"Die komplexen Bewegungen," Berichte der sächsischen Gesellschaft der Wissenschaften (1903), 384-408.

Complex motions

By

E. von WEBER

Translated by D. H. Delphenich

It is known that the complex motions of three-dimensional Euclidian space play a fundamental role in many geometric questions, especially LIE's sphere of ideas, but nevertheless, as far as I know, only their analytical, but not their geometric significance has been appreciated, up to now. In the present article, we shall then attempt to develop a real-geometric interpretation for the aforementioned space transformations, and in fact, on the basis of the two closely-related remarks that, first of all, every complex motion permutes the fourfold-infinitude of minimal planes in space amongst themselves, and secondly, those planes can be associated with the ∞^4 oriented lines in space – i.e., with STUDY's terminology, the *spears* – in a single-valued and invertible way. *We shall then consider the complex motions to be spear transformations of* R_3 .

To that end, in § 1, we first examine which real spatial structures in the geometry of spears correspond to the complex points and lines in R_3 . Naturally, the principle of that map is by no means new. Rather, it is closely related to the well-known representation of the points of R_3 by oriented circles that was studied by M. CHASLES and E. LAGUERRE, and in addition to the work of STUDY that we shall cite later on, we would like to emphasize G. TARRY's theory of imaginaries ([1]). By contrast, the main evaluation of spears that is carried out here will grow out of known ideas in many respects, and for constructive purposes, in particular. The following § 2, in which we investigate spear transformations that are defined by a complex motion more closely, deals with only motions in a narrower sense, while the reversals and conversions of space that one can call "improper" or "anti-motions" that were important in the work of C. JUELS and C. SEGRE shall be reserved for a special treatment. Finally, in the third §, we will answer the fundamental question of the real interpretation of the invariants that a quadruple of spears will exhibit under the group Γ_{12} of all complex motions.

It is hardly necessary to point out that the problems that are treated here all find analogues in the geometry of complex non-Euclidian motions, and that their solutions seem to be free of the asymmetry that their Euclidian consideration would necessitate. That remark by itself will also yield the intrinsic connection between the present investigations and STUDY's recent work ([2]), in which the group of complex non-Euclidian motions was examined repeatedly, and also referred to limiting case that is treated here.

([1]) Assoc. Fr. C. R. Séance XVIII-XXII (1889-93).

([2]) Cf., in particular: "Über nicht-euklidische und Liniengeometrie," Greifswald, 1900, as well as *Geometrie der Dynamen*, Leipzig, 1903.

§ 1. Complex points and lines.

1. We (with STUDY) understand a *spear* to mean an oriented – i.e., endowed with a sense of direction – line in R_3 that does not lie at infinity; any such line then "contains" two opposite spears. One and only one spear is defined by any system of seven real numbers p_{ik}, π that fulfill the conditions:

$$p_{12}\,p_{34} + p_{13}\,p_{42} + p_{14}\,p_{23} = 0, \qquad \pi^2 = p_{41}^2 + p_{42}^2 + p_{43}^2,$$
$$\pi \neq 0.$$

Of those *spear coordinates*, the p_{ik} are nothing but the PLÜCKERian coordinates of the straight line that contains the spear, while the seventh one π is represented in the form of the cosine of the angle that the spear direction makes with the three positive directions of the axes of the basic Cartesian coordinate system. Following LIE, a complex plane:

$$ux + vy + wz + \varpi = 0$$

is called a *minimal plane* when the relation:

$$u^2 + v^2 + w^2 = 0 \tag{1}$$

exists without u, v, w vanishing simultaneously. Any minimal plane contains a finite, real line, and conversely, two conjugate-imaginary minimal planes go through any such line that are associated uniquely with both of the spears that lie on that line. This comes about by means of the formulas:

$$\left\{\begin{aligned} \rho u &= -p_{21}p_{24} - p_{31}p_{24} + i\pi\,p_{23}, \\ \rho v &= -p_{32}p_{34} - p_{12}p_{14} + i\pi\,p_{31}, \\ \rho w &= -p_{13}p_{14} - p_{23}p_{34} + i\pi\,p_{12}, \\ \rho\varpi &= p_{12}^2 + p_{13}^2 + p_{31}^2, \end{aligned}\right. \tag{2}$$

