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Geometric study of anholonomic systems

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INTRODUCTION

The idea of searching for a geometric interpretation of anholonomic systems in mechanics was suggested to me by a question that was posed by Prof. LEVI-CIVITA, namely:

Given an anholonomic mechanical system with constraints that are independent of time, whose *vis viva* has the form:

$$T = \tfrac{1}{2}\sum_{i,j=1}^{n} a_{ij}\frac{dx_i}{dt}\frac{dx_j}{dt},$$

and whose anholonomic constraints are:

$$(\alpha) \qquad \sum_{i=1}^{n}\varphi_{ij}\,dx_i = 0 \qquad (j = 1, 2, \ldots, n-m),$$

there is a metric manifold V_n whose ds^2 is given by the expression:

$$ds^2 = 2T\,dt^2 = \sum_{i,j=1}^{n} a_{ij}\,dx_i\,dx_j,$$

and for which (α) will be the mobility constraints for the representative point of the system in V_n. The spontaneous trajectories of our anholonomic system are curves in V_n. What property must they effectively enjoy in order for their geodetic curvature in V_n to be a minimum subordinate to the constraints (α)?

One should note that HERTZ's ([1]) so-called guiding principal refers that property, not to V_n, but to the (Euclidian) manifold that corresponds to the *vis viva* of the system in Cartesian coordinate, i.e., to:

$$T = \tfrac{1}{2}\sum_{i=1}^{n} m_i\,(x_i'^2 + y_i'^2 + z_i'^2),$$

([1]) See HERTZ, *Die Prinzipien der Mechanik*, pp. 100-119.

which supposes that the system is composed of r material points and the one denotes the mass of the i^{th} point by m_i and its Cartesian coordinates by x_i, y_i, z_i.

It is therefore proper to assume that HERTZ's principle can be interpreted in V_n as exactly the minimum of the geodetic curvature that is compatible with (α). However, it is approporiate to verify that directly by starting with the equations of motion in Lagrangian coordinates and the metric on V_n.

I realized that solving that problem ([1]) would make it possible to develop a geometric study of anholonomic systems that is not devoid of interest in its own right, along with the applications that it has to mechanics.

The goal of this study is precisely that of presenting the results to which I arrived along that direction, while leaving the applications to mechanics to another occasion.

Therefore, in the first part, after having given the definitions of an anholonomic manifold, its fundamental congruences, and of anholonomity in paragraphs 1 and 2, the notion of parallelism in the LEVI-CIVITA sense and the notion of geodetics in that manifold will be introduced in paragraphs 3 and 4.

As far as the absolute differential calculus is concerned, which is presented in paragraphs 5, 6, 7, 8, and 9, one will see that one can conveniently apply what I have called the absolute differential calculus of congruences to that manifold. For the Riemannian manifold V_n, that calculus is entirely equivalent to the usual absolute differential calculus of coordinates. It consists precisely of the idea that one introduces a system of n orthogonal congruences in V_n, as RICCI and LEVI-CIVITA did in their treatment of various problems, and then searches for the systems that have tensorial properties with respect to the transformations of the given system of congruences to another system of congruences that are also orthogonal.

That last problem was considered explicitly by RENÉ LAGRANGE ([2]) and by myself in relation to anholonomic manifolds ([3]).

Since that calculus of congruences is less known, I believe that it would be opportune to present it succinctly as it is defined in the case of Riemannian manifolds and then in the case of a anholonomic manifolds.

Finally, in **§ 10**, the equations of geodetic variation will be given, and their invariant character will be exhibited.

In the second part of this work, the following arguments will be treated: the second fundamental form of an V_n^m, exterior parallelism, or WEYL parallelism in an V_n^m, the geometric interpretation of the principal tensors, and the equivalence problem of two anholonomic manifolds.

Some of the results that are presented in this work have been published before in short notes and will be cited as appropriate.

Permit me to express my warmest gratitude to Prof. TULLIO LEVI-CIVITA for the kind interest that he took in this work.

([1]) See my note: "Sopra le equazioni del moto di un sistema anolonomo," Rend. della R. Accademia dei Lincei (6), vol. IV, pp. 508.

([2]) Cf., RENÉ LAGRANGE, "Calcul différentiel absolu," published in Mémorial des Sciences Mathématiques.

([3]) See my notes: "Sur le calcul différentiel absolu pour les variétés non holonomes," Comptes rendus, t. 183, pp. 1083 and "Sur quelques tenseurs dans les variétés non holonomes," Comptes rendus, t. 186, pp. 995.

PART ONE

§ 1. – The definition of an anholonomic manifold.

Consider a metric manifold V_n whose line element is defined by the expression ([1]):

$$ds^2 = \sum_{i,j=1}^{n} a_{ij}\, dx_i\, dx_j\,, \tag{1}$$

and suppose, moreover, that one is given the $n - m$ ($m > 1$) equations:

$$\sum_{i=1}^{n} \varphi_{ij}\, dx_i = 0 \qquad (j = 1, 2, \ldots, n - m) \tag{2}$$

in this manifold.

The coefficients a_{ij} and φ_{ij} are assumed to be functions of x that are continuous and differentiable as many times as necessary in a region D that pertains to our considerations. We have excluded the cases $m = 0$ and $m = 1$, because in the first case, equations (2), which are considered to be mutually-independent, have only the trivial solutions x = constant, and in the second case, equations (2) will define a family of curves in V_n .

In this article, we would like to study the properties of the manifold V_n when it is constrained by the relations (2), and which will be denoted by simply V_n^m. Since those spaces present themselves very naturally in the study of the anholonomic systems in dynamics, one also calls them *anholonomic spaces* or *manifolds*.

Equations (2) form a system of $n - m$ total differential equations or, as one sometimes says, *a PFAFF system*.

If that system is completely integrable then from (2), one can express $n - m$ of the x as functions of the other x and $n - m$ arbitrary constants $C_1, C_2, \ldots, C_{n-m}$:

$$x_j = f_j\,(x_1, x_2, \ldots, C_1, C_2, \ldots, C_{n-m}) \qquad (j = m + 1, \ldots, n), \tag{3}$$

in such a way that one and only one of those integral manifolds pass through a point in space. If one fixes the integration constants C arbitrarily – for example, such that the general integral will pass through a given point arbitrarily – then the study of the anholonomic space V_n^m will reduce to the study of a metric manifold V_m whose ds^2 is obtained by introducing the values that are given in (3) into formula (1). The study of V_n^m reduces to the study of ∞^{n-m} metric manifolds V_m .

Now suppose that the system (2) is not completely integrable. In that case, it can also admit a certain number of integrable combinations, and one can show ([2]) that one can

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([1]) The notations that relate to metric manifolds that are used in the course of this work are the ones that Prof. LEVI-CIVITA adopted in his *Lezioni di Calcolo differenziale assoluto*, which were compiled by E. PERSICO (Rome, A. Stok, 1923), which will be indicated by simply *Lez. Levi-Civita* in what follows.

([2]) Cf., GOURSAT, *Leçons sur le problème de Pfaff*, Hermann, Paris, pp. 296.

exhibit those integrable combinations by transforming the system into another one of the form:

(4)
$$\begin{cases} df_i = 0 & (i = 1, 2, \ldots, n - p), \\ w_j = 0 & (j = n - p, \ldots, n - m). \end{cases}$$

The first equations in (4) are exact total differentials, and the last ones are Pfaffian equations that do not admit any integrable combinations. From the first equations (4), one can express $n - p$ of the x as functions of the other x and $n - p$ integration constants, and by the same argument as before, the study of V_n^m will reduce to the study of ∞^{n-p} metric manifolds V_p, for which the last of (4) will persist, or to ∞^{n-p} anholonomic manifolds V_n^m.

Upon introducing the language that is typically used for metric manifolds and Euclidian spaces, one can say that the anholonomic manifold V_n^m is *immersed* in the metric manifold V_n. That last result can be expressed by saying that the smallest number of dimensions for a metric manifold in which V_n^m can be considered to be immersed is p. Therefore, the number $p - m$ expresses the degree of anholonomity of the manifold V_n^m.

If the relations (2) do not admit any integrable combinations then one will be inclined to guess that V_n^m contains all of the points of V_n, in the sense that one can go from an arbitrary point of V_n to another arbitrary point along an integral curve [which satisfies (2)] or always remains in V_n^m. A general proof of that fact does not exist, but for certain systems (2), the proof is immediate. The converse is true, and in fact, if equations (2) admit an integrable combination then two points can be joined by an integral curve only when they belong to the same hypersurface that is determined by the integrable combination.

From what was said above, one can always refer to the manifold V_n^m whenever the equations of anholonomity have not integrable combinations. However, given a system of total differential equations (2), the question of how to put that system into the form (4) is very difficult, in general.

We shall therefore refer our considerations to the system (2) without thinking about whether it can admit integral combinations.

§ 2. – On the fundamental and anholonomity congruences.

We begin by recalling some well-known notions about systems of n orthogonal congruences in a metric manifold V_n ([1]).

When one is given a system of n contravariant quantities A^i, the equations:

(4′)
$$\frac{dx^1}{A^1} = \frac{dx^2}{A^2} = \ldots = \frac{dx^n}{A^n}$$

([1]) In particular, see *Lez. Levi-Civita*, chap. X.

will define a system of curves in V_n that one calls a *congruence*.

One and only one of those curves pass an arbitrary point at which not all of the A^i are zero. Since the multiplication of the A^i by a non-zero factor will not change our congruence, we can always suppose that it is defined by the quantities:

$$\lambda^i = \frac{A^i}{\sigma} \qquad (i = 1, 2, \ldots, n),$$

which satisfies the relation:

$$\sum_{i,j=1}^{n} a_{ij}\,\lambda^i \lambda^j = 1,$$

which is equivalent to determining σ from the formula:

$$\sigma^2 = \sum_{i,j=1}^{n} a_{ij}\,A^i A^j .$$

The λ^i thus-defined are called the *parameters of the congruence*. The covariant quantities:

$$\lambda_j = \sum_{i=1}^{n} a_{ij}\,\lambda^i$$

are called the *momenta* of the congruence.

