\dot{q}_s$:

$$T - U = \text{const.}$$

That important theorem, which assumes the independence of the constraints and the force function on time, is called the principle of vis viva ([244]). Its meaning will be treated thoroughly in no. **45**.

The investigations that are connected with **Weber**'s law have also allowed us to consider force functions that depend upon velocity and higher differential quotients of the coordinates, i.e., *look for values of X, Y, Z for which:*

$$\sum (X\, \delta x + Y\, \delta y + Z\, \delta z)$$

is the *complete differential* of a single-valued function (cf., footnote 306, moreover) *under these extended circumstances*. That investigation, which was begun by **Riemann** ([245]) and **C. Neumann** ([246]), was carried out by **Schering** ([247]) in the most-general way. For force functions of that kind, one can also give a form to **Hamilton**'s principle for which an *actual* variation will take place under the integral ([248]).

Due to its simplicity, **Hamilton**'s principle can used as a foundation for most investigations to great advantage ([249]). In particular, one can also take the position that one can employ it with no prior deductive basis with the help of force functions that are defined by certain analogies for the derivation of equations for systems *whose applied forces admit explicit representation* ([250]). One then arrives at the fundamental equations of elasticity, hydrodynamics (cf., Band IV 15), and **Maxwell**'s equations of electrodynamics by means of it; see the specialized investigations.

([244]) For more details, see no. **45**.

([245]) **B. Riemann,** *Schwere, Elektrizität, und Magnetismus*, ed. by **K. Hattendorf**, Hannover, 1880.

([246]) **C. Neumann**, *Die Prinzipien der Elektrodynamik*, Tübingen, 1868, = Math. Ann. **17** (1880), pp. 400; also *ibid.* **1** (1869), pp. 317.

([247]) **E. Schering**, "Hamilton-Jacobi'sche Theorie der Kräfte, deren Maass von der Bewegung der Körper abhängt," Gött. Abh. **18** (1873), pp. 32; cf., **E. Voigt**, *Kompendium*, v. 1, pp. 24; as well as **L. Koenigsberger**, *Die Prinzipien der Mechanik*, Leipzig, 1901.

([248]) In addition to the cited papers, cf., **G. Holzmüller**, Zeit. Math. Phys. **15** (1870), pp. 69; **C. Neumann**, *Allgemeine Untersuchungen über das Newton'sche Prinzip*, Leipzig, 1896, pp. 227, *et seq.*; **E. Budde**, *Mechanik*, v. 1, pps. 339 and 372.

([249]) For the investigation of relative motion, cf., e.g., **C. Neumann**, Leipz. Ber. **51** (1899), pp. 371.

([250]) In addition to the known papers of **W. Thomson** [Edinb. Roy. Soc. Trans. (1863), *Math. and Phys. Papers*, v. **3**, pp. 386; **Kirchhoff**, *Mechanik*, pps. 57, 118], cf., *inter alia*: **A. Walther**, "**Hamilton**'s Methode und die Grundgleichungen der Elastizität," Diss. Berlin, 1868; **Boltzman**, "Über der Prinzip von Hamilton," J. f. Math. **73** (1871), pp. 111, which includes a consideration of multiply-connected spaces filled with fluids; **C. Neumann**, *Beiträge zur mathematischen Physik*, Leipzig, 1893, pp. 193, *et seq.*; likewise, *Die elektrische Kräfte*, v. **2**, pp. 347; **W. Wien**, *Hydrodynamik*, pp. 47, Leipzig, 1900; **H. von Helmholtz**, "Das Prinzip der kleinsten Wirkung in der Elektrodynamik," Berl. Ber. (1892), pp. 187; likewise, "Über die physikalische Bedeutung des Prinzips der kleinsten Wirkung," J. f. Math. **100** (1887), with the remark on pp. 143: "In any event, it seems to me that the general validity of the principle has been verified to such an extent that it can used to great advantage as a heuristic principle and a guideline for the ambition to formulate the laws of new classes of phenomena." **C. Neumann** [Ann. di mat. (2) **2** (1868), pp. 2] also referred to the principle as *forma suprema et sacrosancta, nullis exceptionibus obvia* (supreme and sacrosanct form with no obvious exception).

The fact that Hamilton's principle is equivalent to d'Alembert's, under the given assumptions, emerges immediately from the presentation above. In general, one must observe the virtual character of the variations for non-holonomic constraints precisely. Since **Hertz** did not observe that in his *Mechanik* (1894), he arrived at the idea that **Hamilton**'s integral could not be applied to systems of that type, although **Voss** had already referred to the way that would have to be applied in the context of virtual variations in 1884 ([251]).

43. The principle of least action ([252]). – A correct understanding of the *principle of least effect (Wirkung)*, or as one should properly say, *least action*, raises even more difficulties, since one will then be dealing with a concept of variation that does not find application in true isoperimetric problems.

For a system whose initial and final positions, which are associated with times t_0 and t_1, are fixed, a variation of the actual motion will consist of a path that begins at, say, t_0, and each of point of which $P + \delta P$ is associated with a point P of the original path. In that way, not only a variation of the coordinates δx, δy, δz will occur, but also one of time, namely, the difference between the times of the system positions P and $P + \delta P$. Moreover, if $(P\,P_1) = ds$, $(P + \delta P, P_1 + \delta P_1) = ds + \delta\, ds$ then $ds + \delta\, ds$ will be the time it takes to go through the varied path element, so:

$$\frac{ds+\delta\, ds}{dt+\delta\, dt}$$

will be the varied velocity, which is established arbitrarily for *one* point of the system but can be subject to *one* arbitrary constraint more generally. In particular, one can choose it in such a way that the variation of the total energy:

$$\delta T - \delta' U$$

is zero, in which one understands $\delta' U$ to mean the expression:

$$\sum (X\,\delta x + Y\,\delta y + Z\,\delta z) \ .$$

Hamilton's principle will again follow from the identity:

$$\int_{t_0}^{t_1} (2T\,d\,\delta t + (\delta T + \delta' U)\,dt = \int_{t_0}^{t_1} \sum [(X - m\,\ddot{x})\,\delta x + (Y - m\,\ddot{z})\,\delta y + (Z - m\,\ddot{z})\,\delta z]\,dt$$

when time is *not* varied, i.e., for $\delta t = 0$. By contrast, if one adds δt by means of the admissible constraint:

([251]) **A. Voss**, Math. Ann. **25** (1885), pp. 263; cf., **O. Hölder**, "Über die Prinzipien von **Hamilton** und **Maupertuis**," Gött. Nachr. (1896).

([252]) The more socio-political history of this principle, which is connected with the name of **Maupertuis**, will not be considered in this treatise. See, in addition, **Montucla**, *Histoire*, v. **3**, pp. 645; **A. Mayer**, "Zur Geschichte des Prinzips der kleinsten Aktion," Leipzig (1877). **E. du Bois-Reymond**, *Maupertuis*, Berl. Ber. (1892), pp. 393; likewise, **M. Cantor**, *Vorlesungen über Geschichte der Mathematik*, v. **3**, pp. 579.

$$\delta T = \delta' U ,$$

according to the remark above, then that will give:

$$\delta\, 2\int_{t_0}^{t_1} T\, dt = \int_{t_0}^{t_1} \sum [(X - m\,\ddot{x})\,\delta x + \cdots]\, dt = 0 .$$

That is the principle of least action, whose complete *equivalence with d'Alembert's principle* is obvious from this derivation. However, it is an *extended form* of it that assumes neither a force function that is explicitly free of t for the forces nor independence of time for the holonomic or non-holonomic constraints, while generally the variations δx, δy, δz are treated with no regard for time, so in general it will correspond to a change in position that has nothing whatsoever to do with the constraints on the motion that actually ensues ([253]).

The principle of least action then reads ([254]): *The variation of the integral:*

$$\int_{t_0}^{t_1} T\, dt$$

is itself equal to zero relative to all virtual variations of the path of the system that actually results and satisfies the variational constraint that is required by **d'Alembert**'s principle, along with the boundary condition, and for which the variation of the total energy is zero at every moment, *and conversely, the system of differential equations of dynamics can be derived from that demand*, just as it can from **d'Alembert**'s principle.

In particular, if a force function U exists, so:

$$\delta' U = \delta U ,$$

and if U is independent of time, moreover, and that is also true of the constraints then one will have the principle of least action in the form that was expressed by **Lagrange**.

Meanwhile, with the introduction of the *most-general* concept of a variation, which really first takes on its well-defined form when one intends to obtain the differential equations of motion, the special form of the expression that appears under the integral will become completely irrelevant. As **Voss** remarked, one can ultimately replace it with an entirely-*arbitrary* function of the

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([253]) Here, those virtual paths are (which is *always* rooted in the character of more general virtual displacements) generally *impossible* then, which is a direct contradiction to **Ostwald**'s principle of the distinguished case (cf., footnote 27).

