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ANHOLONOMIC SPACES

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ANHOLONOMIC SPACES

AND

THEIR MECHANICAL APPLICATIONS

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INTRODUCTION.

One knows that the division of mechanical systems into holonomic and anholonomic ones is imposed by analytical considerations. Indeed, holonomic systems are characterized by the property that one may choose the parameters on which the position of the system depend in such a manner that all of the constraints of the systems are expressed by relations in finite terms in these parameters, whereas for anholonomic systems, at least a subset of these constraints is given by a system of Pfaff equations that is not completely integrable.

As is well known, this fact entails that there are essential differences between the analytical study of holonomic systems and that of anholonomic systems. On the one hand, this is because it is only the holonomic systems that are subject to Lagrange's and Hamilton's equations of motion (and one knows that almost all of the results of analytical mechanics are obtained by starting with these equations). On the other hand, this is because it is for the holonomic systems that one may give a very natural geometric interpretation, with the aid of a Riemann space whose metric is defined by the vis viva of the system, in such a manner that the unforced trajectories of a holonomic system with time-independent constraints are also the geodesics of the corresponding Riemann space.

For anholonomic systems, one either seeks to find equations of motion that are applicable all mechanical systems (and we then have the equations of Maggi, Volterra, Appell, etc. [**7**], v. II, para. I, pp. 393, [**4**]), but these equations are loath to have the malleability and the properties of Lagrange's and Hamilton's equations, or one seeks to find geometrical properties of these systems. The latter path has led to the concept of an anholonomic space, which is a generalization of the concept of a Riemann space, but which also is intimately related to spaces with affine connections.

The original idea of applying geometrical considerations to the study of an anholonomic system is due to A. Voss ([**1**], 1885), who was concerned with the unforced trajectories of the motion of a point in ordinary space whose coordinates satisfy a Pfaff equation that is not completely integrable. Later, ([**2**], 1888-1889), the abbot of Issaly extended this to manifolds in ordinary space that are defined by a Pfaff equation that is not completely integrable, manifolds that one calls *pseudo-surfaces,* and that have many of the properties of surfaces. Nevertheless, this extension is almost always formal, and one cannot really see its significance, and perhaps because of this fact these works remain isolated.

The concept of an anholonomic space was introduced in 1926 by G. Vranceanu ([**9**], 1926), who showed that if, in a Riemann space V_n, one is given a system of $n - m$ Pfaff equations that is not completely integrable then one defines an anholonomic space V_n^m in which it is possible to introduce a parallelism in the sense of Levi-Civita, in such a manner that to each anholonomic system with time-independent constraints one may attach an anholonomic space whose geodesics (auto-parallel curves) are also the unforced trajectories of the mechanical system under consideration.

In an independent fashion, Z. Horak ([**12**^{2}], 1927) showed how one may generalize the concept of manifold by introducing anholonomic manifolds as the configuration spaces of anholonomic mechanical systems.

In 1928, J. A. Schouten [**16**] introduced anholonomic spaces with affine connections, and the work that was done by E. Cartan, J. L. Synge, P. Franklin, and C. L. E. Moore, E. Bortollotti, A. Wundheiler, J. A. Schouten, Z. Horak, G. Vranceanu, etc., developed the study of anholonomic spaces considerably. The object of this monograph is precisely that of presenting most important results that were obtained in that direction, as well as some applications to holonomic and anholonomic mechanical systems.

CHAPTER I.

THE ABSOLUTE DIFFERENTIAL CALCULUS OF CONGRUENCES.

1. Systems of *n* independent congruences. – Consider, in the space X_n of n real variables $x^1, x^2, \ldots, x^n$, a contravariant vector λ that has the components $\lambda^i(x)$ ($i = 1, 2, \ldots, n$), where the functions λ^i are continuous and differentiable, as are all of the functions that will be considered in the sequel. Having said this, the differential equations:

$$\frac{dx^1}{\lambda^1} = \frac{dx^2}{\lambda^2} = \cdots = \frac{dx^n}{\lambda^n}$$

define a congruence of curves in the space X_n. Through each point $P(x^1, x^2, \ldots, x^n)$, where the quantities λ^i are not all null there passes one and only one curve of the congruence, such that the tangent to the curve at the point P has the same direction as the vector λ at this point.

Now consider n independent contravariant vectors λ_a^i ($a = 1, 2, \ldots, n$); i.e., the determinant of their components:

$$\Delta = |\lambda_a^i|$$

is non-zero, at least in a certain region of the space X_n in which our considerations are valid. These n vectors determine a system of n independent congruences in X_n, in such a fashion that through each point P there pass n curves of the system whose tangents at P have the same directions as the n independent vectors (λ_a) that pass through that point.

Since the determinant Δ is non-zero, one may consider its inverse λ_i^a, which is related to the elements of Δ by the well-known formulas of the theory of determinants ([1]):

$$\lambda_a^i \lambda_j^a = \delta_j^i, \qquad \lambda_i^a \lambda_b^i = \delta_b^a,$$

where the δ are equal to zero or one according to whether the indices are different or not. These formulas show that the quantities $\lambda_1^a, \lambda_2^a, \ldots, \lambda_n^a$ may be regarded as the components of n covariant vectors (λ^a) in the space X_n, and one sees that these n covariant vectors are determined once one is given the system of independent congruences (λ).

Now, let $P(x^1, x^2, \ldots, x^n)$ be a point and let:

$$P(x^1 + dx^1, x^2 + dx^2, \ldots, x^n + dx^n)$$

([1]) We employ the convention that two repeated indices indicate the sum over these indices. Likewise, a, b, c, d, e, f, g are indices that relate to the congruences (λ), whereas i, j, r, s, t, u, v are indices that relate to the variables (x).

be a point that is infinitely close to P. The infinitesimal displacement PP′ is a contravariant vector whose origin is at P and whose components are $dx^1, dx^2, \ldots, dx^n$. Its projections ds^a on the congruences (λ) that pass through P are given by the formulas:

$$ds^a = \lambda_i^a \, dx^i \qquad (dx^i = \lambda_a^i \, ds^a) \,. \tag{1}$$

It then results that the displacement PP′ is determined either by its components dx^i relative to the system of variables (x), or by its components ds^a relative to the system of congruences (λ).

Following E. Cartan, the Pfaff forms $ds^1, ds^2, \ldots, ds^n$ may be interpreted as the coordinates of a point P′ with respect to a Cartesian frame that is determined at P by the tangents to the congruences (λ) that pass through P. Moreover, the s^a may be taken to be the new variables in the space X_n only if the total differential equations (1) are completely integrable, and for this to be true it is necessary that $\frac{d\lambda_i^a}{dx^j} = \frac{d\lambda_j^a}{dx^i}$. If these conditions are not satisfied, which is obviously the general case, then the formulas (1) do not define a true transformation of variables, because only the differentials ds^a have meaning, and not the s^a. One may say, with J. A. Schouten, that in the general case the formulas (1) define a transformation in the space X_n that takes the variables $x^1, x^2, \ldots, x^n$ to the anholonomic variables $s^1, s^2, \ldots, s^n$.

2. The Riemann space associated with the congruences (λ). – One may give a geometrical interpretation to these anholonomic variables $s^1, s^2, \ldots, s^n$ if one associates our system of congruences with a Riemann space V_n that has the metric:

$$ds^2 = (ds^1)^2 + (ds^2)^2 + \ldots + (ds^n)^2 \,; \tag{1′}$$

i.e., the space V_n in which the (λ) are orthogonal congruences. In this case, as has been known since the work of Ricci and Levi-Civita ([**5**], 1901), the s^a are arcs of the congruences (λ_a), as measured in the associated space V_n, and the Pfaff forms ds^a are the differentials of the arcs of these congruences. As for the quantities λ_a^i and λ_i^a, they are called the *parameters* and *moments* of the congruences (λ).

It is obvious that the Riemann space V_n will vary with the chosen system of congruences in X_n. Indeed, consider another system of independent congruences $(\bar{\lambda})$ that has the quantities:

$$d\bar{s}^a = \bar{\lambda}_i^a dx^i \tag{1″}$$

for its differential arcs, where the $\bar{\lambda}_i^a$ are functions of the variables (x). Due to the fact that the n independent vectors $(\bar{\lambda})$ may always be expressed linearly with the aid of the n independent vectors (λ), and conversely, we will have formulas of the form:

(2) $$d\bar{s}^a = c^a_b ds^b ,$$

where the c^a_b are convenient functions of the variables (x) whose determinant $|c^a_b|$ is non-zero. It then results that the space $\bar{\mathrm{V}}_n$, which is associated with $(\bar{\lambda})$, coincides with V_n, which is associated with (λ), only in the case where the determinant of the c^a_b is orthogonal, that is, if the c^a_b satisfy the orthogonality conditions:

(2′) $$c^a_b c^a_d = \delta^d_b = \begin{cases} 0 & (b \neq d), \\ 1 & (b = d). \end{cases}$$

3. Transformations of congruences. – The formulas (2) may be interpreted as defining a transformation of the congruences, and, more precisely, the transformation that takes the congruences (λ) to the congruences $(\bar{\lambda})$. By that transformation, the moments and parameters of the congruences (λ) and $(\bar{\lambda})$ are related by the formulas:

(2″) $$\bar{\lambda}_i^a = c^a_b \lambda_i^a , \qquad \bar{\lambda}_a^i = c^b_a \lambda_b^i .$$

The transformations of the congruences (2) form a group, in the sense that it contains the identity transformation, each transformation has an inverse, and the product of transformations is also a transformation (2), and this is due to the linearity of the transformations. This group depends upon n^2 arbitrary functions c^a_b of the variables (x), and it contains, as a particular case, the invertible pointlike transformations:

(3) $$x'^i = x'^i (x^1, x^2, \ldots, x^n)$$

that one considers in the absolute differential calculus. For that reason, certain authors (R. Lagrange [**10**], pp. 17; J. A. Schouten, [**16**], [**21**]; Horak [**14**], etc.) have agreed to generalize the absolute differential calculus by associating the transformations (3) with the transformations (2). Nevertheless, one may remark that the property of the group (2) that it must contain the pointlike group (3) as a subgroup might not be true for a subgroup of the group (2). Indeed, the orthogonal subgroup (2′) might not contain any subgroup of the pointlike group if the space V_n that is associated with the congruences (λ) is not Euclidian.

As in the sequel, we will have to occupy ourselves with certain subgroups of the linear groups (2), it is convenient to make a clear distinction between the transformations of the variables (3) and the transformation of the congruences (2). As for the definition of the vectors or tensors, we must take into account the following results:

If we are given a tensor, which, to simplify things, we suppose to be of second order – once contravariant and once covariant – and has the quantities R^i_j for its components relative to the systems of variables (x), then its components with respect to the congruences (λ) are given by the formulas:

$$r^a_b = \mathrm{R}^i_j \lambda^a_i \lambda^j_b .$$

These components, which one also calls *intrinsic*, are invariant under the transformation of variables (3); however, under a transformation of the congruences (2) they transform according to the formula:

(3′) $$\bar{r}^a_d c^b_a = r^b_a c^a_d .$$

Conversely, if we have a system of n^2 quantities that are invariant under the transformations (3) and that transform under (2) according to (3′) then they define a tensor of second order that is contravariant in the index *a* and covariant in the index *b*, and whose components relative to the variables (*x*) are:

$$\mathrm{R}^i_j = r^a_b \lambda^i_a \lambda^b_j .$$

Moreover, if one remarks that the R^i_j are invariant under the transformations (2) and that they transform under (3) according to the well known formulas from the absolute differential calculus of coordinates the *one sees that there exists a complete duality between the calculus of coordinates and that of congruences;* i.e., that one may define vectors, tensors, and, as we verify later on, also affine connections, by the way that they transform under either a transformation of coordinates or a transformation of congruences, while the variables (*x*) remain the same.

As examples of contravariant and covariant vectors relative to the congruences (λ), we have the displacement ds^a and the vector that has as its components, the intrinsic derivatives of an arbitrary function:

$$\frac{\partial f}{\partial x^a} = \lambda^i_a \frac{\partial f}{\partial x^i} .$$

It is useful to remark that the second intrinsic derivatives are not generally symmetric; we have the following formula ([**7**[1]], pp. 290):

(3″) $$\frac{\partial^2 f}{\partial s^a \partial s^b} - \frac{\partial^2 f}{\partial s^b \partial s^a} = w^d_{ab} \frac{\partial f}{\partial x^d}$$

for the commutation of the second intrinsic derivatives, where the w^d_{ab} are defined by the formulas (4′) that are given in the following section.

4. Fundamental formulas and identities ([30], pp. 180). – We return to the formulas (1) in order to calculate the bilinear covariants of the forms ds^a. If one considers another displacement δx^i, which is different from dx^i, then we have:

$$\delta\, ds^a - d\,\delta s^a = \left(\frac{\partial \lambda_i^a}{\partial x^j} - \frac{\partial \lambda_j^a}{\partial x^i}\right) dx^i \delta x^j + \lambda_i^a (\delta\, dx^i - d\,\delta x^i)\,,$$

and if, in that formula, one introduces, in place of dx^i and δx^i, their values as functions of ds^a and δs^a then one finds the formulas:

(4) $$\delta\, ds^a - d\,\delta s^a = w_{bc}^a ds^b \delta s^c + \lambda_i^a \Delta x^i \qquad (\Delta x^i = \delta\, dx^i - d\,\delta x^i\,),$$

(4′) $$w_{bc}^a = \left(\frac{\partial \lambda_i^a}{\partial x^j} - \frac{\partial \lambda_j^a}{\partial x^i}\right) \lambda_b^i \lambda_c^j\,.$$

These quantities w_{bc}^a, which play an important role in the calculus of congruences, are obviously invariant under the transformation of variables. They are also skew-symmetric in b and c, and are all null if the s^a may be regarded as the true variables. In particular, if one of the forms ds^a – for example, ds^n – is an exact total differential then the are w_{bc}^a all null.

If one now takes the bilinear covariants of the forms (2) then one has:

$$\delta\, d\bar{s}^a - d\,\delta \bar{s}^a = \left(\frac{\partial c_b^a}{\partial s^c} - \frac{\partial c_c^a}{\partial s^b}\right) ds^b \delta s^c + c_b^a (\delta\, ds^b - d\,\delta s^b)\,,$$

and if one takes (4) into account here, as well as their analogues for the forms (1′) then one arrives at the following formulas, *which are fundamental to the calculus of congruences:*

(5) $$\frac{\partial c_b^a}{\partial s^c} - \frac{\partial c_c^a}{\partial s^b} = \bar{w}_{ef}^a c_b^e c_c^f - w_{bc}^e c_e^a\,.$$

As one sees, these formulas express the relationships between the quantities w for the congruences (λ), the quantities $\bar{w}$ for the congruences $(\bar{\lambda})$, the coefficients c_b^a, and the first order partial derivatives of these coefficients.

If one differentiates the formulas (5) along an arc s^a and then, after permuting the indices b, c, d, one takes the sum then one finds without difficulty, upon taking into account formula (3″) for the commutation of the second derivatives c_b^a, that the quantities w_{bc}^a must satisfy *the fundamental identities:*

(5′) $$\frac{\partial w_{bc}^a}{\partial s^d} + \frac{\partial w_{db}^a}{\partial s^c} + \frac{\partial w_{cd}^a}{\partial s^b} + w_{df}^a w_{bd}^f + w_{cf}^a w_{db}^f + w_{bf}^a w_{cd}^f = 0\,.$$

Moreover, these identities may also be found by saying that the n partial differential equations $X_a f = \lambda_a^i \dfrac{\partial f}{\partial x^i} = 0$ satisfy the Jacobi identities. As one sees, they also express the fact that the fundamental equations (5), when considered to be partial differential equations in c_b^a, satisfy their integrability conditions identically.

5. Affine connections. – Suppose that we have an affine connection A_n in the space X_n whose coefficients in the systems of variables (x) are Γ_{ij}^k. One knows that under a transformation of variables (3) these coefficients transform according to the well known law of affine connections ([**12**], pp. 3; [**13**], pp. 35). If one takes the coefficients of the connection A_n in the system of congruences (λ) to be the quantities:

$$\gamma_{bc}^{\cdot\,a} = \left(\frac{\partial \lambda_i^a}{\partial x^j} - \Gamma_{ij}^r \lambda_r^a\right)\lambda_b^i \lambda_c^j ,$$

which are invariant under coordinate transformations, then the parallel transport of a vector v^a or v_a along a displacement ds^a will be defined by the equations:

(5″) $$dv^a = \gamma_{bc}^{\cdot\,a} v^b\, ds^c, \qquad dv_a = -\gamma_{ac}^{\cdot\,b} v_b\, ds^c.$$

Having said this, one easily verifies that under a change of congruences the $\gamma^\bullet$ must transform according to the formula:

(6) $$\frac{\partial c_b^a}{\partial s^c} = \bar{\gamma}_{ef}^{\cdot\,a} c_b^e c_c^f - \gamma_{bc}^{\cdot\,e} c_e^a ,$$

which constitutes the transformation law for affine connections in the absolute differential calculus of congruences.

If one introduces the formulas (6) into the fundamental formulas (5) then one finds:

$$\bar{\tau}_{ef}^{\cdot\,a} c_b^e c_c^f = \tau_{bc}^{\cdot\,e} c_e^a , \qquad (\tau_{bc}^{\cdot\,a} = \bar{\gamma}_{bc}^{\cdot\,a} - \gamma_{cb}^{\cdot\,a} - w_{bc}^a),$$

in which the quantities $\tau_{bc}^{\cdot\,a}$ are the components of the *torsion* tensor of the affine connection A_n relative to the congruences (λ).

6. Infinitesimal parallelogram and pentagon. – Consider a point $P(x^i)$ and an infinitely close point $Q(x^i + dx^i)$. One may say that the point Q is obtained by applying the operator d to P. Let $R(x^i + dx^i)$ be another point that is obtained by applying the operator δ to P. If one now applies the operator d to the point and the operator δ to the point R then one finds two other points $S[x^i + dx^i + \delta(x^i + dx^i)]$ and $T[x^i + \delta x^i + d(x^i + \delta x^i)]$.

This being the case, the vector TS has the quantities $\Delta x^i = \delta\, dx^i - d\, \delta x^i$ for its components in the system of coordinates (x). If d and δ are operators defined by parallel transport under the connection A_n then the components of the vector TS relative to the congruences (λ) have, by virtue of formula (4), the expressions:

$$\Delta s^a = \lambda_i^a \Delta x^i = (\,\bar{\gamma}_{bc}^{\cdot\, a} - \gamma_{cb}^{\cdot\, a} - w_{bc}^a\,)\; ds^b\, \delta s^c\,. \tag{6″}$$

It then results that the point T coincides with S if the torsion tensor $\tau_{bc}^{\cdot\, a}$ is null; in this case, the figure PQSR constitutes what one refers to as the *infinitesimal parallelogram* of a space with an affine connection with vanishing torsion.

