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On an anholonomic unitary theory of physical fields ([1])

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Summary. – One sees how one may construct a unitary theory of fields, both gravitational and electromagnetic, by starting with an anholonomic hypersurface V_5^4 that is totally geodesic. This amounts to supposing that physical space is locally four-dimensional, like the space of the relativity theory, but that when one starts at a point P, one may not return to it by an infinitesimal circuit, but only reach a point P´, where the direction PP´ is normal to the local space at P, and the segment PP´ thus defines the torsion of the space. With the aid of the curvature tensor of the local spaces and the torsion tensor, which is considered to be the electromagnetic tensor, one writes the Einstein and Maxwell equations, and one assumes that the geodesic equations of V_5^4 represent the equations of an electrically charged particle, such that the trajectories of light are the special case of auto-parallel geodesics of null length.

In his celebrated general theory of relativity, Einstein gave an interpretation of gravitation as a characteristic property of physical space, which he considered to be a four-dimensional Riemann space with an indefinite quadratic form:

$$ds^2 = a_{ij}\, dx^i\, dx^j \qquad (i, j = 1, 2, 3, 4) \tag{1}$$

in which x^1, x^2, x^3 are treated like the spatial coordinates and x^4 is treated like the time coordinate. Since then, physics has sought to find an analogous interpretation for electromagnetic phenomena, or, more precisely, *a unitary theory* that is capable of explaining Einstein's gravitational equations and Maxwell's equations of electromagnetism by means of the same geometrical principle.

That is why Weyl ([2]), starting with the observation that the electromagnetic field may be defined in spacetime V_4, in the absence of true magnetism, by the rotation:

$$\varphi_{ij} = \frac{\partial \varphi_i}{\partial x^j} - \frac{\partial \varphi_j}{\partial x^i} \tag{1′}$$

of a covariant vector φ_i (i = 1, 2, 3, 4), has proposed that one consider spacetime V_4 to be a space such that when one passes from one point to another, the unit of length for the metric possesses a coefficient of dilatation that is given by the Pfaff form:

[1] This anholonomic unitary theory was the object of a conference *Les espaces non holonome et leurs applications mécanique* at the Institute H. Poincaré, 3 June 1935, at the invitation of the Faculté des Sciences of the University of Paris. It was also summarized in the notes: *La théorie unitaire des champs et les hypersurfaces non holonomes,* Comtes rendus, 1935, **200**, pp. 2056, and in *Sur une théorie unitaire…*, C.R. Ac. Sc. Roumanie, tome I, 1936.

[2] See H. Weyl, *Raum, Zeit, Materie,* 5th ed., Berlin, Springer, 1923, pp. 121.

$$d\varphi = \varphi_i \, dx^i \tag{1''}$$

that one may construct from the electromagnetic potential world-vector.

Another unitary theory was proposed by Kaluza ([1]), who considered the physical continuum to be a five-dimensional Riemann space V_5 whose metric may be written:

$$d\sigma^2 = ds^2 + 2\varphi_i \, dx^i + a_{55} \, (dx^5)^2 , \tag{2}$$

in which ds^2 is given by the metric (1) of the spacetime V_4 of relativity theory. One sees that in Kaluza's theory the metric on the space V_5 is defined when one is given the metric on spacetime V_4, the electromagnetic potential four-vector φ_i , and the coefficient a_{55}.

This hypothesis that the physical world is five-dimensional, when it only seems to have four, and the difficulty of finding a natural interpretation for the coefficient a_{55} have led Einstein and Mayer to propose a unitary theory ([2]) that seeks to avoid these inconveniences of Kaluza's theory, while preserving the principal hypothesis of the existence of a metric space V_5 that is associated with the spacetime V_4 of relativity theory, with the difference that one now supposes that the space V_5 is only a vector space, which avoids the direct consideration of the coefficient a_{55}. As for the method, which plays a major role in the theory of Einstein and Mayer, it was greatly inspired by the method of congruences (tétrapodes) that was considered in the unitary theory that has been proposed by Einstein several times before and was given a remarkable systematization by Levi-Civita ([3]) with the aid of Ricci's notion of pseudo-orthogonal congruences. As one knows, the unitary theory of Einstein, which was also subjected to a profound study by Cartan ([4]), consists of considering the physical world to be a four-dimensional Riemann space that is also endowed with absolute parallelism, and the *torsion* of this absolute parallelism that measures the electromagnetic field.

In this new unitary theory, Einstein and Mayer suppose that the physical world is four-dimensional, namely, the spacetime V_4 of relativity theory, but that this space is found to be embedded in a five-dimensional metric vector space V_5. This is realized in such a manner that at each point of V_4 we have, in addition to four independent directions on V_4, which one may call "interior," or tangent, directions, also an "exterior," or normal, direction to V_4. Moreover, one supposes that one has a metric, not only for the tangent directions, which is the metric of V_4, but also a metric for the non-tangent direction.

One may also say that one associates the spacetime V_4 with a vector space V_5, in such a fashion that the linear space that is tangent to V_4 may be defined at each point of V_4 by a

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[1] See. Th. Kaluza, *Zum Unitätsproblem der Physik*, Sitzungsberichte der preuss. Ak. der Wiss., 1921, pp. 966.

[2] See A. Einstein and W. Mayer, *Einheitliche Theorie von Gravitation und Electrizität*, Sitzungsberichte Akademie, Berlin, 1931, 1.

[3] See T. Levi-Civita, Vereinfachte Herstellung der Einsteinschen einheitlichen Feldgleichungen, *ibid.*, 1929, pp. 137.

[4] See E. Cartan, *Sur la théorie des systemes en involution et ses applications à la relativité*, Bull. de la Soc. Math. de France, 1931, **59**, pp 88.

hyperplane in V_5 that one calls the *distinguished hyperplane*. Moreover, one may arrange this in such a fashion that the components g_{ab} (a, b = 1, 2, … , 5) of the metric tensor of this vector space V_5 are given by the formulas:

$$g_{ij} = a_{ij}\,, \quad g_{i5} = 0, \qquad g_{55} = 1 \; (i, j = 1, 2, 3, 4), \tag{2′}$$

where a_{ij} are the components of the metric tensor of V_4 [Einstein and Mayer formulas (50), (51), (52)]. Having said this, if one considers the Levi-Civita parallel transport of vectors along paths tangent to V_4, which are, moreover, the only possible paths in this theory, and if one imposes the condition that *when a vector that is tangent to V_4 is displaced along its proper direction remains a vector tangent to* V_4, then one finds an anti-symmetric tensor F_{ij} of second order that Einstein and Mayer considered to be the tensor generator of the electromagnetic field.

In order to arrive at this association of the spacetime V_4 with the vector space V_5 in a natural manner, at least from the mathematical point of view, Veblen ([1]), Schouten ([2]), etc., have also considered the projective properties of spacetime V_4, and thus created an important *projective unitary theory* by starting with the theory of Einstein and Mayer.

However, without leaving the metric domain, one may remark that on account of formula (2´) the metric of the space V_5 may be written in the form:

$$d\sigma^2 = ds^2 + (ds^5)^2, \tag{2″}$$

where ds^2 is the metric on V_4 and ds^5 is an arbitrary Pfaff form in the variables x^1, x^2, x^3, x^4, and a new variable x^5. This new variable is determined only by abstraction on a transformation:

$$x^{5'} = f(x^1, x^2, x^3, x^4, x^5), \tag{2‴}$$

which allows us to write the form ds^5 in the form:

$$ds^5 = dx^5 - \varphi_i\, dx^i \qquad (i = 1, 2, 3, 4). \tag{3}$$

This being the case, if we suppose that φ_i depends explicitly only upon x^5 then one finds that the electromagnetic tensor F_{ij} is related the rotation φ_{ij} of the covariant vector φ_i by the following very simple formula:

$$\varphi_{ij} = 2F_{ij}\,. \tag{3′}$$

One thus sees that this unitary theory of Einstein and Mayer may be related to the Pfaff form (3), or, more precisely, to the Pfaff equation:

$$ds^5 = dx^5 - \varphi_i\, dx^i = 0, \tag{3″}$$

[1] See O. Veblen. *Projektive Relativitätstheorie*, Springer, Berlin, 1933.

[2] See J. A. Schouten, *La théorie projective de la relativité*, Annales de l'Institut H. Poincaré, volume V, 1935, pp. 49.

which represents the distinguished hyperplane in V_5, moreover.