([254]) Thus, as **A. Mayer** correctly observed about the variational problem: "If the motion of a system…obeys the principle of *vis viva*, and if the positions at time t_0 and an unknown time t_1 are given then the problem of determining the motion of the system will coincide with the problem of determining the values of the coordinates and the value of t_1 that satisfy the differential equation:

$$T = U + h$$

and make $\delta\int_{t_0}^{t_1} T\, dt = 0$"; Leipz. Ber. **38** (1886), pp. 354.

coordinates and velocities, although the historical forms $T + U$ and T are obviously distinguished by their simplicity and generality ([255]).

44. Historical remarks regarding the principle of least action. – **Euler** was the first to derive the principle that **Maupertuis** ([256]) had expressed so unclearly and incorrectly from a teleological viewpoint ([257]), and applied it to central forces ([258]), in particular. It was **Lagrange** ([259]) who first proved that in general under the assumption of a force function and constraints that were independent of time. However, since **Lagrange** did not define the variational process that he employed more precisely, misunderstandings soon arose regarding the possibility of arriving at the differential equations of motion from **Lagrange**'s integral. Namely, insofar as the variation was performed at constant energy, while the variation of time was not brought under consideration explicitly, the δx, δy, δz did not seem to be *independent of each other* anymore ([260]). In general, **Rodrigues** ([261]) has already also varied the time in the **Lagrange** integral in a completely-applicable way and arrived at the differential equations of motion with the help of the method of multipliers; however, his work has remained unnoticed.

Jacobi ([262]), who likewise started from the assumption that time should not be varied, then gave the question an entirely new twist by eliminating time from the integral completely with the help of the principle of *vis viva*. A so-to-speak *new* variational principle will then arise that one might refer to as **Jacobi**'s principle. For him, only the *geometric elements* of the path appeared under the integral ([263]). Now, if the integral is varied in such a way that the coordinates are varied,

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([255]) **A. Voss**, "Bemerkungen über die Prinzipe der Mechanik," Münch. Ber. (1901), pp. 167.

([256]) **Moreau de Maupertuis** (footnote 252) first published it in Mém. de l'Acad. de Paris (1740), and then in the paper "Des lois de movement et de repos deduites d'un principe métaphysique," Mém. de l'Acad. (1745), pp. 276, esp. 286.

([257]) **Euler**, in "Additamentum II de motu projectorum," pp. 309 in "methodus inveniendi," Lausannae (1744): "Quoniam omnes naturae effectus sequuntur quondam maximi minimive legem, dubium est nullum, quin in lineis curvis, quas corpora projecta describunt si a viribus *quibusconque* sollicitantur, quaepiam maximi minimive proprietas locum habeat." (Since all the effects of nature follow the law of greatest or least, there is no doubt that in the curved lines which projected bodies describe when they are disturbed by forces of any kind, any property of greatest or least exists.)

([258]) *Ibid.*, but with the remark: "Tam late ergo hoc principium patet, ut solus motus a resistentia medii perturbantis excipiendus videatur." (So widely then is this principle evident that the only movement to be received by the resistance of the disturbing medium seems to be received.)

([259]) **Lagrange**, "Application de la méthode des maxima et minima à la resolution de différents problèmes de la dynamique," Misc. Taur. **2** (1760/61), and *Œuvres*, t. 1, pp. 353. Furthermore, the principle of least action takes on only a peripheral position in **Lagrange**'s *Mécanique*, although **Lagrange** had perhaps originally regarded it as closer in spirit to a (not entirely general) basic principle of dynamics, analogous to the principle of virtual velocities in statics. Later on, he gave it even less attention. **Poisson** said of the application of **Hamilton**'s principle of varied action in J. de math. **2** (1837), pp. 333: "that principle of least action, which is only a useless rule today."

([260]) See **A. Mayer**, Leipz. Ber. **28** (1886), pp. 343.

([261]) **Olinde Rodrigues**, Correspond. sur l'éc. polyt. Paris **3** (1815), pp. 159.

([262]) **Jacobi**, *Werke*, *Suppl.*, pp. 44: "In almost all textbooks, the principle is presented in a way that cannot be understood, in my opinion. Indeed, it is said that this law can be true only as long as the law of *vis viva* is true, but they forget to say that one must eliminate time from the integral above by the law of *vis viva* and reduce everything to *spatial elements*." However, in that way, the circumstances that opposed one's understanding *at that point in time* were not even mentioned. The fact that the variation of the integral would vanish due to the equations of motion *was never doubted*. One addressed only the converse of the law (**Lagrange**, *Méc. anal.*, t. **1**, pp. 211).

([263]) The **Jacobi** principle then has an entirely-specialized character. Also, it only takes the *form* of a new statement. (See footnote 174) which might also correspond to possible changes in the path, due to the independence of the constraints of time, then that will, in fact, imply the equations of dynamics.

Namely, starting from:

$$T - U = h ,$$

when one sets:

$$\tfrac{1}{2} \sum a_{ik}\, dq_i\, dq_k = S ,$$

to abbreviate, one will have:

$$dt = \sqrt{\frac{S}{U+h}} .$$

In that way, one will get:

$$J = \int T\, dt = \int \sqrt{S\,(U+h)} ,$$

and in fact:

$$2\, \delta J = \int \sum \left\{ \tfrac{1}{2} \sqrt{\frac{U+h}{S}} \sum \frac{\partial a_{sk}}{\partial q_i}\, dq_s\, dq_k + \frac{\partial U}{\partial q_i} \sqrt{\frac{S}{U+h}} \right\} \delta q_i - \int \sum d\left(\sum a_{ik}\, dq_k \sqrt{\frac{U+h}{S}} \right) \delta q_i = 0 ,$$

in which one introduces:

$$dt = \sqrt{\frac{S}{U+h}} ,$$

to abbreviate, will yield the equations that arise from **Lagrange**'s by eliminating time.

By contrast, **M. Ostrogradsky**, who likewise vigorously emphasized the incorrect foundation of the principle, was of the opinion that **Lagrange**'s minimum principle has to be understood in the sense of **Hamilton** in order to correctly treat the principle ([264]). **Sloudsky** ([265]), recalling **Rodrigues**, was the first to once more emphasize that the latter was essentially different from the principle of least action. **A. Mayer** ([266]) then gave the true state of affairs connected with those papers, which was long-known in England by, e.g., **Routh** ([267]), moreover. **Helmholtz** ([268]) attracted new interest to the principle. It was **Hölder** who first removed all doubt concerning it and expressed the principle in its broadest context ([269]).

([264]) **M. Ostrogradsky**, "Equations différentielles dans le problème des isopérimetres," Pétersb. Mém. de l'Acad. (6) **4** (1850), pp. 385; cf., especially, pp. 415, *et seq.*

([265]) **Th. Sloudsky**, Nouv. ann. de math. (2) **18** (1866), pp. 198.

([266]) **A. Mayer**, "Die beiden allgemeinen Sätze der Variationsrechnung, welche den beiden Formen des Prinzips der kleinsten Wirkung entsprechen," Leipzig. Ber. **38** (1886), pp. 343.

([267]) **E. J. Routh**, *Dynamics of a system of rigid bodies*, 4th ed., v. 2, pp. 244, like **Rodrigues**, used the method of multipliers. **K. Uckermann**, Diss. Marburg, 1893, derived the principle without them.

([268]) **H. von Helmholtz**, "Das Prinzip der kleinsten Aktion." Berl. Ber. (1887), pp. 225. Many objections can be raised against his presentation; cf., **Hölder**'s paper in footnote 269.

([269]) **O. Hölder**, "Über die Prinzipien von **Hamilton** und **Maupertuis**," Gött. Nachr. (1900). Further, see **E. Mathieu**, *Dynamique analytique*, 1877, pp. 42; **G. Sabinine**, "Sur le principle de la moindre action." Ann. di mat. (2) **12** (1883), pp. 237; "Sur le minimum d'une intégrale," *ibid.* **14** (1887), pp. 13; "Sur les considérations d'**Ostrogradsky** et de **Jacobi** relatives au principe de la moindre action," *ibid.* **15** (1888), pp. 27; **M. Réthy**, "Über das Prinzip der kleinsten Wirkung und das Hamilton'sche Prinzip," Math. Ann. **48** (1897), pp. 514; **A. Voss**, Gött. Nachr. (1900).

From the principle of least work, in its original form, one has $\delta \int T\,dt = 0$, so the maximum-minimum condition for the integral is fulfilled. We shall not touch upon an examination of the *sign of the second variation* ([270]) here, i.e., the proof that one is, in fact, dealing with a *minimum* for sufficiently-small intervals, since it properly belongs to the realm of pure mathematics.