If the torsion tensor is not null then the figure PQSTR constitutes the *infinitesimal pentagon* of A_n, and one sees that the fifth side TS of the pentagon is a second order infinitesimal with respect to the two sides PQ, PR from which the pentagon is constructed.

If one considers the parallel transport of a vector along the infinitesimal parallelogram or pentagon then the variations of the components of this vector will be expressed with the aid of curvature tensor of A_n ([**7**[1]], pp. 197).

7. The group of the Riemann space. – Suppose that the space A_n is a Riemann space V_n. In this case, one may choose the congruences (λ) to be an orthogonal system of congruences in V_n ([**7**[1]], chap. X), and if one desires that the $(\bar{\lambda})$ be orthogonal in V_n then it is necessary that the c_b^a satisfy the orthogonality conditions (2′). The transformations of the congruences (2), (2′) obviously form a group, the *orthogonal group*. The properties that are invariant under this group are, at the same time, invariant properties of the space V_n, and conversely. It is interesting to remark that when dealing with the orthogonal group the concept of covariance coincides with the concept of contravariance. As for the affine connection of V_n, or the orthogonal group, it is defined, with respect to the (λ), by the Ricci rotation coefficients γ_{bc}^a $(= -\gamma_{ac}^b)$, which are related to the w_{bc}^a by the formulas:

$$w_{bc}^a = \gamma_{bc}^a - \gamma_{cb}^a\,, \qquad \gamma_{bc}^a = \frac{1}{2}(w_{bc}^a + w_{ca}^b + w_{ba}^c)\,. \tag{5″}$$

Indeed, if one differentiates the orthogonality formulas (2′) along an arc s^c and then one takes into account the fundamental formulas (5) then one finds that the rotation coefficients γ_{bc}^a and $\bar{\gamma}_{bc}^a$ satisfy the law of affine connections (6). Obviously, if they are no longer orthogonal congruences in V_n then the connection on V_n will not be represented by the rotation coefficients for the $(\bar{\lambda})$ relative to the $(\bar{\lambda})$.

It useful to remark that the equations of the geodesics in V_n, which are also the auto-parallel curves in V_n, may be written:

$$(6'') \qquad \frac{du^a}{ds} = \gamma^a_{bc}\, u^b\, u^c\,,$$

where s is the arc length along the curve and the u^a are the direction cosines that the curve makes with the congruences (λ).

Naturally, one may associate the last of equations (1), divided by ds, with these equations; one thus obtains a system in the normal form of $2n$ first order equations in the n unknowns x^i and the n unknowns u^a (Carpanèse, [**5**[1]]).

CHAPTER II

ANHOLONOMIC GROUPS AND SPACES.

8. The group of a Pfaff system. – Now suppose that we have, in the space X_n, a system of $n - m$ Pfaff equations:

$$(7) \qquad ds^{h'} = \lambda_i^{h'} dx^i = 0 \qquad (h' = m + 1, \ldots, n) \qquad (^2).$$

If this system is completely integrable then one may, by a convenient change of the variables (x) and the forms $ds^{h'}$, reduce this to the form:

$$(5^{\text{IV}}) \qquad ds^{h'} = dx^{h'} = 0 \qquad [x^h = c^{h'} \text{ (const.)}],$$

in such a fashion that this system defines a family of ∞^{n-m} spaces X_m in the space X_n. At each point $P^o(x_o^i)$ of the space X_n there passes one and only one space X_m, and this happens precisely when the constants of integration $c^{m+1}, \ldots, c^n$ have the values x_0^{m+1}, $\ldots, x_0^n$. If the Pfaff system (7) is not completely integrable then one may no longer reduce it to the form (5^{IV}); thus, it no longer defines a family of space X_m. In this case, one says that the system (7) defines an *anholonomic space* X_n^m in X_n.

Upon associating the forms $ds^{h'}$ with other forms ds^h ($h \leq m$), subject to only the condition that they form a system of n independent forms, together with the $ds^{h'}$, one easily sees that the most general transformations of the congruences that preserve the system (7) are given by the formula:

$$(7') \qquad \begin{cases} d\bar{s}^h = c_k^h ds^k + c_{k'}^h ds^{k'}, \\ d\bar{s}^{h'} = c_{k'}^{h'} ds^{k'}. \end{cases}$$

They are obtained from the general formulas (2) upon supposing that the coefficients $c_k^{h'}$ are null. These transformations (7′) obviously form a group themselves; it is the group of the anholonomic space X_n^m. This group has the property of preserving the character of the contravariant vector R^i that satisfies equations (7) ($\lambda_i^{h'} R^i = 0$). Such a vector, which is also referred to as a contravariant vector that is *interior* or *tangent* to X_n^m, is characterized by its components $r^h = \lambda_i^h R^i$ relative to the congruences (λ^h), which one also calls the *fundamental* congruences on X_n^m, because its components relative to the congruences $(\lambda^{h'})$ (which are *anholonomic* congruences on X_n^m) are null. The group (7′) also preserves the character of the covariant *exterior* or *normal* vector ($r^h = 0$), and, in

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(2) Unless stated to the contrary, we make the convention that the indices $h, k, l, \alpha, \beta, \gamma, \delta, \mu$ vary from 1 to m, whereas the same indices, when accented, vary from $m + 1$ to n.

particular, the system of partial differential equations that is associated with the Pfaff system (7):

$$X_h = \lambda_h^i \frac{\partial f}{\partial x^i} = 0 .$$

By contrast, the group (7′) does not preserve the character of the interior covariant vector ($r^{h'} = 0$), or that of the exterior contravariant vector ($r^h = 0$), because for the latter, for example, we have:

$$\bar{r}^h = c_{k'}^h\, r^{k'}, \qquad\qquad \bar{r}^{h'} = c_{k'}^{h'}\, r^{k'},$$

and one sees that, in general, the $\bar{r}^h$ are non-null.

One obtains an important property of the anholonomic group (7′) if, in the fundamental formulas (5) that relate to that group ($c_{k'}^{h'} = 0$), one sets $\alpha = h'$, $\beta = k$, $c = l$, which leads us to the formulas:

(7″) $$\bar{w}_{\alpha\beta}^{h'}\, c_k^{\alpha} c_l^{\beta} - w_{kl}^{\alpha'}\, c_{\alpha'}^{h'} = 0 ,$$

which expresses that the quantities $w_{kl}^{h'}$ are the components of a third order tensor relative to the congruences (λ) that is once contravariant exterior and twice covariant interior.

One may obtain an interpretation for that tensor by considering the bilinear covariants of equations (7) for two displacements that satisfy these equations. One finds:

(7‴) $$\Delta s^{h'} = \delta\, ds^{h'} - d\, \delta s^{h'} - w_{kl}^{h'} ds^k ds^l + \lambda_i^{h'} \Delta x^i \qquad (\text{mod } ds^{h'}).$$

These covariants are null at the same time as the Δx^i only if the tensor $w_{kl}^{h'}$ is null, but in this case one knows that the system (7) is completely integrable, and for that reason one calls the tensor $w_{kl}^{h'}$ the *integrability tensor* for the equations of anholonomity (7).

One calls minimum number independent equations in the system $w_{kl}^{h'}\, ds^k = 0$ the *rank* of the covariant $\Delta s^{h'}$ (for a fixed h'), which is always an even number.

9. The group of a Pfaff system and its derived systems. – One knows that the search for integrable combinations of a system of Pfaff equations may be carried out with the aid of the derived systems of the given system ([**6**], pp. 294). One defines the *first derived system* of (7) to be the system that is formed from all of the combinations of these equations such that the bilinear covariants (7‴) are null if $\Delta x^i = 0$. If there exist $p - m$ linearly independent combinations among these then one may take them to be the first equations (7), and in this case the components $w_{kl}^{m'}$ ($m' = m + 1, \dots, p$) of the integrability tensor are null. As for the other components, one knows, moreover, that the equations:

$$w_{kl}^{h'} \lambda_{h'} = 0 \qquad (h' > p)$$

might not have non-zero solutions in the unknowns $\lambda_{h'}$, since, otherwise, the derived system would contain more than $p - m$ equations. If one know desires to obtain the group that contains the system (7) and its derived system $ds^{m'} = 0$ then it is necessary that the coefficients $c_{k'}^{m'}$ $(k' > p)$ be null in the latter formulas (7′).

In an analogous manner, one may consider the derived system of the system $ds^{m'} = 0$, or even the second derived system of the system (7). If this second derived system possesses $q - m$ equations, which are chosen to be the first equations in (7), then one must have:

$$w_{kl}^{\sigma'} = w_{kl'}^{\sigma'} = w_{k'l'}^{\sigma'} = 0 \qquad (\sigma' = m + 1, \ldots, q;\ l', k' = p + 1, \ldots, n).$$

One knows that if it so happens that a derived system that one might form coincides with its proper derived system then it is formed from integrable combinations of our system (7). As a consequence, if (7) does not admit any integrable combinations then one must arrive at a derived system that is identically null. In any case, the group that preserves the system (7) and its derived system is a well defined subgroup of the group (7′).

Along with the concept of derived system, we will find it useful to define the concept of the *class* of a Pfaff system, which is the minimum number of variables that figure in the system under a transformation of variables. This number is equal to the number of independent equations in the Pfaff system:

$$(7^{\mathrm{IV}}) \qquad ds^{h'} = 0\,, \qquad w_{kl}^{h'}\, ds^k = 0\,.$$

If the class is $n - p$ then one may suppose that the forms ds^k for $k \leq p$ do not appear in this system; i.e., that $w_{kl}^{h'} = 0$ $(k \leq p)$.

10. The affine connection of two complementary Pfaff systems. – The anhomolomic group (7′) may be decomposed into a product of two groups. The group:

$$(8) \qquad \begin{cases} d\bar{s}^h = c_l^k\, ds^k, \\ d\bar{s}^{h'} = c_{l'}^{k'}\, ds^{k'}, \end{cases}$$

which completely preserves the character of the interior (tangent) vector or the exterior (normal) to the anholonomic space X_n^m, and the group:

$$(8') \qquad \begin{cases} d\bar{s}^h = ds^h + c_{k'}^k ds^{k'}, \\ d\bar{s}^{h'} = ds^{h'}. \end{cases}$$

One sees that the group (8) is also characterized by the fact that it preserves the system (7), along with the system that is obtained by equating the ds^h to zero. As a consequence, if one sets $a = h$, $b = k'$, $c = l'$ in the fundamental formula for this group ($c_{k'}^h = c_k^{h'} = 0$)

then obtains *analogous* formulas to (7″) (the accented indices having been changed into unaccented indices, and conversely), in such a fashion that the quantities $w^h_{k'l'}$ themselves define a third order tensor with respect to the group (8); it is the *integrability tensor* of the fundamental congruences of X^m_n.

This amounts to insuring that the group (8) also possesses a (partial) *affine connection*). Indeed, if one sets $a = h$, $b = k$, $c = l'$ in the fundamental formula (5) for the group then one finds:

$$\frac{\partial c^h_k}{\partial s^{l'}} = \bar{w}^h_{\alpha\beta} c^\alpha_k c^\beta_{l'} - w^\alpha_{kl'} c^h_\alpha , \tag{9}$$

and if one sets $a = h'$, $b = k'$, $c = l$ then one finds *analogous* formulas. These formulas express, in accord with (6), the fact that the quantities $w^h_{kl'}$ and $w^{h'}_{k'l}$ are the components of an affine connection relative to the congruence (λ), which permits us to transport an interior vector v^h or v_h along an exterior displacement $ds^{l'}$ according to the formulas:

$$dv^h = w^h_{kl'} v^k ds^{l'} , \qquad dv_h = -w^k_{hl'} v_k ds^{l'} , \tag{9′}$$

and an exterior vector $v^{h'}$ or $v_{h'}$ along an interior displacement ds^l by *analogous* formulas.

It is interesting to see the significance of that connection in the particular case where the two integrability tensors $w^h_{kl'}$ and $w^{h'}_{k'l}$ are null; i.e., if the anholonomic space X^m_n is composed of ∞^{n-m} spaces X_m , and likewise if the *complementary* anholonomic space X^{n-m}_n ($ds^h = 0$) is composed of ∞^m spaces X_{n-m} . In this case, one may arrange this in such a fashion that $w^h_{kl'}$, $w^{h'}_{k'l}$ are all null, and the parallel transport (9′) amounts to transporting a vector that is situated in an X_m (X_{n-m} , resp.) along a path that is situated in an X_{n-m} (X_n , resp.) while leaving the vector invariant.

The affine connection $w^h_{kl'}$, $w^{h'}_{k'l}$ *is a connection without torsion.* Indeed, the parallelogram that is constructed from an interior displacement ds^h and an exterior displacement $ds^{h'}$ closes because we have, in accord with the first formulas (9′) and their analogues:

$$\delta\, ds^h = w^h_{kl'}\, ds^k \delta s^{l'} , \qquad d\, \delta s^{h'} = w^{h'}_{k'l}\, \delta s^{k'} ds^l ,$$

and then $\delta\, ds^h = d\, \delta s^{h'} = 0$, because $\delta s^h = ds^{h'} = 0$. Now, the values that were introduced in (6″) show us that the tensor TS is null. Moreover, our connection is *characteristic* by the condition that it forms a parallelogram that is constructed from $ds^h = \delta s^{h'}$.

Along with the (partial) affine connection $w^h_{kl'}$, $w^{h'}_{k'l}$, which exists for any Pfaff systems $ds^h = 0$, $ds^{h'} = 0$, the group (8) might possess a connection that is related *to the non-integrability of these systems.* Indeed, suppose, for example, that the system (7) is not completely integrable. In this case, the commutation formulas (3″) for the second derivatives of the $c^{h'}_{k'}$ along the arcs s^k and s^l, where the first derivatives are given by formulas that are *analogous* to (9), lead us, taking into account the fundamental identities, to the formulas:

$$(8'') \qquad w_{kl}^{\alpha'} \frac{\partial c_{k'}^{h'}}{\partial s^{\alpha'}} = \bar{w}_{\alpha\beta,\alpha'}^{h'} c_k^{\alpha} c_l^{\beta} c_{k'}^{\alpha'} - w_{kl,k'}^{\alpha'} c_{k'}^{h'},$$

where we have set:

$$w_{kl,k'}^{h'} = \frac{\partial w_{kl}^{h'}}{\partial s^{k'}} + w_{k\alpha}^{h'} w_{lk'}^{\alpha} + w_{\alpha l}^{h'} w_{kk'}^{\alpha} + w_{kl}^{\alpha} w_{k'\alpha'}^{h'} .$$

Since the tensor $w_{kl}^{h'}$ is null, the equations (8″) are not all null, and they permit us to deduce the values of at least some of the derivatives $\frac{\partial c_{k'}^{h'}}{\partial s^{\alpha'}}$.

Suppose, to begin with, that the first derived system of (7) is null. In this case, among the equations (8″), one finds at least one system of independent equations in the unknowns $\frac{\partial c_{k'}^{h'}}{\partial s^{\alpha'}}$. Upon solving this system, one may write the solution in the form:

$$(9'') \qquad \frac{\partial c_{k'}^{h'}}{\partial s^{\alpha'}} = \bar{\delta}_{\alpha'\beta'}^{h'} c_{k'}^{\alpha'} c_{l'}^{\beta'} - \delta_{k'l'}^{\alpha'} c_{\alpha'}^{h'} .$$

In order to account for this fact, we introduce these values into (8″), the δ and d being arbitrary, for the moment. If one takes into account (7″) then one obtains the tensorial formulas:

$$(8'') \qquad (\bar{w}_{\alpha\beta}^{\beta'} \bar{\delta}_{\alpha'\beta'}^{h'} - \bar{w}_{\alpha\beta,\alpha'}^{h'})\, c_k^{\alpha} c_l^{\beta} c_{k'}^{\alpha'} = (w_{kl}^{\beta'} \delta_{k'\beta'}^{\alpha'} - w_{kl,k'}^{\alpha'})\, c_{\alpha'}^{h'} .$$

If one lets r, s denote the values of k, l that correspond to the chosen independent system in the $\frac{\partial c_{k'}^{h'}}{\partial s^{\alpha'}}$ then the δ and $\bar{\delta}$ are solutions of the system:

$$w_{rs}^{\alpha'} \delta_{k'\alpha'}^{h'} = w_{kl,k'}^{h'} ,$$
$$w_{rs}^{\alpha'} \bar{\delta}_{k'\alpha'}^{h'} = \bar{w}_{\alpha\beta,k'}^{h'}\, c_r^{\alpha} c_k^{\beta} ,$$

and consequently they are also well defined. If one chooses another independent system then one will have another solution (9″), but the difference between the corresponding δs will be a tensor. Equations (9″), when compared to (6), express the fact that the quantities $\delta_{k'l'}^{h'}$ are the components of an affine connection relative to the congruences (λ) that allows us to transport an exterior vector along an exterior path. Obviously, if the system $ds^h = 0$ then one also has that its first derived system is null, so one may apply the same considerations to it and obtain analogous formulas in such a fashion that in this case the group (8) will possess a complete affine connection. If we let $\overset{*}{\gamma}$ denote the components of that connection relative to the congruences (λ) and take into account the fact that it must preserve the character of the interior and exterior vectors of X_n^m then we have the formulas:

$$(9''')\qquad \begin{cases} \gamma_{kl}^{\bullet h} = \delta_{kl}^{h}, & \gamma_{kl'}^{\bullet h} = w_{kl'}^{h}, \quad \gamma_{k'\alpha}^{\bullet h} = 0, \\ \gamma_{k'l'}^{\bullet h'} = \delta_{k'l'}^{h'}, & \gamma_{k'l}^{\bullet h'} = w_{k'l}^{h'}, \quad \gamma_{k\alpha}^{\bullet h'} = 0. \end{cases}$$

This connection has torsion, because among the components of the torsion tensor we have:

$$\tau_{kl}^{h'} = -w_{kl}^{h'}, \qquad \tau_{k'l'}^{h} = -w_{k'l'}^{h},$$

in such a fashion that one is sure that the parallelogram that is constructed from the two interior (exterior, resp.) displacements does not close. Moreover, one sees that the integrability tensor defines part of the torsion tensor for the connection on the group (8). As far as the tensor (8‴) is concerned, it defines part of the curvature tensor for the connection, i.e., the tensor that is obtained by the condition that the $\frac{\partial c_b^a}{\partial s^c}$ that are given by (6) satisfy the commutation relations for the second derivatives. This shows us that the affine connection (9‴) contains all of the invariants of the Pfaff system (7) and the system $ds^h = 0$.