This fact has led me to look for an interpretation of this unitary theory with the aid of the anholonomic hypersurface V_5^4 that is defined in V_5 by this Pfaff equation. One thus arrives at an interpretation that gives us the possibility of constructing all of that unitary theory by starting with only two invariants: the metric (1) and the Pfaff form (3); i.e., with only the knowledge of two fields: the gravitational field and electromagnetic field. The geometry of the anholonomic hypersurface that one thus considers possesses two fundamental tensors in our case: a *curvature tensor*, namely, the curvature tensor of V_4, and a *torsion tensor*, which is defined by the electromagnetic tensor F_{ij}. It is with the aid of these two tensors that one forms the gravitational equations and the Maxwell equations.

Our anholonomic interpretation ([1]) amounts to saying that the physical world is locally four-dimensional, as is natural, but that the totality of these local spaces may not be regarded as the set of local tangent spaces to the same space V_4. These local spaces are the tangent spaces to anholonomic hypersurface V_5^4, just as the planes in a linear complex are the planes that are tangent to a anholonomic surface, since the total differential equation of the linear complex is not completely integrable. Moreover, one may remark that the use of anholonomic spaces as a basis for a unitary theory of our physical world may be considered to be very natural if one thinks that the anholonomic spaces are obtained by the geometrical interpretation of anholonomic systems in mechanics.

We have divided this memoir into three chapters. In the first chapter, we shall see how one may study the Riemann spaces whose metric is not a positive-definite form on the basis of the notation of the group of transformations of a Pfaff form, by a method that is analogous to Ricci and Levi-Civita's method of orthogonal congruences in spaces with positive-definite metrics. The notation of transformation group for a Pfaff form is, moreover, at the basis for all of our considerations.

In the second chapter, we shall recall a certain number of properties of anholonomic spaces, and in the third chapter we shall give an anholonomic geometrical interpretation of the unitary theory of Einstein and Mayer, while also indicating how one might possibly modify or generalize this unitary theory.

1. Group of a V_n with an indefinite metric. – Suppose that we have an n-dimensional Riemann space V_n whose metric may be given by the formula:

$$ds^2 = a_{ij}\, dx^i\, dx^j \qquad (i, j = 1, 2\ldots, n), \tag{4}$$

in which $x^1, x^2, \ldots, x^n$ are real variables and the a_{ij} are real functions of these variables whose determinant $|\, a_{ij}\,|$ is non-zero. If the quadratic form (4) is a positive-definite form then one may deduce it from a sum of squares:

$$ds^2 = (ds^1)^2 + (ds^2)^2 + \ldots + (ds^n)^2. \tag{4′}$$

[1] As far as the theory of anholonomic spaces is concerned, see G. Vranceanu. *Les espaces non holonome et leurs applications mécaniques*, Mémorial des Sciences Matématiques, fascicule 76, 1936.

If this quadratic form is not positive-definite then it may be reduced to the canonical form:

$$ds^2 = (ds^1)^2 + (ds^2)^2 + \ldots + (ds^p)^2 - (ds^{p+1})^2 - (ds^{p+2})^2 - \ldots - (ds^n)^2, \qquad (4'')$$

i.e., the sum of p positive squares and $n - p$ negative squares. In both cases, the quantities $ds^1, \ldots, ds^n$ are Pfaff forms in conveniently chosen differentials ([1]):

$$ds^a = \lambda_i^a \, dx^i \qquad (a = 1, 2, \ldots, n) \qquad (5)$$

in which the λ_i^a are functions of n variables $x^1, \ldots, x^n$ whose determinant $\Delta = |\lambda_i^a|$ is non-zero. Formula (5) may be solved for the differentials dx^i:

$$dx^i = \lambda_a^i \; ds^a \qquad (i = 1, 2, \ldots, n), \qquad (5')$$

in which λ_a^i are the reciprocals of the determinant Δ.

If one considers the system of n congruences of curves that is defined in the space X_n of the variables $x^1, x^2, \ldots, x^n$ by the differential equations:

$$\frac{dx^1}{\lambda_a^1} = \frac{dx^2}{\lambda_a^2} = \cdots = \frac{dx^n}{\lambda_a^n} \qquad (a = 1, 2, \ldots, n)$$

then one says that the quantities λ_a^i are the *parameters* and the λ_i^a are the *moments* of these congruences. One thus sees that each system of n independent Pfaff forms (5) determines a system of n independent congruences (λ), and conversely.

One knows that in a Riemann space with positive-definite metric one may give the name of *orthogonal congruences* to those congruences (λ) whose parameters satisfy the conditions:

$$a_{ij}\lambda_a^i\lambda_b^j = \delta_{ab} \qquad \begin{cases} = 0, & a \neq b, \\ = 1, & a = b, \end{cases} \qquad (6)$$

which amounts to supposing that the forms ds^a, which are the differentials of the arcs of these congruences, reduce the metric of V_n to the canonical form (4□). In this case, we also have the following formulas between the parameters λ_a^i and the moments λ_i^a of the orthogonal congruence (λ), and the coefficients a_{ij} of the metric on V_n:

$$a_{ij} = \lambda_i^a \lambda_j^b, \qquad \lambda_i^a = a_{ij} \; \lambda_a^i . \qquad (5'')$$

If the metric of our space V_n is not positive-definite then the parameters of the congruences whose differentials ds^a reduce this metric to the canonical form (4″) obviously satisfy the conditions:

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[1] One imposes the well-known convention that when two indices are repeated then this indicates that the sum is taken over those indices. Likewise, one employs indices i, j for the variables (x) and indices $a, b, c, d, e, f, g, h, k, l, r$ for the congruences (λ).

$$a_{ij}\lambda_a^i\lambda_b^j = \varepsilon_{ab}\,, \tag{6'}$$

in which ε_{ab} is equal to zero if a is different from b, ε_{hh} equals 1 ($h \leq p$), and $\varepsilon_{\alpha\alpha}$ equals –1 ($\alpha > p$). In this case, the coefficients a_{ij} are expressible as functions of the moments λ_i^a by the formulas:

$$a_{ij} = \lambda_i^h\lambda_i^h - \lambda_i^\alpha\lambda_j^\alpha \qquad (h = 1, 2, \ldots, p,\ \alpha = p + 1, \ldots, n).$$

As for the formulas that give us the moments as a function of the parameters, by taking into account equations (5″) they may be written:

$$\lambda_i^a = \varepsilon_{ab}\, a_{ij}\, \lambda_b^i \qquad [\lambda_i^h = a_{ij}\, \lambda_h^i,\ \lambda_i^\alpha = -\, a_{ij}\, \lambda_\alpha^i\,]. \tag{6''}$$

We shall call congruences (λ) that satisfy conditions (5″) and (6′) *pseudo-orthogonal congruences* of the space V_n.

We must remark that one may also introduce, with Eisenhart ([1]), pseudo-orthogonal congruences in V_n in another manner by supposing only that the parameters satisfy conditions (6′) and then determine the moments by formulas:

$$\lambda_i'^a = a_{ij}\, \lambda_a^i\,.$$

If $a > p$ then these moments $\lambda_i'^a$ are, as one sees, different from the moments λ_i^a because from formulas (6″) we have $\lambda_i'^a = -\, \lambda_i^\alpha$ ($\alpha > p$). It is Eisenhart's pseudo-orthogonal congruences that Levi-Civita considered in his systematization of Einstein's first unitary theory ([2]).

If one now considers a transformation of n Pfaff forms ds^a into n other Pfaff forms $d\bar{s}^a$ then this transformation may be written in the form:

$$d\bar{s}^a = c_b^a\, ds^b, \tag{7}$$

in which the c_b^a are arbitrary functions of the variables $x^1, x^2, \ldots, x^n$ whose determinant $|c_b^a|$ is non-zero. The totality of these linear transformations form a group (the general linear group) in the sense that it contains the identity transformation, each transformation has an inverse, and the product of transformations of the form (7) is also a transformation of the form (7).

Having said this, if the metric on the space V_n is positive-definite then a transformation of the Pfaff forms, or, more precisely, the congruences (7), preserves the canonical form (4′) only if the coefficients c_b^a of this transformation satisfies the orthogonality conditions:

$$c_b^a\, c_c^a = \delta_{bc}\,. \tag{7'}$$

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[1] See L. P. Eisenhart, *Riemannian Geometry*, Princeton University Press, 1926, chap. III.