C) True integral principles.

45. The principle of *vis viva*. – From the fundamental equations of dynamics [no. **37**, (3)]:

$$m\,\ddot{x}_i = X_i + \sum \lambda_s\, a_{is}\ , \text{ etc.,}$$

and the equations of constraint [*loc. cit.*, (2)]:

$$\sum (a_{si}\,\dot{x}_i + b_{si}\,\dot{y}_i + c_{si}\,\dot{z}_i) + c_s = 0\ ,$$

when one multiplies them by $\dot{x}_i$, $\dot{y}_i$, $\dot{z}_i$, and sums, while introducing the *vis viva* or kinetic energy T, one will get the equation:

$$\frac{dT}{dt} = \sum (X_i\,\dot{x}_i + Y_i\,\dot{y}_i + Z_i\,\dot{z}_i) - \sum \lambda_s\, c_s\ ,$$

in which X_i, Y_i, Z_i are all understood to mean forces, which the exception of the ones that emerge from the equations of constraint. In particular, if the c_s are all equal to zero then it will follow upon integration that:

$$T - T_0 = \sum \int_{t_0}^{t_1} (X_i\,dx_i + Y_i\,dy_i + Z_i\,dz_i)\ .$$

This general *law of vis viva or kinetic energy* ([271]), *namely: the increase in vis viva or kinetic energy is equal to the work done by all forces during that time interval*, is then true for all non-holonomic constraints only in the event that the $c_s = 0$ ([272]). Thus, it will be true when constraints take the form of finite equations:

$$f_1 = 0\ , \ldots, f_k = 0$$

in any event when they are explicitly independent of t.

If one imagines that the X_i, Y_i, Z_i are decomposed into the components:

([270]) See **J. A. Serret**, C. R. Acad. Sci. Paris **72** (1871), pp. 697, or Bull. sciences math. **2** (1871), pp. 97; likewise, **G. Darboux**, *Leçons sur la théorie Générale des surfaces*, t. 2, Paris, 1896, pp. 480. **D. Bobylev**, Peterst. Abh. d. Akad. **59** (1889); **G. Kobb**, "Sur le principle de la moindre action." Toul. Ann. **5** (1891), pp. 1-3.

([271]) It is simpler for one to obtain that equation directly from d'Alembert's principle under the assumption that the actual path-elements dx_i, dy_i, dz_i are included among the virtual ones, in which the holonomic or non-holonomic character is entirely irrelevant.

([272]) That was probably first shown by **A. Voss**, "Über die Differentialgleichungen der Mechanik," Math. Ann. **25** (1885), pp. 266.

$$X_i = X_i' + X_i'' ,$$
$$Y_i = Y_i' + Y_i'' ,$$
$$Z_i = Z_i' + Z_i'' ,$$

the first of which possesses a force function U that is independent of time explicitly, then under those assumptions, one will have:

$$T - T_0 = U - U_0 + \int \sum (X'' dx + Y'' dy + Z'' dz) ,$$

when one introduces the function:

$$V = - U$$

in place of U, one will have:

$$T + V = T_0 + V_0 + \int \sum (X'' dx + Y'' dy + Z'' dz) .$$

In particular, if the work integral on the right-hand side is equal to zero then one will get:

$$T + V = T_0 + V_0 .$$

For all mechanical systems with *one* degree of freedom on which only *conservative* forces act. i.e., ones that arise from a force function that is independent of t explicitly, and under assumptions on the constraints that were given before, this last theorem, which again expresses *the principle of the conservation of vis viva or kinetic energy*, will yield the solution by quadrature, and is, above all, the fundamental theorem that will lead to any further discussion of every mechanical problem.

Meanwhile, it would seem that what is much more important for seeing the overall mechanical picture is the conversion that was first introduced by **Helmholtz** ([273]), which regarded the negative force function as the *potential*, T as the *kinetic*, $T + V$ as the *total energy*, so the law of *vis viva* will lead to the *energy principle*, i.e., the study of the *conservation of energy*.

The consideration of *energy* in a purely-mathematical context offers certain advantages that emerges especially clearly in the *questions of stability*. Here, we shall mention only the far-reaching generalization that **Routh** ([274]) introduced into **Lagrange**'s stability criterion, which was first proved completely by **Dirichlet** ([275]).

On the other hand, all of that is closely connected with the viewpoint that emerges when one extends analytic geometry to a multidimensional conception of space. Above all, let us recall the investigations of the *line element* ([276]), which takes the form of the square root of the differential of (twice) the kinetic energy, the theory of quadratic forms in the *problem of small oscillations*

([273]) **Helmholtz**, "Erhaltung der Kraft," **Ostwald**, K. B., no. 1, pp. 11; **Clausius** was also led back to the same general form of the concept as **Helmholtz**, which probably already existed in isolation [Ann. Phys. Chem. **150** (1873), pp. 109].

([274]) **Routh**, "Essay on the stability of motion," *Dynamik*, v. 2, pp. 75, *et seq.*

([275]) **P. G. Lejeune-Dirichlet**, "Über die Stabilität des Gleichgewichts," J. f. Math. **32** (1846), pp. 85; **A. Liapounoff**, "Sur l'instabilité de l'équilibre dans certains cas où la function n'est pas maximum." J. de math. (5) **3** (1897), pp. 81; **J. Hadamard**, "Sur certaines propriétés des trajectoire en dynamique," *ibid.*, pp. 364.

([276]) Which began with **J. Liouville**'s treatise: "Expression remarquable de la quantité qui est un minimum en vertu du principe de la moindre action," J. de math. (2) **1** (1856), pp. 297.

([277]), and the *study of the equivalence* of mechanical problems ([278]), as well as the presentations of the *theories of groups and transformations* ([279]).

Finally, that is most important is the appearance of the total energy $E = T + V$ in the **Poisson-Hamilton** transformation of the **Lagrange** equations [no. **37**, (4)]. If one decomposes variables q_s, $s = 1, 2, \ldots, r$ into two groups q_{s_1} and r_σ, $s_1 = 1, 2, \ldots, l$; $\sigma = 1, 2, \ldots, l'$; $l + l' = r$, and sets:

$$\frac{\partial T}{\partial \dot{q}_{s_1}} = p_{s_1} ,$$

then it will follow for $T' = (T) - \sum p_{s_1} \dot{q}_{s_1}$, when (T) is the value of T that is produced by replacing the $\dot{q}_{s_1}$ with p_{s_1} in it, such that T' is a function of the q_{s_1}, p_{s_1}, r_σ, $\dot{r}_\sigma$:

$$\frac{dp_{s_1}}{dt} = \frac{\partial T'}{\partial q_{s_1}} + Q_{s_1} ,$$

$$\frac{dq_{s_1}}{dt} = - \frac{\partial T'}{\partial p_{s_1}} ,$$

$$\frac{d}{dt}\left(\frac{\partial T'}{\partial \dot{r}_\sigma}\right) - \frac{\partial T'}{\partial r_\sigma} = Q_\sigma ,$$

or when the groups q_s and q_{s_1} coincide:

$$\frac{dp_s}{dt} = \frac{\partial T'}{\partial q_s} + Q_s ,$$

$$\frac{dq_s}{dt} = - \frac{\partial T'}{\partial p_s} .$$

In the special case where T is a homogeneous quadratic function of the $\dot{q}_s$ (no. **37**), one will have $T' = -(T)$. If one also assumes that:

$$Q_s = - \frac{\partial V}{\partial q_s}$$

then for:

$$E = (T) + V,$$

one will get the *canonical* form for the differential equations of mechanics:

([277]) Namely, see **E. J. Routh**, *Dynamics of a system of rigid bodies*, vols. 1 and 2.

([278]) **P. Stäckel**, J. f. Math. **107** (1891), pp. 319.

([279]) In addition to all of the works of **S. Lie**, cf., the investigations of **P. Painlevé**, **P. Stäckel**, *et al.*, as well as the articles 11-14 in Bd. IV.

$$\frac{dq_s}{dt} = + \frac{\partial E}{\partial p_s},$$
$$\frac{dp_s}{dt} = - \frac{\partial E}{\partial q_s},$$

in which the whole problem will depend upon only the energy function E ([279.a]).

46. Historical remarks about work, *vis viva*, and energy. – The law of *vis viva*, in its simplest form, was already found in **Galilei** ([280]), who recognized that the final velocity of a body falling on a skew plane depends upon only the altitude. However, the *principle of the conservation of vis viva* appeared in a more definitive form in **Huygens** ([281]) as an *axiom*. **Joh. Bernoulli** ([282]) already spoke of the *conservatio virium vivarum*, namely, the capacity of the *vis viva* to do work in various forms.