One proves that one can always choose n mutually-orthogonal congruences in a V_n (and also in an infinitude of ways), and if λ_h^i ($h = 1, 2, \ldots, n$) denote the parameters of that congruence then one will have the following relations at any point of V_n :

$$(5) \qquad \sum_{i.\,j=1}^{n} a_{ij}\,\lambda_h^i\,\lambda_k^j = \delta_h^k = \begin{cases} 1 & h = k \\ 0 & h \neq k. \end{cases}$$

One deduces from them, the following relations between the parameters and the moments:

$$\sum_{i=1}^{n} \lambda_{h|i}\,\lambda_k^i = \delta_h^k ,$$

and the relations between the momenta:

$$\sum_{i,j=1}^{n} a^{ij}\lambda_{h|i}\,\lambda_{k|j} = \delta_h^k ,$$

in which the a^{ij} are the inverses of the a_{ij} .

A noteworthy fact to observe is that if one is given either the momenta or the parameters of a system of n orthogonal congruences V_n then the metric on that manifold will remain completely determined on the basis of the formulas:

$$(5') \qquad a_{ij} = \sum_{i=1}^{n} \lambda_{h|i}\,\lambda_{h|j}\,, \qquad\qquad a^{ij} = \sum_{i=1}^{n} \lambda_h^i\,\lambda_k^j\,.$$

Therefore the metric on V_n can be considered to be epitomized by the n orthogonal congruences λ.

If one is given an infinitesimal displacement ds whose components are dx_1, dx_2, …, dx_n then its projections onto the congruence (λ_h) will be given by the formulas:

$$(6) \qquad ds_h = \sum_{i=1}^{n} \lambda_{h|i}\,dx_i\,,$$

where s_h is nothing but the arc-length of the congruence (λ_h). If one divides by ds then one will have the relations:

$$(6') \qquad u_h = \frac{ds_h}{ds} = \sum_{i=1}^{n} \lambda_{h|i}\,\frac{dx_i}{ds}\,,$$

in which u_h are the cosines that the displacement ds forms with the congruence (λ_h).

$$(7) \qquad dx_i = \sum_{h=1}^{n} \lambda_h^i\,ds_h\,;$$

i.e., the displacement ds is determined completely by the differentials of the arc-length ds_h . If one introduces those values into the quadratic form (1) then, on the basis of (5), one will get the formula:

$$(1') \qquad ds^2 = ds_1^2 + ds_2^2 + \cdots + ds_n^2\,.$$

Having said that, we move on to our anholonomic manifold V_n^m. From equations (2), we have that we can express $n - m$ of the dx_i in that manifold as functions of the other m, and more generally, we can express the dx_i in the form:

$$(5'') \qquad dx_i = \sum_{h=1}^{m} l_h^i\,d\sigma_h\,,$$

in which the $d\sigma_h$ denote m independent linear combinations of the dx_i ([1]).

([1]) Cf., my note: "Sopra una classe di sistemi anolonomi," Rend. della R. Accademia dei Lincei (6) **3** (1926), pp. 549.

Those formulas (in which the $d\sigma_h$ are regarded as arbitrary) express the idea that the possible displacements of the point $(x_1, x_1, \ldots, x_n)$ in the anholonomic manifold V_n^m will proceed in any of the ∞^{m-1} directions of a subspace (*giacitura*), which can be specified by m of its arbitrary, but mutually-independent, directions. In the form (5″), when one assumes that each of the $d\sigma_h$ are equated to zero in succession, except for the first one, the second one, etc., they will present themselves as those directions that correspond to increments in the x_i that are proportional to l_1^i, l_2^i, …, etc., respectively.

One can introduce the parameters of the corresponding direction by using the fundamental form (1) and writing (5″) in the form:

$$dx_i = \sum_{h=1}^{m} \lambda_h^i \, ds_h \, , \tag{7′}$$

in which one sets:

$$\lambda_h^i = \frac{l_h^i}{\rho_h} , \qquad ds_h = \rho_h \, ds_h \tag{6″}$$

and takes the ρ_h in such a way that:

$$\sum_{i.\,j=1}^{n} a_{ij} \, \lambda_h^i \, \lambda_h^j = 1 \qquad (h = 1, 2, \ldots, m),$$

which is equivalent to determining the ρ_h based upon the form:

$$\rho_h^2 = \sum_{i.\,j=1}^{n} a_{ij} \, l_h^i \, l_h^j \, .$$

Take the m directions (λ) to be mutually orthogonal, which is also permissible. The expressions for the dx_i can then be presented (and also in an infinitude of ways) in the form:

$$dx_i = \sum_{h=1}^{n} \lambda_h^i \, ds_h \, , \tag{8}$$

in which the parameters λ_h^i satisfy the orthogonality relations:

$$\sum_{i.\,j=1}^{n} a_{ij} \, \lambda_h^i \, \lambda_h^j = \delta_h^k \qquad (h, k = 1, 2, \ldots, m). \tag{7″}$$

It results from this that equations (2) do nothing but introduce a subspace of ∞^{m-1} directions at any point of V_n , among which one finds all of the displacements in V_n^m. The m orthogonal directions (λ) that determine that subspace at the point $(x_1, x_2, \ldots, x_n)$ define m congruences in all of V_n that one calls *fundamental*.

When one completes the m congruences λ with $n - m$ other congruences in order to form a system of n orthogonal congruences in V_n, equations (2) will be equivalent to the equations ([1]):

(9) $$\sum_{i=1}^{n} \lambda_{h'|i}\, dx_i = 0 \qquad (h' = m + 1, \ldots, n),$$

which express precisely the idea that the projections of the displacement onto V_n^m will be zero for the last $n - m$ congruences λ. Those last $n - m$ congruences are also called the *anholonomity* congruences.

Since equations (9) are a consequence of formulas (8), one can say that the anholonomic manifold V_n^m is determined completely by formulas (1) and (8). If one divides formulas (8) by ds then one will have the following:

(10) $$\frac{dx_i}{dt} = \sum_{h=1}^{m} \lambda_h^i\, u_h ,$$

in which u_h are the cosines of the displacement in V_n^m that is composed of the first m congruences, while the other ones are always zero. If one takes (8) and (7″) into account then *its will once more result that the metric of* V_n^m *is given by the formula:*

(1″) $$ds^2 = ds_1^2 + ds_2^2 + \cdots + ds_m^2 .$$

Integrability conditions. – One knows that the integrability conditions for a PFAFF system of the form (9) can be expressed by annulling the so-called *bilinear covariant* :

$$\sum_{i,j=1}^{n} \left(\frac{d\lambda_{h'|i}}{dx^j} - \frac{d\lambda_{h'|j}}{dx^i} \right) dx_i\, \delta x_j \qquad (h' = m + 1, \ldots, n)$$

for any choice of the two displacements whose components are dx_i and δx_j and which satisfy equations (9). By virtue of the last condition, the displacements can be expressed by the formulas (8):

(8′) $$\begin{cases} dx_i = \sum_{h=1}^{m} \lambda_h^i\, ds_h , \\ \delta x_i = \sum_{h=1}^{m} \lambda_h^i\, \delta s_h , \end{cases}$$

and the bilinear covariants will assume the form:

([1]) In the course of the article, the indices that vary from $m + 1$ to n will be denoted by primed symbols, for more clarity.

$$\sum_{h,k=1}^{m} ds_h\, \delta s_k \sum_{i,j=1}^{n}\left(\frac{d\lambda_{h'|i}}{dx_j}-\frac{d\lambda_{h'|j}}{dx_i}\right)\lambda_h^i\,\lambda_k^j\ .$$

In that formula, ds_h and δs_k are the components along the m fundamental congruences of two displacements that are situated in V_n^m. Since the ds_h and δs_k are arbitrary, it will result that the necessary and sufficient condition for the integrability of equations (9) are given by the formulas:

$$\sum_{i,j=1}^{n}\left(\frac{d\lambda_{h'|i}}{dx_j}-\frac{d\lambda_{h'|j}}{dx_i}\right)\lambda_h^i\,\lambda_k^j = 0. \tag{9′}$$

In order to give a more expressive form to those conditions, recall the definition of the RICCI rotation coefficients, which we shall always use in what follows. The RICCI rotation coefficients relative to a system of n mutually-orthogonal congruences λ of V_n are given by the expressions:

$$\gamma_{hkl} = \left(\frac{d\lambda_{h'|i}}{dx_j}-\left\{\begin{matrix} i\ j \\ r\end{matrix}\right\}\lambda_{h|r}\right)\lambda_k^i\,\lambda_l^j\ , \tag{11}$$

in which $\left\{\begin{matrix} i\ j \\ r\end{matrix}\right\}$ are the CHRISTOFFEL symbols of the second kind that relate to the quadratic form (1). The RICCI coefficients are antisymmetric with respect to the first indices and are invariants of coordinate transformations. Now consider the quantities:

$$w_{kl}^h = \gamma_{hkl}\ , \tag{10′}$$

which are obviously antisymmetric in the lower indices. If one takes formulas (11) into account and the symmetry of the CHRISTOFFEL symbols then one will have:

$$w_{kl}^h = \left(\frac{d\lambda_{h'|i}}{dx_j}-\frac{d\lambda_{h'|j}}{dx_i}\right)\lambda_h^i\,\lambda_k^j\ . \tag{10″}$$

By virtue of that formula, the integrability conditions (9′) will assume the simple form:

$$w_{hk}^{h'} = \gamma_{h'hk} - \gamma_{h'kh} = 0 \qquad (h = m + 1, \ldots, n\ ;\ h, k = 1, 2, \ldots, m). \tag{12}$$

One obtains the following formula from formulas (10′):

$$\gamma_{hkl} = \tfrac{1}{2}\left(w_{kl}^h + w_{lh}^k - w_{hk}^l\right), \tag{10‴}$$

which will define the γ_{hkl} as functions of the elements of only the congruences λ on the basis of (10″).

§ 3. – Parallelism in V_n^m.

Let R be a vector at a point P of the manifold V_n that is characterized, for example, by its contravariant components R^i. The projections of those vectors onto the n orthogonal congruences (λ) are given by the formulas:

$$r_h = \sum_{i=1}^{n} R^i \lambda_{h|i} \qquad (h = 1, 2, \ldots, n), \tag{9″}$$

in which r_h are invariant under coordinate transformations.