[2] See T. Levi-Civita, *Vereinfachte Herstellung*, …, loc. cit., pp. 144.

Together, these transformations form a group that is a subgroup of the general linear group, namely, the orthogonal group, and one knows that one may study the geometrical properties of the space V_n as properties of this orthogonal group of transformations of the congruences (7), (7′). In particular, one knows that the components of the Levi-Civita affine connection of this space on a set of orthogonal congruence (λ) are given by the rotation coefficients γ^a_{bc} of these congruences. This signifies that if one indicates the components of a contravariant vector **u** by $u^1, u^2, \ldots, u^n$ on the n congruences (λ) then the parallel transport of this vector along an infinitesimal displacement ds^c is given by the formula:

$$du^a = \gamma^a_{bc}\, u^b\, ds^c\,. \tag{8}$$

Like Weyl, one knows that this parallel transport is characterized by the property of preserving the length of the vector, which tells us that the coefficients γ^a_{bc} must be anti-symmetric in the indices a and b, as well as the property that infinitesimal parallelograms must be closed. This latter condition amounts to saying that the components t^a_{bc} of the torsion of our connection are null:

$$t^a_{bc} = \gamma^a_{bc} - \gamma^a_{cb} - w^a_{bc} = 0,$$

in which w^a_{bc} are the coefficients of the bilinear covariants Δs^a of the forms ds^a:

$$\Delta s^a = \delta ds^a - d\delta s^a = w^a_{bc}\, ds^b\, \delta s^c\,,$$

$$w^a_{bc} = \left(\frac{\partial \lambda^a_i}{\partial x^j} - \frac{\partial \lambda^a_j}{\partial x^i}\right)\lambda^i_b \lambda^j_c\,. \tag{9}$$

One finds that the coefficients of rotation γ^a_{bc} are coupled with the coefficients w^a_{bc} of the bilinear covariants Δs^a by the formulas:

$$\gamma^a_{bc} = \frac{w^a_{bc} + w^b_{ca} + w^c_{ba}}{2}\,, \qquad w^a_{bc} = \gamma^a_{bc} - \gamma^a_{cb} \tag{9′}$$

If the metric of the space V_n is not positive-definite then the totality of the transformations (7) that preserve the canonical form of that metric also defines a pseudo-orthogonal group, by the equations:

$$c^h_a c^h_b - c^\alpha_a c^\alpha_b = \varepsilon_{ab}\,, \tag{10}$$

in which the ε_{ab} are defined above. This group may be regarded as a generalization of the Lorentz group, a group that is well known in the special theory of relativity. If one introduces imaginary quantities then it may be reduced to an orthogonal group of transformations of the congruences, and, as a consequence, one may deduce the properties of this group from those of the orthogonal group, but we shall show that *one* *may directly study the geometrical properties of this group* without needing to appeal to imaginary quantities.

Indeed, suppose that u^1, u^2, …, u^p, u^{p+1}, …, u^n are the components of a contravariant vector u on the pseudo-orthogonal congruences (λ) in V_n, whose length is given, by virtue of the canonical form (4″), by the formula:

$$u^2 = (u^1)^2 + (u^1)^2 + \ldots + (u^p)^2 - (u^{p+1})^2 - (u^{p+2})^2 - \ldots - (u^n)^2 \,. \tag{10′}$$

If one lets $\overset{*}{\gamma}{}^{a}_{bc}$ denote the components of an arbitrary affine connection Γ^* on the pseudo-orthogonal congruence (λ) then one obtains the variation of the length u under parallel transport using this connection by differentiating formula (10′):

$$u\,du = u^h\,du^h - u^\alpha\,du^\alpha,$$

and if one takes into account the fact that the du^α are given by the parallel transport formula:

$$du^\alpha = \overset{*}{\gamma}{}^{a}_{bc}\,u^b\,ds^c\,, \tag{11}$$

then one arrives at the formula:

$$u\,du = (\overset{*}{\gamma}{}^{h}_{kc} + \overset{*}{\gamma}{}^{k}_{hc})\,u^h\,u^k\,ds^c + (\overset{*}{\gamma}{}^{h}_{\alpha c} - \overset{*}{\gamma}{}^{\alpha}_{hc})\,u^h\,u^{\square}\,ds^c + (\overset{*}{\gamma}{}^{\alpha}_{\beta c} + \overset{*}{\gamma}{}^{\beta}_{\alpha c})\,u^\alpha\,u^\beta\,ds^c\,.$$

It results from this that our connection possesses the property of preserving the length of the vector that is being transported only if the components $\overset{*}{\gamma}{}^{a}_{bc}$ satisfy the conditions:

$$\left.\begin{array}{ll} \overset{*}{\gamma}{}^{h}_{kc} + \overset{*}{\gamma}{}^{k}_{hc} = 0, & \overset{*}{\gamma}{}^{h}_{\alpha c} + \overset{*}{\gamma}{}^{\alpha}_{hc} = 0, \\ \overset{*}{\gamma}{}^{\alpha}_{\beta c} + \overset{*}{\gamma}{}^{\beta}_{\alpha c} = 0 & (h, k \leq p, \alpha, \beta > p). \end{array}\right\} \tag{12}$$

If one associates these conditions with the conditions that express that the connection closes infinitesimal parallelograms:

$$\overset{*}{\gamma}{}^{a}_{bc} - \overset{*}{\gamma}{}^{a}_{cb} = \overset{*}{w}{}^{a}_{bc} \tag{13}$$

then one may infer the values of the γ^* and state the following theorem:

The Levi-Civita affine connection on a Riemann space V_n with an indefinite metric that has been reduced to the canonical form (4″) *has components relative to the pseudo-orthogonal congruences* (λ) *that are given by the following quantities:*

$$\left.\begin{array}{lll} \overset{*}{\gamma}{}^{h}_{kl} = \gamma^{h}_{kl}, & \overset{*}{\gamma}{}^{h}_{k\alpha} = \gamma^{h}_{k\alpha} + w^{\alpha}_{hk}, & \overset{*}{\gamma}{}^{\alpha}_{hk} = \overset{*}{\gamma}{}^{h}_{\alpha k} = -\overset{*}{\gamma}{}^{\alpha}_{kh}, \\ \overset{*}{\gamma}{}^{h}_{\alpha\beta} = \overset{*}{\gamma}{}^{\alpha}_{h\beta} = -\gamma^{h}_{\beta\alpha}, & \overset{*}{\gamma}{}^{\alpha}_{\beta h} = \gamma^{\alpha}_{\beta h} + w^{h}_{\alpha\beta}, & \overset{*}{\gamma}{}^{\alpha}_{\beta\gamma} = \gamma^{\alpha}_{\beta\gamma}, \end{array}\right\} \tag{14}$$

in which the w^{a}_{bc} are the coefficients of the bilinear covariants of the arclength differential of our pseudo-orthogonal congruences, and the γ^{a}_{bc} are the rotation coefficients of these congruences, which are defined as functions of the w^{a}_{bc} by formulas (9).

Naturally, one may express the values of the $\overset{*}{\gamma}{}^a_{bc}$ as functions of only the quantities w^a_{bc}, and one obtains formulas that are somewhat analogous to formulas (9′).

Once one knows the connection $\overset{*}{\gamma}{}^a_{bc}$ on the space V_n, one may find the curvature tensor of V_n by parallel transporting the vector u^a along an infinitesimal parallelogram that is constructed from two infinitesimal displacements ds^h, δs^c. Indeed, one finds the formula:

$$\Delta u^a = \delta da^h - d\delta s^a = \overset{*}{\gamma}{}^a_{bcd}\ u^b\, ds^c\, ds^d\,, \tag{15}$$

in which $\overset{*}{\gamma}{}^a_{bcd}$ are precisely the components of the curvature tensor of V_n on the congruences (λ), and are given as functions of the $\overset{*}{\gamma}{}^a_{bc}$ by the formula:

$$\overset{*}{\gamma}{}^a_{bcd} = \frac{\partial \overset{*}{\gamma}{}^a_{bc}}{\partial s^d} - \frac{\partial \overset{*}{\gamma}{}^a_{bd}}{\partial s^c} + \overset{*}{\gamma}{}^a_{fc}\overset{*}{\gamma}{}^f_{bd} - \overset{*}{\gamma}{}^a_{fd}\overset{*}{\gamma}{}^f_{bc} + \overset{*}{\gamma}{}^a_{bf} w^f_{cd}\,, \tag{16}$$

from which the analogy with the formulas that give the four-index Ricci coefficients γ^a_{bcd} is obvious. By the fact that our parallel transport preserves length, the components $\overset{*}{\gamma}{}^a_{bcd}$ are anti-symmetric with respect to the first indices a and b. Likewise, from formulas (15) it then results that these components are also anti-symmetric with respect to the last indices c and d.