However, the law was first found in its proper form that belongs to analytical mechanics by **Dan. Bernoulli**, who had already developed it for the problems of celestial mechanics ([283]). For **Lagrange** ([284]), the concept of potential function arose for discrete masses, while for **Laplace** ([285]), it arose for continuous masses.

The entire study of energy also arose from the special formula $m v^2 = 2\, g\, h\, m$. Originally, the expression for the *vis viva* was the quantity $m v^2$. Moreover, a concept that is just as primitive is that of the *work P h* that is done when a weight *P* experiences a change in height *h*. It was gradually adapted to all of the (initially constant) forces that are expressible by weights.

([279.a]) This canonical form of the differential equations of dynamics, into which *every* isoperimetric problem can be brought, according to **Ostrogradsky**, Petersb. Mém. de l'Acad. (6) **4** (1850), pp. 403, was already found in an unpublished paper by **Cauchy** in Turin, Mém. (1831). Compare **A. Cayley**, Brit. Assoc. Rep. 1862), London, 1863, pp. 184; for the appearance of canonical equations in **Lagrange**, **Poisson**, **Hamilton**, **Routh**, cf., article 11.a of Bd. IV.

([280]) Cf., **Mach**, *Mechanik,* pp. 342. Similar considerations are discussed in **P. Varignon**, "Propriétés communes aux chutes rectlignes dans le vuide," Paris Mém. de l'Acad. (1720), pp. 107 (Paris, 1722).

([281]) **Ch. Huygens** in "Horologium oscillatorum," Paris, 1673. Cf., in addition, **Lagrange**, *Mécanique*, v. 1, pp. 249, **Mach**, *Mechanik*, pp. 180: "We hope that this principle (of the center of oscillation) can be put into the correct light here as something that is identical to the law of *vis viva*." For a more-precise investigation, cf., the fundamental papers of **Jacob Bernoulli**, "Démonstration générale du centre de balancement," Paris, Mém. de l'Acad. (1703), pp. 78; *Opera*, two vols., Geneva, 1744, Bd. 1, pp. 930; "Démonstration du principe de **M. Huyghens**," Paris, Mém. de l'Acad. (1704), pp. 136; *Opera*, Bd. 1, pp. 947.

([282]) **Joh. Bernoulli**, "Theoremata selecta pro conservation virium vivarum," Comm. Ac. Petrop. **2** (1729), pp. 200; as well as *Opera*, v. 3, pp. 243 (from the Acta erud. Lips., 1735), "de vera notione virium vivarum," with the noteworthy observation that: "Hinc patet, *vim vivam* quae aptius vocatur *facultas agenda* esse aliquid reale et substantiale quod per se substitit et quantum in se est, non dependet ab alio." (From this, it is clear that the living force, which is more properly called the ability to act, is something real and substantial, which subsists by itself, and insofar as it exists by itself, it does not depend upon anything else.)

([283]) **D. Bernoulli**, "Remarques sur le principe de la conservation des forces vives pris dans son sens général," Berlin, Mém. de l'Acad. (1748), pp. 356; *ibidem,* for the *n*-body problem, pp. 363.

([284]) **Lagrange**, Berlin, Mém. de l'Acad. (1777), pp. 155. The term *potential function* is known to go back to **G. Green** ("An essay on the application of mathematical analysis," Nottingham, 1828), as well as being in his *Math. Papers,* it was also printed in J. f. Math. (1850/54), pps. 39, 44, 47. German transl. by **A. Wangerin**, **Ostwald,** K. B. no. 61.

([285]) **Laplace**, Paris, Mém. de l'Acad. (1782), pp. 119.

That quantity was soon referred to as an *effect* of the force, as the *puissance mécanique* ([286]), the *moment d'activité* ([287]), the *quantité d'action* ([288]) (**Coulomb**); however, others [**Ch. Dupin** ([289]), **Hachette** ([290]), **Prony** ([291])] had already referred to it as *Arbeit*, *travail*, *labour*.

However, it was only under the influence of **Poncelet** that **Coriolis** ([292]) completely established a sharp definition of the concept of the *Arbeit*, *work*, *travail*, *lavoro done by a varying force along an arbitrary path.* Those two French researchers, whose ideas overlapped in many places, and were also probably modified by their reciprocal influence on each other, applied the law of *vis viva* in its full generality to the determinate *motion of machines* ([293]): That implied the **Coriolis-Poncelet** formula:

$$\tfrac{1}{2}\sum m(v^2 - v_0^2) = T_m - T_r - T_f - T_c\,,$$

in which the quantities on the right-hand side refer to the works done by the moving forces and the various resistances (collisions, resp.).

S. Carnot ([294]) was the first to also apply that equation to non-mechanical processes, with their meaning in that era, such as thermodynamic problems, and this laid the groundwork for the modern study of energy, while the mathematical formulation was, in fact, developed further by **Green**'s work and **Hamilton**'s general concept of a *force function* ([295]).

The presentation of **Th. Young** ([296]), which was still restricted to purely-mechanical processes of collisions between moving masses, etc., and which ascribed *energy* to bodies as the means by which they could do work, which was already referred to by **L. N. Carnot** as *force vive virtuelle* (which is now called potential energy), along with *force vive*, was built up by **Coriolis** ([297]) and

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([286]) For example, **J. Smeaton** referred to it as "mechanical power" in London Phil. Trans. **66** (1776), pp. 450.

([287]) **Carnot**, *Principes fondamentaux*.

([288]) As in **G. Monge** and **J. P. Hachette**.

([289]) **Ch. Dupin**, *Géométrie et mécanique des arts*, v. 3, 1826, pp. 477.

([290]) **J. P. Hachette**, *Traité élémentaire des machines*, 4th ed., 1828, pp. 19.

([291]) **R. Prony**, Annales des mines (1826), pp. 33; cf., what **Poncelet** said in *Cours de mécanique*, **§ 6.**

([292]) In the Foreword to the first edition of his *Traité de la mécanique des corps solides et du calcul de l'effet des machines*, 1829 (2nd ed., 1844), **G. Coriolis** said: "I shall refer to the quantity that one quite commonly calls the '*puissance mécanique*' by the term *work*,…" In that same reference, **Poncelet** also defined the fundamental theorem of the *work done by the resultant*.

([293]) Meanwhile, according to **C. L. Navier**, "Details historiques sur l'emploi du principe des forces vives dans la théorie des machines," Ann. de chimie **9** (1818), pp. 146, **L. N. Carnot** had already begun that extension of scope in his *Essai sur les machines en générale*, 1783.

([294]) **Said Carnot**, 1824; published in Ann. éc. norm. (2) **1** (1872), pp. 393.

([295]) **W. R. Hamilton**, "On a general method in dynamics," Lond. Phil. Trans. (1834), pp. 249. The term "force function" was introduced by **Jacobi** in 1836, J. f. Math. **17** (1838), pp. 97. The more-concise term *Ergal* was used by **Clausius**, Ann. Phys. Chem. **150** (1873), pp. 136, and more recently by **E. Budde**, *Mechanik*, Bd. 1, pp. 430.

([296]) **Th. Young**, *A course of lectures on natural philosophy*, v. 1, pp. 78; v. 2, pp. 51; also there, on pp. 79, one finds the remark: "The *labour* expended in producing any motion is proportional not to the momentum, but to the energy, which is obtained." Meanwhile, the *energy* of a moving body was already defined by **d'Alembert**, *Encyclopédie* (four vols., 2nd ed., Paris, 1785), t. 2, pp. 82, "Art. Mathématiques."

([297]) **Coriolis**, *Traité*, ed. 1844, pps. 39 and 114. The current definition of *vis viva* also goes back to **Coriolis** (preface to the first edition: "I shall again permit myself a *slight innovation* by calling the product of the weight with the height the *vis viva*."), which expressed precisely that *equivalence of work and vis viva*. This apparently-only-formal alteration is just as important as the knowledge that is obtained from **Helmholtz**'s way of preserving the force by inverting the sign of the potential function in such a way that the total energy will be constant. It took some time for the expression mv^2 to be abandoned. Even to this day, it still exists in many places, such as **W. Schell**, *Theorie der Bewegung*, v. 2, pp. 530. Especially with the French authors, e.g., **H. Resal**, *Mécanique*, t. 2, pp. 235; **R. Liouville**,

Poncelet ([298]) into the *principe de la transmission du travail*, i.e., the study of the conversion of the quantity of work in machines.