Now suppose that the first derived system of (7) is composed of $p - m$ equations $ds^{m+1} = \ldots = ds^p = 0$. In this case, the equations (8″) permit us to deduce only the values of the derivatives $\frac{\partial c_{k'}^{h'}}{\partial s^{l'}}$ $(l' > p)$. As for the solution, it will have the form (9″) only if $c_{l'}^{\alpha'} = 0$ $(\alpha' \leq p,\ l' > p)$; i.e., if the transformations (8) also preserve the derived system of (7). One arrives without difficulty at the following theorem:

The group of transformations of the congruences that preserves the systems $ds^h = 0$ *and* $ds^{h'} = 0$ *and their derived systems induces a complete affine connection on the space* X_n *if these systems do not admit integrable combinations.*

It is obvious that this theorem contains the case that was considered above as a particular case when the first derived systems are null. Moreover, this theorem may be applied to the case of a group that preserves three or more complementary Pfaff systems and their derived systems, because it suffices that the union of these systems gives us n independent equations ([**32**], pp. 195).

One may also remark that if one is given only the Pfaff system (7) then one may, under certain conditions, reduce its group (7′) by invariant operations in the covariants of the system, to a group that preserves two or more complementary systems. In this case, one says that the system (7) is *geometrizable*, the geometrization being complete if the complementary systems do not admit integrable combinations. For example, systems of two equations in an even number of variables, and, in particular, systems of two equations in six variables, are, in general, completely geometrizable (Vranceanu, [**36**]).

11. Anholonomic metric spaces. – Now suppose that the space X_n, in which the Pfaff system (7) is embedded, is a Riemann space V_n. In this case, one may consider the left-hand side $ds^{h'}$ of (7) to be the differentials of the arc lengths of $n - m$ orthogonal congruences in V_n, and for this to be true it suffices to combine the (7), after multiplying them by suitable factors. Moreover, one may associate these $n - m$ forms $ds^{h'}$ with m other forms ds^h, in such a fashion that the n corresponding congruences constitute an orthogonal system of congruences in V_n, in which the metric on V_n is given by the formula (1′). If one takes equations (7) into account in the metric (1′) for the space V_n then it may be written:

$$ds^2 = (ds^1)^2 + (ds^2)^2 + \ldots + (ds^n)^2 . \tag{10}$$

This metric (10), which is the sum of m squares, but which generally contains all of the n variables $x^1, x^2, \ldots, x^n$, constitutes the *metric of the anholonomic space* V_n^m that is defined in V_n by the Pfaff system (7). The space V_n^m thus possesses two invariants: the metric (1), which applies to the interior of the fundamental congruences (λ_h), and the system (7). The group of transformations of these congruences that preserves these two invariants, namely, the *anholonomic group* of the space V_n^m, is obtained from the group (7′) upon associating it with the orthogonality conditions:

$$c_k^h\, c_l^h = \delta_k^l = \begin{cases} 0 & (k \neq l), \\ 1 & (k = l). \end{cases} \tag{11}$$

Naturally, this anholonomic group (7′), (11) preserves the metric (10) that is abstracted from the terms that it annuls with the $ds^{h'}$.

It is interesting to know the significance of that space V_n^m in the particular case where equation (7) forms a completely integrable system that one may suppose to be written in the form (5^{IV}). In this case, the metric (10) on V_n^m becomes a quadratic form in the m differentials $dx^1, dx^2, \ldots, dx^m$ that has coefficients that depend upon $x^1, x^2, \ldots, x^m$ and $n - m$ constants of integration $c^{h'}$. It results from this that our space V_n^m is composed of ∞^{n-m} Riemann spaces V_m. Due to this fact, we continue to refer to the *intrinsic properties of* V_n^m when we mean the properties that are invariant under the group (7′), (11), in the non-integrable case.

We saw above that the group (7′) possesses a remarkable subgroup: the orthogonal group; i.e., the group for which the $c_{k'}^{h'}$ satisfy the orthogonality conditions:

$$c_{k'}^{h'}\, c_{l'}^{h'} = \delta_{k'}^{l'} . \tag{12}$$

This orthogonal group preserves not only the metric (10) on V_n^m, but also the metric (1′) on V_n. The invariant properties of this group are the *rigid properties of* V_n^m, because if V_n^m reduces to a family of V_m then the properties of the orthogonal group are, at the same time, the rigid properties of these V_m, which are embedded in V_n.

One also sees that the *semi-intrinsic* properties of V_n^m are the properties that are intermediate between the rigid properties and the intrinsic properties. One sees this fact very clearly in the integrable case. Indeed, in this case the metric V_n may be written:

$$ds^2 = a_{\alpha\beta}dx^\alpha dx^\beta + 2a_{\alpha\alpha'}dx^\alpha dx^{\alpha'} + a_{\alpha'\beta'}dx^{\alpha'}dx^{\beta'} \tag{13}$$

and the $ds^{h'}$ are linear combinations of the $dx^{\alpha'}$. It then results that under a linear transformation of the $ds^{h'}$ under the group (8), (11), one may modify only the coefficients $a_{\alpha'\beta'}$, in such a fashion that the semi-intrinsic group preserves the coefficients $a_{\alpha\beta}$ of the metric of the family of V_m ($dx^{\alpha'} = 0$), as the intrinsic group, and also the $a_{\alpha\alpha'}$; i.e., the angles between the directions that belong to the family of V_m and the family of complementary V_{n-m} ($dx^\alpha = 0$). In particular, if the two families are orthogonal ($a_{\alpha\alpha'} = 0$) then they remain orthogonal during the transformations of the semi-intrinsic group (8), (11).

One may say, moreover, that the intrinsic properties dependent on the coefficients $a_{\alpha\beta}$, the semi-intrinsic properties, on the $a_{\alpha\beta}$, $a_{\alpha\alpha'}$, and the rigid properties, on the $a_{\alpha\beta}$, $a_{\alpha\alpha'}$, $a_{\alpha'\beta'}$.

CHAPTER III

THE GEOMETRIC PROPERTIES OF V_n^m.

12. The second fundamental form. – In this chapter, we shall study, in the first place, the semi-intrinsic geometric properties of the anholonomic spaces V_n^m that were defined above; i.e., the properties that are invariant under the anholonomic group (8), (11). To that end, we remark that the fundamental formulas (5) that related to that group decompose into six categories, according to whether the indices *a, b, c* have values between 1 and *m* or between *m* + 1 and *n*. We have already considered four of these categories in formulas (7″), (9), and their analogues relative to the group (8), which are the same for the (8), (11), with the sole difference that the c_k^h now satisfy the orthogonality conditions (11). It remains for us only to consider the two categories that are defined by the formula:

$$\frac{\partial c_k^h}{\partial s^l} - \frac{\partial c_l^h}{\partial s^k} = \bar{w}_{\alpha\beta}^h\, c_k^\alpha\, c_l^\beta - w_{kl}^\alpha\, c_\alpha^h, \tag{14}$$

and their analogues. If one associates (14) with the formula that one obtains by differentiating the orthogonality formula (11) with respect to the fundamental congruence then one finds a system of equations in the derivatives $\frac{\partial c_k^h}{\partial s^l}$ that may be solved in the form:

$$\frac{\partial c_k^h}{\partial s^l} = \bar{\gamma}_{\alpha\beta}^h\, c_k^\alpha\, c_l^\beta - \gamma_{kl}^\alpha\, c_\alpha^h. \tag{15}$$

Likewise, if one differentiates (11) with respect to an arc $s^{l'}$ and eliminates the $\frac{\partial c_k^h}{\partial s^{l'}}$ then, with the aid of (9), one arrives at the formula:

$$(\bar{w}_{\beta\alpha'}^\alpha + \bar{w}_{\alpha\alpha'}^\beta) c_k^\alpha\, c_l^\beta c_{l'}^{\alpha'} = w_{ll'}^k + w_{kl'}^l, \tag{14′}$$

which expresses the fact that the quantities $v_{kl,l'} = w_{ll'}^k + w_{kl'}^l = \gamma_{ll'}^k + \gamma_{kl'}^l$ are the components, relative to the congruences (λ), of a third order tensor that is twice interior covariant and once exterior covariant. We thus have three tensors in V_n^m that are semi-intrinsic and of third order: the two integrability tensors $w_{ll'}^k$ and $w_{kl'}^l$, and the tensor $v_{kl,l'}$.

One may find a geometric interpretation for the latter tensor if one considers the variation of the metric (10) on V_n^m under the transition from a point P to an infinitely close point R that is obtained from P by a normal displacement $\delta x^i = \lambda_{h'}^i \varepsilon^{h'}$, where the $\varepsilon^{h'}$ are to be regarded as infinitesimal constants. Indeed, if we let $ds^h = \lambda_i^h dx^i$ denote the components of a tangential displacement that passes through P then the corresponding displacement that passes through R will have the following components relative to the fundamental congruences:

$$d\sigma^h = \lambda_i^h (x + \delta x)\, d(x^i + \delta x^i) = ds^h + w_{k\alpha'}^h ds^k \varepsilon^{\alpha'} .$$

It then results from this that the length ds of the displacement that passes through P is related to the length $d\sigma$ of the displacement that passes through R by the formula:

$$d\sigma^2 = ds^2 + v_{kl,\alpha} ds^k ds^l \varepsilon^{\alpha'} . \tag{16}$$

One may say that the last term in the right-hand side of this formula represents the variation of the metric (10), or of the *first fundamental form* of V_n^m; i.e., it constitutes the *second fundamental form.* This second fundamental form decomposes into $n - m$ quadratic forms:

$$\varphi_{\alpha'} = v_{kl,\alpha'}\, ds^k\, ds^l,$$

which correspond to the $n - m$ anholonomic congruences. If V_n^m is composed of ∞^{n-m} spaces V_m then:

$$w_{kl}^{\alpha'} = \gamma_{kl}^{\alpha'} - \gamma_{lk}^{\alpha'} = 0, \qquad v_{kl,\alpha'} = 2\gamma_{kl}^{\alpha'} ,$$

and the $\gamma_{kl}^{\alpha'}$ are, in this case, the components of the Eulerian curvature of V_m.

13. The class of the metric on V_n^m. – We have remarked above that the metric (10) on V_n^m may depend upon all n variables $x^1, x^2, \dots, x^n$, but obviously this number might be reduced to m. The minimum number of variables that may appear in this metric is given by the number of independent equations in the system ([**30**], pp. 194):

$$ds^h = 0, \qquad w_{k'l'}^{h'}\, ds^{h} = 0 , \qquad v_{kl,\alpha'}\, ds^{h'} = 0 . \tag{15′}$$

One sees that this number is equal to the class of the system $ds^h = 0$ if the tensor of the second form $v_{kl,\alpha'}$ is null.

14. Interior parallelism. – We now return to formula (15). In accord with (6), they express the fact that the Ricci coefficients γ_{kl}^h determine a connection in the interior of the fundamental congruences that permits one to transport an interior vector v^h along an interior path ds^l by the equations ([**9**], pp. 853):

$$dv^h - \gamma_{kl}^h v^k ds^l = 0 . \tag{17}$$

Upon dividing these equations by the length ds of the displacement ds^l and regarding the $\frac{ds^l}{ds}$ to be the cosines u^l of a certain interior curve (c) [x^i = $\varphi^i(s)$], one obtains the equations of parallel transport along (c). This parallelism may be defined geometrically in the same manner as Levi-Civita defines his parallelism in Riemann spaces; i.e., by the condition that the angle between the vectors v^h, $v^h + dv^h$, which are both considered to be vectors in a Euclidian space in which V_n^m is embedded, be the minimum that is compatible with the constraints ([**20**], pp. 18).

This parallel transport preserves lengths and angles. Indeed, the variation of the length of a vector is given by the formula:

$$l\,dl = v^h\,dv^h = \gamma_{kl}^h\,v^h\,v^k\,ds^l\,,$$

and this variation is null due to the skew symmetry in the rotation coefficients. If one now considers the angle θ between the two vectors v^h and u^h, which one may assume to be unitary, then we have:

$$\sin\theta\,d\theta = v^h\,du^h + u^h\,dv^h = 0 \qquad \text{(i.e., } d\theta = 0).$$

This transport is different from the parallel transport of Levi-Civita in the surrounding V_n, which given by formula (5″), when one uses γ_{bc}^a in place of $\gamma_{bc}^{\cdot\,a}$, because in order for that type of transport to give us an interior vector when it is applied to a vector and an interior path, it is necessary that the $\gamma_{kl}^{h'}$ be null. From this, it results that the necessary and sufficient condition for the interior transport (17) on V_n^m to be, at the same time, a parallel transport on V_n is that the tensors $w_{kl}^{h'}$, $v_{kl,l'}$ both be null.

If the vector v^h is not parallel transported along the interior curve (c) [x^i = $\varphi^i(s)$] then the quantities:

$$\text{(16′)} \qquad \frac{\mathrm{D}v^h}{ds} = \frac{dv^h}{ds} - \gamma_{kl}^h v^k u^l\,,$$

where u^l are the cosines of (c) and are calculated along (c), represent the components of the *derivative* of a vector v^h along (c).

15. Infinitesimal pentagon in V_n^m. – One knows that the parallel transport in V_n enjoys the property that the parallelogram that is constructed from two infinitesimal displacements PQ and PR closes. Moreover, one knows (H. Weyl, [**6**[1]], pp. 88) that Levi-Civita transport on V_n may be defined in an intrinsic manner (i.e., without appealing to the surrounding Euclidian space) as the transport that preserves lengths and closes the parallelogram. We shall see that our interior transport in V_n^m does not close the parallelogram. Indeed, if, in formula (6″), one takes into account that $\gamma_{kl}^{\cdot\,h} = \gamma_{kl}^h$ and that

$\gamma_{kl}^{\cdot h'} = 0$, because our transport must preserve the character of interior vectors, then one finds that the closure vector TS is an exterior vector that has the components:

$$\Delta s^{h'} = -w_{kl}^{h'} ds^k \delta s^l . \tag{17′}$$

One sees that these components are null for any displacements d and δ only if V_n^m decomposes into V_m ($w_{kl}^{h'} = 0$). As a consequence, our V_n^m has an infinitesimal figure that is a pentagon that is constructed from two interior displacements ds^h and δs^h, and whose fifth side is the exterior vector (17′).

One may also give our transport (17) an intrinsic geometric definition: It is the transport that preserves length and the character of interior vectors, and which annuls the interior components of the vector TS. Indeed, the former condition says that the $\gamma_{kl}^{\cdot h}$ must be skew symmetric in h and k, and the latter one says that they must satisfy the condition:

$$\gamma_{kl}^{\cdot h} - \gamma_{lk}^{\cdot h} = w_{kl}^{\cdot h} ,$$

and consequently that the $\gamma_{kl}^{\cdot h}$ are equal to the rotation coefficients γ_{kl}^{h}.

16. Geodesics (auto-parallel curves). – The auto-parallel curves of the interior connection on V_n^m are obviously obtained if, in formula (17), one supposes that the vector v^h, which one may assume to be unitary, is tangent to the curve of transport, and consequently, that its components are equal to the cosines $u^h = \dfrac{ds^h}{ds}$. If one also takes into account formula (1) for an interior curve ($ds^{h'} = 0$) then one finds the equations for the auto-parallel curves in the form:

$$\frac{dx^l}{ds} = \lambda_h^i u^h , \qquad \frac{du^h}{ds} = \gamma_{kl}^h u^k u^l . \tag{18}$$

It constitutes a first order differential system in normal form; i.e., $n + m$ equations in the $n + m$ unknowns x and u. One sees that these curves have the property that at each point P of V_n and tangent to each interior direction of V_n^m, there passes one and only one of these curves. Since ∞^{n-m} of these curves pass through each point P it then results that upon starting at a point P one might not reach all of the points of V_n with the aid of auto-parallel curves in V_n^m that start at P, because in order for this to be true there would have to be ∞^{n-m} curves that start at P.

These curves also satisfy a minimum condition. They are the curves such that the distance between two points that are sufficiently near to one of these curves (G) is the shortest one when compared to all of the neighboring curves (g) that pass through these two points and are obtained from (G) by *interior* displacements ([**20**], pp. 22). In general, it so happens that these neighboring curves are no longer interior curves in V_n^m, and consequently the minimum problem does not coincide with the usual minimum problem, which is that of finding the interior curves whose length is a minimum compared to all of the neighboring *interior* curves. This latter problem will be treated later on (§ **20**).

In order for a fundamental congruence – for example, (λ_m) – to be a geodesic congruence it is necessary that $u^h = 0$ $(h < m)$, $u^m = 1$, be a solution of (18); i.e., it is necessary that quantities γ_{hm}^{m}, which one also calls the components of the *geodesic curvature* of the congruence, be null. Likewise, in order for the auto-parallel geodesics in V_n^m to be, at the same time, geodesics of the surrounding V_n, it is necessary that the latter equations (6‴) be satisfied if one sets $u^{h'} = 0$. This amounts to saying that the tensor $v_{hk,h'}$ is null. In this case, one says that the anholonomic space V_n^m is *totally geodesic* in the Riemann space V_n.

If the tensor $v_{hk,h'}$ is non-null, i.e., if the $n - m$ second fundamental forms $\varphi_{\alpha'}$ are not all null, then the solutions to the equations $\varphi_{\alpha'} = 0$, if they exist, define the *asymptotic curves* of the space V_n^m. In the case where $m = n - 1$, and in particular the case $m = 2$, $n = 3$, one may extend many of the properties and formulas that we obtained in the holonomic case to these asymptotic curves (Hlavaty, [**24**[1]]).

17. Exterior parallelism and the infinitesimal parallelogram. – One refers to the parallelism that is provided by formula (9′) and its analogues as *exterior parallelism*, which permits one to transport an interior (exterior, resp.) vector along an exterior (interior, resp.) path. We have seen that this parallelism is completely defined by the property that it close the parallelogram that is constructed from an interior displacement ds^h and an exterior displacement $ds^{l'}$. It is interesting to remark that this parallelism does not preserve the length of the interior vector v^h, because the variation of that length is given by the formula:

$$l\,\delta l = \tfrac{1}{2}\, v_{hk,h'}\, v^h\, v^k\, ds^{l'},$$

and one sees that this variation may be constantly zero only in the case where V_n^m is totally geodesic in V_n.

In summation, we have a semi-intrinsic affine connection in the anholonomic space V_n^m that has the quantities γ_{kl}^{h}, $w_{kl'}^{h}, w_{k'l}^{h'}$ for its components relative to the congruences (λ). This connection is not complete because it does not give us the possibility of transporting an exterior vector along an exterior curve. However, if the system (7) in V_n^m has its first derived system equal to null then one may associate our connection with the connection $\delta_{k'l'}^{h'}$ [formulas (9″)], and one then obtains a *complete semi-intrinsic connection* that has components relative to the congruences (λ) that are equal to:

$$(18')\qquad \begin{cases} \overset{\bullet}{\gamma}{}_{kl}^{h} = \gamma_{kl}^{h}, & \overset{\bullet}{\gamma}{}_{kl'}^{h} = w_{kl'}^{h}, & \overset{\bullet}{\gamma}{}_{k'a}^{h} = 0, \\ \overset{\bullet}{\gamma}{}_{k'l'}^{h'} = \delta_{k'l'}^{h'}, & \overset{\bullet}{\gamma}{}_{k'l}^{h'} = w_{k'l}^{h'}, & \overset{\bullet}{\gamma}{}_{ka}^{h'} = 0. \end{cases}$$

If the first derived system is non-null, but the system (7) has no integrable combinations, then one may confirm that the subgroup of the semi-intrinsic group (8), (11) that preserves the derived system of (7) possesses a complete affine connection.