Conversely, the vector R is determined completely by those invariants, and the contravariant components R^i are expressed as functions of the r_h by means of the formulas:

$$R^i = \sum_{i=1}^{n} r_h \lambda_h^i \qquad (i = 1, 2, \ldots, n). \tag{11′}$$

The vector R is said to be situated in the anholonomic manifold V_n^m when the $n - m$ projections r_h onto the $n - m$ anholonomity congruences are zero, so one will have:

$$\sum_{i=1}^{n} R^i \lambda_{h'|i} = 0 \qquad (h = m + 1, \ldots, n).$$

If one is given a vector in V_n^m and a displacement that is also in V_n^m, which has components ds_h and links the point P with a neighboring point P' then one would like to transport the vector R from the point P to the point P' by parallelism in the LEVI-CIVITA sense. In order to do that, recall that the equations of parallelism in V_n are obtained by the symbolic equations ([1]):

$$\sum_{k=1}^{n} \tau_k \, \delta x_k = 0, \tag{12′}$$

which must be true for all of the independent displacements δx_h and as a consequence one will have the following equations of parallelism:

$$\tau_k = \sum_{j=1}^{n} a_{kj} \, dR^j + \sum_{i,j=1}^{n} \begin{Bmatrix} j\,i \\ k \end{Bmatrix} R^j dx_i = 0 \qquad (k = 1, 2, \ldots, n). \tag{11″}$$

([1]) Cf., *Lez. Levi-Civita*, pp. 157, formulas (50).

In order to find the equations of parallelism in the case of our V_n^m, we must observe that the δx_k are no longer arbitrary, but are given by formulas (8), so the symbolic equations (12′) are written:

(12″) $$\sum_{h=1}^{m} \delta s_h \sum_{k=1}^{n} \tau_k \, \lambda_h^k = 0,$$

and given the arbitrariness in the ds_h in our case, the equations of parallelism will assume the form:

(13) $$\sum_{k=1}^{n} \tau_k \, \lambda_h^k = 0 \qquad (h = 1, 2, \ldots, m).$$

We shall specify those equations under the double hypothesis that the vector R and the infinitesimal segment PP' whose components are dx_i are found in the anholonomic manifold V_n^m, and therefore the following formulas will both be true:

(13′) $$R^j = \sum_{\alpha=1}^{m} \lambda_\alpha^j \, r_\alpha \,, \qquad dx_i = \sum_{l=1}^{m} \lambda_l^i \, ds_l \,.$$

Differentiate the first of those formulas along PP':

$$dR^j = \sum_{\alpha=1}^{m} dr_\alpha \, \lambda_\alpha^j + \sum_{\alpha=1}^{m} r_\alpha \sum_{i=1}^{n} \frac{\partial \lambda_\alpha^j}{\partial x_i} dx_i \,,$$

and introduce those values into equations (13).

If one takes into account the values of τ_k that are given in (11″) and moves the summation that goes from 1 to m to the first position then one will obtain:

$$\sum_{\alpha=1}^{m} dr_\alpha \sum_{k,j=1}^{n} a_{kj} \, \lambda_\alpha^j \, \lambda_h^k + \sum_{\alpha=1}^{m} r_\alpha \, ds_l \sum_{i,j,k=1}^{n} \frac{\partial \lambda_\alpha^j}{\partial x_i} a_{kj} \lambda_h^k \, \lambda_l^i + \sum_{\alpha=1}^{m} r_\alpha \, ds_l \sum_{i,j,k=1}^{n} \begin{vmatrix} i \; j \\ k \end{vmatrix} \lambda_\alpha^j \lambda_l^i \, \lambda_h^k = 0.$$

The first term in this equation reduces to dr_h , based upon the orthogonal formulas (7″). As for the second summation in the second term, if we introduce the momenta of the congruences h, which are given by the formulas:

$$\lambda_{h\,|\,j} = \sum_{k=1}^{n} a_{kj} \, \lambda_h^k \,,$$

and if we also take into account the formula:

(13″)
$$\frac{d\lambda_\alpha^i}{dx^i}\lambda_{h|j} = -\frac{d\lambda_{h|j}}{dx_i}\lambda_\alpha^j,$$

which one obtains by differentiating formula (5′), which couples the parameters and the momenta. The equations can then be written:

$$dr_h = \sum_{\alpha=1}^{m} r_\alpha\, ds_l \sum_{i,j=1}^{n}\left(\frac{d\lambda_{h|j}}{dx_i} - \sum_{k=1}^{m}\begin{vmatrix} i\ j \\ k \end{vmatrix}\lambda_h^k\right)\lambda_\alpha^j\,\lambda_l^i .$$

By virtue of the formula that couples the CHRISTOFFEL symbols of the first and second kind, one will further have:

$$\sum_{k=1}^{m}\begin{vmatrix} i\ j \\ k \end{vmatrix}\lambda_h^k = \sum_{k=1}^{m}\begin{Bmatrix} i\ j \\ k \end{Bmatrix}\lambda_{h|k},$$

and based upon formula (11), which defines the RICCI rotation coefficients, the equations of parallelism in V_n^m will assume the definitive form:

(14)
$$dr_h = \sum_{\alpha,l=1}^{m}\gamma_{h\alpha l}\, ds_l \qquad (h = 1, 2, \dots, m).$$

Those m equations determine the increments that the m components r_h, which specify the vector R in V_n^m, must experience when one transports that vector along the infinitesimal segment PP' in such a way that the angle between the vector R' at P' and the vector R at P is a minimum that is compatible with the anholonomity constraints (9).

We must observe that if no anholonomity relation exists then all of the calculations that we are about to do will be valid in the system of n orthogonal congruences in V_n by simply putting n in place of m, and the equations of parallelism in the metric manifold V_n will have the form ([1]):

(14′)
$$dr_h = \sum_{\alpha,l=1}^{n}\gamma_{h\alpha l}\, r_\alpha\, ds_l \qquad (h = 1, 2, \dots, n).$$

When one compares those equations with the usual equations of parallelism that refer to the coordinates x, one will see that one has introduced the RICCI rotation coefficients into the calculation of the congruences in place of the CHRISTOFFEL symbols, and that observation will be true in general, as one will see in what follows.

It is clear that the parallelism that is defined by equation (14) is different from the parallelism in V_n that is defined by equations (14′), so one can call the former *constrained parallelism*, or simply *parallelism in* V_n^m.

([1]) See CARPANESE, "Parallelismo e curvature in una varieta qualunque," Annali di Matematica **27** (1919), 147-169.

Now, if one considers a finite segment of a curve in V_n^m then the way that one varies r_h when one transports the vector R by parallelism along that curve will be defined by the differential equations [equations (14) divided by ds]:

$$\frac{dr_h}{ds} = \sum_{\alpha,l=1}^{m} \gamma_{h\alpha l}\, r_\alpha\, u_l \qquad \left(u_l = \frac{ds_l}{ds}\right), \tag{15}$$

which define a system of m linear differential equations in the r and will serve to define the r_h as a function of s. One must take the x into account, and consequently, the γ, like the cosines u_l , are well-defined functions of the arc-length s along T.

One easily proves that parallel transport does not change the length of the vector or the angle between two vectors.

Therefore, it is enough to know how the cosines of the vector R change under parallel transport, in which case, equations (15) can be written:

$$\frac{dv_h}{ds} = \sum_{\alpha,l=1}^{m} \gamma_{h\alpha l}\, v_\alpha\, u_l\,. \tag{15′}$$

These equations obviously have the quadratic first integral:

$$\sum_{h=1}^{m} v_h^2 = 1.$$

They then form a system of differential equations with an antisymmetric determinant, due to the antisymmetry property of the coefficients γ with respect to the first two indices.

It results from the properties of the linear differential equations in the normal form (15) that when one is given a direction at a point and a curve that passes through that point, its parallel constraint will remain determined uniquely along the curve.

§ 4. – Geodetics in V_n^m.

In order to find the equations of the geodetics in the manifold V_n^m, we start with the autoparallel curves. Those are the curves whose directions at each point form a system of parallel vectors in V_n^m along that curve. In order to get the equations of those curves, we set $v = u$ in the equations of parallelism (15′). If we associate those equations with equations (10) then we will arrive at the system:

(16)
$$\begin{cases} \dfrac{dx_i}{ds} = \sum_{h=1}^{m} \lambda_h^i u_h & (i = 1, 2, \ldots .m), \\ \dfrac{du_h}{ds} = \sum_{h=1}^{m} \gamma_{h\alpha l}\, u_\alpha\, u_h & (h = 1, 2, \ldots, m). \end{cases}$$

That system of $m + n$ differential equations, in a normal form, will serve to define the $m + n$ unknown x and u as functions of the arc-length s. One sees from (16) that when one chooses an arbitrary point (x_1, x_2, …, x_n) and a direction (u_1, u_2, …, u_n), there will exist one and only one autoparallel curve that passes through that point in that direction.

We would now like to prove that equations (16) provide us, at the same time, with the geodetics of the manifold V_n^m. For the sake of brevity, recall that when one is given two points A and B on a curve C that is in the manifold V_n^m, the variation of the arc-length AB for an infinitesimal variation of the curve C to a neighboring curve c, while the endpoints A and B are fixed, is expressed by the formula ([1]):

(16′)
$$\delta l = -\int_A^B \sum_{k=1}^{n} p_k\, \delta x_k\, ds,$$

in which the quantities p are given by the expressions:

(16″)
$$p_k = \sum_{j,k=1}^{n} a_{jk}\, \ddot{x}_j + \sum_{j,l=1}^{n} \begin{vmatrix} j\,l \\ k \end{vmatrix} \dot{x}_j\, \dot{x}_l\, .$$

The variation of the arc-length must be zero for the geodetics, so the symbolic equation of the geodetic will take the form:

(17)
$$\sum_{k=1}^{n} p_k\, \delta x_k = 0,$$

which must be valid for all displacements that are compatible with the constraints.

Given the arbitrariness of the δx_k, one will get the equations of the geodetics in the manifold V_n by equating the p_k to zero. In the case of V_n^m, one obviously supposes that the curve C is found in V_n^m and then that the neighboring curve c is obtained from C by displacements δx_k that are in V_n^m, so they are given by the expressions:

(17′)
$$\delta x_k = \sum_{h=1}^{m} \lambda_h^k\, \delta s_h\, ,$$

([1]) See *Lez. Levi-Civita*, pp. 153, formula (44).

in which the δs_h are regarded as arbitrary. In general, the neighboring curve c will be in V_n^m, and one will see later on that this is generally a characteristic property of the manifolds V_n^m for which the anholonomity relations are not completely integrable.

By virtue of (17′), the symbolic equation (17) will become:

$$\sum_{h=1}^{m} \delta s_h \sum_{k=1}^{n} p_k \lambda_h^k = 0. \tag{17″}$$

It will result with no further analysis that the geodetics of V_n^m will satisfy the m equations:

$$\sum_{k=1}^{n} p_k \lambda_h^k = 0 \qquad (h = 1, 2, \ldots, m). \tag{18}$$

If one takes into account the first formula in (16), which is always valid V_n^m, and formula (16″) then one will easily find that those equations are nothing but the last of (16).