As a result of **R. Mayer**'s ([299]) bold and entirely-original train of thought, that led to an entirely-general conception of things, by means of which all phenomena were subsumed by the unified picture of mechanically-equivalent works that could be transformed into each other. Those ideas were expressed by Helmholtz ([300]) in a mathematically more-precise form, and independently of **R. Mayer**, as the general *law of conservation of energy:*

$$\tfrac{1}{2}\sum mv^2 + V = \text{const.},$$

into which Helmholtz introduced representation of the *tension V* ([301]) in place of minus the potential function, and at the same time enriched the most far-reaching applications to thermodynamics, electrodynamics, etc.

Those ideas were further developed conceptually by, above all, **Rankine** and **W. Thomson**, whose terminology is finally coming into general use.

For **Rankine** ([302]), the present or *sensible* energy (*vis viva*, heat, light, electrical motion, etc.), which was called dynamical or *kinetic* energy by **W. Thomson** soon afterwards had to be contrasted with the *potential* (latent) energy (molecular forces, gravitation, chemical affinity, electrical charge, etc.). All phenomena are based upon an ongoing transformation of those two forms of energy, whose total amount is conserved, and the problem of physical mechanics is to find the law that makes those conversions result. (For the further development of those ideas by **Ostwald**, see no. **49**.)

47. The energy principle. – The law of conservation of energy, in the older sense (no. **42**), is a purely-dynamical one. However, things are quite different ([303]) with the energy principle in modern physics, which regards it as an axiom that is founded upon a substantial induction.

C. R. Acad. Sci. Paris **114** (1892), pp. 1171; **P. Appell**, J. de math. **12** (1896), pp. 5. **J. Boussinesq** distinguished between the *énergie actuelle* and **Leibniz**'s *force vive* (Acta erudit. Lips. 1695). It is almost universal in the English literature; see the remark by **Routh**, *Dynamik*, v. 1, pp. 315.

([298]) **J. V. Poncelet**, *Cours de mécanique*, pp. 17: "The sum of the elementary works that are developed, whether by the various forces that produce the modification of the motion or by the forces of inertial that are created by that modification, is contantly equal to zero."

([299]) **R. Mayer**, manuscript from 1841 for Ann. Phys. Chem. in: *R. Mayer, kleinere Schriften und Briefe*, ed., by **J. Weyrauch**, Stuttgart, 1893.

([300]) **H. von Helmholtz**, *Über die Erhaltung der Kraft*, Berlin, 23 July 1847 = *Wiss. Abh.*, Bd. 1, pp. 12-75; also **Ostwald**, K. B., no. 1.

([301]) **Ostwald**, K. B., no. 1., pp. 12.

([302]) **W. J. M. Rankine**, "On the general law of the transformation of energy," Glasgow, Phil. Soc. Proc. **3** (1853) = *Papers*, 1881, pp. 203; "Outlines of the sciences of energetics," *ibid.*, 1855 = *Papers*, pp. 209 with the remark: "Any kind of energy may be made by the means of performing any kind of work," pp. 218; likewise, **W. Thomson**, "On the origin and transformations of motive power," 1856 = *Papers*, v. 2, pp. 182.

([303]) Cf., e.g., **P. Duhem**, *Traité élémentaire de mécanique chimique*, Paris, 1897, pp. 25.

The *energy of a material system* ([304]) is the amount, measured in mechanical units of work, of all effects that are produced "externally" to the system when it goes from its state ([304.a]) to a certain normal state in any way. That amount is completely independent of the type of transition.

We assume that a material system whose particles not only exhibit dynamical phenomena, but also exist in various states (thermal, elastic, magnetic, chemical affinities, …), and that total state is defined by a series of parameters $q_1, q_2, \dots, q_k$, and their velocities $\dot{q}_1, \dot{q}_2, \dots, \dot{q}_k$. If the system now goes from any *normal state* $Z_0(q_s^0, \dot{q}_s^0)$ to a new state $Z(q_s^0, \dot{q}_s^0)$ then a certain amount of mechanical work ([305]) A would be produced outside of the system, which generally assumes that it is possible to measure all of those effects by *equivalent mechanical work*. Now, if the work done along the first path W_1 from Z_0 to Z is equal to A_1, and the work done along a second path W_2 is equal to A_2, and if the path W_3 from Z to Z_0 corresponds to the work A_3 then one has two closed paths:

$$W_1 + W_3 \qquad \text{and} \qquad W_2 + W_3$$

that correspond to the works $A_1 + A_3$, $A_2 + A_3$, resp. If one now makes the *assumption that a perpetuum mobile is impossible*, namely, that the work done along a closed path is always equal to zero ([306]), then it will follow that:

$$A_1 = A_2 ,$$

i.e., the total work is a function of the parameters that depends upon only the initial and final state:

$$A = F(q_s, \dot{q}_s \mid q_s^0, \dot{q}_s^0) \ .$$

Moreover, since one must also have:

$$A = F(\bar{q}_s, \dot{\bar{q}}_s \mid q_s^0, \dot{q}_s^0) + F(q_s, \dot{q}_s \mid \bar{q}_s, \dot{\bar{q}}_s)$$

upon introducing another state $\bar{Z}(\bar{q}_s, \dot{\bar{q}}_s)$, one must generally have:

$$A = \Phi(q_s, \dot{q}_s) + \Psi(q_s^0, \dot{q}_s^0) \ ,$$

or when one imagines that:

$$0 = \Phi(q_s^0, \dot{q}_s^0) + \Psi(q_s^0, \dot{q}_s^0) ,$$

([304]) **W. Thomson**, 1851, Phil. Mag. (4) **9** (1855), pp. 523: "The total mechanical Energy of a body might be defined as the mechanical value of all the effects it would produce if heat were omitted and resistances overcome, if it were cooled to the utmost. But…it is convenient to choose a certain state as standard," likewise, Quart. J. of Math. **1** (1857), pp. 57. Cf., **M. Planck**, *Energie*, pp. 99; **G. Helm**, *Grundzüge de math. Chemie*, Leipzig, 1894, pp. 1; **Planck**, *Vorlesungen über Thermodynamik*, Leipzig, 1897, pp. 34, *et seq*.

([304.a]) For the terminology here, which was chosen by **Planck**, cf.: **Planck**, *Prinzip der Erhaltung der Energie*, pp. 93.

([305]) **P. Duhem** referred to that as *Oeuvre* in *Commentaire aux principes de la thermodynamique*, J. de math. (4) **8** (1892), pp. 290.

([306]) Naturally, as soon as one deals with general manifolds, the theorems of *analysis situs* will come into play, which were developed by **E. Betti**, "Sopra gli spazii d'un numero qualunque di dimensioni," Ann. di mat. (2) **4** (1870/71), pp. 140, likewise, **E. Lemmi**, "Sur les cas d'exception du théorème des forces vives," J. de math. (3) **2** (1876), pp. 233. Cf., **Maxwell**, *Elektrizität and Magnetismus*, Bd. 1, pp. 19.

one must have:

$$A = \Phi(q_s, \dot{q}_s) - \Phi(q_s^0, \dot{q}_s^0)\,.$$

If one now assumes that Φ splits into two parts, one of which includes only the q_s, while the other one is a homogeneous quadratic function of the $\dot{q}_s$ whose coefficients might depend upon the q_s then one will have:

$$-\Phi = V(q_s) + T(q_s, \dot{q}_s)\ ,$$

so one must have:

$$-A = V - V_0 + T - T_0\ ,$$

in which A is expressed by an ordinary work integral or a sum of thermal, electrical, chemical, … works, when expressed in mechanical units. Since ([307]):

$$-\ \bar{d}A = dV + dT\ ,$$

or when one denotes the left-hand side, which does not need to be a complete differential in the q_s, by:

$$\sum_1^m P_s\, dq_s + \sum_{m+1}^k E_s\, Q_s\, dq_s$$

(the first part of which refers to the mechanical forces, while the second refers to the other contributions, expressed in terms of suitable equivalent numbers):

$$dV + dT - \sum_1^m P_s\, dq_s - \sum_{m+1}^k E_s\, Q_s\, dq_s = 0$$

will be the expression for the general law of conservation of energy.

If one now defines:

$$\delta P = \sum P_s\, \delta q_s\ ,$$
$$\delta Q = \sum Q_s\, E_s\, \delta q_s$$

then one can finally regard the *extended Hamilton principle:*

$$\delta \int (T - V + P + Q)\, dt = 0$$

for all mechanical processes.

([307]) This appropriate notation for incomplete differentials of **C. Neumann** [Leipz. Ber. **46** (1894), pp. 1] is also used in **W. Voigt**, *Kompendium*, Bd. 1, pp. 22.

In the past ([308]), one sought to represent the general viewpoint of energetics with the help of the axiom of the impossibility of a *perpetuum mobile*. **Helmholtz** ([309]) had already gone down that path in his well-known treatise, with the remark that the energy principle coincides with the law of conservation of *vis viva* when one attributes all processes to pure action-at-a-distance ([310]).