By contracting over the indices a and c, which is possible since the first index is contravariant and the third one is covariant, one obtains the Ricci tensor:

$$R_{bd} = \overset{*}{\gamma}{}^a_{bad}\,, \tag{17}$$

and if one lets ε^{ab} denote the reciprocals of the coefficients ε_{ab} of the metric ($\varepsilon^{ab} = \varepsilon_{ab}$) then one may consider the mixed components R^a_d of the tensor R_{bd}, i.e., $R^a_d = \varepsilon^{ab} R_{bd}$, and finally, by contraction, the value R of the Ricci scalar ($R = R^a_a$).

The connection and the curvature on V_n are thus defined by their components $\overset{*}{\gamma}{}^a_{bc}$ and $\overset{*}{\gamma}{}^a_{bcd}$ on the pseudo-orthogonal congruences. If one considers a transformation of the congruences or the Pfaff forms (7) then the new components $\overline{\overset{*}{\gamma}}{}^a_{bc}, \overline{\overset{*}{\gamma}}{}^a_{bcd}$ of our connection and curvature are related to $\overset{*}{\gamma}{}^a_{bc}$ and $\overset{*}{\gamma}{}^a_{bcd}$ by well-known transformation formulas for affine connections and tensors ([1]).

2. Anholonomic spaces with indefinite metrics. – Now suppose that we have a certain number $n - m$ of Pfaff equations:

$$ds^{h'} = \lambda^{h'}_i\ dx^i = 0 \quad (h' = m + 1, \ldots, n) \tag{18}$$

in the Riemann space V_n with the metric (4).

[1] As far as the absolute differential calculus of congruences is concerned, see G. VRANCEANU, *Les espaces non holonome et leurs applications mécaniques*, loc. cit., chap I.

If the covariants $\Delta s^{h'}$ (mod $ds^{h'}$) of these equations (in which mod $ds^{h'}$ indicates that one accounts for equations of the form $ds^{h'} = 0$ in the $\Delta s^{h'}$):

$$\Delta s^{h'} \ (\text{mod}\ ds^{h'}) = w_{kl}^{h'} ds^k \, ds^l \tag{18′}$$

are null, which happens only when the coefficients $w_{kl}^{h'}$ are null, then the Pfaff system (18) will be completely integrable. In this case, equations (18) may be written by conveniently choosing the variables:

$$ds^{h'} = dx^{h'} = 0 \qquad (x^{h'} = \text{constants}), \tag{18″}$$

and our system (18″) determines a family of ∞^{n-m} V_m's in the space V_n.

If the system (18) is not completely integrable then one says, by analogy with anholonomic mechanical systems, that the system defines an *anholonomic space* V_n^m in the Riemann space V_n. In order to study the properties of anholonomic spaces one may commence by associating the $n - m$ Pfaff forms $ds^{h'}$ with m other forms:

$$ds^h = \lambda_i^h \, dx^i \qquad (h = 1, 2, \ldots, m),$$

in such a fashion that the n forms ds^h, $ds^{h'}$ constitute a system of n independent forms. If one expresses the n differentials dx^i with the aid of these n forms ds^a then the metric (4) of the space V_n may be written:

$$ds^2 = g_{ab} \, ds^a \, ds^b \qquad (g_{ab} = a_{ij} \, \lambda_a^i \lambda_b^j),$$

and if one takes the equations of the system (18) into account in this metric then one obtains the metric of the anholonomic space V_n^m:

$$ds^2 \ (\text{mod}\ ds^{h'}) = g_{hk} \, ds^h \, ds^k \quad (h, k = 1, 2, \ldots, m). \tag{19}$$

One sees that this metric can only be applied to the directions that satisfy system (18), or, more precisely, the directions that are tangent to V_n^m.

We thus have two fundamental invariants of the anholonomic space, the metric (19) and the Pfaff system (18). One uses the term "intrinsic properties" of the space V_n^m to describe properties that depend only upon these two invariants, which is due to the fact that if the system (18) is completely integrable then these properties coincide with the intrinsic properties, in the Riemann sense, of the ∞^{n-m} V_m's into which we decomposed our anholonomic space V_n^m in this case.

One may relate the study of the intrinsic properties of V_n^m to that of the properties of a group of transformations of Pfaff forms. Indeed, the most general transformations of Pfaff forms that preserve the system (18) are given by the formulas:

$$\left.\begin{aligned} d\bar{s}^h &= c_k^h ds^k + c_{k'}^h ds^{k'} \quad && (h, k = 1, 2, \cdots, m), \\ d\bar{s}^{h'} &= c_{k'}^{h'} ds^{k'} && (h', k' = m+1, \cdots, n), \end{aligned}\right\} \tag{20}$$

in which $c_k^h, c_{k'}^{h'}, c_{k'}^h$ are real functions of n variables x^1, x^2, …, x^n, such that the determinants $|c_k^h|$ and $|c_{k'}^{h'}|$ are non-zero. These transformations define the group of the Pfaff system (18), and the intrinsic properties of V_n^m are the properties of this group, to which one may associate the metric (19) of V_n^m. If one now reduces this metric to the canonical form:

$$ds^2 \text{ (mod } ds^{h'}) = (ds^1)^2 + \ldots + (ds^p)^2 - (ds^{p+1})^2 - \ldots - (ds^m)^2 , \tag{19′}$$

in which the number p is equal to m if the metric is positive-definite, then the transformations (20) that preserve this canonical form are obtained by imposing the conditions:

$$c_k^h c_l^h - c_k^\alpha c_l^\alpha = \varepsilon_{ab} \qquad (h \leq p,\ \alpha > p), \tag{20′}$$

in which ε_{ab} is null if $k \neq l$, $\varepsilon_{hh} = 1$ and $\varepsilon_{\alpha\alpha} = -1$.

From this, it results that one may define the intrinsic properties of V_n^m to be the properties of the group of transformations of the Pfaff forms (20), (20′), a group that one calls the *intrinsic group of the anholonomic space* V_n^m.

In the case in which the Pfaff system is not completely integrable, one proves that this intrinsic group is not generally geometrizable; i.e., that the knowledge of the two fundamental invariants of V_n^m is not generally sufficient to give the important geometric properties of the space.

We associate these two invariants with the condition that the coefficients $c_{k'}^h$ have well-defined values, which one may assume to be zero after a convenient transformation of the ds^h, which geometrically amounts to fixing the normal space to V_n^m, a space that is then defined by the system:

$$ds^h = 0 \qquad\qquad (h = 1, 2, \ldots, m).$$

With this condition the subgroup of our intrinsic group, which also preserves this Pfaff system ($c_{k'}^h = 0$), constitutes a semi-intrinsic group of the space, and groups like this are called *geometrizable groups*.

Under certain conditions, one may, in a manner of speaking, reduce the study of the intrinsic group of V_n^m to the study of one of its *semi-intrinsic* subgroups, and likewise to one of the *rigid* subgroups of the intrinsic group ([1]), subgroups that are obtained by supposing that the $c_{k'}^h$ satisfy conditions that are analogous to the conditions (20′). These rigid groups thus possess a metric in all directions, and not only for the tangent directions.

The reduction of the intrinsic group to the rigid group is always possible if the Pfaff system (18) is composed of only one non-completely integrable Pfaff equation:

$$ds^n = \lambda_i^n dx^i = 0, \tag{21}$$

[1] See my work: *Sur quelques points*, …, loc. cit., pp. 184-191.

i.e., if our anholonomic space is an anholonomic hypersurface.