The existence of the semi-intrinsic affine connection naturally entails the possibility of the tensorial derivation of tensors. We remark only that, as long as one is limited to the connection γ^h_{kl}, $w^h_{kl'}$, $w^{h'}_{k'l}$, which one may call *regular* [because it exists for any system (7)], one does not have the possibility of obtaining another tensor from an exterior or mixed tensor by differentiating along an arc of the anholonomic congruence ([**20**], pp. 35).

18. Curvature tensors ([**18**], pp. 65; [**20**], pp. 38). – We shall now find two semi-intrinsic tensors of order four: one of them is interior and the other one is once covariant exterior. These two tensors may be obtained by calculating the variation of the components of an interior vector under parallel transport along the infinitesimal pentagon and parallelogram in V^m_n. Indeed, if one first transports the vector v^h along the pentagon PQRSTRP then the variations of the components v^h, taking into account only terms of second order, is obtained by taking the difference of the components of the vector v^h when it is transported, in one case, along PQS, and in another case, along PRTS, and we have:

$$\mathrm{D}v^h = \delta\, dv^h - d\, \delta v^h - \Delta v^h ,$$

in which Δ denotes transport along TS. Upon performing the calculations, one arrives at the formula:

$$\mathrm{D}v^h = \lambda^h_{klr}\, v^k\, ds^l\, \delta s^r , \tag{19}$$

where we have set:

$$\lambda^h_{klr} = \frac{\partial \gamma^h_{kl}}{\partial s^r} - \frac{\partial \gamma^h_{kr}}{\partial s^l} + \gamma^h_{k\alpha} w^\alpha_{lr} - \gamma^h_{\alpha r}\gamma^\alpha_{kl} + \gamma^h_{\alpha l}\gamma^\alpha_{kr} + w^h_{k\alpha'} w^{\alpha'}_{lr} . \tag{19′}$$

This formula may also be written:

$$\lambda^h_{klr} = \gamma^h_{klr} + \omega^h_{k\alpha'}\omega^{\alpha'}_{lr} , \tag{19″}$$

where the γ^h_{klr} are the four-indexed Ricci coefficients relative to the fundamental congruences.

The quantities λ^h_{klr} are obviously the components of a fourth order interior tensor; it is the interior curvature tensor of V^m_n. If V^m_n is composed of V_m then the λ^h_{klr} are equal to the four-indexed Ricci coefficients relative to the fundamental congruence, and consequently our tensor coincides with the Riemann curvature tensor of the V_m.

The quantities λ^h_{klr} are skew-symmetric in the indices l and r. As far as the indices h and k are concerned, they satisfy the formulas:

$$\lambda^h_{klr} + \lambda^k_{hlr} = v_{hk,\alpha'} w^{\alpha'}_{lr} .$$

Since the variation Dl of the length l of the vector v^h along the pentagon is provided by the formula l Dl = v^h Dv^h, it results from this that the length of the vector v^h is preserved along the pentagon if the curvature λ^h_{klr} is also skew-symmetric in the first two indices h and k, which comes about, in particular, when V^m_n is composed of V_m, or if V^m_n is totally geodesic in V_n.

If one denotes the angle between a unitary vector u^h and the vector v^h by θ, the length by l, and the angle between the vectors u^h and v^h + Dv^h by θ + Dθ then the variation Dθ, when one has taken into account only the first order terms, is given by the formula:

$$(20) \qquad -\sin\theta \; \mathrm{D}\theta = \lambda^h_{klr} u^h v^k ds^l \delta s^r - \cos\theta \frac{\mathrm{D}l}{l} .$$

This formula is analogous to that of Pérès ([**7**[1]], pp. 219) for the V_n, but one sees that for the V^m_n it is no longer symmetric in the vectors (u) and (v), in general ([**33**], § 6).

If one now transports the vector v^h along the infinitesimal parallelogram in then one finds:

$$(21) \qquad \mathrm{D}' v^h = \lambda^h_{klr'} v^k \, ds^l \, \delta s^{r'} ,$$

where the quantities:

$$(20') \qquad \lambda^h_{klr'} = \frac{\partial \gamma^h_{kl}}{\partial s^r} - \frac{\partial w^h_{kr'}}{\partial s^l} + \gamma^a_{k\alpha} w^\alpha_{lr'} + \gamma^h_{\alpha l} w^\alpha_{kr'} - w^h_{\alpha r'} \gamma^\alpha_{kl} + w^h_{k\alpha'} w^{\alpha'}_{lr'}$$

are the components of the exterior curvature tensor of V^m_n, which is, as one sees, a fourth order tensor that is once exterior covariant. One may consider the tensors λ^h_{klr} and $\lambda^h_{klr'}$ to be just one tensor λ^h_{kla}, in which the index a varies from 1 to n. This tensor is obtained when one looks for the variations of the vector v^h along the infinitesimal circuit that is constructed from an interior displacement ds^h and an arbitrary ds^a.

19. Geometrizable anholonomic groups. – The results of this chapter show that the space V^m_n possesses some remarkable semi-intrinsic geometric properties. If the space V^m_n is composed of ∞^{n-m} copies of V_m ($w^{h'}_{kl}$ = 0) then a subset of these properties, more precisely, interior parallelism, auto-parallel geodesics, and the interior curvature tensor are also intrinsic properties of V^m_n, or rather, the ∞^{n-m} copies of V_m that it is composed of. We shall now show, on the one hand, that if V^m_n is an anholonomic space, properly speaking (i.e., $w^{h'}_{kl}$ ≠ 0), then none of these geometric properties is an intrinsic property of V_n^m, and, on the other hand, that there exist, in general, other subgroups of the intrinsic group that are larger than the semi-intrinsic subgroup, and which preserve some of these properties (Cartan [**17**]; Vranceanu [**30**]). In order to do this, one starts with the remark that if V_n^m is composed of V_m then the intrinsic properties of these V_m are also intrinsic properties of V_n^m. Now, as intrinsic properties of the V_m, we have, along with the metric (10), the Levi-Civita parallelism of these V_m, which is defined by the interior connection γ_{kl}^h on V_n^m, and the Riemann curvature of these V_m, which coincides, as we have already remarked, with the intrinsic curvature tensor of V_n^m [**23**].

Consequently, in the non-integrable case, it is natural to demand that the interior connection and the interior curvature are, moreover, invariants of the intrinsic group; however, since this group is the product of the semi-intrinsic group and the group (8′) it suffices to see whether they are invariants of this latter group. From the fundamental equations (5), relative to the group (8′), one deduces that:

$$\bar{w}_{kl}^h = w_{kl}^h + w_{kl}^{\alpha'} c_{\alpha'}^h . \tag{20″}$$

Since the γ_{kl}^h are defined as functions of the w_{kl}^h by formula (5″), it then results that the interior connection will be an intrinsic invariant only if V_n^m is composed of V_m ($w_{kl}^{\alpha'} = 0$). One also sees that the γ_{kl}^h are invariant under the transformations (8′) only when the coefficients $c_{\alpha'}^h$ satisfy the equations:

$$w_{kl}^{\alpha'} c_{\alpha'}^h = 0 . \tag{21′}$$

Now, *if the first derived system* (7) *is null* then these equations might only have the solution $c_{\alpha'}^h = 0$, in such a fashion that in this case *the largest subgroup of the intrinsic group that preserves the interior connection is the semi-intrinsic subgroup.*

If the derived system of (7) is composed of $p - m$ equations $ds^{m+1} = \ldots = ds^p = 0$ then the general solution of (21′) is:

$$c_{k'}^h = 0 \ (k' > p) \qquad \text{and} \qquad c_{m'}^h \text{ arbitrary } (m' \leq p),$$

in such a fashion that the largest subgroup of (8′) that preserves the interior connection is given by the formulas:

$$\begin{cases} d\bar{s}^h = ds^h + c_{m'}^h ds^{m'} \quad (m' = m+1, \cdots, p), \\ d\bar{s}^{h'} = ds^{h'}. \end{cases} \tag{21″}$$

Obviously, the largest group that preserves the interior connection is the product of this group (21″) with the semi-intrinsic group, and if one so desires, since it is preferable, to preserve the character of the $ds^{m'}$ as forms of the derived system then one obtains the group:

(22)
$$\begin{cases} d\bar{s}^h = c^h_k ds^k + c^h_{m'} ds^{m'}, & \\ d\bar{s}^{m'} = c^{m'}_{n'} ds^{n'} & (m', n' = m+1, \cdots, p), \\ d\bar{s}^k = c^k_{\alpha'} ds^{\alpha'} & (k' = p+1, \cdots, n). \end{cases}$$

One may say that this group *is the largest geometrizable subgroup of the intrinsic group.*

In an analogous manner, one finds that the transformations (8′), which preserve the interior curvature, are analogous to (21″), with the difference that the $ds^{m'}$ must belong to the *second derived system* of (7). It then results that the group (22″) also preserves the interior curvature of V^m_n only if the first derived system is completely integrable. In the general case, the largest group that preserves both the connection and the interior curvature is a subgroup of the group (22) that one may easily write down.

One may also pose the problem of finding the group that preserves the auto-parallel geodesics of V^m_n. This problem is of considerable mechanical interest because, as will be proven later on, these curves are also the unforced trajectories of an anholonomic mechanical system. For a long time now ([**3**], 1895) we have known the following result, which is due to J. Hadamard, that the equations of motion of an anholonomic system do not remain the same if one modifies the vis viva of the system in an arbitrary manner with the aid of the equations of anholonomity (intrinsic case), but they do remain the same if one modifies the vis viva by a quadratic form in these equations (semi-intrinsic case [**33**], Introduction).

In order to find the group of auto-parallel geodesics, one remarks that their coefficients $\gamma^h_{kl} + \gamma^h_{lk} = w^k_{lh} + w^l_{kh}$ are invariant under the transformations (8′) only when the $c^h_{\alpha'}$ satisfy the equation:

(22′)
$$w^{\alpha'}_{lh} c^k_{\alpha'} + w^{\alpha'}_{kh} c^l_{\alpha'} = 0 .$$

These transformations define a group. Indeed, if we have two solutions of (22′) then the product of the corresponding transformations (8′) has the sum of the solutions for its coefficients, which is also a solution of equation (22′), due to the fact that these equations are linear and homogeneous.

The group that preserves the auto-parallel geodesics is, in general, the largest group (22) that preserves the interior connection, as one may see, for example, in a space V^m_n that is defined by the following forms and equations:

$$\begin{aligned} ds^h &= dx^h, \qquad (h = 1, 2, 3) \\ ds^4 &= dx^4 - x^2\, dx^3 = 0 , \\ ds^5 &= dx^5 - x^3\, dx^4 = 0 , \\ ds^6 &= dx^6 - x^4\, dx^2 = 0 . \end{aligned}$$

Indeed, in this case the components of the integrability tensor $w_{kl}^{\alpha'}$ are all null, except for $w_{23}^4, w_{31}^5, w_{12}^6$, which are all unity. It then results that the group (21″) reduces to the identity, because the first derived system of the equations of anholonomity is null, whereas the subgroup of (8′) that preserves the auto-parallel geodesics is given by the formulas:

$$(22'') \qquad \begin{cases} d\bar{s}^h = ds^h + \rho\, ds^{h+s} & (h = 1, 2, 3), \\ d\bar{s}^{h'} = ds^{h'} & (h' = 4, 5, 6), \end{cases}$$

where ρ is an arbitrary function of the variables $x^1, x^2, \ldots, x^n$.

These considerations show us that the group that one may attach to an anholonomic mechanical system in a natural manner, as the group that preserves the unforced equations of motion of the system, may be geometrically realized if it coincides with the group that preserves the interior connection.

20. Geodesics of minimum length. – If the intrinsic group of V_n^m possesses no geometric invariants in the non-integrable case, aside from the metric on V_n^m and the tensor $w_{kl}^{h'}$, then it will nevertheless possess an important analytical invariant: *the geodesics of minimum length;* i.e., the interior curves in V_n^m such that the length:

$$(23) \qquad I = \int_{\mathrm{A}}^{\mathrm{B}} ds = \int_{\mathrm{A}}^{\mathrm{B}} \sqrt{(u^1)^2 + (u^2)^2 + \cdots + (u^m)^2}\, ds$$

between two of their points A and B that are sufficiently close is the smallest when compared to all of the neighboring *interior* curves that pass through the A and B (Voss [**1**], pp. 280; Franklin and Moore [**29**], pp. 189).

Obviously, the curves of minimum length in V_n^m are, at the same time, the extremal curves of the integral (23); i.e., the interior curves such that the variation δI of the integral (23) under the transition from one of these curves to an infinitely close neighboring interior curve is null. This fact permits us to find their equations by the method of Lagrange multipliers, which consists of seeking the extremal curves of the integral that one obtains from (23) by adding the term $v_{\alpha'}\, ds^{\alpha'}$ to ds, where the $v_{\alpha'}$ are the multipliers, which one must regard as functions of the arc length s. Upon performing the calculations, one will arrive at the equations for the geodesics of minimum length in the form:

$$(24)\qquad \begin{cases} \dfrac{dx^i}{ds} = \lambda^i_h u^h, \\ \dfrac{du^h}{ds} - \gamma^h_{kl} u^k u^l = w^{h'}_{kh} u^k v_{h'}, \\ \dfrac{dv_{h'}}{ds} + w^{\alpha'}_{h'k'} v_{\alpha'} u^k = \tfrac{1}{2} v_{kl,h'} u^k u^l, \end{cases}$$

of a first order normal differential system that consists of $2n$ equations in $2n$ unknowns x^i, u^h, $v_{h'}$.

It is easy to see that this system is an invariant of the semi-intrinsic group, if one regards the multipliers $v_{\alpha'}$ to be the components of a *covariant* exterior vector relative to the congruences (λ). In this case, in effect, the left-hand sides of the latter equations are the components of the derivative of the vector $v_{\alpha'}$ along the curve of shortest distance, this derivative being performed with respect to the connection $\omega^{h'}_{k'l'}$. As a consequence, in order to conclude that the system (24) is an invariant of the intrinsic group of V_n^m, it suffices to see that the system (24) is an invariant of the group (8′), which is effectively the case if one takes new multipliers to be the quantities:

$$\bar{v}^{\alpha'} = v_{\alpha'} - c^h_{\alpha'} u^h .$$

In the non-integrable case, the geodesics of minimum length are different from the auto-parallel geodesics, as well as being more numerous. If $p - m$ is the number of equations in the derived system of (7) then there are $n - p$ of these geodesics that pass through each point P and tangent to each interior direction; one may see this in the right-hand side of the second equation in (24). One may also show that the system (24) may be decomposed into two subsets, the first of which will suffice to determine the unknowns x^i, u^h, $v_{h'}$ ($h' = m + 1, \ldots, q$) only if the equations $ds^{\alpha'}$ ($\alpha' > q$) are integrable combinations of the system (7).

Equations (24) also show us that in the non-integrable case the auto-parallel geodesics are, at the same time, the geodesics of minimum length if V_n^m is totally geodesic in V_n ($v_{hk,l'} = 0$). This signifies that in this case one may suppose that the multipliers $v_{\alpha'}$ are null in equations (24).

21. Rigid connections on V_n^m. – We shall now study the rigid geometric properties of V_n^m, that is, the properties of V_n^m that are invariant under the orthogonal group (8), (11), (12). We remark that this group coincides with the rigid group of the anholonomic space V_n^{n-m} that is complementary to V_n^m, which one obtains by equating the ds^h to zero, in such a fashion that the rigid properties of V_n^m and V_n^{n-m} coincide. Nevertheless, there obviously exist properties that one attaches to V_n^m more than one does to V_n^{n-m}, such as the geodesics of V_n^m, interior parallelism, etc. Moreover, one can easily account for the fact that if one adds the semi-intrinsic properties of V_n^{n-m} to the semi-intrinsic properties of V_n^m then one obtains all of the rigid properties of V_n^m or V_n^{n-m}. Indeed, if one takes into account the fact that the $c_{k'}^{h'}$ satisfy the orthogonality conditions (12) then the formulas that are *analogous* to (14) may, like (14) themselves, be solved with respect to the derivatives $\frac{\partial c_k^{h'}}{\partial s^{l'}}$:

$$\frac{\partial c_k^{h'}}{\partial s^{l'}} = \gamma_{\alpha'\beta'}^{h'} c_{k'}^{\alpha'} v_{l'}^{\beta'} - \gamma_{k'l'}^{\alpha'} c_{\alpha'}^{h'} \ . \tag{25}$$

These formulas express precisely the last that the rotation coefficients $\gamma_{k'l'}^{h'}$ are the components of an affine connection that permits us to transport an exterior vector in V_n^m along a path that is also exterior. This connection is nothing but the interior connection on the space V_n^{n-m} that is complementary to V_n^m. Consequently, if one associates the connection $\gamma_{k'l'}^{h'}$ to the semi-intrinsic connection on V_n^m then one obtains a rigid connection on V_n^m, in the form of the complete connection that is defined by the formula (Schouten and Kampen [**24**], pp. 771; Vranceanu [**30**], pp. 199):

$$\gamma_{kl}^{\bullet h} = \gamma_{kl}^{h}, \qquad \gamma_{kl'}^{\bullet h} = w_{kl'}^{h}, \qquad \gamma_{k'l}^{\bullet h'} = w_{k'l}^{h'}, \qquad \gamma_{k'l'}^{\bullet h'} = \gamma_{k'l'}^{h'}, \qquad \gamma_{k'a}^{\bullet h} = \gamma_{ka}^{\bullet h'} = 0 \ . \tag{26}$$

This connection obviously has the property of preserving the character of interior and exterior vectors ($\gamma_{k'a}^{\bullet h} = \gamma_{ka}^{\bullet h'} = 0$). It is characterized by the property of preserving length under the transport of an interior (exterior, resp.) vector along an interior (exterior, resp.) path and closing the parallelogram *as much as is possible.* Indeed, this connection is composed of interior connections on V_n^m and V_n^{n-m} and the exterior connection on V_n^m, which is, at the same time, an exterior connection V_n^{n-m}, and all of these connections have that property. It is also obvious that as examples of third order rigid tensors, we have the two integrability tensors $w_{kl}^{h'}$, $w_{k'l'}^{h}$ the second fundamental form $v_{kh,l'}$ of V_n^m, and the second fundamental form $v_{k'h',l}$ for V_n^{n-m}.

