"Über die Transformation des Zwanges in allgemeine Coordinaten," Sitz. Kais. Akad. Wiss. **104** (1895), 281-285.

On the transformation of the constraint into general coordinates

Prof. Dr. **A. Wassmuth**

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Translated by D. H. Delphenich

The principle of least constraint that **Gauss** gave in 1829 (1), which he himself called a new general basic law of mechanics, has suffered a strange fate in its later evolution. Although the importance of that principle was recognized by many (2), not much has been added to the presentation in the recent textbooks on mechanics than what was given originally. The grounds for that might lie in the fact that **Gauss** himself did not give an "analytical" formulation of his principle. In1858, **Schefler** (3) already derived the general expression for the constraint Z (German *Zwang*), that is, the function that is to be minimized, and found that:

$$Z = \sum m\left[\left(x'' - \frac{X}{m}\right)^2 + \left(y'' - \frac{Y}{m}\right)^2 + \left(z'' - \frac{Z}{m}\right)^2\right],$$

in which the sum is extended over all mass-points m with the accelerations x'', y'', z'' and the force components X, Y, Z. However, it was in 1877 that **Lipschitz** (4)first confirmed that one must regard the accelerations x'', y'', z'' as variable when one imposes the minimum condition upon Z, but the coordinates x, y, z and the velocities x, y, z must be regarded as constants. **Lipschitz** (5) also introduced general variables (i.e., ones that fulfill the condition equations identically) in place of the rectangular coordinates when he exhibited a certain covariant that would be minimized by the principle of least constraint. However, the study of that very significant paper required quite a bit of effort, time, and foreknowledge, since it assumes that one is familiar with two papers by the same author on his investigations into homogeneous functions of n differentials (6). That only explains the conspicuous fact that the physical literature has not pursued the concept any

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(1) **Gauss**, *Werke V*, Crelle's Journal **4** (1829).
(2) **Lagrange**, *Mécanique analytique*, II, pp. 360. Note by Bertrand.
(3) **Scheffler**, Zeit. Math. Phys. **3** (1858).
(4) **Lipschitz**, Borchardt's Journal **82** (1877), pp. 323.
(5) **Lipschitz**, *loc. cit.*, pp. 330.
(6) *ibid.*, Bd. 70 and 72.

further. The excessively short derivation of the transformation equation that is given below, which is based entirely upon physical foundations, might not be unwelcome then.

The problem at hand is to introduce the mutually-independent variables $p_1, p_2, \ldots, p_k$ that fulfill the condition equations identically in place of the rectangular coordinates x_i, y_i, z_i for the n points into the expression for the constraint:

$$Z = \sum_{i=1}^{n} Z_i = \sum_{i=1}^{n} m_i \left[\left(x_i'' - \frac{X_i}{m_i} \right)^2 + \left(y_i'' - \frac{Y_i}{m_i} \right)^2 + \left(z_i'' - \frac{Z_i}{m_i} \right)^2 \right],$$

in which one sets, say, $x_i = f_1^i(p_1, p_2, \ldots, p_k)$, $y_i = f_2^i(p_1, p_2, \ldots, p_k)$, and $z_i = f_3^i(p_1, p_2, \ldots, p_k)$. If one sets $\frac{\partial x_i}{\partial p_\mu} = f_{1\mu}^i$, $\frac{\partial y_i}{\partial p_\mu} = f_{2\mu}^i$, …, to abbreviate, then the variations will be:

$$\delta x_i = f_{11}^i \cdot \delta p_1 + f_{12}^i \cdot \delta p_2 + \ldots + f_{1k}^i \cdot \delta p_k, \qquad \delta y_i = \ldots, \qquad \delta z_i = \ldots,$$

and analogously, the velocities will be:

$$x_i' = f_{11}^i \, p_1' + \ldots + f_{1k}^i \, p_k', \qquad \ldots,$$

but only when the conditions do not contain time explicitly, and the *vis viva* will be:

$$L = \sum_{i=1}^{n} L_i = \tfrac{1}{2} \sum a_{\mu\nu} \, p_\mu' \, p_\nu' \qquad (\mu, \nu = 1, 2, \ldots, k)$$

If one now considers the variation of a function H (in the case of a force function, it will coincide with energy) that is given by:

$$\delta H = \sum_{i=1}^{n} [(m_i \, x_i'' - X_i) \, \delta x_i + \cdots]$$

then when one introduces the values above for δx_i, δy_i, δz_i, and inverts the sequence of summations, one will get:

$$\delta H = \sum_{\mu=1}^{k} \delta p_\mu \sum_{i=1}^{n} [(m_i \, x_i'' - X_i) \, f_{1\mu}^i + \cdots] \; .$$

As is known, when $P_{i,\mu} = X_i f_{1\mu} + Y_i f_{2\mu} + Z_i f_{3\mu}$, one will have the identity:

$$(m_i \, x_i'' - X_i) \, f_{1\mu}^i + (m_i \, y_i'' - Y_i) \, f_{2\mu}^i + (m_i \, z_i'' - Z_i) \, f_{3\mu}^i = \frac{d}{dt} \frac{\partial L_i}{\partial p_\mu'} - \frac{\partial L_i}{\partial p_\mu} - P_{i\mu} = Q_{i\mu} \, ,$$

such that when one sets:

$$\sum_{i=1}^{n} Q_{i\mu} = \frac{d}{dt}\frac{\partial L_i}{\partial p'_\mu} - \frac{\partial L_i}{\partial p_\mu} - \sum_{i=1}^{n} P_{i\mu} = Q_\mu ,$$

immediately after one has performed the summation, one will have the known equation:

$$\delta H = \sum_{\mu=1}^{k} \delta p_\mu \cdot Q_\mu = Q_1\ \delta p_1 + \ldots + Q_k\ \delta p_k .$$