Indeed, the intrinsic group of that hypersurface may obviously be written:

$$\left.\begin{aligned} d\bar{s}^h &= c_k^h ds^k + c^h ds^n \\ d\bar{s}^n &= \lambda ds^n, \end{aligned}\right\} \tag{21′}$$

in which the c_k^h satisfy the pseudo-orthogonality or orthogonality conditions (20′), and we have set $c_n^h = c^h$, $c_n^n = \lambda$ to simplify. The coefficients w_{kl}^n of the hypersurface covariant in equation (21):

$$\Delta s^n \;(\mathrm{mod}\; ds^n) = w_{kl}^n \, ds^k \, ds^l, \tag{22}$$

form a third-order tensor with respect to the group whose transformation law is given by the formula:

$$\overline{w}_{\alpha\beta}^n c_k^\alpha c_l^\beta = \lambda w_{kl}^n . \tag{22′}$$

Having said this, if the rank of covariant Δs^n (mod ds^n), which is always an even number $2q \leq n - 1$, is equal to $n - 1$, which happens only if n is odd, then we have the formula:

$$\delta^2 \overline{\Delta} = \lambda^{2q} \Delta , \tag{21″}$$

in which Δ is the determinant of the covariant Δs^n (mod ds^n) and δ is the determinant $|c_k^h|$, which is equal to +1. This formula shows us that if one reduces the determinant Δ to unity with the aid of the coefficient λ of the group (21′) then it will remain equal to unity only for the transformations (21′) with $\lambda = 1$. The form ds^n is invariant under these transformations, and, as a consequence, the covariant:

$$\Delta s^n = w_{kl}^n \, ds^k \, \delta s^l + w_{nk}^n \, (ds^k \, \delta s^n - ds^k \, \delta s^n) \tag{22″}$$

is also an invariant. Now, one may choose the coefficients c^h in one and only one manner that annuls all of the coefficients w_{nl}^n in this covariant and consequently reduces the study of the intrinsic group (21′) to the study of a rigid group. This result may be considered to be a particular case of a theorem of Schouten ([1]) on affine anholonomic hypersurfaces.

If the rank $2q$ of the covariant Δs^n is less than $n - 1$, which always happens if n is even, then one may arrange that this covariant involves only the first $2q$ forms ds^h. Having said this, the group that preserves this situation is a subgroup of our intrinsic group that separately transforms the first $2q$ forms ds^h and the last $n - 2q - 1$ forms ds^k, and this is true because subgroups of an orthogonal or pseudo-orthogonal group have the property of being completely integrable. If we now let ν denote the determinant of the transformation of the first $2q$ forms ds^h and let Δ denote the determinant of the covariant Δs^n thus reduced then we have a formula that is analogous to (1″), in such a fashion that if we reduce Δ to unity then one may choose the coefficients c^h ($h \geq 29$) in one and only

[1] See J. A. Schouten, *On non-holonomic connections*, Koninklijke Ak. Wetenschappen Amsterdam, 1928, **31**, no. 3, pp. 299.

one manner that annuls the coefficients w_{nh}^{n} ($h \leq 2q$). The group that preserves the covariant, thus normalized, obviously preserves the two Pfaff systems:

$$\left.\begin{array}{c} ds^1 = ds^2 = \cdots = ds^{2q} = 0, \\ ds^{2q+1} = ds^{2q+2} = \cdots = ds^{n-1} = ds^n = 0. \end{array}\right\} \tag{22'''}$$

Now, in the latter of these systems the covariant Δs^n (ds^{2q+1}, … , ds^n) also preserves its rank $2q$, which is the maximum rank of this system. Consequently, if one considers the covariant of one of the equations in this system that takes the form:

$$ds^k + c^k\, ds^n \qquad\qquad (k = 2p + 1, \ldots, n - 1)$$

then one may choose the quantities c^k to be the roots of the characteristic equation of this covariant in such a fashion that the rank of the this covariant is less than $2q$ in the interior of our system. It then results that one may generally choose the c^k in several different ways. If the rank of the covariant Δs^n is equal to two then this manner of choosing the c^k is obviously unique, and the $ds^k = 0$ then constitute the equations of the derived system of our second system (22′′′). In particular, this happens in the case in which the hypersurface is a V_5^4 and the covariant is not of maximum rank $n - 1 = 4$. We may then state the following theorem:

If one is given an anholonomic hypersurface V_n^{n-1} *then one may always, and in several different manners, reduce the study of its intrinsic group to the study of a rigid group, or, in other words, one may always fix the direction of the normal* ($c^h = 0$) *and the metric on this normal* ($\lambda = 1$). *If the hypersurface is a* V_5^4 *then this reduction is always unique.*

The importance of this theorem consists in the fact that it reduces the study of the invariants of the intrinsic group of V_n^{n-1} to the study of a rigid group, a group for which one knows a *complete system of invariants*. Obviously, one arrives at the same results if one considers *the form* ds^n *to be an invariant* ($\lambda = 1$) from the outset, or, more precisely, if Δ is an invariant, a case that seems to also have an interesting physical interpretation, as we shall show in the third part.

Now consider a rigid group of an anholonomic hypersurface V_n^{n-1}:

$$d\bar{s}^h = c_k^h\, ds^k, \qquad d\bar{s}^n = ds^n, \tag{23}$$

or, more precisely, the group of an anholonomic hypersurface that possesses a normal direction and a metric on that that normal.

This rigid group possesses two affine connections, each of which preserves the character of the tangent vector and the normal vector, and precisely one affine connection whose principal characteristic is that it closes infinitesimal parallelograms as much as possible. This latter connection has the following quantities for its components on the congruences (λ):

$$\left.\begin{array}{r} \Gamma_{kl}^h = \overset{*}{\gamma}{}_{kl}^h, \quad \Gamma_{kn}^h = w_{kn}^h, \quad \Gamma_{nk}^n = w_{nk}^n, \qquad \Gamma_{nn}^n = 0, \\ \Gamma_{na}^h = \Gamma_{ha}^n = 0. \end{array}\right\} \tag{24}$$

One sees that the parallel transport of a tangent vector v^h along a tangent path ds^l is given by the formulas:

$$dv^h = \overset{*}{\gamma}{}^h_{kl}\, v^k\, ds^l \tag{25}$$

and formally coincides with the Levi-Civita parallel transport of the metric on the hypersurface. I say "formally" because our transport preserves length, but it does not close the parallelogram that one abstractly constructs with a fifth side that is directed along the normal and has the components:

$$w^n_{kl}\, ds^k\, \delta s^l,$$

in which ds^k and δs^l are the components of the two displacements on which one constructs the parallelogram. From this, it results that the integrability tensor w^n_{kl} is, at the same time, the *torsion tensor of the anholonomic hypersurface* V^{n-1}_n.

If one now considers parallel transport of a tangent vector v^h along the normal then we have:

$$dv^h = w^n_{kl}\, v^k\, ds^n\,, \tag{26}$$

in such a fashion that the variation of the square of the length of this vector:

$$v\, dv = v^h\, dv^h - v^n\, dv^h$$

is given by the formula:

$$2v\, dv = (v_{hk,n}\, v^h\, v^k + 2v_{h\alpha,n}\, v^h\, v^\alpha + v_{\alpha\beta,n}\, v^\alpha\, v^\beta)\, ds^n\,,$$

in which the quantities $v_{kl,n}$ are the components of the second fundamental form for the hypersurface. One may express the components of this tensor with the aid of the rotation pseudo-coefficients γ^* by the formula:

$$v_{kl,n} = \overset{*}{\gamma}_{kl} + \overset{*}{\gamma}{}^n_{lk} \quad (k, l = 1, 2, \ldots, n-1). \tag{27}$$

If one transports a tangent vector along the infinitesimal parallelogram that is constructed from two tangent displacements ds^l, δs^r then the variation of the components of this vector is given by the formula:

$$Dv^h = \lambda^h_{klr}\, v^k\, ds^l\, \delta s^r\,,$$

in which the quantities λ^h_{klr} represent the components of the curvature tensor (interior, or tangent) of the hypersurface. These components are given by the formula:

$$\lambda^h_{klr} = \overset{*}{\gamma}{}^h_{klr} + w^h_{kn} w^n_{lr}, \tag{25$'$}$$

in which the $\overset{*}{\gamma}{}^h_{klr}$ are the quantities (16) when one allows the index f to vary only from 1 to $n - 1$. Consequently, in the integrable case these quantities define the Riemann curvature tensor of the V_{n-1} into which we have decomposed our V_n^{n-1}. One thus sees that our curvature tensor λ_{klr}^h coincides with the Riemann curvature tensor γ_{klr}^{*h} in the integrable case ($w_{lr}^h = 0$) or in the case where the parallelism (26) along the normal preserves the values of the components v^h ($w_{kn}^h = 0$).