$$\left\{\begin{aligned} &\sigma p_{12} = \varpi' w'' - \varpi'' w', \quad \sigma p_{41} = v'w'' - w'v'', \\ &\sigma p_{23} = \varpi' u'' - \varpi'' u', \quad \sigma p_{42} = w'u'' - u'w'', \\ &\sigma p_{31} = \varpi' v'' - \varpi'' v', \quad \sigma p_{43} = u'v'' - v'u'', \\ &\sigma\pi = u'^2 + v'^2 + w'^2 = u''^2 + v''^2 + w''^2; \end{aligned}\right. \tag{3}$$

in which one has set $u = u' + i\,u''$, etc. Any spear is therefore assigned to a minimal plane in an invertible, single-valued way, and should be regarded as the real representation of the latter. From now on, we can then also understand "spear coordinates" to mean any four complex numbers that are coupled by equation (1) without u, v, w vanishing simultaneously.

2. We call the totality of ∞^2 spears that represent the minimal planes that go through a real or complex point P a *spear cycle*, or more briefly, the *cycle P*. If P is a real point then the cycle P will be identical with the *spear bundle P;* viz., the totality of all spears that go through P. If P is a complex point at infinity then the minimal planes that go through it will define an *improper cycle*; the associated spears distribute themselves on two different bundles of syntactic ([1]) spears. Those bundles are opposite when P is a real point at infinity, while they will coincide when P lies in the spherical circle at infinity.

If $a + i\,a'$, $b + i\,b'$, $c + i\,c'$ are the coordinates of P, referred to our Cartesian axis-cross, and A, A' are the real spatial points with the coordinates a, b, c ($a + a'$, $b + b'$, $c + c'$, resp.) then we would like to refer to the point-pair AA' as the *arrow* that belongs to the cycle P, where A is the starting point, and A' is the end point. A is also called the *center*, the positive line segment $AA' = r$ is called the *radius*, and the line AA' is called the *axis* of the cycle P. Since it is defined completely by the real points A, A', we could probably also speak of *the cycle* $[AA']$. If the real point A'' is determined in such a way that A is the midpoint of the line segment $A''A'$ then the arrow AA'' will belong to the conjugate-imaginary point $\bar{P}$, and A will be the midpoint of the pure-imaginary line segment $P\bar{P}$.

In the plane that is perpendicular to the axis AA' at the point A, we construct the connected circle K with its center at A and radius r, and endow it with the sense of traversal that, when seen from the point A', appears the same as the rotation that takes the positive X-axis to the positive Y-axis in the shortest way would appear when considered from a point of the positive Z-axis. That oriented circle K is called the *equator* of the cycle P. Now, as one verifies easily, the spears of that cycle consist of the totality of all generators of the confocal one-sheeted hyperboloid of rotation that has the equator K for its focal circle, and in fact those generators are oriented such that the vertical projections of the point A that lie in the equatorial plane will circle in the same sense as the equator itself. Among the spears of the cycle P, one also finds the ∞^1 oriented tangents of K. Naturally, one will obtain the conjugate cycle $\bar{P}$ by inverting the sense of all spears of a cycle P.

3. Two spears σ_1, σ_2 of the cycle P go through any real point Q. They are the generators of the one-sheeted hyperboloid of rotation that goes through Q and has the focal circle K, so it is easy to construct when A, A', Q are given. They will go to the oriented tangents that are drawn to K when Q is chosen to be on the equatorial plane outside of K, and will coincide if and only if Q lies in K itself. The two spears that lie on the axis will then be the only opposite ones in the cycle.

4. If a spear σ is given arbitrarily then there will exist one and only one spear in the cycle P that is syntactic to it. In order to find it, one draws the spear σ' that is syntactic to the given one through the point A and chooses the point Q along it in the direction of the spear such that $AQ = r$. If Q' is the projection of Q onto the equatorial plane, and furthermore, Q'' is determined on that plane such that the line segment AQ' is equal and

([1]) That is, spears that are parallel and directed the same way (STUDY).

perpendicular to AQ'', and the rotation through 90° that brings AQ'' to the position AQ' in the sense of the equator then the spear that is drawn through Q'' syntactic to σ will be the desired one.