Instead of regarding the rigid group as a subgroup of the semi-intrinsic group of V_n^m, one may also regard it as a subgroup of the orthogonal group on V_n. In this case, in order to obtain the fundamental formulas of our rigid group one must assume that $c_{k'}^{h} = c_k^{h'} = 0$ in the fundamental formula (6) for the orthogonal group ($\overset{*}{\gamma} = \gamma$) on V_n. Along with the formula (15) and its analogue (25), one finds the following formulas:

$$\begin{cases} \dfrac{\partial c_k^h}{\partial s^{l'}} = \overline{\gamma}_{\alpha'\beta'}^{h} c_k^{\alpha} c_{l'}^{\beta'} - \gamma_{kl'}^{\alpha} c_{\alpha}^{h}, \\ \quad 0 = \overline{\gamma}_{\alpha'\beta'}^{h} c_k^{\alpha} c_{l'}^{\beta'} - \gamma_{kl'}^{\alpha} c_{\alpha}^{h}, \end{cases} \tag{27}$$

and analogous formulas. The first of these formulas and their analogues for (15) and (25) show that the anholonomic space V_n^m also possesses the complete rigid connection that was considered by Schouten ([**16**], pp. 294; [**24**], pp. 770; [**30**], pp.198):

$$\gamma_{ka}^{\bullet h} = \gamma_{ka}^{h}, \qquad \gamma_{k'a}^{\bullet h'} = \gamma_{k'a}^{h'}, \qquad \gamma_{k'a}^{\bullet h} = \gamma_{ka}^{\bullet h'} = 0 . \tag{28}$$

As one sees, this connection differs from the rigid connection (26) by the quantities $\gamma_{kl}^{h'}, \gamma_{k'l'}^{h}$, which are, by virtue of the last of (27), the components of two rigid tensors of third order. Moreover, one may remark that the four rigid tensors of third order that were considered above can be expressed with the aid of these two tensors.

The connection (28) itself preserves the character of interior and exterior vectors. It is characterized by the property that it preserve lengths like the Levi-Civita connection of the surrounding space V_n ; however, it closes the parallelogram only if the tensors $\gamma_{kl}^{h'}, \gamma_{k'l'}^{h}$ are both null. This signifies that V_n^m and V_n^{n-m} must be holonomic and totally geodesic. Moreover, parallel transport under the connection (28) may be obtained from the one on V_n by following construction (Enéa Bortollotti [**27**], pp. 7):

One obtains the vector that is parallel to an interior (exterior, resp.) vector by taking the projection of the vector that is parallel in V_n *onto the fundamental (and anholonomic) congruences*.

From the rigid viewpoint, the two connections (26) and (28) have the same value and both of them preserve the interior or exterior character of a vector; however, if one also considers the semi-intrinsic properties then one sees that the connection (26) is related to the properties of V_n^m in an intimate manner.

We also have another rigid connection, which is due to Synge ([**15**], pp. 745; [**27**], pp. 2; [**30**], pp. 202), that only preserves the character of interior vectors, and which was the first rigid connection that was considered in the study of anholonomic spaces. The components of that connection on the congruences (λ) are given by the formula:

$$\gamma_{ka}^{\bullet h} = \gamma_{ka}^{h}, \qquad \gamma_{k'a}^{\bullet h'} = \gamma_{k'a}^{h'}, \qquad \gamma_{k'a}^{\bullet h} = 2\gamma_{k'a}^{h}, \qquad \gamma_{ka}^{\bullet h'} = 0 . \tag{29}$$

One sees that this connection preserves the character of exterior vectors only if $\gamma_{k'a}^{h} = 0$, that is, if V_n^m and V_n^{n-m} are holonomic and totally geodesic.

22. Rigid curvature tensors. – The infinitesimal circuits for the connection (26) are obviously two pentagons in V_n^m and V_n^{n-m}, and the parallelogram in V_n^m that is also common to V_n^{n-m}. If one transports an interior vector and an exterior vectors around these three circuits then one obtains six curvature tensors. Four of these tensors are already known to us: they are the two curvature tensors $\lambda_{klr}^{h}, \lambda_{klr'}^{h}$ on V_n^m, and the two curvature tensors $\lambda_{k'l'r'}^{h'}, \lambda_{k'l'r}^{h'}$ on V_n^{n-m}. In order to obtain the other two curvature tensors, one must transport an exterior vector around the pentagon in V_n^m and an interior vector around the pentagon in V_n^{n-m}. Upon performing the calculations, one easily perceives that one does not arrive at any new tensors $w_{kl,r'}^{h'}$, $w_{k'l',r}^{h}$, but only the derived tensors and the integrability tensors ([**30**], pp. 200).

Naturally, the curvature tensors of the connection (28) and (29) are expressed as functions of the curvature tensors of the connection (26) and the tensors $\gamma_{kl}^{h'}$, $\gamma_{k'l'}^{h}$, which represents the difference between the two connections. In particular, (Horak, [**19**]) if one transports an interior vector around the pentagon in V_n^m with the aid of the connection (28) then one obtains the curvature tensor $R_{klr}^{\cdot\cdot\cdot h}$ that was found by J. A. Schouten:

$$R_{klr}^{\cdot\cdot\cdot h} = \lambda_{klr}^{h} + \gamma_{hk}^{\alpha'} w_{lr}^{\alpha'} .$$

One sees that the tensor $R_{klr}^{\cdot\cdot\cdot h}$, like the tensor λ_{klr}^{h}, reduces to the Riemann tensor on V_m if the system (7) is completely integrable. In the non-integrable case, the tensor $R_{klr}^{\cdot\cdot\cdot h}$ is only a rigid tensor, and not a semi-intrinsic tensor like the tensor λ_{klr}^{h} of interior curvature on V_n^m.

In concluding this chapter, we would like to remark, in the first place, that one may also consider anholonomic spaces V_n^p embedded in an anholonomic space V_n^m; i.e., spaces that are defined in V_n^m by a Pfaff system $ds^{\alpha} = 0$ ($\alpha = p + 1, \ldots, m$). One obviously obtains the groups of these spaces if, in the groups of V_n^m, one separates the transformations of the fundamental congruences of V_n^m into two subsets, according to whether the indices h, k have values between 1 and p or between $p + 1$ and m.

In the second place, we would like to remark that one obtains an interesting subgroup of the semi-intrinsic group in V_n^m, by supposing that the $c_{k'}^{h'}$ are constant. Indeed, this anholonomic group possesses a *complete* affine connection whose only non-zero components are the following ones:

$$\overset{\bullet}{\gamma}{}_{kl}^{h} = \gamma_{kl}^{h}, \qquad \overset{\bullet}{\gamma}{}_{kl'}^{h} = w_{kl'}^{h} .$$

This connection preserves the character of interior and exterior vectors; however, it is rigidly related to the anholonomic congruences. As for the third order tensors of this connection, they are given by the second fundamental form tensor $v_{kl,h'}$ and the four torsion tensors $w_{kl}^{h'}$, $w_{k'l'}^{h}$, $w_{k'l}^{h'}$, $w_{k'l'}^{h'}$.

CHAPTER IV.

ANHOLONOMIC SPACES WITH AFFINE CONNECTION.

23. Geometric properties. – Up till now, we have considered the properties of anholonomic spaces V_n^m to be the invariants of certain groups of transformations of the congruences; however, obviously, *one may also express these properties in an arbitrary system of congruences or coordinates* ([**14**], [**15**], [**16**], [**21**], [**24**]). In particular, one may easily pass to the condition that the congruences be orthogonal with respect to the metric (10) on V_n^m, because it suffices to introduce this metric, not as an invariant of the group, but explicitly in the form:

$$ds^2 = a_{\alpha\beta}\, ds^\alpha\, ds^\beta \qquad (\alpha, \beta = 1, 2, \ldots, m). \tag{28′}$$

In this case, the intrinsic properties of V_n^m will be invariants of the group (7′) and the metric (28′), and the semi-intrinsic properties will be invariants of the group (8) and the metric (28′). It results from this that the semi-intrinsic exterior connection $w_{kl'}^h, w_{k'l}^{h'}$ is preserved, whereas the components $\gamma_{kl}^{\cdot\,h}$ of the interior connection on V_n^m will be represented by the coefficients of rotation γ_{kl}^h only if the metric (28′) reduces to the sum of squares of the ds^α. As for the rigid properties, they are the invariants of the group (8), the metric (28′), and a metric that is analogous to the interior one on a system of anholonomic congruences.

These considerations are useful if one wishes to study anholonomic spaces with affine connection by the same methods as the groups of transformations of congruences. Indeed, suppose that we have the Pfaff system (7) in a space A_n with affine connection. If one associates the forms $ds^{h'}$ with *m* complementary forms ds^h and one lets $\gamma_{bc}^{\cdot\,a}$ index the components of the connection on A_n relative to the *n* congruences (λ) then one may refer to the connection that has the components $\gamma_{kl}^{\cdot\,h}$ as the *connection induced* by A_n on the interior of the congruences (λ^h). One easily perceives that in this case, as well, this induced connection is an invariant of the intrinsic group (7′) only if the system (7′) is completely integrable and only if it is an invariant of the group (8) because we have the formula:

$$\frac{\partial c_k^h}{\partial s^l} = \gamma_{\alpha\beta}^{\cdot\,h}\, c_k^\alpha\, c_l^\beta - \gamma_{kl}^{\cdot\,\alpha} c_\alpha^h \;. \tag{29′}$$

Consequently, the semi-intrinsic properties of the anholonomic space A_n^m, which is defined in A_n by the system (7) and the congruences (λ^h), are invariants of the group (8) that one associates with the induced connection $\gamma_{kl}^{\cdot\,h}$. The regular semi-intrinsic connection on the anholonomic space A_n^m has the quantities $\gamma_{kl}^{\cdot\,h}, w_{kl'}^h, w_{k'l'}^h$ for its components relative to the congruences (λ). As for the irregular connection $\delta_{k'l'}^{h'}$, if it exists, it depends, as one knows, only on the system (7).

If the connection A$_n$ is not symmetric then the interior components $\tau_{kl}^h = \gamma_{kl}^{\cdot h} - w_{kl}^h$ of the torsion tensor might not be null, and in this case they determine an interior tensor of third order with respect to the group (8).

In order to find this semi-intrinsic curvature tensor $\lambda_{kla}^{\cdot h}$ on A_n^m, one may, as in the case of a V_n^m, seek the variations of the components of an interior vector under parallel transport along the infinitesimal pentagon that is constructed from an interior displacement ds^h and an arbitrary displacement δs^a. We remark only that the fifth side of the pentagon also has, in this case, the interior components $\Delta s^h = \tau_{kl}^h\, ds^k\, ds^l$ if τ_{kl}^h is non-null. From this, it results that in order to obtain the components $\lambda_{klr}^{\cdot h}$ of the curvature tensor of A_n^m, it is necessary to substitute $\gamma_{kl}^{\cdot h}$ in place of γ_{kl}^h in (19′), and add the term $-\gamma_{k\alpha}^{\cdot h}\tau_{lr}^{\alpha}$.

In order to obtain the rigid properties of A_n^m, one must associate the connection $\gamma_{k'l'}^{\cdot h'}$ that is induced from the surrounding space A$_n$ on the interior of the anholonomic congruences. The connection that has $\gamma_{kl}^{\cdot h}, w_{kl'}^h, w_{k'l}^{h'}, \gamma_{k'l'}^{\cdot h'}$ for its non-null components thus constitutes a first complete rigid connection on the space A_n^m, which is equivalent to the connection (26) for a V_n^m. In order to obtain the equivalence with the rigid connection (28), one must take into account that $c_{k'}^h = c_k^{h'} = 0$ in formulas (6), and one finds the connection that has the quantities $\gamma_{ka}^{\cdot h}, \gamma_{k'a}^{\cdot h'}$ for its non-null components (Schouten and Kampen [**24**], pp. 770, 775).

24. Equations of variation for auto-parallel curves. – The auto-parallel connection of the interior connection on the anholonomic affine space A_n^m is obviously given by the equations:

$$\text{(28″)} \qquad \begin{cases} \dfrac{dx^i}{ds} = \lambda_h^i u^h, \\ \dfrac{du^h}{ds} = \gamma_{kl}^{\cdot h} u^k u^l. \end{cases}$$

The parameter s that is determined along one of these curves by these equations will be called the *affine arc length* of the curve. In order to find the equations of variation of these curves (Vranceanu, [**20**], pp. 41; Wundheiler [**28**]), consider, in the space X$_n$ of variables x^i, an arbitrary curve (C):

$$x^i = \varphi^i(\sigma),$$

σ being a parameter whose values determine the different points of the curve, and let:

$$x^i = \varphi^i(\sigma) + \lambda_a^i \varepsilon^a ,$$

be the equations of a curve (c) that is close to (C). In these equations, the parameters λ_a^i appear with their values as a function of σ along (C), and ε^a are considered to be first order quantities. Moreover, they represent the components, relative to the congruences (λ), of the vector that connects the points that correspond to the same value of the parameter σ on the two curves (C) and (c). If we let s denote an arbitrary parameter upon which the points of the curve (c) depend then we have, neglecting terms of higher than first order in the ε^a, the formula:

$$u^a = (\lambda_i^a)_c \frac{dx^l}{ds} = \frac{d\sigma}{ds}\left(c^a + \frac{d\varepsilon^a}{d\sigma} + w_{bd}^a c^b \varepsilon^d\right),$$

where the w_{bd}^a appear with their values along (C), and c^a are the cosines of the curve (c).

Now, suppose that the curve (C) is an interior curve in the anholonomic space X_n^m ($c^{h'}$ = 0). In order for the curve (c) to be itself an interior curve in X_n^m, one must satisfy the equation:

$$u^{h'} = \frac{d\varepsilon^a}{d\sigma} + w_{kl'}^{h'} c^k \varepsilon^{l'} + w_{kl}^{h'} c^k \varepsilon^l = 0 . \tag{29''}$$

One sees that if one supposes that the ε^l are known then these equations constitute a first order differential system in the $\varepsilon^{h'}$. We also remark that if the quantities $w_{kl}^{h'} c^k$ are null, in particular, if the space X_n^m is composed of ∞^{n-m} copies of X_m ($w_{kl}^{h'}$ = 0), then equations (29″) do not depend upon the ε^l, in such a fashion that they possess the solution $\varepsilon^{h'} = 0$ in this case; that is, one may obtain the neighboring curves to (C) by displacements that are interior to X_n^m. If one introduce the vector derived from the vector $\varepsilon^{h'}$ along (C):

$$\frac{\mathrm{D}\varepsilon^{h'}}{d\sigma} = \frac{d\varepsilon^{h'}}{d\sigma} - w_{kl'}^{h'} \varepsilon^{k'} c^l$$

then one may give equations (29″) the semi-intrinsically invariant form:

$$\frac{\mathrm{D}\varepsilon^{h'}}{d\sigma} + w_{kl}^{h'} \varepsilon^k c^l = 0 . \tag{29'''}$$

If we are in a space with affine connection A_n^m then we also have the formula:

$$w_{kl}^h = \gamma_{kl}^{\bullet h} - \gamma_{lk}^{\bullet h} - \tau_{kl}^h ,$$

and if we are interested only in auto-parallel curves then one may change the connection without changing these curves, in such a fashion that the components τ_{kl}^{h} of the torsion are null. This being the case, if one introduces the vector that is derived from the interior vector ε^h along (C):

$$\frac{\mathrm{D}\varepsilon^h}{d\sigma} = \frac{d\varepsilon^h}{d\sigma} - \gamma_{kl}^{\cdot h}\varepsilon^k c^l ,$$

then one may write the components of the tangent vector to (c) in the form:

$$u^h = \frac{d\sigma}{ds}\left(c^h + \frac{\mathrm{D}\varepsilon^h}{ds} + \gamma_{kl}^{\cdot h} c^k \varepsilon^l + w_{kl'}^{h} c^k \varepsilon^{l'} \right).$$

Now suppose that the curve (C) is an auto-parallel curve of the anholonomic space A_n^m and that σ is its affine arc length. If one wishes that (c) itself should be an auto-parallel curve in A_n^m with the affine arc length then it is necessary that the u^h satisfy the formulas (28″). If we assume, as is, moreover, natural, that:

$$\frac{d\sigma}{ds} = 1 - \mu ,$$

in which μ is a first order quantity, and we introduce the values of the u^h into (28″), and neglect terms greater than first, then we find the following formula without difficulty:

(30) $$\frac{\mathrm{D}^2\varepsilon^h}{d\sigma^2} - \frac{d\mu}{d\sigma} c^h = \lambda_{kla}^{\cdot\cdot h} c^k c^l \varepsilon^a ,$$

where the $\dfrac{\mathrm{D}^2\varepsilon^h}{d\sigma^2}$ are the components of the second vector derivative of ε^h along (C) and the $\lambda_{kla}^{\cdot\cdot h}$ are the components of the curvature tensor of A_n^m. These equations, which obviously have a semi-intrinsic invariant character and are associated with (29‴), constitute a system of n differential equations in the n + 1 unknowns ε^a, μ that is second order in the ε^h and first order in the $\varepsilon^{h'}$, μ.

In order to determine the unknowns ε^a, μ, it suffices to associate this system with a law of correspondence between curves (C) and (c). If our anholonomic space is a V_n^m then we may choose the displacement vector ε^a to be orthogonal to the curve (C) ($\varepsilon^h c^h = 0$), and since the sum of the squares of the u^h must be, as with cosines, equal to unity, we have the value of μ as:

$$\mu = c^h \frac{\mathrm{D}\varepsilon^h}{d\sigma} + \tfrac{1}{2} v_{hk,l'} c^h c^k \varepsilon^{l'} .$$

Moreover, if one takes the fundamental congruences in such a fashion that the congruence (λ) is tangent to the curve (C) then all of the cosines c^h are null, except for c, which is unity. It then results that the last m – 1 equations (30) no longer contain μ and take on the form:

$$\text{(30′)} \qquad \frac{\mathrm{D}^2\varepsilon^h}{d\sigma^2} = \lambda_{1ra}^{\cdot\, h}\varepsilon^a, \qquad (h = 2, 3, \ldots, m).$$

If the displacement ε^a is orthogonal to the curve (C) ($\varepsilon^1 = 0$) then these equations that are associated with (29‴) constitute a differential system of n – 1 equations in the n – 1 unknowns $\varepsilon_2, \ldots, \varepsilon_n$. One may further simplify equations (30′) by taking the congruences (λ^2), …, (λ^m) to be congruences that are transported by parallelism along (C), because in this case the components $\frac{\mathrm{D}^2\varepsilon^h}{d\sigma^2}$ ($h \geq 2$) of the second vector derivative coincide with the second derivatives $\frac{d^2\varepsilon^h}{d\sigma^2}$. Likewise, one may simplify equations (29‴) by taking the anholonomic congruences to be congruences that are transported by semi-intrinsic parallelism along (C), since the components $\frac{\mathrm{D}\varepsilon^{h'}}{d\sigma}$ in this case are also equal to $\frac{d\varepsilon^{h'}}{d\sigma}$.