Likewise, if one transports a normal vector v^n along the two circuits – viz., the pentagon and parallelogram that were considered – then one obtains a curvature tensor in the form of the tensor w_{kln}^h that is the derivative of the torsion tensor w_{kl}^h along the normal, and the tensor $\frac{\delta w^n_{pk}}{\delta s^n}$, but these tensors are only rigid tensors, whereas the tensors w_{kl}^h, $v_{kl,n}$, λ_{klr}^h, λ_{kln}^h are also semi-intrinsic tensors.

We shall now show that one may choose the variables in such a fashion that the metric and the equation of the hypersurface reduce to simple forms. Indeed, one may always suppose that the $n - 1$ forms ds^h ($h \leq n - 1$) are expressed as functions of only the $n - 1$ differentials $dx^1, dx^2, \ldots, dx^{n-1}$. In this case, the metric of the hypersurface will be a quadratic form in the $n - 1$ differentials:

$$ds^2 = a_{ij}\, dx^i\, dx^j \qquad (i, j = 1, \ldots, n-1), \tag{26′}$$

in which the coefficients a_{ij} are general functions of the variable x^n. As for the equation of the hypersurface, since it must contain the differential dx^n, one may, by a convenient change, write it in the form:

$$ds^n = dx^n - \varphi_i\, dx^i = 0 \qquad (i = 1, \ldots, n-1) \tag{26″}$$

in which the φ_i are general functions of all of the variables $x^r, \ldots, x^n$.

If this is true then one finds that in this case the metric tensor (26′) does not depend on the variable x^n, and the second fundamental form and the second curvature tensor $\lambda_{k,ln}^h$ are both null. As for the curvature tensor λ_{klr}^h of the hypersurface, it reduces to the curvature tensor of the metric (26′). In this case, one says that our hypersurface is *totally geodesic*. If the functions φ_i do not depend upon the variable x^n either then the curvature tensors $w_{kl,n}^h$, $\frac{\partial w_{nk}^n}{\partial s^l}$ are also null, in such a fashion that the only tensors that may be non-zero in this case are the torsion tensor and interior curvature tensor, whose components are obviously functions of only the variables $x^1, x^2, \ldots, x^{n-1}$.

The case in which a_{ij} and φ_i do not depend on the variable x^n is characterized by the fact that the hypersurface admits a continuous one-parameter group:

$$Xf = \frac{\partial f}{\partial x^n} \qquad (\bar{x}^n = x^n + t)$$

that represents a translation along the normal. Indeed, if an anholonomic hypersurface *that is intrinsically defined by its metric and its equation* admits a one-parameter group that is not tangent (one may not say "normal" since the direction of the normal is not fixed) then one may reduce it to this case by choosing the normal to be the direction of the group trajectories. The normal direction thus determined might not coincide with its determinant when one reduces the determinant Δ of the covariant of the equation (26″) to unity.

3. Non-holonomic unitary theory. – Suppose that we now concern ourselves with the case of physics: i.e., we have, on the one hand, the spacetime V_4 of the theory of relativity, which is a four-dimensional Riemann space whose metric is an indefinite quadratic form that has three negative squares and one positive square:

$$ds^2 = -(ds^1)^2 - (ds^2)^2 - (ds^3)^2 + (ds^4)^2 \tag{26‴}$$

in which the forms ds^1, ds^2, ds^3, ds^4 are functions of three variables x^1, x^2, x^3, and time t, or, more precisely, the four variables x^1, x^2, x^3, $x^4 = ct$, where c is the velocity of light, and, on the other hand, the electromagnetic field, which may be determined in the space of these four variables by the rotation of a covariant vector φ_i (i = 1, 2, 3, 4). Indeed, if the space V_4 is reduced to the space of special relativity, in which we have:

$$ds^1 = dx^1, \qquad ds^2 = dx^2, \qquad ds^3 = dx^3, \qquad ds^4 = dx^4 = c\,dt,$$

in which x^1, x^2, x^3, are the orthogonal Cartesian coordinates, the components (e_x , e_y , e_z) of the electric vector **e** and the components (m_x , m_y , m_z) of the magnetic vector **m** are given with the aid of the components of this rotation by the formulas:

$$e_x = \varphi_{14}, \quad e_y = \varphi_{24}, \quad e_z = \varphi_{34}, \quad m_x = \varphi_{23}, \quad m_y = \varphi_{31}, \quad m_z = \varphi_{12},$$
$$\varphi_{ij} = \frac{\partial \varphi_i}{\partial x^j} - \frac{\partial \varphi_j}{\partial x^i}.$$

From the fact that the φ_{ij} are the components of a rotation, they satisfy the formulas:

$$\frac{\partial \varphi_{ij}}{\partial x^j} + \frac{\partial \varphi_{jl}}{\partial x^l} + \frac{\partial \varphi_{li}}{\partial x^j} = 0 \qquad (i, j, l = 1, 2, 3, 4), \tag{27′}$$

which constitute the first Maxwell equations for the vectors **e** and **m** in the case where there exists no true magnetism.

From this, it then results that, whenever the electromagnetic field is not identically null the form:

$$dx^5 = \varphi_i\, dx^i$$

is not an exact total differential, or, more precisely, the Pfaff equation:

$$ds^5 = dx^5 - \varphi_i\, dx^i = 0 \tag{28}$$

is not completely integrable. One must then observe that, whereas the knowledge of the vector φ_i uniquely determines the vectors **e** and **m**, the converse is not true. The components φ_i are determined by **e** and **m** alone, which is an abstraction from terms of the form $\frac{\partial f}{\partial x^i}$ where f is a function of the variables x^1, x^2, x^3, x^4. This amounts to saying that the variable x^5 that appears in equation (28) only through its differential is determined by abstracting from a transformation:

$$x'^5 = x^5 + f(x^1, x^2, x^3, x^4).$$

This transformation plays a significant role in the projective unitary theory. In our case, in the manner in which we posed our problem all of our results are invariant under this transformation. One may likewise observe that the determinant Δ of the covariant:

$$\Delta s^5 = -\Delta x^5 = \varphi_{ij}\, dx^i\, \delta x^j ,$$

is equal to the square of the scalar product of the vectors **e** and **m**:

$$\Delta = (e_x\, m_x + e_y\, m_y + e_z\, m_z)^2 ,$$

and consequently the rank of this covariant is two or four depending on whether these vectors are orthogonal or not.

In the general case where V_4 does not reduce to the Euclidian space of special relativity, one also agrees to consider the electromagnetic field as being defined by the rotation of a covariant vector, or, more, precisely, by an anti-symmetric tensor F_{ij} of second order that satisfies the first Maxwell equations (27′) because these equations represent the necessary and sufficient condition for the F_{ij} to be regarded as the components of a rotation.

The question at hand now is to see how one must relate these two invariants in order to arrive at a unitary theory that satisfies the various conditions that are imposed by physics.

In Weyl's theory, the form $\varphi_i\, dx^i$ appears as a coefficient of dilatation for the metric on spacetime V_4, i.e., as something that is superposed on the space V_4 and determined in the same manner by which the metric on V_4 is defined or known.

In Kaluza's theory ([1]), one makes the form $\varphi_i\, dx^i$ play an external role, and, more precisely, one supposes that it serves to determine the metric (2) of the physical continuum, which is supposed to be five-dimensional, but this form is not sufficient to completely determine the metric on V_5 when starting with the one on V_4.

In the theory of Einstein and Mayer, the form $\varphi_i\, dx^i$ does not appear explicitly. What one uses for the determination of the electromagnetic field is an anti-symmetric tensor F_{ij} of second order. One arrives at this tensor upon considering the parallel transport in the Levi-Civita sense in the vector space V_5 that is associated with our space V_4, whose metric may written in the form (2″), or, more precisely, reduced to the canonical form:

$$d\sigma^2 = -(ds^1)^2 - (ds^2)^2 - (ds^3)^2 + (ds^4)^2 + (ds^5)^2 ,$$

[1] See also O. Klein, *Quantentheorie und fünfdimensionale Relativitätstheorie*, Z. für Physik, 1926, **37**, pp. 895.

when this metric is referred to a system of five pseudo-orthogonal congruences in V_5 that consists of four pseudo-orthogonal congruences in V_4 and a fifth congruence that is normal to V_4. Indeed, if one lets $v^1, v^2, \ldots, v^5$ denote the components of a contravariant vector (v) on the five pseudo-orthogonal congruences of V_5 then the parallel transport of this vector along an infinitesimal path that is tangent to V_4 ($ds^5 = 0$), which are the only possible paths from the physical viewpoint, is defined by the equation:

$$dv^a = \overset{*}{\gamma}{}^a_{bl}\, v^b\, ds^l \qquad (a, b = 1, 2, \ldots, 5, \qquad l = 1, 2, 3, 4), \tag{29}$$

in which the rotation coefficients $\overset{*}{\gamma}{}^a_{bl}$ of our five pseudo-orthogonal congruences are determined by formula (14).