5. Conversely, if a spear σ and a real point A are given then one understands B to mean the base point of the perpendicular that is dropped from A to σ and determines the point B by the demand that the line segment AB' must be perpendicular to the plane (A, σ) and equal to AB, and in fact such that the direction of σ that points from B to A and direction that points from A to B' will follow each other in a manner that is analogous to the X, Y, and Z directions of our coordinate system. The line through B' that is drawn parallel to σ will then be the locus of all points A' with the property that the cycle $[AA']$ contains the spear σ. We then obtain a simple geometric definition of the ∞^4 arrows that belong to the points of a given minimal plane.

6. The ∞^2 complex points that lie on a real line γ will be represented by the totality of arrows that have their starting and ending points on γ. Secondly, we consider a *sub-imaginary* line γ – i.e., a complex line that possesses one real point and lies in a real plane, and indeed initially under the assumption that the real point is at infinity. Two minimal planes will then go through γ that are represented by anti-syntactic spears σ, σ'. Conversely, a complex line with a real direction is defined by two such spears as the line of intersection of the two associated minimal planes. The ∞^1 oriented circles that are contacted by the spears σ and σ' are then associated with just as many arrows, whose starting (ending, resp.) points fill up a line h (h', resp.) that is parallel to σ and σ'. h lies in the plane of the spears σ, σ' at the same distance d from each of them, h' is at a distance d from h, and the plane ($\sigma\sigma'$) will be perpendicular to the plane of the lines h, h'. The latter plane contains the line γ. The ∞^2 complex points of that line will now be represented by the arrows that have their starting points on h (their end points on h', resp.). Conversely, in order for the connecting line of two complex points $[AA']$ and $[BB']$ to have a given real direction, it is necessary and sufficient that the real lines AB and $A'B'$ should be parallel to that direction.

7. Now, let γ mean a sub-imaginary line with the real point O; let it contain the real plane e that goes through O. The arrows of all complex points of γ will then lie in e. Conversely, if two arrows A_1A_1' and A_2A_2' lie in the same real plane e without the lines A_1A_2 and A_1A_2' being parallel then the connecting line of the complex points $[A_1A_1']$ and $[A_2A_2']$ will contain a real point O that lies at infinity.

8. We think of the plane e as being oriented; i.e., the domains into which it divides space are distinguished as its positive and negative sides. Of the two bundles of syntactic spears that are perpendicular to e, we say the *first* (*second*, resp.) bundle to mean the spears that point into the positive (negative, resp.) side of e. The minimal planes that are defined by the first (second, resp.) bundle of spears then cut out the first (second, resp.) system of minimal lines from e.

If $A_i A_i'$ is an arrow that lies in e then one can distinguish a first P_i and a second P_i' among the two points at which the associated equator cuts the plane e according to whether the spear that contacts the point of the equatorial circle in question on the positive or negative side of e, resp. $P_i A_i P_i'$ lie along a line that is perpendicular to $A_i A_i'$, and one will have $P_i A_i = A_i P_i' = A_i A_i'$. Since the pair $P_i P_i'$ can then be constructed in an entirely single-valued way from any point-pair $A_i A_i'$ that lies on e, and conversely, the one object can be employed as the definition of a complex point that lies on e as well as the other, and in fact $P_i P_i'$ represents those complex points of e at which the minimal lines of the first one that go through P_i intersect those of the second system that go through P_i'; the complex-conjugate point will be represented by $P_i' P_i$.

9. We consider the plane to be the GAUSSian number plane and accordingly refer it to a rectangular axis-cross ξ, η such that the first (second, resp.) system of minimal planes is given by the equations:

$$\xi + i\,\eta = \text{const.}, \qquad \xi - i\,\eta = \text{const., resp.}$$

If the complex numbers ξ, η mean the coordinates of the point $[A_i A_i']$, moreover, while the numbers z, z' mean the GAUSSian affix of the aforementioned real points P_i, P_i' then one will obviously have:

(4) $$z = \xi + i\,\eta, \qquad \bar{z}' = \xi - i\,\eta.$$

Let the complex line γ be defined by the equation:

(5) $$a\xi + b\eta + c = 0.$$

One writes that by means of (4) as follows:

$$\bar{z}' = \alpha z + \beta,$$

in which one sets:

$$\alpha = -\frac{a - ib}{a + ib}, \quad \beta = \frac{-2c}{a + ib}.$$

That relation defines an indirect similarity transformation $\mathfrak{A}$ of the plane e into itself. Its fixed point O is laid at a finite point, under the assumption that was made about γ, which is expressed analytically by $|\alpha| \neq 1$, so the ratio of a and b is not real.