That is to say, equations (16) will provide the curve in V_n^m such that the distance between two points on one of the those curves is *minimal* with respect to all of the neighboring curves that through those two points that are obtained from *displacements that are compatible with the anholonomity constraints* (9). In that sense, the autoparallel curves in V_n^m are also the geodetics, and conversely.

One must observe that it is not true that when one is given two points arbitrarily, there will exist a geodetic that passes through those two points.

§ 5. – Absolute differential calculus of congruences in a V_n .

It is obvious that when one is given a metric manifold V_n , the way that one refers that manifold to a system of n orthogonal congruences λ is not unique. One should note that if one is given another system of n orthogonal congruences – call it $\bar{\lambda}$ – then there will exist linear relations between the two systems of the form:

$$\bar{\lambda}_{h|i} = \sum_{k=1}^{n} c_h^k \lambda_{k|i} \qquad (h = 1, 2, \ldots, n), \tag{19}$$

where the c_h^k are invariant under coordinate transformations.

Formulas (19) define a transformation of congruences. If one takes formulas (6) into account then one will find the transformation formulas for the differentials of arc-length:

$$d\bar{s}_h = \sum_{k=1}^{n} c_h^k ds_k . \tag{19′}$$

Since our transformations (19′) must leave the quadratic form (1′) invariant, it will result that the c_h^k are the coefficients of an orthogonal substitution and that they will therefore satisfy the relations:

$$\sum_{h=1}^{n} c_h^k\, c_h^l \;=\delta_k^l = \begin{cases} \lambda & l = k \\ 0 & l \neq k. \end{cases} \tag{19″}$$

That fact can also be expressed by saying that, based upon formulas (5′), formulas (19) will not change the values of a_{ij} and consequently, the quadratic form (1). If one takes into account the formulas that couple the parameters and momenta then one will find the same law of transformation. One will find the inverse formulas by virtue of formulas (19″):

$$\lambda^{l\,|\,i} = \sum_{\alpha=1}^{n} c_\alpha^l\, \overline{\lambda}_{\alpha|i}\;.$$

If one now solves formulas (19) for the c_h^k then one will get:

$$c_h^k \;=\; \sum_{i=1}^{n} \overline{\lambda}_{h|i}\, \lambda_k^i\;. \tag{20}$$

We would like to see that the c_h^k satisfy a system of first-order differential equations that involve the RICCI rotation coefficients relative to the λ and $\overline{\lambda}$. In order to do that, we recall the formula the gives the intrinsic derivative of an arbitrary function u of position:

$$\frac{du}{ds^\alpha} = \sum_{j=1}^{n} \frac{du}{dx^j}\,\lambda_k^j\;,$$

and differentiate (20) with respect to the arc-length s_l , so we will have:

$$\frac{dc_h^k}{ds^l} = \sum_{i,\,j=1}^{n} \frac{d\overline{\lambda}_{h|\,i}}{dx_j}\,\lambda_k^i\lambda_l^j + \sum_{i,\,j=1}^{n} \frac{d\lambda_k^i}{dx_j}\,\lambda_l^j\;.$$

The second term in the right-hand side of that formula can be transformed based upon (19) and then based upon formula (13″), and one can write:

$$\frac{dc_h^k}{ds^l} = \sum_{i,\,j=1}^{n} \frac{d\overline{\lambda}_{h|\,i}}{dx_j}\,\lambda_k^i\lambda_l^j - \sum_{i,\,j=1}^{n} \frac{d\lambda_{\alpha|i}}{dx_j}\,\lambda_k^i\,\lambda_l^j\;. \tag{20′}$$

Since one can infer from the defining formula for the rotation coefficients (11) that:

$$\frac{d\lambda_{h|i}}{dx_j} = \sum_{\alpha=1}^{n} \begin{Bmatrix} i\ j \\ \alpha \end{Bmatrix} \lambda_{h|\alpha} + \sum_{\alpha,\beta=1}^{n} \gamma_{h\alpha\beta}\,\lambda_{\alpha|i}\,\lambda_{\beta|\,j}\,,$$

and an analogous form for the $d\bar{\lambda}_{\alpha|i}/dx_j$, one will obtain the desired formula by introducing terms into (20′) that contain the CHRISTOFFEL symbols and reducing:

$$\frac{dc_h^k}{ds_l} = \sum_{\alpha,\beta=1}^{n} \bar{\gamma}_{h\alpha\beta}\, c_\alpha^k\, c_\beta^l - \sum_{\alpha=1}^{n} c_h^\alpha\, \gamma_{\alpha kl}\,. \tag{21}$$

Vectors. – If one is given a vector R in V_n then its (invariant) components r_h along the congruences λ will be given by formulas (9″). If $\bar{r}_h$ denote the components of that vector along the congruences $\bar{\lambda}$, which are coupled with those of the λ by formulas (19), then one will have the transformations formulas:

$$\bar{r}_h = \sum_{k=1}^{n} c_h^k\, r_k\,. \tag{20″}$$

Conversely, if one is given a system of n quantities that are invariant under the coordinate transformations and one changes them by a transformation of congruences according to the law (20″) then those quantities will specify a vector in V_n whose covariant (or contravariant) components are given by the expressions:

$$A^i = \sum_{h=1}^{n} r_h\, \lambda_h^i\,, \qquad A_i = \sum_{h=1}^{n} r_h\, \lambda_{h|i}\,.$$

Tensors. – If one is given an arbitrary tensor R (and for simplicity, suppose that it is a second-order mixed tensor with components R_i^j) then its invariant components in the system of congruences λ will be given by the expressions:

$$r_{hk} = \sum_{i,j=1}^{n} R_i^j\, \lambda_h^i\, \lambda_{k|j}\,.$$

If one denotes the components of that tensor R in the system of congruences λ by $\bar{r}_{hk}$ then one will immediately have the transformation formulas:

$$\bar{r}_{hk} = \sum_{\alpha,\beta=1}^{n} r_{\alpha\beta}\, c_h^\alpha\, c_k^\beta\,. \tag{21′}$$

Conversely, if one is given a system of n^2 quantities that are invariant under coordinate transformations and one changes them by a transformation of congruences according to the law (21′) then that will specify a second-order tensor in V_n. Therefore, the RIEMANN symbols of the first kind (which are, as one knows, the components of a covariant tensor of order four) will determine the invariants:

(21″) $$\gamma_{hk,\,lr} = \sum_{i,j,\alpha,\beta=1}^{n} (ij,\alpha\beta)\,\lambda_h^i\,\lambda_k^j\,\lambda_l^\alpha\,\lambda_r^\beta$$

in the system of congruences λ, which are just the four-index RICCI coefficients. It will result that those RICCI coefficients form a fourth-order tensor in the absolute differential calculus of congruences. Those coefficients expressed as functions of the rotation coefficients by the known formulas:

(22) $$\gamma_{hk,\,lr} = \frac{d\gamma_{khl}}{ds^r} - \frac{d\gamma_{hkr}}{ds^l} + \sum_{\alpha=1}^{n}[\gamma_{\alpha hr}\,\gamma_{\alpha kl} - \gamma_{\alpha hl}\,\gamma_{k\alpha r} + \gamma_{hk\alpha}\,(\gamma_{\alpha lr} - \gamma_{\alpha rl})].$$

One must observe that not all systems of invariants can specify a tensor, and an example of that would be the rotation coefficients, which are invariants, but do not form a tensor, as formula (21) shows.

Tensorial derivation. – Given the vector R with (invariant) components r_h, differentiate the transformation law (20″) of that vector with respect to $\overline{s}_l$, while taking into account the formulas:

$$\frac{du}{d\overline{s}_l} = \sum_{\alpha=1}^{n} \frac{du}{ds_\alpha}\,c_l^\alpha\,.$$

The derivatives of the coefficients c_h^k appear in the result. If one eliminates those derivatives with the help of formulas (21) and lets $r_{h\,|\,l}$ denote the quantities ([1]):

(22′) $$r_{h\,|\,l} = \frac{dr_h}{ds_l} - \sum_{\alpha=1}^{n} r_\alpha\,\gamma_{h\alpha l}\,,$$

one will find the formula:

(22″) $$\overline{r}_{h|l} = \sum_{\alpha,\beta=1}^{n} r_{\alpha|\beta}\,c_h^\alpha\,c_l^\beta\,.$$

That signifies that the $r_{h\,|\,l}$ specify a second-order tensor that one calls the *tensorial derivative* of the vector R. If one now differentiates formula (22″) and takes into account (21) then one find the tensorial second derivatives of the vector R, which are given by the formulas:

(23) $$r_{h\,|\,lk} = \frac{dr_{h|l}}{ds_k} + \sum_{\alpha=1}^{n} r_{\alpha|l}\,\gamma_{\alpha hk} + \sum_{\alpha=1}^{n} r_{h|\alpha}\,\gamma_{\alpha lk}\,.$$

([1]) That method of forming the tensorial derivatives is analogous to the one that is used in *Invariants of quadratic differential forms* by OSWALD VEBLEN, Cambridge University Press, London, 1927, pp. 36-40.

One can find the tensorial derivatives of an arbitrary tensor in an analogous way. Finally, if one considers the difference between the second derivatives then one will find the formulas:

$$r_{h\,|\,lk} - r_{h\,|\,kl} = \sum_{\alpha=1}^{n} r_\alpha\, \gamma_{\alpha h, lk}\,, \tag{23′}$$

in which $\gamma_{ah,\,lk}$ are the four-index RICCI coefficients that were indicated above.

We can find the tensorial derivatives by making use of the equations of parallelism, and that is the method that we prefer, because one can easily extend it to anholonomic manifolds.

One sees that the two points of view for considering the absolute differential calculus are equivalent for a metric manifold V_n ; that is to say, one can pass from one to the other at any moment.

The calculus of congruences presents a certain simplicity, albeit formally, because one can consider only the orthogonal transformations. It is therefore unnecessary to distinguish between covariance and contravariance, because they coincide.

§ 6. – Absolute differential calculus on anholonomic manifolds.