Among the rotation coefficients $\overset{*}{\gamma}{}^a_{bl}$, the coefficients $\overset{*}{\gamma}{}^h_{kl}$ ($h, k, l \leq 4$) are completely determined by the metric on V_4, whereas the other rotation coefficients $\overset{*}{\gamma}{}^5_{kl}, \overset{*}{\gamma}{}^k_{5l}$ that also appear in the parallel transport equation (20) *are undetermined if the form ds^2 that is defined by formula* (3) *is not given.* In any case, one may remark that if one applies the transport law (29) to a tangent vector v^b ($v^5 = 0$) then this vector remains a tangent vector under transport if one has the formula:

$$dv^b = \overset{*}{\gamma}{}^5_{kl}\, v^k\, ds^l = 0.$$

We suppose that this is not the case in general, but that a tangent vector, which one may assume is unitary, remains tangent to V_4 if it is transported along its proper direction. We thus assume that one has:

$$dv^5 = (\overset{*}{\gamma}{}^5_{kl} + \overset{*}{\gamma}{}^5_{lk})\, u^k\, u^l\, ds = 0 \qquad (ds^l = u^l\, ds) \tag{29′}$$

in which u^1, u^2, u^3, u^4 are the cosines that the vector makes with the pseudo-orthogonal of V_4 and ds is the length of the displacement ds_l, which constitutes condition III of Einstein and Mayer. In this case, formula (20) shows us that the components $\overset{*}{\gamma}{}^5_{kl}$ must satisfy the anti-symmetry conditions:

$$\overset{*}{\gamma}{}^5_{kl} + \overset{*}{\gamma}{}^5_{lk} = 0, \tag{30}$$

and it suffices to consider the projections of the tensor $\overset{*}{\gamma}{}^5_{kl}$ onto the directions of the system of variables (x) of the spacetime V_4, in order to arrive at the tensor:

$$F_{ij} = \overset{*}{\gamma}{}^5_{kl}\, \lambda^k_i\, \lambda^l_j \qquad (\overset{*}{\gamma}{}^5_{kl} = F_{ij}\, \lambda^i_k\, \lambda^j_l)$$

that one takes to be the definition of the electromagnetic tensor.

This being the case, if one considers the anholonomic hypersurface V^4_5 that is defined in V_5 by the Pfaff equation (28) and has a metric that is defined by the metric on V_4 and has a normal in the direction V_5 that is orthogonal to V_4 then, as we saw in the preceding section, that hypersurface possesses two tensors of third order: the second fundamental form, *which is null in this case,* by virtue of conditions (30), and the integrability tensor w^5_{kl}, which, by virtue of the same condition, can be written:

$$w_{kl}^5 = \overset{*}{\gamma}{}_{kl}^5\, \overset{*}{\gamma}{}_{lk}^5 = 2\overset{*}{\gamma}{}_{kl}^5 . \tag{30′}$$

It suffices to project these formulas into the system of variables (x) in order to arrive at the important formula (3′):

$$\varphi_{ij} = 2F_{ij},$$

which leads us to relate the unitary theory of Einstein and Mayer to the properties of the anholonomic surface V_5^4.

Moreover, from the fact that the parallel transport (29) does not generally preserve the character of the tangent vector to V_4, it seems to us preferable to consider one of two rigid transport laws on our anholonomic hypersurface instead of this parallel transport, laws that have the property that they preserve the character of the tangent vector and, in particular, exhibit the transport law (25), which is more intrinsically linked with V_5^4. We shall now see that if we consider this transport law (25) and the anholonomic interpretation, in general, then, from the physical viewpoint, one may arrive at the same consequences as in the initial theory of Einstein and Mayer. More precisely, we shall show how, by starting with the anholonomic hypersurface V_5^4, one may arrive, on the one hand, at the equations of the trajectories of a charged electric particle, and, on the other hand, at the equations of gravitation and electromagnetism.

Einstein and Mayer arrived at the equations of the trajectories of charged electric particle by considering the curves in spacetime V_4 that are auto-parallel curves of the transport law (29), and, as a consequence, if one lets u^1, u^2, u^3 denote cosines that the tangent to one of these curves makes with the four pseudo-orthogonal congruences of V_4, then we also have that $ds^l = u^l\, ds$ in such a fashion that these cosines must satisfy the following equations:

$$\left.\begin{array}{c} \dfrac{du^h}{ds} - \overset{*}{\gamma}{}_{kl}^h u^k u^l = \overset{*}{\gamma}{}_{hl}^5 u^l u^5 \\ (h = 1, 2, 3) \quad \left(\overset{*}{\gamma}{}_{5l}^h = \overset{*}{\gamma}{}_{hl}^5, (h \leq 3)\right) \\ \dfrac{du^4}{ds} - \overset{*}{\gamma}{}_{kl}^4 u^k u^l = -\overset{*}{\gamma}{}_{4l}^5 u^l u^5 \\ \left(\overset{*}{\gamma}{}_{5l}^4 = -\overset{*}{\gamma}{}_{4l}^5\right). \end{array}\right\} \tag{31}$$

The quantity v^5, which is a constant by virtue of condition (29′), is, by definition, associated with the ratio:

$$v^5 = \rho = \frac{e}{m_0},$$

in which e is the electric charge and m_0 is the rest mass of the particle.

If one considers our anholonomic interpretation then one knows that we have two types of tangent geodesics for an anholonomic space: geodesics that are defined as auto-parallel curves and geodesics that are defined as curves of shortest distance ([1]). In the

[1] See G. Vranceanu, *Sur quelques points de la théorie des espaces non holonomes*, loc. cit., pp. 192.

present case, in which the anholonomic hypersurface is totally geodesic and the components of the electromagnetic vector φ_i are independent of the variable x^5 ($\overset{5}{w}_{5k} = 0$), the latter curves have the equations:

$$\left.\begin{aligned} &\frac{du^h}{ds} - \overset{*h}{\gamma}_{kl} u^k u^l = \overset{5}{w}_{hl} u^l u^5 \qquad \frac{dv_5}{ds} = 0, \\ &\frac{du^4}{ds} - \overset{*4}{\gamma}_{kl} u^k u^l = -\overset{5}{w}_{4l} u^l u^5, \end{aligned}\right\} \tag{32}$$

in which v_5 is now the normal component of a covariant vector. One sees that if we take (30′) into account then it suffices to set:

$$v^5 = 2v_5 = \rho$$

in order to identify equations (31) with equations (32).

This being the case, one may remark that equations (31), in which v^5 is the normal component of a contravariant vector, are not invariant under modifications of the metric on the normal to V_4, i.e., changes of the form ds^5 into $\lambda\, ds^5$, where λ is an arbitrary function of the variables x^1, x^2, x^3, x^4, a property that equations (32) exhibit. It thus seems preferable to take equations (32) to be the equations of a charged particle of electricity, equations that describe the geodesics of shortest distance on the anholonomic hypersurface V_5^4, and to assimilate the coefficient ρ into the component v_5 of a covariant normal vector. This amounts to saying that the form $\rho\, ds^5$ is independent of changes of the metric on the normal.

One may remark that if one supposes that this coefficient ρ is null then equations (32) for the charged particle of electricity become:

$$\frac{du^h}{ds} - \overset{*h}{\gamma}_{kl}\, u^k\, u^l = 0. \tag{30″}$$

Now, these equations are precisely the equations of the auto-parallel geodesics of the hypersurface V_5^4, and thus also coincide with the geodesics of spacetime V_4, in such a fashion that if one takes into account the fact that the velocity of light is such that:

$$ds^2 = a_{ij}\, dx^i\, dx^j = -(ds^1)^2 - (ds^2)^2 - (ds^3)^2 + (ds^4)^2 = 0,$$

then it results that one finds the trajectories of light that are given by the theory of relativity, i.e., *the null-length geodesics* ([1]) of spacetime V_4, trajectories that are, as one sees, *independent of the electromagnetic field* if one supposes that the factor ρ is null. This amounts to supposing that that electric charge is zero for light, or, more precisely,

[1] See T. Levi-Civita. *The Absolute Differential Calculus*, Blackie, London, 1927, pp. 330.

the photon, and that the ponderable mass m_0 is non-zero, a fact that agrees with the hypotheses of Louis de Broglie ([1]).