10. If a sub-imaginary line γ is then given by two points that lie on it – i.e., by two arrows A_1A_1' and A_2A_2' that lie in a plane e – then one will find its real point O as the fixed point of the indirect similarity transformation $\mathfrak{A}$ that takes the points P_1, P_2 that are constructed according to no. **8** to P_1' and P_2'. Two minimal planes go through γ whose spears σ, σ' contain the point O the cycle can be easily constructed from no. **3** as the common spear. Now, in order to find the arrow A_iA_i' of all complex points of γ, one determines the point A_3' that is the common point of intersection of the plane e and two lines that are parallel to σ and σ' for an arbitrarily chosen point A_3 on e according to no. **5**. One can also ascertain the fixed point P_3 of the indirect similarity transformation that arises from the succession of $\mathfrak{A}$ and the reflection through the point A_3 , and constructs the points P_3' and A_3' from the prescription that was given in no. **8**.

11. The analysis of no. **9** admits an exception when the coefficient b in equation (5) is equal to ia ($-ia$, resp.), so γ means a minimal line of the first (second, resp.) system. The first (second, resp.) point of every pair P_iP_i' will then be identical to the real point O of γ, so the spears σ, σ' will coincide in the spear of the first (second, resp.) bundle that goes through O perpendicular to e, and one will then obtain the point A_i' as the point of intersection of e with the parallels to σ that are constructed as in no. **5** for every point A_i that lies in e. A sub-imaginary minimal line γ is then characterized completely by the spear σ of the minimal plane that goes through it and the real point O that lies on γ and σ.

12. Now, let a super-imaginary line γ be given as the connecting line of two complex points $[A_1\ B_1]$ and $[A_2\ B_2]$. The four points A_i , B_i will not lie in the same plane then. Let e be the plane that goes through A_1 and A_2 and is parallel to the line $B_1\ B_2$, and likewise let e' be the plane that goes through B_1 and B_2 parallel to $A_1\ A_2$. The arrows $A_i\ B_i$ of the ∞^2 points of γ then lie such that the points A_i fill up the plane e, while the B_i fill up the plane e'. The plane e is parallel to γ and the line $\bar{\gamma}$ that is complex-conjugate to it.

The vertical projection of the arrow that belongs a complex point P that lies in e is obviously identical with the arrow of the vertical projection of P onto e. Hence, if the complex points $[A_i\ B_i]$ lie on γ, and A_i' means the vertical projection of B_i onto e then the complex points $[A_iA_i']$ will lie on the sub-imaginary line γ' to which γ projects. We construct the real point O in the line γ from no. **10** by means of the given points A_1A_1' and A_2A_2', as well as the base point P of the perpendicular that is dropped from O to e'. The line OP cuts γ and $\bar{\gamma}$ perpendicularly, and $[OP]$ is its base point on γ. Furthermore, if σ and σ' are two spears that go through O and belong to the minimal planes that lie along γ', while τ and τ' are spears that represent the minimal planes that go through γ, then obviously σ will be syntactic to τ, and σ' is syntactic to τ', due to the parallelism of γ and γ'. One then constructs τ and τ' easily according to no. **4** by means of the remark that this spear must belong to any of the cycles $[A_i\ B_i]$; e.g., to the cycle $[OP]$, as well.

13. If the connecting line γ of the points $[A_i\ B_i]$ is a super-imaginary minimal line then the construction of the planes e, e', as well as the points O, P will stay the same, while the spears τ, τ' will coincide with one of the spears that fall along the line OP. A super-imaginary minimal line can then be characterized completely by a spear τ and an arrow OP that lies along it. The planes e and e' will then be perpendicular to τ at the points O and P, resp., and one will obtain the points B_i that are associated with arbitrarily-chosen points A_i of e as the intersection of e' and a line that is parallel to τ (no. **5**).

14. From the foregoing, two cycles $[A_1\ B_1]$ and $[A_2\ B_2]$ possess two different common spears whose construction will emerge immediately from what was said in nos. **10-12**. The points $[A_i\ B_i]$ lie on a minimal line, so their separation distances will vanish if and only if the two cycles have a single, doubly-counted spear in common.