As we saw in § **2**, an anholonomic manifold V_n^m is characterized, on the one hand, by the first *m* congruences λ, which we also called *fundamental*, and on the other hand, by the last $n - m$ congruences λ, which we have also called *anholonomity congruences*, due to the fact that the moments of the latter congruences specify the anholonomic constraints (9). It will then result, with no further discussion, that the congruences of V_n^m are divided into two distinct groups. It is also obvious that the way that one chooses the congruences that define one of the two groups is not unique.

In the first place, one can replace the *m* congruences λ, which define the first group, with *m* other congruences $\bar{\lambda}$ of that same subspace, which are therefore given by the transformation formulas:

$$\bar{\lambda}_{h|i} = \sum_{k=1}^{m} c_h^k\, \lambda_{k|i} \qquad (h = 1, 2, \ldots, m), \tag{24}$$

in which the *m* quantities c_h^k are invariants. Analogous formulas for the transformation of the differentials of the arc-length length will result from those formulas:

$$d\bar{s}_h = \sum_{k=1}^{m} c_h^k\, ds_k \qquad (h = 1, 2, \ldots, m). \tag{24′}$$

Since the quadratic form (1″) that defines the metric on V_n^m must remain invariant under those transformations, as in the case of V_n , it will result that c_h^k are the coefficients of an orthogonal substitution. Therefore, as in the case of V_n , one will also have the formulas:

$$(24'')\qquad \begin{cases} \overline{\lambda}_h^i = \sum_{k=1}^{m} c_h^k \lambda_k^i, \\ \lambda_{l|i} = \sum_{k=1}^{m} c_k^l \overline{\lambda}_{k|i}, \\ \lambda_l^i = \sum_{k=1}^{m} c_k^l \lambda_k^i, \end{cases} \qquad (h, l = 1, 2, \ldots, m).$$

We now move on to the second group of congruences that are determined by equations (9), and the situation is completely different for them.

Indeed, the system of equations (9) admits the following group of transformations:

$$(25)\qquad \overline{\lambda}_{h'|i} = \sum_{k'=m+1}^{n} c_{h'}^{k'} \lambda_{k'|i} \qquad (h' = m+1, 2, \ldots, n),$$

where the invariant quantities satisfy $c_{h'}^{k'}$ only the condition that their determinant must be non-zero. Because of that, formulas (25) will be invertible, and one will consequently have:

$$(25')\qquad \lambda_{h'|i} = \sum_{k'=m+1}^{n} \overline{c}_{l'}^{k'} \overline{\lambda}_{l'|i},$$

as well, in which $\overline{c}_{l'}^{k'}$ is the inverse of the determinant of the $c_{h'}^{k'}$.

It is interesting to observe that although the transformations (24) and (25) do leave the metric in V_n^m invariant, they do not generally leave the one in V_n invariant, and therefore they do not leave the coefficients a_{ij} of the quadratic form invariant either. In fact, if $\overline{a}_{ij}$ denote the coefficients of the metric that corresponds to the n congruences $\overline{\lambda}$ then, based upon (5), one will have:

$$\overline{a}_{ij} = \sum_{h=1}^{m} \overline{\lambda}_{h|i} \overline{\lambda}_{h|j} + \sum_{h'=m+1}^{n} \overline{\lambda}_{h'|i} \overline{\lambda}_{h'|j}.$$

By virtue of (24) and (25) and the orthogonality of c_h^k, that formula can be written:

$$\overline{a}_{ij} = \sum_{h=1}^{m} \lambda_{h|i} \lambda_{h|j} + \sum_{h',k',l'=m+1}^{n} c_{k'}^{h'} c_{h'}^{l'} \lambda_{k'|i} \lambda_{l'|j},$$

and it is enough to take into account those values of a_{ij} if one is to put those relations into the form:

$$(25'')\qquad \overline{a}_{ij} = a_{ij} + \sum_{h',k',l'=m+1}^{n} (c_{k'}^{h'} c_{h'}^{l'} - \delta_{h'}^{l'}) \lambda_{k'|i} \lambda_{l'|j}.$$

From this, one sees that the metric on V_n will remain invariant only in the case where $c_{k'}^{h'}$ can also be the coefficients of an orthogonal substitution.

In order to now find the transformation formulas for the parameters of the anholonomity congruences $\bar{\lambda}$ and λ, one takes into account the relations between the momenta and the parameters:

$$\sum_{i=1}^{n} \bar{\lambda}_{h'|i'} \bar{\lambda}_{l'}^{i} = \delta_{h'}^{l'} \qquad (h', l' = m + 1, \ldots, n)$$

that derive from the fact that the parameters are the inverses of the determinants that are composed of the momenta. One will easily find the following transformation formulas then:

$$(25''') \qquad \left\{ \begin{aligned} \bar{\lambda}_{l'}^{i} &= \sum_{\alpha'=m+1}^{n} \bar{c}_{l'}^{\alpha'} \lambda_{\alpha'}^{i}, \\ \lambda_{\alpha'}^{i} &= \sum_{l'=m+1}^{n} \bar{c}_{l'}^{\alpha'} \bar{\lambda}_{l'}^{i}. \end{aligned} \right.$$

It is probably superfluous to point out that the transformations (24) and (25) do not disturb the orthogonality of the two groups of congruences, in the sense that the relations:

$$\sum_{i=1}^{n} \bar{\lambda}_{h|i} \lambda_{h'}^{i} = 0 \qquad (h \leq m, h' > m)$$

will always be satisfied.

Tensors. – A system of certain quantities that are functions of the x that are invariant under coordinate transformations will be said to form a ***tensor*** *relative to the manifold* V_n^m *when the new quantities that result from performing the two transformations* (24) *and* (25) *in succession can be expressed as linear, homogeneous functions of the old ones whose coefficients are homogeneous functions of the same degree in* c_h^k *and* $c_{h'}^{k'}$ *that do not involve the derivatives of the c.*

One will be better able to see what that definition means after some examples that will be considered in what follows.

Let R be a vector in V_n^m (see § **3**) that is determined by its projections r_h onto the m fundamental congruences of V_n^m. It is obvious that the transformation (25) does not change those projections at all, because it does not change the fundamental congruences, and the transformations (24) will change the r_h according to the formulas:

$$(26) \qquad \bar{r}_h = \sum_{\alpha=1}^{m} c_{h'}^{\alpha} r_{\alpha} \qquad (h = 1, 2, \ldots, m),$$

Vector derived from a vector along a curve. – Let C be a curve in V_n^m whose cosines are u_l and whose arc-length is s, and consider the quantities:

(26′) $$(Dr_h) = \frac{dr_h}{ds} - \sum_{k,l=1}^{m} \gamma_{hkl}\, r_k\, u_l \qquad (h = 1, 2, \ldots, m),$$

in which r_h are the components of the vector R in V_n^m. In order to do that, in the first place, observe that the transformations (25) do not disturb (26′).

Indeed, it will obviously not change the components r_h and the cosines u_l , and by virtue of the formula (10), it will also leave invariant the RICCI rotation coefficients γ_{hkl} (h, k, $l \leq m$).

Therefore, consider the transformations (24) and let:

$$(D\bar{r}_h) = \frac{d\bar{r}_h}{ds} - \sum_{k,l=1}^{m} \bar{\gamma}_{hkl}\, \bar{r}_k\, \bar{u}_l$$

denote the quantities (26′) that correspond to the congruences $\bar{\lambda}$. If one takes into account formula (26), when differentiated with respect to the arc-length s, and that of the curve C and the formula:

$$\bar{\gamma}_{hkl} = \sum_{\rho,\delta=1}^{m} \left(\frac{dc_h^\delta}{ds_\rho} + \sum_{\alpha=1}^{m} c_k^\alpha\, \gamma_{\alpha\delta\rho} \right) c_h^\delta\, c_l^\rho \qquad (h, k, l \leq m),$$

which will be proved in § **7**, formula (30), then one will arrive with no difficulty with the relations:

$$(D\bar{r}_h) = \sum_{\alpha=1}^{m} c_h^\alpha (Dr_\alpha),$$

which express the (Dr_h) precisely and define a vector in V_n^m. Call that vector the *derived vector* to R along the curve C, just as one called its analogue in the case of a manifold V_n . It results from the equations of parallelism (15) that the derived vector is zero when the vector R is transported by constrained parallelism along C. It also results from this that the equations of parallelism in V_n^m have an invariant character with respect to the transformations (24) and (25).

Geodetic curvature. – Now suppose that the vector R is a unit vector that is tangent to the curve C, in which the case, formula (26′) will assume the form:

(26″) $$u_h = \frac{du_h}{ds} - \sum_{k,l=1}^{m} \gamma_{hkl}\, u_k\, u_l ,$$

and will obviously continue to form a vector in V_n^m that one calls the *geodetic curvature*. When one compares that equation with the geodetic equation (16), it will result that the geodetics of V_n^m are the curves in V_n^m *that have zero geodetic curvature*, and that the equations of those geodetics have an invariant form with respect to the transformations (24) and (25). That geodetic curvature vector is precisely the projection into V_n^m of the geodetic curvature vector of C in V_n, but calculated based upon formulas (10), which are true in V_n^m. If one multiplies (26″) by u_h and sums then one will get zero, which is to say that the geodetic curvature vector of the curve C is normal to that curve.

Directly or inversely interior and exterior tensors. – Only the coefficients c_h^k of (24) enter into the transformation laws that were considered so far. As a result of that fact, the vectors or tensors that have that property will be called *interior* vectors or tensors in V_n^m.

A tensor whose transformation laws are of degree p in the c_h^k and degree q in the $c_{h'}^{k'}$ will be called *interior* of order p and *exterior* of order q. Moreover: A tensor that is supposed to be exterior of order one will be called *directly exterior* if its components change like the momenta of the congruence λ [formulas (25)], and on the contrary, one will call it *inversely exterior* if its components change like parameters [formulas (25‴)] ([1]).

It is interesting to observe that those anholonomic manifolds are metric manifolds only with respect to the fundamental congruences, but not with respect to the anholonomity congruences, which can be subjected to a general linear transformation of the type (25). Therefore, one cannot speak of the directly exterior or inversely exterior components of those tensors.

The $n - m$ components of an arbitrary vector in V_n with respect to the anholonomity congruences provide an example of a directly exterior vector in V_n^m, and the derivatives of a function of position along the anholonomity congruence will form an inversely exterior vector.