25. The equivalence of two anholonomic spaces. – Consider two spaces X_n and X'_n, one of which refers functions of the variables $x^1, \ldots, x^n$ to the congruences (λ), and the other of which refers functions of the variables $x'^1, x'^2, \ldots, x'^n$ to the congruences (λ'). If one carries out the transformation of variables (3) in the space X'_n then the forms ds'^a of the congruences (λ') become linear forms in the variables (x) and in that form they may be expressed linearly with the aid of the forms ds^a of the congruences (λ) of X_n:

$$\text{(31)} \qquad ds'^a = c_b^a\, ds^b,$$

where the c_b^a are suitable functions of the variables (x) whose determinant is non-zero. If one considers the transformation (3) to be unknown then equations (31) constitute a system of total differentials in the unknowns x'^i, as functions of x^i, which may also be written:

$$\text{(32)} \qquad \frac{\partial x'^i}{\partial x^j} = c_b^a \lambda'^i_a \lambda^b_j.$$

As for the integrability conditions for this system, which express the fact that the second derivatives of the x'^i are symmetric, they may be written in the form:

$$\frac{\partial c_b^a}{\partial s^c} - \frac{\partial c_c^b}{\partial s^b} = w'^a_{ef} c_b^e c_c^f - w_{ef}^e c_e^a . \tag{33}$$

One sees that these conditions differ from the fundamental formulas (5) only by the fact that here the w'^a_{ef} are considered to be functions of the variables (x').

Equations (32), (33) constitute a first order system of partial differential equations in the n unknowns x'^i and the n^2 auxiliary unknowns c_b^a, which are considered to be functions of the n variables x^i. As long as one imposes no conditions on the c_b^a, this system obviously possesses all of the transformations (3) as its solutions, because this signifies that the spaces X_n and X'_n are equivalent, the equivalence group being the point group (3).

Now suppose that the c_b^a satisfy the equations $c_k^{h'} = 0$, which express the fact that the Pfaff system $ds'^h = 0$ that is associated with (32) and (33) provides us with intrinsic equations of equivalence for the anholonomic spaces X_n^m and X'^m_n, which are defined in X_n and X'_n by the Pfaff systems $ds^{h'} = 0$ and $ds'^h = 0$. Among the equations (33), we also have, in this case, the relations in finite terms:

$$\bar{w}_{\alpha\beta}^{h'} c_k^\alpha c_l^\beta - \bar{w}_{kl}^\alpha c_\alpha^h = 0, \tag{32$'$}$$

which express the fact that the integrability tensors of our two Pfaff systems are equivalent. If one exhibits the derived systems to these systems then one finds that a necessary condition that the equations of equivalence for X_n^m and X'^m_n have solutions is that the c_b^a must belong to the group that preserves the system (7) and its derived systems ([**3**], § 7). If one also imposes the conditions $c_{k'}^h = 0$ on the c_b^a then obtains the equations of equivalence for the anholonomic spaces X_n^m and X'^m_n, which can be regarded from either the semi-intrinsic or the rigid viewpoint, since the two coincide for the X_n^m. Obviously, in this case the c_b^a must also belong to the group that preserves the derived systems of the system $ds^h = 0$.

It then results from this that if the systems $ds^h = 0$, $ds^{h'} = 0$ have no integrable combinations then our equivalence problem, in accordance with the theorem in section 10, reduces *to the equivalence problem for two complete spaces with affine connection.* One knows that such a problem reduces to the study of *a system of mixed total differentials,* and consequently it may be regarded as completely solved ([**12**], pp. 14). In particular, one knows that the equivalence transformations, if they exist, might only depend upon arbitrary constants.

If our complementary system has integrable combinations then one may no longer assert that the integration of the semi-intrinsic or rigid equivalence equations for X_n^m and X'^m_n reduces to a system of total differentials, in such a fashion that, in this case, and *a fortiori* in the intrinsic case, the equivalence transformations may also depend on arbitrary functions.

One may now pass to the problem of the equivalence of two anholonomic spaces with affine connection A_n^m and $A_n'^m$. If one takes the rigid viewpoint then the equivalence equations are obviously composed of equations (32). Then, in place of equations (33), we have: equations (29′) (where the $\bar{\gamma}_{\alpha\beta}^{\cdot h}$ are replaced by $\gamma_{\alpha\beta}'^{\cdot h}$) and their analogues, equations (9) and their analogues, in which the $\bar{w}$ are replaced with w', and for the relations in finite terms, we have (32′) and their analogues, and finally, the relations in finite terms that is provided by the torsion tensor τ_{kl}^h and its analogue. We thus have a system of mixed total differentials to determine the n unknowns x'^i, m^2 unknowns c_k^h, and the $(n - m)^2$ unknowns $c_{k'}^{h'}$ as functions of the n variables. If one considers the integrability conditions of the equations in c then one finds relations in finite terms that is provided by the curvature tensors $\lambda_{kla}^{\cdot\cdot h}, \lambda_{k'l'a}^{\cdot\cdot h'}$. Naturally, upon deriving these relations in finite terms one finds relations in which the tensor derivatives of our tensors appear. Consequently, one may assert that the only relations in finite terms that must be satisfied for there to be equivalence are the ones that are provided by the four tensors of third order $w_{kl}^{h'}, w_{k'l'}^{h}, \tau_{kl}^h, \tau_{k'l'}^{h'}$, the curvature tensors, and by the tensor derivatives of the six tensors. One may also say that these six tensors and their tensor derivatives constitute *a complete system of invariants for the rigid anholonomic space* A_n^m.

If one takes the semi-intrinsic viewpoint then one must first renounce equations that are analogous to (30′), in such a fashion that the system of equivalence is no longer composed of total differentials. Nevertheless, one knows that it reduces to such a system if the system (7) has no integrable combinations. In any case, the semi-intrinsic tensors $w_{kl}^{h'}, w_{k'l'}^{h}, \tau_{kl}^h, \lambda_{kla}^{\cdot\cdot h}$, and the derived tensors, using the regular semi-intrinsic connection on A_n^m, do not constitute a complete system of semi-intrinsic invariants on A_n^m.

These results on the A_n^m are obviously also valid for the V_n^m, with the sole difference that in this case one must also associate the relations that are provided by the metric tensor $a_{\alpha\beta}$ in the semi-intrinsic case, and also by the tensor $a_{\alpha'\beta'}$ in the rigid case. In the case of V_n^m, one may also take the intrinsic viewpoint, and the relations in finite terms are then $c_k^{h'} = 0$, equations (32), and the ones that are given by the tensor $a_{\alpha\beta}$. However, in the case of the V_n^m one may also simplify the problem by the use of orthogonal congruences. In this case, the intrinsic equivalence equations for V_n^m are (32), (33), $c_k^{h'} = 0$, and (11); to obtain those of semi-intrinsic equivalence, one must associate the $c_{k'}^{h} = 0$ ([**33**], Chap II), and, finally, for rigid equivalence, one must associate (12).

26. The transformation groups of anholonomic spaces. – One arrives at the equations of transformation of an anholonomic space onto itself by supposing that the congruences (λ') are the same functions of the (x') as the congruences (λ) are of the (x). In this case, the equations of equivalence that were considered above always have the identity solution:

$$(33') \qquad x'^i = x^i, \qquad c_b^a = \delta_b^a = \begin{cases} 0 & (a \neq b), \\ 1 & (a = b), \end{cases}$$

and the question amounts to seeing whether these equations also have other solutions besides (33′).

To find the defining equations for the infinitesimal transformations of the transformation group of the space in a neighborhood of the identity transformation one must set

$$(33'') \qquad x'^i = x^i + \xi^i\, dt, \qquad c_b^a = \delta_b^a + \varepsilon_b^a\, \delta t\,,$$

in the equations of equivalence, where the ξ^i, ε_b^a are new unknowns and δt is to be considered as a constant quantity of first order. One will thus find equations and relations in finite terms that are linear in the unknowns ξ^i and ε_b^a.

If we are dealing with the case of a V_n^m in which we have chosen orthogonal congruences then equations (11), the following equations:

$$c_k^{h'} = c_{k'}^{h} = 0\,,$$

and equations (12), which represent the defining equations for rigid anholonomic group for V_n^m, give us, quite simply, the defining equations of the infinitesimal transformations as:

$$(34) \qquad \varepsilon_k^h + \varepsilon_h^k = 0\,, \quad \varepsilon_k^{h'} = 0, \qquad \varepsilon_{k'}^{h} = 0, \qquad \varepsilon_{k'}^{h'} + \varepsilon_{h'}^{k'} = 0\,.$$

Obviously, in order to have defining equations for the semi-intrinsic infinitesimal transformations of V_n^m, one must limit oneself to the first three groups of equations (34), and to have those of the intrinsic case on V_n^m, one must limit oneself to only the first two groups.

Since the unknowns ε_b^a are the auxiliary unknowns of our problem, the principal unknowns being the ξ^i – or rather, the projections $\varepsilon^a = \lambda_i^a \xi^i$ of the vector (ξ) onto the congruences (λ) – one may eliminate them from (34), because we have the formulas:

$$(34') \qquad \varepsilon_b^a = \frac{\partial \varepsilon^a}{\partial s^b} + w_{bc}^a \varepsilon^c\,.$$

Upon taking into account these formulas, one easily verifies that (34), just like the equations relative to the semi-intrinsic or intrinsic V_n^m, have an invariant character with respect to the corresponding group on V_n^m.

Now suppose that our V_n^m possesses the one-parameter transformation group G_1. One may always suppose that this group is generated by the infinitesimal transformation:

$$X_1 f = \frac{\partial f}{\partial x^1}. \tag{34′}$$

In this case, equations (34) become linear homogeneous equations in the derivatives $\frac{\partial \lambda_i^a}{\partial x^1}$, in such a fashion that (34) are satisfied identically if the congruences (λ) do not depend upon x^1 explicitly. Conversely, one may prove that if a rigid, semi-intrinsic, or intrinsic V_n^m, resp., possesses the group (34′) then one may, by a suitable transformation that belongs to the rigid, semi-intrinsic, or intrinsic group, resp., refer the V_n^m to a system of congruences that does not contain the variable x^1 explicitly (Vranceanu [**33**], § 10).

If the congruences (λ) in the space V_n^m have constants for the quantities w_{bc}^a then the equations of transformation have the solution $c_b^a = \delta_b^a$ for any x'^i, which then satisfy the completely integrable system of total differentials (32) ($c_b^a = \delta_b^a$). It then results that in this case V_n^m possesses a simply transitive transformation group. Moreover, this group is the reciprocal of the simply transitive continuous group that is determined in this case by the congruences (λ). One may also prove that *if the transformation group of* V_n^m *possesses a simply transitive subgroup then it may be referred to a system of congruences that has constant rotation coefficients.*

27. Anholonomic hypersurfaces. – An anholonomic space A_n^{n-1} that is defined by just one Pfaff equation that is not completely integrable may be called an *anholonomic hypersurface*. If the connection on A_n preserves volumes, and if the equation $ds = 0$ has its covariant of rank $n - 1$, which can happen only if n is an odd number, then one may always reduce the intrinsic group of A_n^{n-1} to the semi-intrinsic group, which amounts to saying that one may fix an affine normal to A_n^{n-1} in an invariant manner (Schouten [**16**], pp. 299).

If we have an anholonomic hypersurface V_n^{n-2} then one may always, and generally in several different ways, reduce the intrinsic group to a rigid group, which amounts to saying that one may always fix the normal and the metric on that normal. Now consider an anholonomic space V_3^2 that is defined in a three-dimensional Riemann space by a Pfaff equation that is not completely integrable ($w_{12}^3 \neq 0$), a space that one may also call an *anholonomic surface*. The transformation group of such a space may contain four parameters at maximum ([**33**], §14).

If the space V_3^2 admits a G_4 then this G_4 must contain a simply transitive G_3 because, on the one hand, one may prove that a V_3^2, properly speaking ($w_{12}^3 \neq 0$), might not have an intransitive G_3, and, on the other hand, one knows that a G_4 always possesses a G_3. It then results that the V_3^2 that possess a G_4 may be referred to a system of congruences with constant rotation coefficients. Having done so, equations (32′), which, due to the orthogonality of the c_k^h $(h, k = 1, 2)$ become $\bar{w}_{12}^3 = c_3^3 w_{12}^3$, tell us that $c_3^3 = 1$, or rather, that the rigid and semi-intrinsic transformation groups of our V_3^2 coincide. One also proves that the V_3^2 ($w_{12}^3 \neq 0$) that possess a G_4 are totally geodesic and the exterior curvature tensor, just like the tensor derivative of the interior curvature tensor, is null. As for the interior curvature tensor itself, it has only one component $\lambda_{12,12} = K$, which might be a positive, negative, or null constant. If we write down the equation of anholonomity for V_3^2 in the form:

$$dx^3 + u\, dx^1 = 0 ,$$

where u is a functions of the single variable x^2, which is always possible ([**6**], pp. 39), one may give the metric on V_3^2 the form:

$$ds^2 = \left(\frac{du}{ds}\right)^2 (dx^1)^2 + (dx^2)^2 ,$$

in which the function u has the values x^2, $\frac{1}{\sqrt{k}} \sin \sqrt{k}\, x^2$, $\frac{1}{\sqrt{-k}} e^{\sqrt{-k}\, x^2}$ according to whether our anholonomic surface has curvature that is null, positive, or negative, resp. One sees that in the space of variables x^1, x^2 the metric on V_3^2 is the metric of a surface of constant curvature.

28. Anholonomic planes. – One knows that the curves in ordinary space that have the lines of a linear complex for their tangents satisfy a Pfaff equation that is not completely integrable, which, if one takes the axis of the complex to be the z-axis, may be written:

$$x\, dy - y\, dx - k\, dz = 0 ,$$

where x, y, z are orthogonal Cartesian coordinates and the constant k is the parameter of the complex. This equation, which is also called the *equation of the complex*, defines one or more simple anholonomic spaces V_3^2 in Euclidian space. Indeed, if one takes cylindrical coordinates then one may take the forms on V_3^2 to be the following ones:

$$(34'') \qquad \begin{cases} ds^1 = d\rho, \quad ds^2 = \rho_1(k\rho d\theta + \rho dz), \quad ds^3 = \rho_1(-\rho^2 d\theta + kdz) \\ \qquad\qquad \left(\rho_1 = \dfrac{1}{\sqrt{\rho^2 + k^2}}\right). \end{cases}$$

It then first results that V_3^2 is totally geodesic because the second fundamental form is identically null:

$$(\gamma_{11}^3 = \gamma_{22}^3 = \gamma_{12}^3 + \gamma_{21}^3 = 0),$$

which is obviously true *a priori*, since V_3^2 contains the lines of the complex at each point. This property is characteristic, in such a fashion that, by analogy with the two-dimensional planes that constitute the totally geodesic surfaces in ordinary space, one may call the V_3^2 that are defined by linear complexes *anholonomic planes* (G. Moisil [**25**], pp. 17).

Since the forms (34″) do not contain the variables θ and z explicitly it then results that V_3^2 admits a transformation group that takes the form of the Abelian group:

$$\text{(34″)} \qquad X_1 = \frac{\partial f}{\partial \theta}, \qquad X_2 = \frac{\partial f}{\partial z},$$

which is composed of a rotation around the axis of the complex and a translation around the same axis. This group constitutes the total semi-intrinsic or rigid transformation group of V_3^2. Indeed, on the one hand, a V_3^2, properly speaking ($w_{12}^3 \neq 0$), might not have an intransitive three-parameter group. On the other hand, our V_3^2 might not have a simply transitive group, because the interior curvature, which has, in this case, just the one component:

$$\lambda_{12,\,12} = \tfrac{1}{2} k^2 \rho_1^2,$$

must, by the transformation conditions, remain invariant, which is true only if ρ invariant.

Upon appealing to the well known properties of linear complexes, one may give the parallelism on our V_3^2 an interesting geometric interpretation (D. Hulubei [**26**]).

CHAPTER V.

ANHOLONOMIC MECHANICAL SYSTEMS.

29. Systems with time-independent constraints. – Consider a holonomic mechanical system S_n with time-independent constraints, and let:

$$T = \frac{1}{2} a_{ij} \dot{x}^i \dot{x}^j \qquad\qquad \left(\dot{x}^i = \frac{dx^i}{dt}\right),$$

be the vis viva of the system, in which t is time and the a_{ij} are functions of the Lagrangian parameters $x^1, x^2, \ldots, x^n$ that the positions in the system S_n depend upon. One may, as is well known (Ricci and Levi-Civita [**5**]), associate the holonomic system S_n with the Riemann space V_n that is defined in the space of variables $x^1, x^2, \ldots, x^n$ by the metric:

$$ds^2 = 2T\, dt^2 = a_{ij}\, dx^i\, dx^j .$$

If we introduce a system of orthogonal congruences (λ) into V_n then we will have the formulas:

$$\text{(35)} \qquad \begin{cases} \delta x^i = \lambda_a^i \delta s^a, \qquad \delta s^a = \lambda_i^a \delta x^i, \\ \dfrac{dx^i}{dt} = \lambda_a^i u^a, \quad u^a = \dfrac{ds^a}{dt} = \lambda_i^a \dfrac{dx^i}{dt}, \\ T = \frac{1}{2}[(u^1)^2 + (u^1)^2 + \cdots + (u^n)^2], \end{cases}$$

in which the δx^i denote virtual displacements of S_n . As for the u^a, one calls them the *kinetic characteristics of the motion* (Volterra [**3**[1]]).

Having said this, if, in the symbolic equation of dynamics for the system S_n :

$$\left(\frac{d}{dt}\frac{\partial T}{\partial \dot{x}^i} - \frac{\partial T}{\partial x^i} - P_i\right)\delta x^i = 0 ,$$

where P_i is the component in the x^i direction of the resultant of the forces that are applied directly, one takes into account formulas (35) and the following formulas:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}^i} = \frac{d}{dt}\left(\frac{\partial T}{\partial u^a}\lambda_i^a\right) = \frac{du^a}{dt}\lambda_i^a + u^b \frac{\partial \lambda_i^b}{\partial x^j}\lambda_c^j u^c ,$$

$$\frac{dT}{dx^i} = \frac{\partial T}{\partial u^a}\frac{\partial u^a}{\partial x^i} = u^b \frac{\partial \lambda_j^b}{\partial x^i}\lambda_c^j u^c ,$$

then it may be written:

$$\left(\frac{du^a}{dt} - w^b_{ca}u^b u^c - \mathrm{P}_a\right)\delta s^a = 0, \tag{36}$$

in which P_a denotes the component of the force in the direction of the congruence (λ_a). As for the w^b_{ca}, they are defined by formula (4′).

Since this symbolic equation must be true for any increments δs^a, it then results, upon also taking into account that $w^b_{ca} + w^c_{ba} = \gamma^a_{bc} + \gamma^a_{cb}$, that one may write the equations of motion in S_n in the form:

$$\begin{cases} \dfrac{dx^i}{dt} = \lambda^i_a u^a, \\ \dfrac{du^a}{dt} = \gamma^a_{bc}u^b u^c + \mathrm{P}_a, \end{cases} \tag{37}$$

One sees that the system (37) constitutes a first order differential system in normal form for the n unknowns x^i and the n unknowns u^a, which are to be determined as functions of t.