48. The virial theorem and the second law of thermodynamics. – Upon multiplying the **Lagrange** equations (no. **37**) by x, y, z and summing, one will get:

$$\frac{dR}{dt} = \frac{1}{2}\frac{d}{dt}\sum m_i\, \ddot{r}_i^2 = T + \tfrac{1}{2}\sum (X\,x + Y\,y + Z\,z) \tag{1}$$

for a free system (i.e., one in which all constraints are replaced by forces), in which r is the distance from the point to the coordinate origin. If one now assumes that the left-hand side remains unchanged during the course of time from t_0 to t then one will have:

$$T - T_0 = V - V_0\,,$$

when one lets:

$$V = -\,\tfrac{1}{2}\sum (X\,x + Y\,y + Z\,z)$$

denote the *virial* of the forces ([311]). According to Clausius, this very special law can be generalized when one integrates (1) over time:

$$R - R_0 = \int_{t_0}^{t_1} T\,dt - \int_{t_0}^{t_1} V\,dt\,,$$

from which, it will follow that when the left-hand side fluctuates within relatively-narrow limits, the mean values T_m, V_m of T, V for sufficiently-large $t_1 - t_0$ will satisfy:

$$T_m = V_m\,,$$

([308]) One cf., **P. Duhem**, "Commentaire aux principes de la thermodynamique," J. de math. (4) **8** (1892), pp. 269; *ibid.*, **9** (1893), pp. 293; *ibid.*, **10** (1894), pp. 207. likewise, *Traité de mécanique chimque*, t. 1, pp. 25. Similarly, there is a sharper application of the axiom of the *perpetuum mobile* that **Helmholtz**'s in **M. Planck**, *Energie*, pp. 140; **L. Natanson**, "Über die Gesetze nicht umkehrbarer Vorgänge," Zeit. phys. Chemie **21** (1896), pp. 193.

([309]) Moreover, the same argument is already found in **G. Green**'s 1837 treatise on the work done by elastic forces: "Indeed, if $\delta\varphi$ were not an exact differential, a *perpetual motion* would be possible, and we have every reason to think that the forces of nature are so disposed as to render this a natural impossibility," (**Green**, *Papers*, pp. 248).

([310]) For that dogmatic formulation, which was already contested by **Clausius**, Ann. Phys. Chem. **91** (1854), pp. 604, and referred to as awkward (*misslich*) by **Planck** (*Energie*, pp. 137), but has probably been generally abandoned nowadays, see, e.g., **H. Klein** "Deduktion des Satzes von der Erhaltung der Kraft," Schul.-Progr. Dresden, 1889, no. 508.

([311]) **R. Clausius**, Ann. Phys. Chem. **141** (1870), pp. 124; Jubelband 1874, pp. 411; **Y. Villarceau**, "Sur un nouveau principe de mécanique," C. R. Acad. Sci. Paris **75** (1872), pp. 232, 377. For the virial, see **Jacobi**'s *Dynamik*, pp. 22; c f., **R. Lipschitz**, Bull. sciences math. **3** (1872), pp. 349. The virial already appeared in statics in 1837, and for **Möbius**, it was the *certainty function* (*Sicherheitsfunktion*) (*Statik*, Bd. 1, pp. 230). Later, it was discussed in **F. Schwein**, J. f. Math. **38** (1849), pp. 77, and *ibid.*, **47** (1854), pp. 238, as torque (*Fliehmoment*), in analogy with the usual moments.

i.e., *the mean vis viva is equal to the (mean) virial.*

The virial shifts the consideration of the *mean state* of a system to the foreground, as one actually does in the kinetic theory of gases and gives an expression for the closely-connected study of the calculation of probabilities over the periodic recurrence of certain states. We shall pursue that topic here only to the extent that it relates to the *second law of thermodynamics*.

If one sets:

$$T + U = H$$

for generalized coordinates q_s then one will have:

$$\delta\int_{t_0}^{t_1} H\,dt = \left| H\,\delta t + \sum \frac{\partial H}{\partial \dot{q}_s}(\delta q_s - \dot{q}_s\,\delta t)\right|_{t_0}^{t_1} + \int_{t_0}^{t_1} \sum \frac{\partial U}{\partial c_s}\,\delta c_k\,dt \;,$$

in the event that the potential energy parameter, which is not varied in **Hamilton**'s integral, experiences a variation that is denoted by δc. If one denotes the last integral by W then one will have:

$$\delta\int_{t_0}^{t_1} 2T\,dt = \delta\int_{t_0}^{t_1} E\,dt - \left| E\,dt\right|_{t_0}^{t_1} + \left|\sum \frac{\partial H}{\partial \dot{q}_s}\delta q_s\right|_{t_0}^{t_1} + W$$

for the total energy $T - U = E$ with time-independent constraints when T is a homogeneous function of second degree of $\dot{q}_s$. Now, if the system moves in such a way that the points assume the same positions and velocities at times t_0 and t_1 ([312]) then:

$$\delta\int_{t_0}^{t_1} 2T\,dt = (t_1 - t_0)\;\delta E + W\,,$$

so when one replaces the integral with its value $2\,T_m\,(t_1 - t_0)$:

$$2\;\delta[T_m\,(t_1 - t_0)] = (t_1 - t_0)\;\delta E + W\,.$$

In particular, it will follow for $W = 0$ that:

$$\frac{2\,\delta[T_m\,(t_1 - t_0)]}{T_m\,(t_1 - t_0)} = \frac{\delta E}{T_m}\,.$$

Therefore, if a series of states of motion are traversed that define a *complete cyclic process*, such that one ultimately returns to the original state of motion then:

([312]) The very-specialized assumptions that were made in this treatise are found in extended form in **Clausius**, "Über die Zurückführung des zweiten Hauptsatzes der mechanischen Wärmetheorie auf allgemeine mechanische Principien," Ann. Phys. Chem. **142** (1871), pp. 433; *ibid.*, Suppl. **7** (1876); for a new mechanical law, Ann. Phys. Chem. **150** (1873), pp. 106; for the connection between the second law… and **Hamilton**'s principle, Ann. Phys. Chem. **146** (1872), pp. 585, cf., also **C. Szily**, "Das dynamische Prinzip von **Hamilton** in der Thermodynamik," Ann. Phys. Chem. **149** (1873), pp. 74.

$$\int \frac{\delta E}{T_m} = 0 ,$$

which will correspond to the second law of thermodynamics for complete cyclic process when T_m is replaced by the absolute temperature T, and δE is replaced with the quantity of added heat dQ ([313]).

Now, the two fundamental laws of **Gibbs** ([314]), which reproduce the principle of virtual velocities, are also true for *energy E* and *entropy S:*

$$S = \int \frac{dQ}{T} .$$

For the equilibrium of a material system that is free from external influences, it is necessary and sufficient that the change in entropy must be:

$$\delta S \leq 0 ,$$

and that the change in energy for unchanged S must be:

$$\delta E \geq 0$$

for all possible changes in its state that leave the energy unchanged.

Here, we can only suggest the further development of the concepts of energy and entropy, e.g., the distinction between free and bound energy; for that, cf., Band V.

49. The localization of energy. – It would be natural to regard the energy of a system as a primitive quality of it that expresses *only* the value of mechanical work, in addition to space and time quantities. **Ostwald** ([315]) had developed **Rankine**'s ideas regarding that into a *system of energetics* whose problem consists of distinguishing the different forms of energy by their "capacity" and "intensity" factors and, at the same time, giving the ground rules for their conversions. The amount of energy in an independent system is unvarying, and of all the conversions that can occur, the ones that will happen will be the ones that produce the greatest change in potential energy in a given time interval.

Those ideas, which are subsumed by a set of different phenomena from more-general viewpoint by way of an interesting analogy (which is also generally more formal, in part), are indeed repeatedly challenged from various angles nowadays ([316]), but they seem to have an

([313]) **Boltzmann**, "Über die mechanische Bedeutung des zweiten Hauptsatzes in der Wärmetheorie," Wien. Ber. **53** (1866), pp. 195.

([314]) **J. W. Gibbs**, *Thermodynamischen Studien*, pp. 66.

([315]) **W. Ostwald**, "Die Energie und ihre Wandlungen," Leipz. Antrittsrede 1888; "Studien zur Energetik," Leipz. Ber. **43** (1891), pp. 271; *ibid.*, **44** (1892), pp. 211; likewise, *Lehrbuch der allgemeinen Chemie*, Leipzig, 1893, Bd. 2^1, pp. 1-39.