Tensorial character of the integrability conditions. – Suppose that the transformations (24) and (25) have been performed, and let [see formulas (10″)]:

$$\bar{w}_{kl}^{h'} = \sum_{i,j=1}^{n} \left(\frac{d\bar{\lambda}_{h'|i}}{dx_j} - \frac{d\bar{\lambda}_{h'|j}}{dx_i} \right) \bar{\lambda}_k^i \, \bar{\lambda}_l^j$$

be the integrability conditions for the system of congruences $\bar{\lambda}$. By virtue of the derivatives of (25) and the first of formulas (24″), one will find the relations:

$$\bar{w}_{kl}^{h'} = \sum_{a,b=1}^{m} c_a^k \, c_b^l \sum_{\alpha'=m+1}^{n} c_{h'}^{\alpha'} \, w_{ab}^{\alpha'} , \tag{27}$$

([1]) In the interests of greater clarity, the primed indices that are intended to indicate the property of a tensor being directly exterior will be placed above, while the ones that relate to the inversely exterior tensors will be placed below.

which expresses precisely the idea that the integrability conditions (12) represent a tensor of order three in V_n^m that is twice interior and once directly exterior.

§ 7. – Fundamental formulas in V_n^m.

We have seen that in the case of a manifold V_n, the coefficients of the transformations (19) satisfy equations (21). We would now like to find the analogous equations for the coefficients c_h^k and $c_{h'}^{k'}$ in (24) and (25).

We begin with the transformations (24), from which we will get the c_h^k from the formulas:

$$(28)\qquad c_h^k = \sum_{i=1}^{n} \overline{\lambda}_{h|i}\, \lambda_k^i .$$

If one differentiates this with respect to s_l ($l \leq m$) then one will arrive at the formula:

$$(28)\qquad \frac{dc_h^k}{ds_l} = \sum_{i,j=1}^{n} \frac{d\overline{\lambda}_{h|i}}{dx_j}\lambda_k^i \lambda_l^j - \sum_{\alpha=1}^{m} c_h^\alpha \sum_{i,j=1}^{n} \frac{d\lambda_{\alpha|i}}{dx_j}\lambda_k^i \lambda_l^j$$

in an analogous way [see (20′)].

We shall also make use of the formulas:

$$(28'')\qquad \begin{cases} \dfrac{d\lambda_{h|i}}{dx_j} = \displaystyle\sum_{\alpha=1}^{n} \begin{Bmatrix} i\ j \\ \alpha \end{Bmatrix} \lambda_{h|\alpha} + \sum_{\alpha,\beta=1}^{n} \gamma_{h\alpha\beta}\, \lambda_{\alpha|i}\, \lambda_{\beta|j}, \\ \dfrac{d\overline{\lambda}_{h|i}}{dx_j} = \displaystyle\sum_{\alpha=1}^{n} \overline{\begin{Bmatrix} i\ j \\ \alpha \end{Bmatrix}} \lambda_{h|\alpha} + \sum_{\alpha,\beta=1}^{n} \overline{\gamma}_{h\alpha\beta}\, \overline{\lambda}_{\alpha|i}\, \overline{\lambda}_{\beta|j}, \end{cases}$$

but when we take into account the fact that the CHRISTOFFEL symbols in this case refer to two different metrics that are coupled by formulas (26).

By virtue of those formulas and (28), (28′) can be written as:

$$(29)\qquad \frac{dc_h^k}{ds} = \sum_{\alpha,\beta=1}^{m} \overline{\gamma}_{h\alpha\beta}\, c_h^k\, c_\beta^l - \sum_{\alpha=1}^{m} c_h^\alpha\, \gamma_{\alpha kl} + \sum_{i,j,r=1}^{n} \rho_{ij}^r\, \overline{\lambda}_{h|r}\, \lambda_k^i\, \lambda_l^j ,$$

in which the ρ_{ij}^r denote the third-order tensor (twice covariant and once contravariant):

$$(29)\qquad \rho_{ij}^r = \overline{\begin{Bmatrix} i\ j \\ r \end{Bmatrix}} - \begin{Bmatrix} i\ j \\ r \end{Bmatrix} .$$

Without performing all of the calculations that pertain to that tensor on the basis of formulas (26), one can account for the fact that all terms in that tensor have at least one of the parameters or momenta of the anholonomity congruence for a factor, which will always amount to zero when combined with the product $\bar{\lambda}_{h\,|r}\lambda_k^i\,\lambda_l^j$ (h, k, $l \leq m$).

Hence, (29) will take on the aspect of (21), and therefore:

$$\frac{dc_h^k}{ds^l} = \sum_{\alpha,\beta=1}^{m} \bar{\gamma}_{h\alpha\beta}\,c_h^k\,c_\beta^l - \sum_{\alpha=1}^{m} c_h^\alpha\,\gamma_{\alpha kl} \qquad (h, k, l \leq m). \tag{30}$$

Formulas (30) give only the first m intrinsic derivatives of the c_h^k. In order to find the other ones, one must differentiate with respect to $s_{l'}$ ($l' < m$), and therefore put l' in place of l in formula (28′). In that case, the product:

$$\rho_{ij}^r\,\bar{\lambda}_{h|r}\,\lambda_k^i\,\lambda_{l'}^j$$

will no longer be zero. In order to arrive at the result directly, observe that when one is given the orthogonality of the two groups of congruences, one will have the formula:

$$\sum_{j=1}^{n} \bar{\lambda}_{h|j}\,\lambda_{l'}^j = 0 \qquad (h \leq m\ ;\ l' = m + 1, \ldots, n),$$

which can be written:

$$\sum_{i,j,=1}^{n}\left(\frac{d\bar{\lambda}_{h|j}}{dx_i}\lambda_{l'}^j\,\lambda_k^i - \sum_{\alpha=1}^{m}\frac{d\lambda_{h|j}}{dx_i}\lambda_{l'}^j\,\lambda_k^i\right) = 0$$

when it is differentiated with respect to s_k ($k \leq m$).

When one subtracts that formula from (28′), in which one sets $l' > m$ in place of l, then when one also takes into account formulas (10″) and (28), one will obtain the desired formula:

$$\frac{dc_h^k}{ds_{l'}} = \sum_{\alpha'=1}^{n}\sum_{\alpha=1}^{m} \bar{w}_{\alpha\alpha'}^h\,c_\alpha^k\,c_{\alpha'}^{l'} - \sum_{\alpha=1}^{m} w_{kl}^\alpha\,c_h^\alpha\ , \tag{31}$$

which is essentially different from (30), as one sees.

We shall now move on to the transformations (25), from which we will get the values of $c_{h'}^{k'}$ in the form:

$$c_{h'}^{k'} = \sum_{i=1}^{n} \bar{\lambda}_{h'|i}\,\lambda_{k'}^i\,.$$

Here, we must also first look for the first m intrinsic derivatives, and we will find the formulas:

$$\frac{dc_{h'}^{k'}}{ds^l} = \sum_{\alpha=1}^{m}\sum_{\alpha'=1}^{n} \bar{w}_{\alpha'\alpha}^{h'}\,c_{\alpha'}^{k'}\,c_\alpha^l - \sum_{\alpha'=1}^{m} c_{h'}^{\alpha'}\,w_{k'l}^{\alpha'} \tag{32}$$

from a calculation that is analogous to the preceding one.

One cannot simplify the form of (29) as far as the last $n - m$ intrinsic derivatives of the $c_{h'}^{k'}$ are concerned, and therefore it will be written:

(33) $$\frac{dc_{h'}^{k'}}{ds_{l'}} = \sum_{\alpha',\beta'=m+1}^{n} \bar{\gamma}_{h'\alpha'\beta'}\, c_{\alpha'}^{k'}\, c_{\beta'}^{l} - \sum_{\alpha'=m+1}^{n} c_{h'}^{\alpha'}\, \gamma_{\alpha' k' l'} + \sum_{\alpha'=m+1}^{n} c_{h}^{\alpha'}\, \varepsilon_{k'l'}^{\alpha'} ,$$

in which one sets:

(32′) $$\varepsilon_{k'l'}^{\alpha'} = \sum_{i,j,r=1}^{n} \rho_{ij}^{r}\, \lambda_{\alpha|r}\, \lambda_{k'}^{i}\, \lambda_{l'}^{j} ,$$

for simplicity.

The quantity $\varepsilon_{k'l'}^{\alpha'}$, which cannot be eliminated from (33), will be especially interesting in what follows. We shall try to prove that no metric exists inside of the anholonomity congruence.

If we now look for the interior derivatives of the $\bar{c}_{h'}^{k'}$, instead of the $c_{h'}^{k'}$, then we will arrive at the formulas:

(32″) $$\frac{d\bar{c}_{h'}^{k'}}{ds_{l'}} = -\sum_{\alpha=1}^{m}\sum_{\alpha'=1}^{n} \bar{w}_{h'\alpha}^{\alpha'}\, \bar{c}_{\alpha'}^{k'}\, c_{\alpha}^{l} + \sum_{\alpha'=m+1}^{n} w_{\alpha' l}^{k'}\, \bar{c}_{h'}^{\alpha} ,$$

which will also be useful in what follows.

Given the symmetry of the $\varepsilon_{k'l'}^{\alpha'}$ in the lower indices, we have the formulas:

(33′) $$\frac{dc_{h'}^{k'}}{ds_{l'}} - \frac{dc_{h'}^{l'}}{ds_{k'}} = \sum_{\alpha,\alpha'=m+1}^{n} \bar{w}_{\alpha'\beta'}^{h'}\, c_{\alpha'}^{k'}\, c_{\beta'}^{l'} - \sum_{\alpha'=m+1}^{n} w_{k'l}^{\alpha'}\, c_{h'}^{\alpha'} ,$$

which determine the difference between the exterior derivatives of the $c_{h'}^{k'}$ and the formulas in which the $\varepsilon_{k'l'}^{\alpha'}$ no longer appear.

§ 8. – Tensorial derivation.

Definitions ([1]). – If one is given an interior tensor in V_n^m and the transformation law for its components with respect to the transformations (24) then one can differentiate that law along an arc of the fundamental congruence. The first *m* intrinsic derivatives of the coefficients c_h^k will appear in the result. If one eliminates those derivatives with the help of formulas (30) and combines, on the one hand, the quantities that refer to the congruences λ and, on the other, the ones that refer to the congruences $\bar{\lambda}$ then one will find the transformation law for a new interior tensor whose components will be called

([1]) See the footnote on page 18.

interior tensorial derivatives of the given tensor. The order of that derived tensor is greater by one than that of the given tensor.

If one differentiates that tensor along an arc of the anholonomity congruence and eliminates the derivatives of the c_h^k with the help of (31) then one will find the transformation laws of exterior tensorial derivatives that specify a tensor that is as many time interior as the given tensor and more than once inversely exterior.