Let the point B_2' be chosen such that the arrows A_1B_2' and $A_2\ B_2$ are equal and equally-directed. In order for the points $[A_1\ B_1]$ and $[A_2\ B_2]$ to have a zero separation, it is necessary and sufficient that the line segments $A_1\ A_2$ and B_1B_2' have the same length and that the lines that contain them are perpendicular ([1]).

15. We understand the *middle plane* of two non-parallel spears to mean the real plane that goes through the common normal secant to both spears and defines the same angles with them in such a way that the vertical projections of the spears onto the aforementioned plane will be anti-tactic. If the spears themselves are anti-tactic then we will refer to the plane that is perpendicular to both spears and the plane that is parallel to them and has the same distance to both of them as its middle plane.

Obviously, the plane that was called e in no. **12** is the middle plane for the spear-pair σ, σ', as well as the pair τ, τ'. We would like to briefly call the complex line of intersection γ of two minimal planes that are represented by the spears τ and τ' the *line* $(\tau\tau')$; the line $\bar{\gamma}$ will then be defined by the opposite spear-pair $(\bar{\tau}\,\bar{\tau}')$. From nos. **10-12**, the starting points A_i of the arrows that belong to the points of γ will lie on the middle plane of τ and τ' and fill it up completely when those spears are not anti-tactic. From no. **5**, one obtains the point B_i from any point A_i as the point of intersection of certain lines that are parallel to τ and τ'.

16. The results of the previous no. yield the solution to the fundamental problem of constructing the arrow $[AA']$ of a cycle when three of its spears σ_1, σ_2, σ_3 are given ([2]); i.e., of finding the point of intersection of three given minimal planes. It suffices to determine the center A of our cycle; one will then obtain the point A' as the intersection point of three lines that are parallel to the respective spears σ_i (no. **5**).

([1]) For more on this so-called "minimal position" of two cycles, see my note "Zur Geometrie der Kreise in Raum," which will appear soon in Arch. f. Math.

([2]) The assumption that two of the spears σ_i , or all three of them, are syntactic, leads to improper cycles.

Now, A is the intersection point of the middle planes of the three spear-pairs σ_i, σ_k, In general, The three aforementioned planes will intersect along a line h that is perpendicular to the same real plane e if and only if the three spears of e are parallel. In that case, let P_i be the point where the plane that goes through h and perpendicular to σ_i meets that spear; A will then be the intersection point of h with the plane $P_1 P_2 P_3$.

The latter construction will also break down when the three spears σ_i are intersected perpendicularly by the same line h. We then let M_i denote intersection point of σ_i with h, and think of the line h as being oriented such that the line segments AM_i, $M_i M_k$ are also determined up to sign. Now, if ϑ_i is one of the angles that the equatorial plane of the desired cycle through h makes with σ_i, and r is the (positive) radius of our cycle, moreover, then one will have:

$$AM_i = r \cos \vartheta_i, \tag{6}$$

and therefore:

$$M_i M_k = r (\cos \vartheta_i - \cos \vartheta_k), \tag{7}$$

$$M_2 M_1 (\cos \vartheta_3 - \cos \vartheta_1) = M_3 M_1 (\cos \vartheta_2 - \cos \vartheta_1). \tag{8}$$

The latter equation yields the following construction: Let e be a real plane that is perpendicular to h at the point M, and let σ_i' be the vertical projection of σ_i on e. One then carries the point Q_i from M along σ_i' in the direction of that spear in such a way that $MQ_i = 1$ and lays a line l in the plane e through M in such a way that the ratio of the projections of the line segments $Q_2 Q_1$ and $Q_3 Q_1$ onto l will be equal to $M_2 M_1 : M_3 M_1$. The latter elementary geometric construction obviously possesses one and only one solution. The plane of the lines h and l is the equatorial plane of the desired cycle, and one knows the absolute magnitude of $\cos \vartheta_i$ and $\cos \vartheta_i - \cos \vartheta_k$ from now on, so one can construct the (absolutely-taken) line segments r and AM_i from (7) and (6), and therefore, A itself.

§ 2. Complex motions as spear transformations.