If one now wishes to integrate the equations of motion (31) or (32) of a charged particle of electricity then one must associate them with the equations:

$$\frac{dx^i}{dt} = \lambda^i_h u^h, \tag{31′}$$

and one thus obtains a system of eight equations in eight unknowns x^i, u^h, a system that possesses the particular quadratic first integral:

$$-(u^1)^2 - (u^2)^2 - (u^3)^2 + (u^4)^2 = 1. \tag{32′}$$

We now pass on to the equations of gravitation and electromagnetism, to which Einstein and Mayer arrived by a very complicated method, by considering the curvature tensor of the affine connection (29) of the space V_5 and following a path that is somewhat analogous to the one that led to the Einstein tensor (15′). In the case of our anholonomic interpretation one may arrive at these equations in a manner that is very simple and, in our opinion, quite natural.

Indeed, if one is given, on the one hand, the spacetime V_4, which has the metric (26), and, on the other hand, the Pfaff form (3), or, more precisely, the electromagnetic field and the gravitational field, then, as we saw in the second part of this memoir, one determines a hypersurface V_5^4 whose normal direction may be fixed in an invariant manner as being defined by the four equations:

$$ds^1 = ds^2 = ds^3 = ds^4 = 0.$$

One arrives at this invariant determination of the normal, either by starting with the fact that the form ds^2 is an invariant, or by starting with the fact that V_5^4 must possess a continuous one-parameter group of transformations that are not tangent. This anholonomic hypersurface, which is *totally geodesic* under these conditions, possesses only two non-zero tensors: the curvature tensor λ^h_{klr} of V_5^4, which coincides with the curvature tensor of V_4, and the torsion tensor w^5_{kl} of V_5^4, which we write as w_{kl} to simplify, and which determines the electromagnetic field. We remark that Einstein and Mayer considered this tensor w_{kl} to be a part of the curvature tensor of space V_5, which is not true in our case, at least if one considers the curvature to be the fourth order tensor that one obtains by parallel transporting a vector along an infinitesimal circuit, as is customary.

This being the case, with the aid of the curvature tensor, one may, by the well-known method of the general theory of relativity, construct the Einstein tensor:

$$R_{kl} - \tfrac{1}{2}\varepsilon_{kl} R \qquad\qquad (R_{kl} = \lambda^h_{khl}), \tag{32″}$$

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[1] See Louis de Broglie. *Une nouvelle conception de la lumiére*, Hermann, Paris, 1934, pp. 47-48.

in which the ε_{kl} are equal to the coefficients of the metric (26), and one may write the equations of gravitation in the form:

$$R_{kl} - \tfrac{1}{2}\varepsilon_{kl}\, R = \mu\, T_{kl}\,, \tag{33}$$

in which μ is a constant and the T_{kl} are the components of the energy tensor. Likewise, with the aid of the covariant derivatives:

$$w^k_{h,l} = \frac{\partial w^k_h}{\partial s^l} + w^k_\alpha \gamma^{*\alpha}_{hl} + w^\alpha_h \gamma^{*k}_{\alpha l}$$

of the mixed components of the torsion:

$$w^k_h = \varepsilon_{k\alpha}\, w_{h\alpha}\,,$$

then, by contracting over the indices k and l one may form the divergence of this tensor and write the second Maxwell equation in the form:

$$w^k_{h,k} = \frac{\partial w^k_h}{\partial s^k} + w^k_\alpha \gamma^{*\alpha}_{hl} + w^\alpha_h \gamma^{*k}_{\alpha l} = 0. \tag{34}$$

As for the first Maxwell equations (27′), they may also be written:

$$\frac{\partial w_{hk}}{\partial s^k} + \frac{\partial w_{kl}}{\partial s^k} + \frac{\partial w_{lh}}{\partial s^k} + w_{h\alpha} w^\alpha_{kl} + w_{k\alpha} w^\alpha_{lh} + w_{l\alpha} w^\alpha_{hk} = 0, \tag{34′}$$

in which the w^{kl}_h are the coefficients of the bilinear covariants of the four forms ds^1, ds^2, ds^3, ds^4 and it is easy to see that the equations (34) and (34′) coincide with the well-known Maxwell equations for empty space if one suppose that the metric (26″) is that of special relativity.

In the case where one supposes that space is not empty, but in which there still exists no true magnetism, one must modify equations (34) by introducing the electric current vector into the right-hand side ([1]). One may also suppose that the energy tensor T_{kl} is the sum of two tensors, one of which is due to the electromagnetic field, and has components equal to the quantities:

$$t_{kl} = w_{hk} w^k_l - \tfrac{1}{4}\varepsilon_{kl} w \qquad (w = \varepsilon^{hl}\, w_{hk}\ w^k_l),$$

which is obtained from the tensor w_{kl} in a manner that is somewhat analogous to the one by which one obtained the Einstein tensor from the curvature tensor.

Conclusion. – One may now summarize the results to which we are led by our anholonomic interpretation. We have associated the equations of motion of a charged

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[1] See T. Levi-Civita. *Vereinfachte*, loc. cit., pp. 149-150.

particle of electricity with geodesics of shortest distance of the anholonomic hypersurface V_5^4, in such a fashion that one may arrive at the equations of motion for light rays, which are defined as null-length geodesics of the spacetime V_4, by annulling the constant factor ($\rho = 2v_5$) in these equations. Likewise, we have constructed the gravitational equations with the aid of the Einstein tensor, which is derived form the curvature tensor, and the Maxwell equations from the divergence of the torsion tensor; the first Maxwell equations are satisfied by definition.

Now, one may remark that these equations, like the equations of gravitation and Maxwell's equations, are independent of the variation of the metric on V_5 along the normal to V_4, which is obviously preferable. Moreover, our anholonomic physical space is locally four-dimensional, i.e., the points of that space are determined by the values of four variables x^1, x^2, x^3, x^4, and the distance between two of these points is given by the metric (26‴), but the totality of these local tangent spaces is not to be considered as the totality of the local tangent spaces to the same space V_4, and this is due to the fact that one supposes that the parallelogram that is constructed from infinitesimal displacements closes only by imagining that the fifth side points in a direction that is exterior to our space V_4, and that this fifth side is measured essentially by the electromagnetic tensor. It then results that our physical space also coincides globally with the space V_4 of the theory of relativity only if the electromagnetic field is null. One may say that the fact that physical space seems to us to have only four dimensions is a result of what we perceive only locally.

However, if the equations considered up till now are invariant under changes of metric on the normal then they are not invariant under changes of the normal direction, except for the equations of motion for an electrically charged particle. This question leads us to think about the case in which one supposes that the normal deviates from that invariant position, a deviation that one may suppose is due to certain physical phenomena. This deviation will be obviously measured by the values of four functions c^h in the intrinsic group (20) of the hypersurface V_5^4. In this case, the curvature tensor of V_5^4 does not coincide with the curvature of V_4 because the w_{kn}^h in formula (25′) are no longer null, in such a fashion that the electromagnetic tensor also appears in the left-hand side of equation (33) for gravitation, and consequently, the interaction of the two fields is very complex, which might be more in conformity with the nature of things.

Moreover, in the case of the deviation of the normal we have some new tensors. For example, the second fundamental form, which is no longer null, in general, and the second curvature tensor:

$$\lambda_{kl5}^h = \frac{\partial \overset{*}{\gamma}{}_{kl}^h}{\partial s^5} - \frac{\partial \overset{*}{\gamma}{}_{k5}^h}{\partial s^l} + \overset{*}{\gamma}{}_{k\alpha}^h w_{l5}^\alpha + \overset{*}{\gamma}{}_{\alpha l}^h w_{k5}^\alpha - w_{\alpha 5}^h \overset{*}{\gamma}{}_{kl}^\alpha + w_{k5}^h w_{l5}^5, \tag{35}$$

which is a tensor of the third order in the indices *h, k, l*; perhaps these new tensors might serve to account for other physical phenomena.

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