If one is now given a tensor that is one or more times exterior then one can find the interior tensorial derivative of that tensor with the same method. However, if one would like to find the exterior derivatives then one must also use formulas (33) and therefore the quantities $\varepsilon_{k'l'}^{\alpha'}$ will appear in the result, which means that the exterior derivatives are not tensors. It then results that when one starts from an interior tensor, it is not possible to find tensors that are one or more times exterior by derivation, and when one starts from an exterior or mixed tensor, one cannot increase the exterior order by derivation. One knows only one tensor in V_n^m that is twice exterior, which is specified by the integrability conditions of the fundamental congruences and which transforms according to the law:

$$w_{p'q'}^h = \sum_{\alpha=1}^m c_\alpha^h \sum_{\alpha',\beta'=m+1}^n \overline{w}_{\alpha'\beta'}^\alpha \, c_{\alpha'}^{p'} \, c_{\beta'}^{q'} \quad ,$$

and according to the last observation, its interior tensorial derivatives will also have that property.

Example. – If one is given an interior vector R then determine its components r_h . First, differentiate the transformation law (26) for that vector with respect to one arc-length $\overline{s}_k$ of the fundamental congruences $\overline{\lambda}$, and if one takes formulas (30) into account then one will find the tensorial relations:

$$\overline{r}_{h|k} = \sum_{\alpha,\beta=1}^m r_{\alpha|\beta} \, c_h^\alpha \, c_k^\beta \,, \tag{34}$$

in which the interior derivatives $r_{h\,|\,k}$ are given by the formulas:

$$r_{h\,|\,k} = \frac{dr_h}{ds_k} - \sum_{\alpha=1}^m r_\alpha \, \gamma_{h\alpha k} \,, \tag{34′}$$

and are in the same form as if space were completely Riemannian [see (22′)].

In order to find the exterior derivatives of R, differentiate (26) with respect to an arc-length $\overline{s}_k$, so that one has:

$$\frac{d\overline{r}_h}{ds_{k'}} = \sum_{\alpha=1}^m c_h^\alpha \, \frac{dr_\alpha}{d\overline{s}_{k'}} + \sum_{\alpha=1}^m r_\alpha \, \frac{dc_h^\alpha}{d\overline{s}_{k'}} \,.$$

If one takes into account the fact that the exterior intrinsic derivatives are inversely exterior vectors, along with formulas (31), one can give (35) the tensorial form:

(35′) $$\bar{r}_{h\,|\,k'} = \sum_{\alpha=1}^{m} c_h^{\alpha} \sum_{\alpha'=m+1}^{n} \bar{c}_{k'}^{\alpha'}\, r_{\alpha|\alpha'} \;.$$

In that formula, the $r_{h\,|\,k'}$ denote the last $n - m$ tensorial derivatives of the vector R, which are determined on the basis of the expressions:

(36) $$r_{h\,|\,k'} = \frac{dr_h}{ds_{k'}} - \sum_{\alpha=1}^{m} w_{\alpha k}^{h}\, r_{\alpha} \;.$$

As one sees, those derivatives, which are the components of a second-order tensor that is once interior and once inversely exterior, are essentially different from the derivatives that one would have if space were Riemannian.

We now move on to the determination of the tensorial second derivatives of the vector R, that is to say, the first derivatives of the tensors $r_{h\,|\,k}$ and $r_{h\,|\,k'}$.

We first focus our attention on the first m derivatives of the tensor (34′). If we differentiate that formula with respect to s_l and make use of (30) then we will easily find the expressions for the interior tensorial second derivatives:

(36′) $$r_{h\,|\,k\,l} = \frac{dr_{h|k}}{ds_l} - \sum_{\alpha=1}^{m} r_{\alpha|k}\, \gamma_{h\alpha l} - \sum_{\alpha=1}^{m} r_{h|\alpha}\, \gamma_{k\alpha l} \;,$$

which are obviously the components of a third-order interior tensor.

If we now differentiate formulas (34′) with respect to $s_{k'}$ then it will be enough to take (31) into account if we are to arrive at the following second-order tensorial derivatives:

(37) $$r_{h\,|\,k\,k'} = \frac{dr_{h|k}}{ds_{k'}} - \sum_{\alpha=1}^{m} r_{\alpha|k}\, w_{\alpha k'}^{h} - \sum_{\alpha=1}^{m} r_{h|\alpha}\, w_{\alpha k'}^{k} \;.$$

They are the components of a third-order tensor in V_n^m that is twice interior and once inverse exterior.

We now move on to the tensor $r_{h\,|\,k'}$ and differentiate (35′) with respect to s_k. If we take (32″) into account then we will get the following tensorial derivatives with no difficulty:

(37′) $$r_{h\,|\,k'k} = \frac{dr_{h|k'}}{ds_k} - \sum_{\alpha=1}^{m} r_{\alpha|k'}\, \gamma_{h\alpha k} - \sum_{\alpha'=m+1}^{n} r_{h|\alpha}\, w_{k'k}^{\alpha'} \;,$$

which are the components of a third-order tensor that is twice interior and once inversely exterior.

Formulas (36′), (37), and (37′) represent all of the second-order tensorial derivatives of the vector R, which specify tensors in V_n^m.

§ 9. – Principal tensors in a V_n^m.

We would now like to find the expressions for two fourth-order tensors by considering the differences of the second derivatives of the vector R that were found in the preceding paragraph. One of those tensors is interior and the other one is three-times interior and once inversely exterior.

In order to arrive at the expressions for the first tensor, consider the difference of the second derivatives in (36′):

$$\Delta_{kl}^h = r_{h\,|\,kl} - r_{h\,|\,lk} = \frac{d^2 r_h}{ds_l\, ds_k} - \frac{d^2 r_h}{ds_k\, ds_l} + \ldots \tag{38}$$

In order to avoid excessively long calculations, observe that the difference in question is expressed by formula (23′) in the case of a V_n. In that case, the unwritten terms on the right-hand side of formula (38) will be the same as the ones for a manifold V_m that is determined by the m fundamental congruences λ of V_n^m. However, that is no longer true for the written terms, which can be written:

$$\frac{d^2 r_h}{ds_l\, ds_k} - \frac{d^2 r_h}{ds_k\, ds_l} = \sum_{\alpha=1}^{m} \frac{dr_h}{ds_\alpha} w_{kl}^\alpha - \sum_{\alpha'=m+1}^{n} \frac{dr_h}{ds_{\alpha'}} w_{kl}^{\alpha'}, \tag{38′}$$

from a known formula in V_n, where in the right-hand side we have divided the sum into two parts in order to exhibit the elements that relate to the anholonomity congruences. If the anholonomity relations (9) are completely integrable then the integrability conditions $w_{kl}^{\alpha'}$ will all be zero, and one will have the formula:

$$\Delta^h = \sum_{\alpha=1}^{m} r_\alpha\, \gamma_{\alpha h,kl},$$

in which the four-index RICCI coefficients refer to only the m fundamental congruences, and are therefore given by the expressions:

$$\gamma_{\alpha h,\,kl} = \frac{d\gamma_{\alpha hk}}{ds_l} - \frac{d\gamma_{\alpha hl}}{ds_k} + \sum_{i=1}^{m} [\gamma_{\alpha hi}(\gamma_{ikl} - \gamma_{ilk}) + \gamma_{i\alpha l}\, \gamma_{ihl} - \gamma_{i\alpha k}\gamma_{ikl}]. \tag{38″}$$

In the general case, (38) can be written:

$$\Delta_{kl}^h = \sum_{\alpha=1}^{m} r_\alpha\, \gamma_{\alpha h,kl} - \sum_{\alpha'=m+1}^{n} \frac{dr_h}{ds_{\alpha'}} w_{kl}^{\alpha'}, \tag{39}$$

and it is obviously intended that the $\gamma_{\alpha h,kl}$ are always determined by (38″). It is also obvious that the difference Δ_{kl}^h specifies a fourth-order interior tensor, but in order to exhibit that fact in the right-hand side of (39), as well, recall formula (36), which gives the exterior derivatives of the vector *R*, from which one has:

$$\frac{dr_h}{ds_{\alpha'}} = r_{h\,|\,\alpha'} + \sum_{\alpha=1}^{m} w^h_{\alpha\alpha'}\, r_\alpha\,.$$

By virtue of that relation, (39) assumes the definitive form:

$$(39')\qquad \Delta^h_{kl} = \sum_{\alpha=1}^{m} \lambda_{\alpha h,kl}\, r_\alpha - \sum_{\alpha'=m+1}^{n} r_{h|\alpha'}\, w^{\alpha'}_{kl}\,,$$

in which one sets:

$$(40)\qquad \lambda_{\alpha h,\,kl} = \gamma_{\alpha h,\,kl} - \sum_{\alpha'=m+1}^{n} w^h_{\alpha\alpha'}\, w^{\alpha'}_{kl}\,,$$

for simplicity.

Given the fact that the last term in the right-hand side of (39′) is obviously the component of a third-order interior tensor, the same thing will be true for the left-hand side, since the quantities $\lambda_{\alpha h,kl}$ specify a fourth-order interior tensor. That tensor is antisymmetric in the last two indices, because the $w^{\alpha'}_{kl}$, as well as the $\gamma_{\alpha h,kl}$, are antisymmetric in the indices *k*, *l*. However, the latter are no longer antisymmetric in the first indices, so we will have the formula:

$$(39'')\qquad \gamma_{\alpha h,\,kl} + \gamma_{h\alpha,\,kl} = \sum_{\alpha'=m+1}^{n} v_{h\alpha,\alpha'}\, w^{\alpha'}_{kl}\,,$$

in which we have set:

$$(40')\qquad v_{h\alpha,\,\alpha'} = \gamma_{\alpha h,\,kl} + \gamma_{h\alpha,\,kl}\,.$$

It will then result that the quantity $v_{h\alpha,\,\alpha'}$ also specifies a fourth-order tensor that is twice interior and once inversely exterior.