Now consider the anholonomic mechanical system S^m_n that is obtained from S_n by imposing $n - m$ constraints of the form (7) on the variables (x). One may always consider the left-hand sides $ds^{h'}$ of these constraints to be the differentials of the arc lengths of $n - m$ orthogonal congruences in the Riemann space V_n that is associated with S_n, and for this to be true, one need only combine (7), after multiplying them by suitable factors. One may also associate the $ds^{h'}$ with m other forms ds^h in such a fashion that the congruences (λ_h) and ($\lambda_{h'}$) are orthogonal congruences in V_n . Consequently, to each anholonomic system S^m_n one may associate the anholonomic space V^m_n that is defined in V_n by the system (7) of equations of anholonomity in S^m_n, which may be written:

$$\left(\frac{du^h}{dt} - \gamma^h_{kl}u^k u^l - \mathrm{P}_h\right)\delta s^h = 0 \qquad (h \leq m), \tag{37′}$$

since the increments $\delta s^{h'}$ and the characteristics $u^{h'}$ are null by virtue of the constraints (7).

Since this symbolic equation in S^m_n must be true for any δs^h, it then results that the equations of motion of S^m_n may be written in the form:

$$\begin{cases} \dfrac{dx^i}{dt} = \lambda^i_h u^h, \\ \dfrac{du^h}{dt} = \gamma^h_{kl}u^k u^l + \mathrm{P}_h. \end{cases} \tag{38}$$

They constitute a differential system of first order in normal form for the n unknowns x^i and the m kinetic characteristics u^h (Vranceanu [**11**], Horak [**14**]).

If the force is derived from a potential U ($\mathrm{P}_i = \frac{\partial \mathrm{U}}{\partial x^i}$, $\mathrm{P}_a = \frac{\partial \mathrm{U}}{\partial s^a}$) then the system S_n admits the vis viva integral. In order to find this integral by starting with equations (37), one must multiply the latter equations with u^a and sum, upon taking into account the fact that the Ricci rotation coefficients γ^a_{bc} are skew-symmetric in the indices a and b. One finds the integral:

$$\mathrm{T} = \tfrac{1}{2}[(u^1)^2 + (u^2)^2 + \ldots + (u^n)^2] = \mathrm{U} + \text{const.}$$

If one passes to the anholonomic system S^m_n, then the vis viva integral obviously takes the form:

$$\tfrac{1}{2}[(u^1)^2 + (u^2)^2 + \ldots + (u^n)^2] = \mathrm{U} + \text{const.}$$

Now suppose that the system S_n is unforced ($\mathrm{P}_i = 0$). In this case, the vis viva integral tells us that T is constant and that one may conveniently choose the unit of time in such a fashion that one has $ds = dt$. Having done so, one may verify that the unforced trajectories of S_n are, at the same time, the geodesics (6‴) of the associated space V_n.

In an analogous manner, one easily sees that *the unforced trajectories of an anholonomic system* S^m_n *are also the auto-parallel geodesics* (18) *of the anholonomic space* V^m_n *associated with the system* S^m_n.

30. Systems with independent characteristics. – The integration of the equations of motion (37) in S_n or (38) in S^m_n (unforced) decompose into two subsets if the coefficients of the latter of these equations, $\gamma^a_{bc} + \gamma^a_{cb}$ or $\gamma^h_{kl} + \gamma^h_{lk}$, which are generally functions x^1, x^2, …, x^n, do not depend upon these variables. These systems were studied by V. Volterra ([**3**[1]], 1898) and were called *independent characteristics*, since in order to obtain the values of the kinetic characteristics of S_n or S^m_n as functions of time it suffices to integrate only the last of equations (37) or (38).

Ultimately, upon taking into account the values that were found for these characteristics as functions of time, the integration of the first of equations (37) or (38) provides us with the values of the parameters x^i as functions of time.

We do not have, moreover, a geometric characterization of systems with independent characteristics, but we do know an important class of these systems. They are the mechanical systems such that one may choose a system of congruences (λ) that has constant rotation coefficients.

These mechanical systems are characterized by the property that the space V_n or V^m_n that is associated with them possesses a simply transitive group of transformations (§ 26).

We also have another class of mechanical systems that may be regarded as a generalization of the class of systems with independent characteristics. That class is mechanically remarkable because many of the usual anholonomic mechanical systems are included in that class.

Suppose that the force on S_n^m derives from a potential function of x^1 and that in the space V_n^m that is associated with S_n^m one may choose a system of congruences (λ) such that the parameters and moments of the fundamental congruences are functions of only the position coordinate x^1, and that, moreover, the direction of x^1 may be chosen to be the direction of a fundamental congruence – for example, (λ_1). Having said this, the first of equations (38) may be written:

$$(37'') \qquad \begin{cases} \dfrac{dx^1}{dt} = au^1, \\ \dfrac{dx^i}{dt} = \lambda_h^i u^h, \quad (i = 2, \cdots, n), \end{cases}$$

a and λ_h^i being functions of x^1.

In these formulas, λ_1^i may be regarded as null, because otherwise one may reduce it to zero by a transformation $x'^i = x^i + f^i(x^1)$. In this case, the w_{kl}^h $(h \geq 2)$ are null if k, l are both different from unity, and likewise $w_{kl}^h = 0$, in such a fashion that the last $m - 1$ equations in (38) assume the form:

$$(38') \qquad \frac{du^h}{dt} = (\gamma_{k1}^h + \gamma_{1k}^h) u^k u^1 \qquad (h, k = 2, 3, \ldots, m).$$

If x^1 is not constant during some interval of time ($u^1 \neq 0$) then one may divide these equations by $\dfrac{dx^1}{dt}$ and obtain the system:

$$(39) \qquad \frac{du^h}{dx^1} = \frac{1}{a} (\gamma_{k1}^h + \gamma_{1k}^h) u^k \qquad (h, k = 2, 3, \ldots, m).$$

One sees that by integrating this homogeneous linear differential system one may obtain the values of the $m - 1$ kinetic characteristics u^h $(h \geq 2)$ as functions of the variable x^1.

From the vis viva integral, which may written:

$$\frac{1}{a^2}\left(\frac{dx^1}{dt}\right)^2 + (u^2)^2 + \ldots + (u^m)^2 = 2\,\mathrm{U}(x^1) + \text{const.},$$

one deduces the value of x^1 as a function of time by a single quadrature. Finally, by introducing the values of x^1 and the u^h as functions of time into the last of equations (37″) one finds the values of the variables $x^2, \ldots, x^n$ by $n - 1$ quadratures.

It then results that the integration of the equations of motion of our system S_n^m reduce to the integration of the linear system (39) and to n quadratures.

We have left aside the case where x^1 is constant, which corresponds to the stationary solution of S_n^m:

$$x^1 = c^1, \qquad u^h = c^h, \qquad (h \geq 2);$$

one may study the complete stability of this system (and not just in the first approximation) with the aid of the vis viva integral (Vranceanu [**8**[1]]).

31. Linear first integrals. – One may now demand to know the conditions under which the equations of motion (38) in S_n^m admit the first integral:

$$f(x^1, x^2, \ldots, x^n; u^1, u^2, \ldots, u^m) = \text{const.}$$

Since the derivative of that integral with respect to time must be null, by virtue of (38), one finds that the function f must satisfy the partial differential equation:

$$\frac{\partial f}{\partial s^h} u^h + \frac{\partial f}{\partial u^h} \gamma_{kl}^h u^k u^l + \frac{\partial f}{\partial u^h} \mathrm{P}_h = 0 . \tag{39′}$$

If the function f is a polynomial of degree p in the kinetic characteristics then equation (39′) decomposes into $p + 2$ equations upon first equating to zero the set of terms of degree $p + 1$, then the terms of degree p, etc. In particular, when the set of terms of degree $p + 1$ in the u^h are equated to zero, that will give us the equation:

$$\frac{\partial f^p}{\partial s^h} u^h + \frac{\partial f^p}{\partial u^h} \gamma_{kl}^h u^k u^l = 0 ,$$

where f^p denotes the set of terms of degree p in the first integral f = const. This latter equation shows us that f^p = const. must be the first integral of the unforced equations of motion in S_n^m, or rather, the auto-parallel geodesics of the anholonomic space V_n^m that is associated with S_n^m. Here, we consider only the case where the integral is linear. Such an integral, by a convenient transformation of the fundamental congruences, may be written:

$$a u^m = \text{const.}, \tag{39″}$$

in which a is a functions of the variables $x^1, x^2, \ldots, x^m$. In order for (39″) to be a first integral of equations (38), it is first necessary that the equation $u^m = 0$ that is obtained from (39″) by giving the constant the value zero be an invariant equation of (38), and for this to be true, one must satisfy the following conditions:

$$\gamma_{kl}^m + \gamma_{lk}^m = 0 \qquad (k, l = 1, 2, \ldots, m - 1).$$

They express the fact that the anholonomic space V_n^{m-1} ($u^m = 0$), which is embedded in V_n^m, is totally geodesic in V_n^m.

If that invariance condition is satisfied and if the quantity u is constant then in order for u^m = const. to be a first integral of the geodesics in V_n^m it is necessary that the coefficients of rotation γ_{hm}^m be null, or, what amounts to the same thing, that the congruence (λ_m) be a geodesic congruence in V_n^m. If a is not constant then we have the conditions:

$$\frac{\partial \log a}{\partial s^h} = -\gamma_{hm}^m,$$

and the congruence $\bar{\lambda}_m$ ($d\bar{s}^m = a\, ds^m$) is then a geodesic congruence ($\bar{\gamma}_{hm}^m = 0$). One may remark that this congruence $\bar{\lambda}_m$ defines one of the spaces V_n^1 that is complementary to the space V_n^{m-1} ($u^m = 0$) and defined semi-intrinsically in V_n^m.

Now suppose that we have a certain number $m - p$ of homogeneous linear first integrals of the form:

$$a_h^\alpha u^h = c^2 \qquad (a = p + 1, \ldots, m). \tag{40}$$

It is obvious that one may arrange this in such a fashion that only the cosines $u^{p+1}, \ldots, u^m$ appear in these equations. It then results that the equations $u^{p+1} = \ldots = u^m = 0$ are invariant equations, and consequently that the anholonomic space V_n^p defined in V_n^m by these equations is totally geodesic in V_n; i.e., that we have:

$$\gamma_{kl}^\alpha + \gamma_{lk}^\alpha = 0 \qquad (\alpha = p + 1, \ldots, m, k, l \leq p).$$

Obviously, these invariance conditions are not sufficient for the existence of first integrals (40). In particular, if one desires that the $u^\alpha = c^\alpha$ ($\alpha = p + 1, \ldots, m$) be the first integrals then one must satisfy the conditions:

$$\gamma_{h\beta}^\alpha + \gamma_{\beta h}^\alpha = 0 \quad (\beta > p).$$

32. The equations of trajectories of S_n^m. – From a result of Painlevé ([**7**], vol. II[1], pp. 414), one knows that the totality of the trajectories of a holonomic mechanical system with time-independent constraints depends upon $2n - 1$ arbitrary constants instead of $2n$, and if the system is unforced then only $2n - 2$ constants are arbitrary. We shall see that this result may be also extended to anholonomic systems. Indeed, suppose that during the motion of the system one of these variables – for example, x^1 – is not constant ($dx^1 \neq 0$), and that one of the characteristics, which one may suppose to be u^1, by a suitable change of indices, is different from zero. In that case, if one takes the new independent variable to be x^1 instead of t, and one sets:

$$u^h = \frac{ds^h}{dt} - \frac{ds^h}{ds^1}\frac{ds^1}{dt} = v^h\, u^1 \qquad (h = 2, \ldots, m)$$

then one may consider u^1, v^2, …, v^m to be the new characteristics and the first of equations (38) may be written:

$$(40') \qquad \begin{cases} \dfrac{dx^1}{dt} = (\lambda_1^i + \lambda_1^i v^h), \\ \dfrac{dx^i}{dx^1} = \dfrac{\lambda_1^i + \lambda_1^i v^h}{\lambda_1^1 + \lambda_1^1 v^h}, \qquad (i = 2, \cdots, n). \end{cases}$$

If we now take into account the fact that we have:

$$\frac{du^h}{dt} = \frac{dv^h}{dt}u^1 + v^h\frac{du^1}{dt} = \frac{dv^h}{dt}(u^1)^2(\lambda_1^1 + \lambda_h^1 v^h) + v^h\frac{du^1}{dt} \;\; ,$$

then the last of equations (38) take on the form:

$$(40'') \quad \begin{cases} \dfrac{du^1}{dx^1} = \dfrac{(\gamma_{k1}^1 v^k + \gamma_{kl}^1 v^k v^l)(u^1)^2 + \mathrm{P}_1}{(\lambda_1^i + \lambda_1^i v^h)u^1}, \\ \dfrac{dv^h}{dx^1} = \dfrac{[\gamma_{11}^h + (\gamma_{1k}^h + \gamma_{k1}^h)v^k + \gamma_{kl}^h v^k v^l](u^1)^2 + \mathrm{P}_1 - v^h(\gamma_{k1}^1 v^k + \gamma_{kl}^1 v^k v^l)(u^1)^2 - v^h \mathrm{P}_1}{(\lambda_1^1 + \lambda_1^1 v^h)(u^1)^2}. \end{cases}$$

Upon taking into account the fact that time does not appear explicitly in our equations one sees that equations (40′), except for the first one, and equations (40″) constitute a differential system in normal form, namely, $n + m - 1$ first order equations for the $n + m - 1$ unknowns x^2, …, x^n, u^1, v^2, …, v^m, and the independent variable x^1. It then results from this that this system provides us with the values of the variables x^2, …, x^n as functions of the variable x^1 and $n + m - 1$ arbitrary constants. It constitutes the differential system of the trajectories in S_n^m.

If the mechanical system is unforced then the characteristic u^1 does not appear in the last $m - 1$ of equation (40″), in such a fashion that, in this case, one may consider the equations of the trajectories in S_n^m to be the system of $n + m - 2$ first order equations that is formed from the last $n - 1$ equations in (40′) and the last $m - 1$ equations in (40″). It then results from this that if the system is unforced then the trajectories depend only upon the $n + m - 2$ arbitrary constants.

One may remark that the method that we followed in order to arrive at the equations of the trajectories may be simplified if the system is holonomic. Indeed, in this case, one may consider the congruence (λ^1) in the direction of the variable x^1 ($dx^1 = \lambda_1^1 ds^1$), which implies as a consequence the fact that the characteristics v^h no longer appear in the denominators of equations (40′), (40″). In the anholonomic case, this simplification is possible only if the system $ds^h = 0$ admits an integrable combination, which one may then take to be dx^1. If this is not true, and if the system is unforced then the right-hand sides of the last $m - 1$ equations (40″) are polynomials of third degree in the characteristics $v^2, \dots, v^m$. In order for them to be polynomials of only second order it is necessary that $\gamma_{kl}^1 + \gamma_{lk}^1 = 0$, or rather, that the integrable combination ($u^1 = 0$) be an invariant equation of the equations of motion in S_n^m.

33. Trigonometric stability of equilibrium. – Suppose that our mechanical system possesses an equilibrium point that is chosen to be the origin of the coordinates x^i. In this case, the equations of motion (38) of the system must possess the solution $x^i = u^h = 0$, in such a fashion that upon developing the right-hand side of equations (38) into a series around the point $x^i = u^h = 0$ the constant terms of this series are null. In order to simplify the terms of the first order in the first of equations (38), one may suppose that the first m coordinates x^i are chosen to be tangent to the origin of the m fundamental congruences in such a fashion that, abstracting from the terms of order higher than the first, equations (38) may be written:

$$\left\{\begin{aligned} \frac{dx^\alpha}{dt} &= a^\alpha, \\ \frac{dx^{\alpha'}}{dt} &= 0, \\ \frac{du^\alpha}{dt} &= a_\beta^\alpha x^\beta + a_{\alpha'}^\alpha x^{\alpha'}. \end{aligned}\right. \tag{41}$$

In these equations, one may further suppose that the coefficients $a_\beta^\alpha = 0$ ($\beta > \alpha$) are null, because otherwise one would reduce them to zero by an orthogonal transformation with constant coefficients of the fundamental congruences and the coordinates x^α. Having said this, one sees that the characteristic equation of our equilibrium point always possesses $n - m$ null roots and m roots that are equal to $\pm\sqrt{a_\alpha^\alpha}$. One sees that the equilibrium point is stable in the first approximation if the non-null roots are all pure imaginary; i.e., in our case, if all of the quantities a_α^α are negative or null. If this happens to be the case then equations (41) can be integrated with the aid of linear polynomials in the sines and cosines of $\sqrt{a_\alpha^\alpha}\,t$, or, if one so desires, with the aid of a trigonometric series. Poincaré has shown that if a mechanical system is holonomic and conservative, and the equilibrium point is stable in the first approximation ($a_\alpha^\alpha = -r_\alpha^2$) then one may, at least formally, satisfy equations (37) by means of trigonometric series if none of the roots $\pm\sqrt{-1}\,r_\alpha$ of the characteristic equation is null and if the r_α satisfy no commensurability relation $p_1 r_1 + \dots + p_m r_m = 0$, in which the p's are integers.

This property was taken by G. Birkhoff ([**12**[1]], pp. 106, 113) as the definition of trigonometric stability of equilibrium by showing that, in a certain sense, *this property is characteristic of conservative (Hamiltonian) holonomic systems.*

The proof of this assertion amounts to showing that this property is common to all mechanical systems with time-independent constraints (Vranceanu [**13**]). Indeed, if one supposes that none of the r_α is null and that they are all different then one can, by a transformation of the form:

$$\begin{aligned} \bar{x}^\alpha &= x^\alpha + c^\alpha_\beta x^\beta + c^\alpha_{\beta'} x^{\beta'} \\ \bar{u}^\alpha &= u^\alpha + c^\alpha_\beta u^\beta \end{aligned} \quad (\beta < \alpha),$$

annul all of the coefficients $a^\alpha_\beta\ (\beta \neq \alpha)$, $a^\alpha_{\beta'}$, and we remark that these transformations preserve the property, which is fundamental for us, that the right-hand side of the first n of equations (38) are odd functions of the characteristics u^α and the right-hand sides of the last m of equations (38) are even functions of the same variables u^α.

Having said this, in place of the variables x^α, u^α, consider the conjugate imaginary variables $\bar{x}^\alpha$, $\bar{u}^\alpha$:

$$(41') \qquad \begin{cases} x^\alpha = \bar{x}^\alpha + \bar{u}^\alpha \\ u^\alpha = \sqrt{-1}\, r_\alpha (\bar{x}^\alpha - \bar{u}^\alpha) \quad (a \text{ fixed}). \end{cases}$$

When the derivatives $\dfrac{dx^i}{dt}$ are expressed with the aid of the variables $\bar{x}^\alpha$, $\bar{u}^\alpha$, $x^{\alpha'}$, they will have only pure imaginary coefficients, because they are odd functions of the u^α, and it is only the last of (41′) that introduces the imaginary numbers. It then results from this that the derivatives $\dfrac{dx^{\alpha'}}{dt}$ might not have terms of the form:

$$(41'') \qquad \mathrm{A}(\bar{x}^1\bar{u}^1)^{\alpha_1} \cdots (\bar{x}^m\bar{u}^m)^{\alpha_m} (x^{m+1})^{\alpha_{m+1}} \cdots (x^n)^{\alpha_n}.$$

Indeed, on the one hand, A must be pure imaginary, and, on the other hand, nothing changes if one changes $\sqrt{-1}$ into $-\sqrt{-1}$, because $\dfrac{dx^{\alpha'}}{dt}$ is real; i.e., A = 0.