([316]) **L. Boltzmann**, "Über d. Entwicklungen der Methoden d. theoretischen Physik," Deutsche Math.-Ver. **8** (1900), pp. 71, esp. pp. 87; "Ein Wort der Mathematik an die Energetik," Ann. Phys. Chem. (2) **57** (1896), pp. 39; **M.**

inductive value that cannot be underestimated, on the whole, and they might lead to many further insights into their intrinsic connection to our present understanding of nature ([317]). As evidence for that, we would only like to emphasize the representation that refers to the *migration of energy*.

In a continuous medium where any sort of process might take place, a certain quantum of energy that might depend upon x, y, z, t ([318]) will exist at every point in time and every location. One can also deal with a change in the energy in the sense of **Euler**'s differential equations of hydrodynamics. One will then get a representation of a *current or migration of localized energy* that is connected with **Lagrange**'s conception of fluid motion, which does not examine the state of motion at an arbitrary location, but the motion of each individual particle, if one would like to also determine the *paths of that current* ([319]). Such representations are not new, moreover. **Coriolis** ([320]) already compared the kinetic energy in a machine to a fluid that flows in it, and similar pictures might also be found in many other places. However, they have appeared in a well-defined form only in the last twenty-five years.

N. Umow ([321]) has already developed the problem of the migration of energy in fluid and elastic media in an entirely-general way in 1874. However, the first to draw attention to this picture in an outstanding way was **Poynting** ([322]), who represented the current of electromagnetic energy on the basis of **Maxwell**'s formulas as something that was governed by very simple laws.

As is known, the *equation of continuity* for any continuous medium whose mass is distributed with a density of ρ and is thought of as invariable reads:

$$\frac{\partial\rho}{\partial t}+\frac{\partial\rho u}{\partial x}+\frac{\partial\rho v}{\partial y}+\frac{\partial\rho w}{\partial z}=0,$$

in which u, v, w are the velocity components of the current. Conversely, for any equation:

$$\frac{\partial E}{\partial t}+\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}+\frac{\partial W}{\partial z}=0, \tag{1}$$

Planck, "Gegen d. neuere Energetik," Ann. Phys. Chem. (2) **57** (1896), pp. 72; **L. Dressel**, "Zur Orientierung in der Energielehre," Natur und Offenbarung 39, Münster 1893; pps. 321, 390, 449.

([317]) Cf., in addition to **Ostwald**, *Lehrbuch der allgemeinen Chemie*, also **Duhem**'s *Mécanique chimique*, **J. H. van't Hoff**, *Vorlesungen über theoretische und physikalische Chemie*, Braunschweig, 1898/99 (also in French by **Corvisy**, 2 vols., Paris, 1899.1900), as well as **Lem**'s *Energetik*. The emerging attempt in the German literature (cf., e.g., **H. Januschke**, *Das Prinzipe d. Erhaltung d. Energie*, Leipzig, 1897) to make a summary treatment of the multifaceted problems by applying the concept of energy, in place of clarity of insight, still seems premature at this point in time.

([318]) Cf., **O. Lodge**, "On the identity of energy," Phil. Mag. (4) **19** (1885), pp. 482.

([319]) Up to now, that analogy can be pursued only in individual cases.

([320]) **Coriolis**, *Traité de mécanique*, pp. 117; "In addition, the fluid can accumulate in certain bodies and stay there…the work that is stored, which we assimilate to a fluid, is what we have called the *vis viva*," pp. 171: "One can compare the transmission of work by the machine to the flowing of a fluid, etc."

([321]) **N. Umow**, "Ableitung d. Bewegungsgleichungen der Energie in kontinuierlichen Medien," Zeit. Math. Phys. **19** (1874), pp. 419.

([322]) **J. H. Poynting**, "On the transfer of Energy in the electromagnetic field," Lond. Phil. Trans. **175** (1884), pp. 343; cf., **O. Heaviside**, Electrician **14** (1885), pp. 178, 306; esp., "On the forces, stresses and fluxes of energy in the electromagnetic field," Lond. Phil. Trans. **183** (1892), pp. 423; **W. Wien**, "Über d. Begriff d. Lokalisierung d. Energie," Ann. Phys. Chem. (2) **45** (1892), pp. 684.

one can refer to U, V, W as the components of the current and U / E, V / E, W / E as the velocity components.

It will follow from the equations of motion for an elastic (fluid, resp.) medium:

$$
(2) \qquad
\begin{aligned}
\rho \ddot{x} &= \rho X + \frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z}, \\
\rho \ddot{y} &= \rho Y + \frac{\partial Y_x}{\partial x} + \frac{\partial Y_y}{\partial y} + \frac{\partial Y_z}{\partial z}, \\
\rho \ddot{z} &= \rho Z + \frac{\partial Z_x}{\partial x} + \frac{\partial Z_y}{\partial y} + \frac{\partial Z_z}{\partial z},
\end{aligned}
$$

and the **Euler** equations for:

$$u = \frac{dx}{dt}, \qquad v = \frac{dy}{dt}, \qquad w = \frac{dz}{dt},$$

$$
(3) \qquad
\begin{aligned}
\frac{du}{dt} &= \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}, \\
\frac{dv}{dt} &= \frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}, \\
\frac{dw}{dt} &= \frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z},
\end{aligned}
$$

when one multiplies equations (2) by $\dot{x}$, $\dot{y}$, $\dot{z}$ that:

$$
(4) \qquad \frac{\rho}{2}\frac{dq}{dt} = \rho\,(X\,u + Y\,v + Z\,z) - \left(\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z}\right) + \Phi,
$$

when one sets:

$$q = u^2 + v^2 + w^2,$$

$$
\begin{aligned}
A &= X_x\,u + Y_x\,v + Z_x\,w, \\
B &= X_y\,u + Y_y\,v + Z_y\,w, \\
C &= X_z\,u + Y_z\,v + Z_z\,w,
\end{aligned}
$$

and finally uses $-\Phi$ as an abbreviation for the expression:

$$X_x\frac{\partial u}{\partial x} + Y_x\frac{\partial v}{\partial x} + Z_x\frac{\partial w}{\partial x} + X_y\frac{\partial u}{\partial y} + Y_y\frac{\partial v}{\partial y} + Z_y\frac{\partial w}{\partial y} + X_z\frac{\partial u}{\partial z} + Y_z\frac{\partial v}{\partial z} + Z_z\frac{\partial w}{\partial z}.$$

It also follows from (3) that:

$$
(4) \qquad \tfrac{1}{2}\rho\frac{\partial q}{\partial t} = \rho\,(X\,u + Y\,v + Z\,z) - \left\{\frac{\partial A}{\partial x} + \frac{\partial B}{\partial y} + \frac{\partial C}{\partial z} + \frac{\rho}{2}\left(u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial w}{\partial z}\right)\right\} + \Phi,
$$

or with the help of the equation of continuity:

(5) $$-\frac{1}{2}\frac{\partial(\rho q)}{\partial t} = \rho\,(X\,u + Y\,v + Z\,z) - \frac{\partial}{\partial x}\left(A + \tfrac{1}{2}\rho\,u\,q\right) - \frac{\partial}{\partial y}\left(B + \tfrac{1}{2}\rho\,v\,q\right) - \frac{\partial}{\partial z}\left(C + \tfrac{1}{2}\rho\,w\,q\right) + \Phi\,.$$

If one assumes conservative forces *X*, *Y*, *Z*, for the sake of clarity, i.e., one sets:

$$X = -\frac{\partial V}{\partial x}, \qquad Y = -\frac{\partial V}{\partial y}, \qquad Z = -\frac{\partial V}{\partial z},$$

then it will follow from (5) that:

(6) $$\frac{1}{2}\frac{\partial(\rho q)}{\partial t} + \frac{\partial \rho}{\partial t}V = -\frac{\partial}{\partial x}\left(\rho u V + A + \tfrac{1}{2}\rho\,u\,q\right) - \frac{\partial}{\partial y}\left(\rho v V + B + \tfrac{1}{2}\rho\,v\,q\right) - \frac{\partial}{\partial z}\left(\rho w V + C + \tfrac{1}{2}\rho\,w\,q\right) + \Phi\,.$$

Now, it can be shown by an application of **Green**'s partial integration that – Φ is the partial differential quotient with respect to *t* of the density *S* of the *deformation energy* ([323]), so one will get from (6) that:

$$\frac{\partial}{\partial t}\left[\tfrac{1}{2}\rho\,q^2 + \rho V + S\right] + \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = 0\,,$$

or, when one assumes, in addition, that the potential energy *V* depends upon not only external forces, but also originates in the action-at-a-distance of the masses in the system upon each other, such that:

$$\tfrac{1}{2}\rho q^2 + \rho V + S$$

can now be regarded as the density *E* of total energy per unit volume, one will have:

$$\frac{\partial E}{\partial t} + \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = 0\,.$$