17. On the basis of a Cartesian coordinate system x, y, z, the formulas:

$$\begin{aligned} x' &= \alpha_1 x + \beta_1 y + \gamma_1 z + a, \\ y' &= \alpha_2 x + \beta_2 y + \gamma_2 z + b, \\ z' &= \alpha_3 x + \beta_3 y + \gamma_3 z + c, \end{aligned} \tag{1}$$

in which the complex numbers α_i, β_i, γ_i fulfill two well-known groups of six relations, define a complex motion when the three-rowed determinant of the quantities α_i, β_i, γ_i have the value + 1. Since the transformation (1) takes any minimal plane to another one, any complex motion can be interpreted as a transformation of the spears of R_3, from no. **2**.

The *real* motions are then characterized within the group Γ_{12} of all complex motions by the fact that they convert any pair of opposite spears into another such pair. Two anti-tactic spears do not generally go to anti-tactic spears under complex motions, while by contrast, one pair of syntactic spears will go to another one.

18. The complex translation:

$$x' = x + a, \qquad y' = y + b, \qquad z' = z + c$$

is called *pure-imaginary* when the a, b, c have the form ia', ib', ic'. It takes the coordinate origin O to the complex point $[OO']$, where O' means the real point with the coordinates a', b', c', and furthermore, the complex point $[AB]$ will go to the point $[AB']$, where B' is determined by the fact that the arrow BB' coincides with OO' in magnitude and direction. From no. **3**, one will obtain the spear σ' to which a given spear σ goes as the spear of the cycle $[BB']$ that is syntactic to σ when B is chosen arbitrarily on σ, and B' is determined as it just was.

19. Any complex translation $\mathfrak{T}$ can be represented in the form $T\,T' = T'T$, where T means a real, and T' means a pure imaginary translation. If it takes the point O to P, and the latter to $[OO']$ then one will obtain the complex point $[A'B']$ into which the point $[AB]$ is converted by $\mathfrak{T}$ when one chooses the real points A', B_1, B' in such a way that the arrow AA' coincides with OP in magnitude and direction, $A'B_1$, with AB, and finally, B_1B' with OO'. Conversely, when A, B, A', B' are given, one will get the points P and O' from this directly; i.e., the translation $T\,T'$ that converts the point $[AB]$ to $[A'B']$.

20. If Q is the real point that the real translation T takes O to, and if σ_1, σ_2 mean the two spears of the cycle $[OO']$ that go through Q then all of the spears that are syntactic to σ_1 or σ_2, and only them, will remain individually invariant under the translation $\mathfrak{T} = T\,T'$. The improper cycle that is defined by those two bundles belongs to the complex fixed point at infinity of our translation. Any other bundle of syntactic spears will be displaced into itself by means of a real translation under the motion $\mathfrak{T}$.

The translation $\mathfrak{T}$ is a minimal translation – i.e., two points $[AB]$ and $[A'B']$ that correspond under $\mathfrak{T}$ have a separation distance of zero when $\sphericalangle O'OP$ is a right angle, and $OP = OO'$. The two bundles of invariant spears will then coincide in a single one.

21. We now turn our consideration to the complex motion (rotations) that fix the coordinate origin O ([1]). The real sphere with center O and radius 1 will be called κ. The two systems of equations:

([1]) For this, one can also confer E. STUDY: "Über Nicht-euklidische und Liniengeometrie."

(1) $$x + i\,y = \lambda\,(1 + z), \qquad 1 - z = \lambda\,(x - i\,y),$$

(2) $$x - i\,y = \mu\,(1 + z), \qquad 1 - z = \mu\,(x + i\,y),$$

define the two systems of minimal lines that lie on κ; we distinguish them as the *first* and *second* system, respectively. The coordinates x, y, z of any point P of the sphere can be expressed rationally in terms of λ and μ by means of those equations; λ is called the *first parameter* of the point P, while μ is called the *second* one. If that point is real then one will have $\mu = \bar{\lambda}$, and conversely.