In order to find the second tensor of which we spoke at the beginning of this paragraph, consider the difference:

$$(41)\qquad \Delta^h_{kl} = r_{h\,|\,k'k} - r_{h\,|\,kk'},$$

in which the $r_{h\,|\,kk'}$ and $r_{h\,|\,k'k}$ are defined by formulas (37) and (37), since that difference can be written:

$$(41')\qquad \Delta^h_{kl} = \frac{dr_{h|k'}}{ds_k} + \sum_{\alpha=1}^{m} r_{\alpha|k'}\,\gamma_{h\alpha k} + \sum_{\alpha'=m+1}^{n} r_{h|\alpha'}\, w^{\alpha'}_{k'k} - \frac{dr_{h|k}}{ds_{k'}} + \sum_{\alpha=1}^{m} r_{\alpha|k}\, w^h_{\alpha k'} + \sum_{\alpha'=m+1}^{n} r_{h|\alpha}\, w^k_{\alpha k'}$$

$$(h,\, k \leq m\,;\, k' > m).$$

Now recall formulas (34′) and (36), which define the first tensorial derivatives of the vector R, and first consider the terms in the right-hand side of (41′) that contain the first and second derivatives of r_h , and which can be written:

$$\frac{ds_k\, ds_{k'}}{d^2 r_h} - \frac{d^2 r_h}{ds_k\, ds_{k'}} + \sum_{\alpha=1}^{m}\left(\frac{dr_\alpha}{ds_k} w^h_{\alpha k'} - \frac{dr_\alpha}{ds_{k'}} \gamma_{h\alpha k}\right) - \sum_{\alpha=1}^{m}\left(\frac{dr_\alpha}{ds_{k'}} \gamma_{h\alpha k} - \frac{dr_\alpha}{ds_k} w^h_{\alpha k'}\right) + \sum_{\alpha=1}^{m} \frac{dr_h}{ds_\alpha} w^k_{\alpha k'} + \sum_{\alpha'=m+1}^{n} \frac{dr_h}{ds_{\alpha'}} w^{\alpha'}_{k' k} \,.$$

If one takes the formula (38′) into account, which provides the difference between the second derivatives, and reduces the similar terms, then all of those terms will reduce to the following one:

$$\sum_{\alpha=1}^{m} \frac{dr_h}{ds_\alpha} v_{\alpha k, k'} \,,$$

in which the quantities $v_{\alpha k,k'}$ are given by (40′).

If one introduces the first tensorial derivatives in place of the dr_h / ds_α , based upon (34″), and performs the calculations then one can give (41) the form:

$$\Delta^h_{kl} = \sum_{\alpha=1}^{m} r_\alpha\, v_{h\alpha, kk'} + \sum_{\alpha=1}^{m} r_{h|\alpha}\, v_{\alpha k, k'} \,, \tag{42}$$

in which one sets:

$$v_{h\alpha k,k'} = \frac{d\gamma_{h\alpha k}}{ds^{k'}} - \frac{dw^h_{\alpha k'}}{ds^k} + \sum_{i=1}^{m} (\gamma_{h\alpha i} w^i_{kk'} + \gamma_{hik} w^i_{\alpha k} - \gamma_{i\alpha k} w^h_{ik'}) + \sum_{\alpha'=m+1}^{n} w^h_{\alpha\alpha'} w^{\alpha'}_{hk'} \,. \tag{40′}$$

It results with no further analysis that the quantities $v_{h\alpha k,k'}$ specify a fourth-order tensor that is three times interior and once inversely exterior.

§ 10. – Equations of geodetic variation ([1]).

In this paragraph, we would like to consider succinctly the equations of geodetic variation for the anholonomic manifold V_n^m in the general form and exhibit their invariant character with respect to the transformations (24) and (25) in V_n^m.

Let a curve (C) in the manifold V_n^m be defined by the equations:

$$x_i = \varphi_i(\sigma), \tag{C}$$

([1]) See my note: "Sullo scostamento geodetico nelle varietà anolonome," Rend. della R. Accad. dei Lincei (6) **7** (1928), pp. 134. In my article: "Sur l'écart géodésique dans les espaces non holonomes," Annales Scientifiques de l'Université de Jassy, t. XI, fasc. 1-2, pp. 7-24, due to the hypotheses that the displacement (Σ) is an interior vector in V_n^m, which is a hypothesis that is not generally compatible with the anholonomity relation, I did not find correct equations for the variations in certain particular cases, but the preliminary calculations and the methodology were valid. See also my note in the following issue of that review, which refers to that article.

in which σ indicates the arc-length of the curve. The cosines of the angles that that curve makes with the congruences λ are provided by the expressions:

$$c_h = \sum_{i=1}^{n} \lambda_{h|i}\, \varphi_i'(\sigma)\,.$$

A curve (c) that is close to (C) can always be determined by equations of the type:

$$(c) \qquad x_i = \varphi_i(\sigma) + \sum_{h=1}^{n} \lambda_h^i\, \varepsilon_h\,,$$

in which ε_h denotes the invariant components of the displacement vector, which takes (C) to the neighboring curve (c).

The cosines of the angles that the curve (c) makes with the congruences λ in V_n are given by the formulas:

$$(42') \qquad u_h = \frac{d\sigma}{ds}\left(c_h + \frac{d\varepsilon_h}{d\sigma} + \sum_{k,l=1}^{n} w_{kl}^h\, c_k\, \varepsilon_l\right) \qquad (h = 1, 2, \ldots, n),$$

in which s denotes the arc-length of (c), and the w have values that they must have along the curve (C). Those formulas are obviously calculated under the hypothesis that the ε_h are first-order quantities.

On the basis of the formulas that define the invariant components of the derived vector of (ε) along (C), those formulas can also be written:

$$(42'') \qquad u_h = \frac{d\sigma}{ds}\left(c_h + (D\varepsilon_h) + \sum_{k,l=1}^{m} \gamma_{hkl}\, c_k\, \varepsilon_l\right).$$

If one introduces those values into the quadratic relations for the cosines then one will get a value for the ratio:

$$\frac{d\sigma}{ds} = 1 - \mu,$$

in which μ denotes the first-order quantity:

$$\mu = \sum_{h=1}^{n} c_h (D\varepsilon_h)\,.$$

Now suppose that (C) is a geodetic in V_n and demand that (c) should also be a geodetic in V_n . In order for that to be true, it is enough to impose the condition on the cosines of (c) that are provided by formulas (42″) that they must satisfy the geodetic equations in V_n . One will then arrive at the following equations of variation:

$$(D^2\varepsilon_h) - \frac{d\mu}{d\sigma} c_h = \sum_{k,lr=1}^{n} \gamma_{hk,lr}\, c_k\, c_l\, \varepsilon_r\,, \tag{43}$$

which are nothing but the invariant components of the vector that provides LEVI-CIVITA's equations of variations ([1]).

We now move on to the anholonomic manifolds and suppose that the curve (C) is situated in V_n^m, that is to say, that the last $n - m$ cosines $c_{h'}$ ($h' > m$) are zero.

The cosines between the curve and a neighboring curve are given by the same formulas (42′), since the displacement is always an arbitrary vector in V_n , and in particular, the last $n - m$ formulas (42′) are written:

$$u_{h'} = \frac{d\sigma}{ds}\left(\frac{d\varepsilon_{h'}}{d\sigma} + \sum_{l'=m+1}^{n}\sum_{k=1}^{m} w_{kl'}^{h'}\, c_k\, \varepsilon_{l'} + \sum_{k,l=1}^{m} w_{kl}^{h'}\, c_k\, \varepsilon_l\right),$$

which say precisely that the cosines between (c) and the anholonomity congruence are first-order quantities.

If one would now desire that the curve (c) should also be situated in V_n^m then one must have the equations:

$$\frac{d\varepsilon_{h'}}{d\sigma} + \sum_{l'=m+1}^{n} \varepsilon_{l'} \sum_{k=1}^{m} w_{kl'}^{h'}\, c_k + \sum_{k,l=1}^{m} w_{kl}^{h'}\, c_k\, \varepsilon_l = 0 \qquad (h' = m + 1, \ldots, n), \tag{43′}$$

which represent $n - m$ linear differential equations in the $n - m$ quantities $\varepsilon_{h'}$ ($h' > m$) whose known terms are linear an homogeneous in the first ε_h ($h \le m$). One immediately sees that these differential equations cannot admit the solution $\varepsilon_{h'} = 0$, *which says that the displacement is itself situated in* V_n^m, in which case, one will have the relations:

$$\sum_{k,l=1}^{m} w_{kl}^{h'}\, c_k\, \varepsilon_l = 0$$

along (C).

Those relations can be valid for an arbitrary curve (C) and displacement (ε) only in the case where the $w_{kl}^{h'}$ are zero, or what amounts to the same thing, when the anholonomity relations are completely integrable. Therefore, in an effectively anholonomic manifold, one must always associate a neighboring curve (c) with the differential equations (43′) in order to determine the $n - m$ $\varepsilon_{h'}$ as functions of the arc-length σ when the m ε_h are known.

Now suppose that (C) is a geodetic in V_n^m, and therefore that the cosines c_h satisfy equations (16). In order for (c) to also be a geodetic in V_n^m, its cosines u_h must also

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([1]) See T. LEVI-CIVITA, *The absolute differential calculus,* edited by E. PERSICO, Blackie, London, 1927, pp. 215, formulas (57).

satisfy equations (16), and when one performs the calculations, one will find the following equations of variation:

$$(D^2\,\varepsilon_h) - \frac{d\mu}{d\sigma}c_h = \sum_{k,l,r=1}^{m}\lambda_{kh,lr}\,c_k\,c_l\,\varepsilon_r + \sum_{k,l=1}^{m}c_k\,c_l\sum_{r'=m+1}^{n}\nu_{hk,lr'}\,\varepsilon_{r'}\,, \tag{44}$$

in which the $(D^2\,\varepsilon_h)$ represent the components of the second derived vector in V_n^m [see formulas (26′)] of the projection of the displacement (ε) into V_n^m along our curve (C), and the $\lambda_{hk,lr}$ and $\nu_{hk,lr'}$ are the components of the tensors (40) and (40′), resp. As for the quantity μ, it is always determined by the quadratic relation between the cosines and can be written:

$$\mu = \sum_{h=1}^{m}c_h\,(D\varepsilon_h) + \tfrac{1}{2}\sum_{l'=m+1}^{n}\varepsilon_{l'}\sum_{h,k=1}^{m}\nu_{hk,l'}\,c_h\,c_k\ .$$

Equations (44), along with (43), define a system of differential equations of order $n - m$ for the determination of the n unknowns ε.

In order to confirm that this system has an invariant character with respect to the transformations of the congruences (24) and (25), it is enough to observe that the left-hand side of (44), as well as the right, gives the components of an interior vector. As for (43′), since they are exterior cosines, they will be the components of a directly exterior vector.