The velocity components of the energy current will then be:

([323]) See footnote 309. Moreover, **W. Thomson** had already shown in 1857 [Quart. J. of Math. **1** (1867), pp. 57], by an application of the second law of mechanical theory of heat, that elastic forces will possess a force function whenever the temperature of the medium either remains constant or changes adiabatically, while nothing beyond that much can be asserted, in general; cf., **Love**, *Elasticity*, v. 1, pp. 117. Thermodynamic effects are ignored completely (constant temperature is assumed, resp.) in this treatise. The extended question does not seem to have been considered up to now.

$$\frac{1}{E}\left(\rho u V + A + \tfrac{1}{2}\rho u q\right),$$
$$\frac{1}{E}\left(\rho v V + B + \tfrac{1}{2}\rho v q\right),$$
$$\frac{1}{E}\left(\rho w V + C + \tfrac{1}{2}\rho w q\right).$$

For an inviscid fluid, the pressure p enters in place of X_x, Y_y, Z_z, so A, B, C will then be proportional to u, v, w, and current will flow in the direction of motion of the fluid ([324]). When **Poynting** developed similar formulas for the electromagnetic energy, that led to **Poynting**'s theorem that the energy in an electromagnetic field flowed with a certain intensity perpendicular to the plane of the lines of magnetic and electric force ([325]). For a conductor with current flowing in it, energy must then flow into it in order for heat to appear ([326]).

In general, there is a difficulty in regard to the real meaning that one can ascribe to those pictures. That is because, as would emerge from the presentation above, one can add arbitrary functions u_1, v_1, w_1 that satisfy the equation:

$$\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y} + \frac{\partial w_1}{\partial z} = 0$$

to the components, i.e., the flow velocity components for an incompressible fluid. It therefore seems that the whole approach will have only the character of a representation that is capable of many modifications as long as one is not in a position to sharply distinguish between a *true* energy current and the innumerable fictitious ones. We shall not go into the question of whether it is possible at this point in time to make such a decision, which is a possibility that **Heaviside** and **Föppl** have both disputed ([327]).

The foregoing considerations refer to *continuous* masses. More recently, **Volterra** ([328]) has extended that picture in several articles to *discrete* masses, between which actions-at-a-distance also appear by means of an extension of **Maxwell**'s equations. There, as well, there are analogous values for U, V, W, since the influence of discontinuity surfaces across which jumps in the density and velocities take place drops out. However, in so doing, it is necessary for one to also assign *negative* values to the energy, which is hard to reconcile with the usual physical assumptions.

50. Energetic foundation of mechanics. – We shall now mention the attempts to arrive at the differential equations of motion from the energy principle, initially in the simple form:

([324]) For viscous fluids, see **Umow**, footnote 321; **Wien**, footnote 322, pp. 698.

([325]) **Poynting**, footnote 322, pp. 348; for similar pictures in mechanics, see **A. Föppl**, *Technische Mechanik*, pp. 213, *et seq.*

([326]) According to **G. Mie**, "Ein Beispiel zum Poynting'schen Theorem," Zeit. f. Phys. Chem. **34** (1900), pp. 522, the direction of motion of the energy current in the immediate neighborhood of the wire is roughly parallel to it.

([327]) Cf., **A. Föppl**, *Einführung*, pp. 293. According to **G. Mie**, "Entwurf einer allgemeiner Theorie der Energieübertragung," Wien. Ber. **107** (1898), pp. 1114, it is possible.

([328]) **V. Volterra**, "Sul flusso di energie meccanica," Torino, Atti dell'Accad, **34** (1899), likewise in Nuovo Cimento (4) 10 (1899).

$$T + V = c\ .$$

If one sets:

$$T = \tfrac{1}{2} \sum m_i (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2)$$

for a completely-free system that is subject to only conservative forces, and one *assumes* that the accelerations are *independent* of the velocities, and that c should be a constant then one will have:

$$0 = \sum \left[m_i\,(\dot{x}_i\,\ddot{x}_i + \dot{y}_i\,\ddot{y}_i + \dot{z}_i\,\ddot{z}_i) + \frac{\partial V}{\partial x_i}\dot{x}_i + \frac{\partial V}{\partial y_i}\dot{y}_i + \frac{\partial V}{\partial z_i}\dot{z}_i \right] .$$

Should that equation be true for *all* values of $\dot{x}_i$, $\dot{y}_i$, $\dot{z}_i$ then it would, in fact, follow that:

$$m\,\ddot{x} + \frac{\partial V}{\partial x_i} = 0\,, \quad \text{etc.}$$

However, one cannot reach a similar conclusion when equations of constraint exist between x, y, z, as **R. Lipschitz** ([329]) remarked before. **Helm** ([330]) then sought to appeal to the variational process by giving the energy principle the form: The change in total energy in each possible direction is equal to zero. However, one can only understand that change to mean the increase in energy that corresponds to a variation of the coordinates x, y, z by δx, δy, δz, resp. In that way, one will now have, in fact:

$$\delta V = \sum \left(\frac{\partial V}{\partial x_i}\delta x_i + \frac{\partial V}{\partial y_i}\delta y_i + \frac{\partial V}{\partial z_i}\delta z_i \right) ,$$

but the change in the kinetic energy is:

$$\sum m \left(\dot{x}_i\,\frac{\delta\,dx}{dt} + \dot{y}_i\,\frac{\delta\,dy}{dt} + \dot{z}_i\,\frac{\delta\,dz}{dt} \right) = d \sum m \left(\dot{x}_i\,\frac{\delta x}{dt} + \dot{y}_i\,\frac{\delta y}{dt} + \dot{z}_i\,\frac{\delta z}{dt} \right) - \sum m\,(\ddot{x}\,\delta x + \ddot{y}\,\delta y + \ddot{z}\,\delta z)\,,$$

and is by no means equal to:

$$\sum m(\ddot{x}\,\delta x + \ddot{y}\,\delta y + \ddot{z}\,\delta z)\,,$$

which is an expression that can be obtained only by an inadmissible substitution of dx, dy, dz with δx, δy, δz that does not apply to both parts of the energy ([331]). However, if one defines the possible change of energy to be that expression from the outset then one will have an arbitrary formalism

([329]) See **Helmholtz**, "Über die Erhaltung der Kraft," **Ostwald**, K. B., no. 1, pp. 55, also *Wiss. Abh.*, Bd. 1, 1882, pp. 12.

([330]) See **J. Boussinesq**, *Recherches sur les principes de la mécanique*, 1872; J. de math. (2) **18** (1873), pp. 315; *Leçons*, pp. 24. Cf., also the remarks of **C. Neumann** (**Helm**, *Energetik*, pp. 229).

([331]) **G. Helm**, Zeit. Math. Phys. **35** (1890), pp. 307; *Energetik*, pp. 232; also Ann. Phys. Chem. (2) **57** (1896), pp. 646. Cf., **Boltzmann**, *ibid.*, pp. 39.

that was invented with the sole purpose of being able to assert the equivalence of the energy principle with that of **d'Alembert**.

The repeatedly-mentioned investigations of **P. Duhem** point in a whole different direction, namely, towards finding a basis for physical mechanics from an energetic standpoint. He sought to give an abstract foundation for the mechanics of material systems, and with regard to the thermodynamic concept of work, in particular, while giving painstaking emphasis to the hypothetical "conventions." However, it has still not been decided at present to what extent that will influence the systematic representation of the foundations of theoretical mechanics.

51. Concluding remarks. – The general mathematical principles of mechanics always prove to be theorems and methods *that are probably based upon the basic intuitions about their mechanical* (i.e., expressed by mathematical concepts) *connection to phenomena in their simplest form, but appear to be inductive, heuristic statements in their advanced form*, and their validity is first tested by the possibility of applying them. Thus, the ideal of a purely-deductive philosophical system, of the type that mechanics had in mind during the Eighteenth Century, and as **Hertz** undertook to present in a completely-abstract way, has not been achieved in reality up to now. Therefore, the present standpoint on the theory offers the possibility that our ongoing knowledge of the facts will not be inhibited by arbitrary deductive principles that are inferred from a restricted sphere of facts. That standpoint is the one that **Galilei** had assumed before, which is characteristic of the mathematical description of nature, and neither asks about unknowable causes nor starts from the notion that all phenomena should be subject to the constraint of one (or at most a few) fundamental physical hypotheses, but with the assumption that a coherent understanding of reality that is free from contradictions is possible at all. It initially seeks the forms that would suffice for describing the simplest processes and reserves the extension and correction of them to the extent that it would expand the scope of the experiments.