Since the derivatives $\dfrac{du^\alpha}{dt}$ are even functions of the u^α, they will have real coefficients in the variables $\bar{x}^\alpha$, $\bar{u}^\alpha$, $x^{\alpha'}$. Consequently, the variables $\bar{x}^\alpha$, $\bar{u}^\alpha$, $x^{\alpha'}$ satisfy the differential system:

$$(42) \qquad \begin{cases} \dfrac{d\bar{x}^\alpha}{dt} = \sqrt{-1}\, r_\alpha \bar{x}^\alpha + \cdots \\ \dfrac{d\bar{u}^\alpha}{dt} = -\sqrt{-1}\, r_\alpha \bar{u}^\alpha + \cdots \\ \dfrac{dx^{\alpha'}}{dt} = 0 + \cdots, \end{cases}$$

where the unwritten terms are of at least second order with respect to the variables $\bar{x}^\alpha, \bar{u}^\alpha, x^{\alpha'}$, with pure imaginary coefficients. It is completely obvious that this system is changed into a system that acts under the same force by the transformation:

$$(42') \qquad \bar{x}^\alpha = y^\alpha + \mathrm{F}^\alpha, \qquad \bar{u}^\alpha = v^\alpha + \mathrm{G}^\alpha, \qquad x^{\alpha'} = y^{\alpha'} + \mathrm{H}^{\alpha'},$$

the F^α, G^α, $\mathrm{H}^{\alpha'}$ being polynomials with real coefficients in the variables y^α, v^α, $y^{\alpha'}$ that are of at least second order in these variables with F^α and G^α being conjugate, i.e., they change into each other when one exchanges y^α with v^α.

If F^α, G^α, $\mathrm{H}^{\alpha'}$ are homogeneous polynomials of second order then the terms of order two that are introduced into the first of equations (42) are given by the formula:

$$\sqrt{-1}\left(r_\alpha \mathrm{F}^\alpha - \frac{\partial \mathrm{F}^\alpha}{\partial y^\beta} r^\beta y^\beta + \frac{\partial \mathrm{F}^\alpha}{\partial v^\beta} r^\alpha y^\beta\right) \qquad (\alpha \text{ fixed}).$$

Now, one sees without difficulty that if the r_α are not mutually commensurable then the only terms in these expressions that are null for any F^α are of the form $\mathrm{A}y^\alpha\varphi$, φ being a function of the pairs $y^\alpha v^\alpha$ and the $y^{\alpha'}$.

For the second of equations (42), y^α is changed into v^α, and for the third one it is the terms (41″) that may not be introduced by the transformations (42′).

It then results, upon taking into account the fact that the $\dfrac{dx^{\alpha'}}{dt}$ are not terms of the form (41″), that upon performing the transformations (42′), where F^α, G^α, $\mathrm{H}^{\alpha'}$ are polynomials of second order, then third order, etc., one may use the coefficients of these transformations to give our equations the form:

$$(43) \qquad \begin{cases} \dfrac{dy^\alpha}{dt} = \mathrm{M}_\alpha y^\alpha, \\[2mm] \dfrac{dv^\alpha}{dt} = \mathrm{N}_\alpha y^\alpha, \\[2mm] \dfrac{dy^{\alpha'}}{dt} = 0, \end{cases}$$

M_α and N_α being functions of the pairs $y^\alpha v^\alpha$ and the $y^{\alpha'}$.

Obviously, one knows nothing about the convergence of the series that is obtained by taking the product of all of the transformations (42″) that have been considered up till now. Since the functions M_α and N_α are, at the same time, conjugate and have pure imaginary coefficients one must have $\mathrm{M}_\alpha = -\mathrm{N}_\alpha$, in such a fashion that the equations (43) have the first integrals $y^\alpha v^\alpha = y^{\alpha'} = \text{const.}$ Consequently, the functions M_α and N_α become constants themselves, and the equations integrate into trigonometric expressions of the form:

(44) $$y^{\alpha} = y_0^{\alpha} e^{\mathrm{M}_{\alpha}^0 (t-t_0)}, \qquad v^{\alpha} = v_0^{\alpha} e^{-\mathrm{M}_{\alpha}^0 (t-t_0)}, \qquad y^{\alpha'} = y_0^{\alpha'} .$$

We only have to introduce these values into the series that expresses the x^{α}, u^{α}, $x^{\alpha'}$ as functions of the y^{α}, v^{α}, $y^{\alpha'}$ to obtain the values of the x^{α}, u^{α}, $x^{\alpha'}$ as formal developments in trigonometric series.

34. Systems with time-dependent constraints.- Consider a holonomic system S_n with time-dependent constraints, and let:

(45) $$\mathrm{T} = \mathrm{T}_2 + \mathrm{T}_1 + \mathrm{T}_0 ,$$

be the vis viva of the system, where T_2 is a quadratic form that is positive definite in the derivatives $\frac{dx^i}{dt}$, T_1 is a linear form, and T_0 is independent of these derivatives:

$$\mathrm{T}_2 = \tfrac{1}{2} a_{ij} \frac{dx^i}{dt} \frac{dx^j}{dt}, \qquad \mathrm{T}_2 = \alpha_i \frac{dx^i}{dt}, \qquad \mathrm{T}_0 = \tfrac{1}{2} \alpha_0 ,$$

the coefficients a_{ij}, α_i, α_0 generally being functions of the variables $x^1, x^2, \dots, x^n$ and t.

Since the form T_2 is positive definite one may associate the system S_n with the family of Riemann spaces V_n that have the metric:

(46) $$ds^2 = 2\, \mathrm{T}_2\, dt^2 = a_{ij}\, dx^i\, dx^j .$$

By the fact that one obtains the various spaces V_n of this family by taking constant values at the time t, one may say that the family of V_n is *virtually associated* with S_n . Obviously, one may, with the aid of the n forms $ds^a = \lambda_i^a dx^i$, reduce the metric (46) to a sum of n squares, and one may define an anholonomic system S_n^m in S_n by equations of anholonomity of the form:

(47) $$ds^{h'} = v^{h'} dt, \qquad (h' = m + 1, \dots, n).$$

If one takes the characteristics of the motion in S_n^m to be the quantities $u^h = \frac{ds^h}{dt}$ then one may arrive at the equations of motion in S_n^m by a method that is analogous to the one that was followed in the case of time-independent constraints. It is clear that these equations will be invariant under the transformations of the variables (3).

We also have equations of motion in S_n^m that are invariant under the transformations (3), which contain time as a parameter (Wundheiler [**31**]). Indeed, consider the family of the virtual V_n , embedded in the Riemann space V_{n+1} whose metric is provided by the total vis viva in S_n:

$$d\sigma^2 = 2\mathrm{T}\, dt^2 = a_{ij}\, dx^i\, dx^j + 2\alpha_i\, dx^i\, dt + \alpha_0\, dt^2.$$

This metric may be reduced to the form:

$$(48) \qquad d\sigma^2 = (d\sigma^1)^2 + (d\sigma^2)^2 + \ldots + (d\sigma^n)^2 + \lambda^2\, dt^2$$

if one sets:

$$d\sigma^a = ds^a + \alpha^a\, dt \qquad (\alpha^a = \lambda_i^\alpha \alpha^i = \lambda_a^i \alpha_i),$$
$$\lambda^2 = \alpha^0 - \alpha^2,$$

where α is the length of the vector α_i under the metric (46). One sees that the metric (48) on the space V_{n+1} is itself positive definite if the quantity $\lambda^2 = \alpha_0 - \alpha^2$ is positive. If λ^2 is not positive then one then one may add a suitable constant to it, because this amounts to adding a constant to the vis viva of the system, which obviously does not change the equations of motion of the system. Moreover, the Lagrange equations show us that the equations of motion on S_n do not change if one adds the derivative $\dfrac{d\varphi}{dt}$ of an arbitrary function of the variables $x^1, x^2, \ldots, x^n$ and t to the vis viva.

From this, it results that one may always associate a system S_n with a Riemann space V_{n+1} with a positive definite metric (48) that is defined by abstracting from a term of the form $d\varphi\, dt$, and in which the family of virtual V_n is defined by the completely integrable Pfaff system $ds_{n+1} = \lambda\, dt = 0$. One sees that this family constitutes an anholonomic space V_{n+1}^m in V_{n+1} whose arc lengths along the fundamental congruences are $\sigma^1, \ldots, \sigma^n$, and whose anholonomic congruence is σ^{n+1} (Vranceanu [**34**]). Obviously, the kinetic characteristics $v^a = \dfrac{d\sigma^a}{dt}$ are invariants of the transformation of variables (3), which also contains time as a parameter. In order to arrive at the equations of motion of the mechanical system S_n that has these characteristics one must set $\sigma^{n+1} = 0$ and $v^{n+1} = \lambda$ in the symbolic equation (36) of the Riemann space V_{n+1} that is associated with the system S_n.

Since we also have:

$$\lambda w_{n+1a}^{n+1} = \frac{\partial \lambda}{\partial \sigma^a}, \qquad w_{ca}^{n+1} = 0,$$

one thus obtains the equations:

$$(49) \qquad \begin{cases} \dfrac{dx^i}{dt} = \lambda_a^i v^a - \alpha^i, \\[2ex] \dfrac{dv^a}{dt} = w_{ca}^b v^b v^c + w_{n+1a}^b v^b \lambda + \lambda \dfrac{\partial \lambda}{\partial \sigma^a} + \mathrm{P}_a, \end{cases}$$

where the quantities w are defined relative to the $n + 1$ forms $d\sigma^1, \ldots, d\sigma^n, d\sigma^{n+1} = \lambda\, dt$ in the $n + 1$ variables $x^1, \ldots, x^n, t$.

In order to exhibit the invariant character of these equations under the rigid group of the virtual anholonomic space V_{n+1}^n that is associated to S_n, we remark that one may replace the w_{ca}^b with γ_{bc}^a, and that the covariant differential of the interior vector v^a of V_{n+1}^n, which is constructed with the aid of the rigid connection (28) may be written:

$$\mathrm{D}v^a = dv^a - \gamma_{bc}^a v^b \sigma^c - \gamma_{m+1b}^a v^b \lambda\, dt\,,$$

in such a fashion that one may give the last of equations (49) the form:

$$(49') \qquad \frac{\mathrm{D}v^a}{dt} + \gamma_{ba}^{n+1} v^b \lambda = \lambda \frac{d\lambda}{d\sigma^a} + \mathrm{P}_a\,.$$

Since $2\gamma_{ba}^{n+1} = 2\gamma_{ab}^{n+1}$ are the components of the second fundamental form tensor on V_{n+1}^n, the invariance that we demanded then results. One may thus take the equations of motion on S_n to be the first of equations (49) and equations (49′) [Wundheiler [**31**], pp. 128, formula (53)].

If one now considers the anholonomic mechanical system S_n^m then its equations of anholonomity may be written:

$$v^{h'} = \frac{d\sigma^{h'}}{dt} = \alpha^{h'} + c^{h'}\,.$$

Upon introducing these values of into the first $n + m$ of equations (49) and taking into account that in the anholonomic space V_{n+1}^m that is defined in the virtual space V_{n+1}^n by the equations $d\sigma^{h'} = 0$, the differential of the interior vector v^h may be written:

$$\bar{\mathrm{D}}v^h = dv^h - \gamma_{kl}^h v^k d\sigma^l - \gamma_{kl'}^h v^k d\sigma^{l'} \qquad (l' = m + 1, \ldots, n + 1,\ d\sigma^{n+1} = \lambda\, dt),$$

one may give the equations of motion in S_n^m the form:

$$(50) \qquad \begin{cases} \dfrac{dx^i}{dt} = \lambda_a^i v^h + q^i \qquad (q^i = \lambda_{h'}^i v^{h'} - \alpha^i), \\ \dfrac{\bar{\mathrm{D}}v^h}{dt} + \gamma_{hk}^{\alpha'} v^k v^{\alpha'} = w_{\beta' h}^{\alpha'} v^{\alpha'} v^{\beta'} + \mathrm{P}_h \\ (\alpha', \beta' = m+1, \cdots, n, n+1, \quad v^{n+1} = \lambda). \end{cases}$$

These equations obviously have an invariant character with respect to the transformations of the rigid group of the space V_{n+1}^m ($d\sigma^{h'} = 0$), which is embedded in V_{n+1}^n ($\lambda\, dt = 0$).

We know that among these holonomic systems with time-dependent constraints, there exists an important class that is comprised of the systems whose vis viva does not depend upon time explicitly. Obviously, this property does not depend upon the system of variables $x^1, x^2, \ldots, x^n$ that are chosen to represent the position of the mechanical system. The necessary and sufficient condition for there to be a transformation of variables that contains time as a parameter, so that one arrives at a vis viva that is independent of time is that the Riemann space V_{n+1} that is associated with S_n possesses a one-parameter group of transformations into itself that is determined by an infinitesimal transformation of the form:

$$\mathrm{X}f = \frac{\partial f}{\partial t} + \beta^i \frac{\partial f}{\partial x^i} \tag{50′}$$

β_i being functions of the $x^1, x^2, \ldots, x^n$.

Indeed, one knows that by a transformation of variables $x^1, x^2, \ldots, x^n$ that contains time one may reduce the β_i to zero, and then the metric on V_{n+1} does not contain time explicitly, and consequently the same is true for the vis viva of S_n. Naturally, in these calculations, one may appeal to the fact that the vis viva on S_n may be modified by the addition of the derivative with respect to time of a certain function φ of $x^1, x^2, \ldots, x^n, t$. Likewise, one may appeal to the fact that time itself may be changed by the formula $d\bar{t} = \psi(t)\,dt$; however, in this case one must take the new vis viva to have the expression:

$$2\mathrm{T} = \psi a_{ij}\dot{x}^i\dot{x}^j + 2\alpha_i\dot{x}^i + \frac{1}{\psi}\alpha_0 .$$

These results may be summarized in the theorem: *the mechanical system* S_n *possesses a system of coordinates and a time t such that the vis viva does not depend upon time explicitly if the metric:*

$$ds^2 = 2\psi\left(\mathrm{T} + \frac{d\varphi}{dt}\right)dt^2$$

admits the infinitesimal transformation:

$$\mathrm{X}f = \frac{1}{\psi}\frac{\partial f}{\partial t} + \beta^i \frac{\partial f}{\partial x^i},$$

with a suitable definition of the function φ of the $x^1, x^2, \ldots, x^n$ and the function ψ of t.

These considerations may be extended to anholonomic systems, in the sense that the vis viva of the system and the equations of anholonomity V_{n+1}^m may be reduced to expressions that do not contain time explicitly if the associated anholonomic space, when considered from the rigid viewpoint, possesses the infinitesimal transformation (50′).

If the vis viva of the system S_n does not contain time explicitly, and if T is the derivative with respect to time of a certain function of $x^1, x^2, \ldots, x^n$, then the system may be considered as time-independent constraint that has vis viva T_2 and force potential T_0.

35. Generalized vis viva integrals. – One knows that if the vis viva of the holonomic system S_n does not contain time and the forces are derived from a potential U that does not contain time either then there exists a generalized vis viva integral:

$$T_2 - T_0 - U = \text{const.} \tag{52}$$

One refers to the terms in the equations of motion in S_n that are provided by the linear part of T_1 as the *gyrostatic* terms, because these terms make no actual contribution to the vis viva integral. In order to exhibit these gyrostatic terms in the equations of motion, it is convenient to consider the u^a to be the kinetic characteristics, instead of the v^a, and in this case the equations of motion may be written:

$$\begin{cases} \dfrac{dx^i}{dt} = \lambda_a^i u^a, \\ \dfrac{du^a}{dt} = w_{ca}^b u^b u^c + g_{ab} v^b + \dfrac{\partial T_0}{\partial s^a} + P_a, \end{cases} \tag{53}$$

in which the g_{ab} denote the quantities:

$$g_{ab} = \left(\frac{\partial \alpha_i}{\partial x^j} - \frac{\partial \alpha_j}{\partial x^i}\right) \lambda_a^i \lambda_b^j .$$

These quantities are precisely the gyrostatic terms, because if one multiplies the last of equations (53) by u^a and takes the sum to arrive at the integral (52) then the part that contains these terms is null.

The integral of the vis viva (52) also continues to exist for the anholonomic system S_n^m, whose equations of anholonomity (47) do not contain time explicitly, and the $e^{h'}$ are null. If the $e^{h'}$ are not null then this integral may exist only if a certain number of conditions are satisfied.

In the general case of a system whose vis viva depend on time, one may demand that there exists a first integral that differs from the integral (52) by at most a polynomial of first degree in the characteristics. Indeed, if one multiplies equations (49′) by v^a and sums then one finds, upon supposing that the forces derive from a potential, that:

$$\begin{cases} \dfrac{1}{2}\dfrac{d}{dt} v^2 + \gamma_{ba}^{n+1} v^b v^a \lambda = \dfrac{1}{2}\dfrac{d(\lambda^2 + 2U)}{dt} - \dfrac{1}{2}\dfrac{d(\lambda^2 + 2U)}{d\sigma^{n+1}} \\ \qquad [v^2 = (v^1)^2 + \cdots + (v^n)^2]. \end{cases} \tag{54}$$

Now, if there exists a polynomial of first degree V that satisfies the condition:

$$\frac{d}{dt}\mathrm{V} = \gamma_{ba}^{n+1} v^b v^a \lambda + \frac{1}{2}\frac{\partial(\lambda^2 + 2\mathrm{U})}{\partial t}$$

then the system S_n possesses the first integral:

$$\frac{1}{2}v^2 + \mathrm{V} - \frac{1}{2}\lambda^2 + \mathrm{U} = \mathrm{T}_2 - \mathrm{T}_0 + \mathrm{T}_1 + \mathrm{V} + \alpha^2 - \mathrm{U} = \text{const.} \tag{55}$$

In particular, such an integral exists if the virtual space V_{n+1}^{n} is totally geodesic ($\gamma_{ba}^{n+1} = 0$) and if $\frac{1}{2}\frac{\partial(\lambda^2 + 2\mathrm{U})}{\partial t} = 0$. In this case, V = 0 and the integral (55) differs from the integral (52) by a polynomial of first degree in the characteristics $\mathrm{T}_1 + \alpha^2$. These considerations may also be extended to anholonomic systems ([**31**], pp. 132-134).

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