Now one can arrive at *another* expression for δH when one introduces the constraint Z. The possibility is based upon the fact that one has:

$$f_{1\mu}^{i} = \frac{\partial x_i}{\partial p_\mu} = \frac{\partial x'_i}{\partial p'_\mu} \text{ and also } = \frac{\partial x''_i}{\partial p''_\mu},$$

which is why:

$$\delta H = \sum_{\mu=1}^{k} \delta p_\mu \sum_{i=1}^{n} [(m_i\, x''_i - X_i)\, f_{1\mu}^{i} + \cdots] = \sum_{\mu=1}^{k} \delta p_\mu \sum_{i=1}^{n} [(m_i\, x''_i - X_i)\frac{\partial x''_i}{\partial p''_\mu} + \cdots]$$

$$= \tfrac{1}{2}\sum_{\mu=1}^{k} \delta p_\mu \sum_{i=1}^{n} \frac{1}{m_i}\frac{\partial}{\partial p''_\mu}[(m_i\, x''_i - X_i)^2 + \cdots],$$

so it follows that:

$$\delta H = Q_1\ \delta p_1 + \ldots + Q_k\ \delta p_k = \tfrac{1}{2}\sum_{\mu=1}^{k} \delta p_\mu \frac{\partial Z}{\partial p''_\mu} = \frac{1}{2}\frac{\partial Z}{\partial p''_1}\delta p_1 + \cdots + \frac{1}{2}\frac{\partial Z}{\partial p''_k}\delta p_k .$$

It follows from this that:

If $\delta H = 0$ *is true – i.e., if d'Alembert's principle is true – then due to the independence of the* δp*, one will get:*

$$\frac{\partial Z}{\partial p''_1} = 0, \qquad \ldots, \qquad \frac{\partial Z}{\partial p''_k} = 0;$$

i.e., ***Gauss****'s principle, and the former statement will follow from that latter conversely. Hence, both principles are completely equivalent.*

In addition, one will get:

$$Q_r = \frac{1}{2}\frac{\partial Z}{\partial p''_r} .$$

Now, it is known ([1]) that:

$$\frac{\partial Q_\rho}{\partial p_r''} = a_{\rho r} = a_{r\rho} ,$$

and therefore:

$$\frac{\partial Z}{\partial p_r''} = \frac{\partial Z}{\partial Q_1}\frac{\partial Q_1}{\partial p_r''} + \ldots = a_{1r}\frac{\partial Z}{\partial Q_1} + \cdots a_{kr}\frac{\partial Z}{\partial Q_k} = 2Q_r .$$

If one sets $r = 1, 2, \ldots, k$ in this, in succession, then one will obtain k linear equations whose determinant $D = (a_{\mu\nu})$ does not vanish. If $A_{\mu\nu} = \partial D / \partial a_{\mu\nu}$ is a sub-determinant then that will give the solution:

$$\frac{1}{2}\frac{\partial Z}{\partial Q_1} = \frac{1}{D}[A_{11}\, Q_1 + A_{12}\, Q_2 + \ldots + A_{1k}\, Q_k], \ \ldots,$$

from which, one concludes that the constraint is:

$$Z = \frac{1}{D}\sum A_{\mu\nu}\, Q_\mu\, Q_\nu + \varphi(p_1, p_1', p_2, p_2', \ldots) .$$

The function φ, which includes only the p and their *first* differential quotients with respect to time, must be added to that, since only the p'' were regarded as variable in the previous differentiation. T*he transformation of the constraint Z into general coordinates is performed* using that formula, which is required for *the application of* ***Gauss****'s principle.* The determination of φ (which is not necessary in that) will not introduce any complications either.

Now, as far as the importance of **Gauss**'s principle is concerned, it might be reiterated ([2]) that when the virtual work and the *vis viva* are given for a physical problem, the minimum property of the constraint Z expresses *a law for the system*. That will become all the more valid as one strives to describe a series of theories by mechanical analogies, as **W. Voigt** called them. I have already applied that law to electrodynamics, and have likewise already presented it for thermodynamics.

Since the constraint Z combines *all of* ***Lagrange****'s equations:* $Q_1 = 0, \ldots, Q_k = 0$, within it, **Gauss**'s principle will then prefer those equations when one strives for precisely that *unification* of them; for example, in the case of a string that one imagines to be composed of discrete points.

The value of that principle as a fundamental law is probably best shown by the fact that **Hertz** ([3]) constructed all of his mechanics from that principle and the law of inertia.

([1]) **Wassmuth**, "Über die Anwendung des Principes des kleinsten Zwanges auf die Elektrodynamik," Wied. Ann. **54**, pp. 166 [or Sitz. Kön. Bayer. Akad. (1894), pp. 226 and 222].

([2]) **Wassmuth**, *loc. cit.*, pp. 167.

([3]) **Hertz**, *Principien der Mechanik*, pp. 185, no. 391.