Any spear σ that goes through O will cut κ at two diametrically-opposite points, namely, the *exit point* P and the *entrance point* P_0 ; we also refer to σ as *the spear OP*. The associated minimal plane cuts out of κ the minimal line of the first system that goes through P and the minimal line of the second system that goes through P_0 . The coordinates ξ, η, ζ of the exit point P are connected with the coordinates p_{ik} , π (u, v, w, o, resp.) of the spear σ by the equations:

(3) $$u : v : w = i\,\xi + \eta\,\zeta : \xi\zeta - i\eta : \zeta^2 - 1,$$

(4) $$\xi = \frac{p_{41}}{\pi}, \qquad \eta = \frac{p_{42}}{\pi}, \qquad \zeta = \frac{p_{43}}{\pi},$$

while the parameters λ, μ of the point P are defined by the formulas:

(5) $$\lambda = \frac{iu - v}{w} = \frac{w}{u + iv} = \frac{\xi + i\eta}{1 - \zeta} = \frac{1 - \zeta}{\xi - i\eta} = \bar{\mu}.$$

22. We consider a complex motion $\mathfrak{B}$ that leaves the point O fixed to be a certain collineation of the sphere κ into itself for which any generating system of κ is transformed into itself. If the spear OP goes to $O\Pi$ under $\mathfrak{B}$ then the first parameters λ, Λ of the sphere points P, Π will be connected by an equation of the form:

(6) $$\Lambda = \frac{\alpha\lambda + \beta}{\gamma\lambda + \delta},$$

in which the $\alpha, \beta, \gamma, \delta$ mean complex constants that satisfy the inequality:

(7) $$\alpha\delta - \beta\gamma \neq 1.$$

P and Π are then corresponding points of a circle conversion V. The spherical point Q that is diametral to P is converted into Q' by means of V, and let Π' be the point of κ that is dimetrically-opposite to Q'. If one then denotes the second parameters of the points P and Π' by μ, N then one will have the equation:

(6′) $$M = \frac{\delta\mu - \gamma}{-\beta\mu + \alpha}.$$

The sphere generator of the first (second, resp.) system that goes through P then goes to the generator of the first system that goes through Π (the generator of the second system that goes through Π′, resp.) under our rotation. In other words: Formulas (6), (6′) together define the collineation of the sphere κ into itself that corresponds to the motion 𝔅. Conversely, if the four complex numbers $\alpha, \beta, \gamma, \delta$ correspondingly give the condition (7) then formulas (6), (6′) will define a collineation of the sphere κ into itself that, when interpreted as a spatial transformation, will take every minimal plane that goes through O to another one, and thus, the spherical circle at infinity will go to itself, so it will be identical with a complex motion 𝔅.

If F, G are the fixed points of the circle conversion V, and one lets σ, τ denote the spear OF, OG, then the complex axis of rotation 𝔅 will be the line (σ, τ) (cf., no. **15**). For the complex rotation angle, one easily finds that:

$$\cos \omega = 1 + \frac{(\alpha - \delta)^2 + 4\beta\gamma}{2(\alpha\delta - \beta\gamma)},$$

and ω is equal to the logarithm of the double ratio that the points F, G define with any pair of points P, Π that correspond under V, multiplied by $-i$.

23. From the foregoing, the group Γ_6 of the complex motions that leave the point O fixed is assigned to the direct circle conversions on the sphere κ in an invertible and single-valued way ([1]). There is then one and only one motion 𝔅 that takes three given spears σ_i through O to three other such spears σ_i', respectively. If P_i, P_i' are the exit points of those spears in κ then one will find the spear σ_4' or OP_4' that corresponds to any spear σ_4 that goes through O by means of 𝔅 when one, following MÖBIUS, constructs the point that corresponds to P_4 under the circle conversion V that associates the three point-pairs $P_i P_i'$.

24. If we preserve the notation of no. **22**, a real point P of κ will correspond in the aforementioned complex motion 𝔅 to a complex point whose equatorial circle π cuts the sphere κ perpendicularly at the points Π, Π′. In that, Π is referred to as the *exit point* of the oriented circle π, and Π′ is referred to as the *entrance point*; namely, the circle will contact the sphere at the point Π of the spear OΠ, and at the point Π′ of the spear OQ' (or Π′O).

([1]) Cf., STUDY, *loc. cit.*

25. One likewise finds the complex point into which an arbitrary real spatial point A will be converted by $\mathfrak{B}$ when one applies the analogous construction to the sphere that is concentric to κ and goes through A.

It is, moreover, easy to construct the spear σ that the motion $\mathfrak{B}$ will take a given spear σ' to. Namely, if one lays the spear τ that is syntactic to σ through O, and if τ is the spear that τ' goes to under $\mathfrak{B}$ then σ will be the spear of the cycle that is syntactic to τ_i' into which any real point of σ will be transformed by the motion $\mathfrak{B}$.

26. A complex rotation $\mathfrak{R}$ that fixes the complex point $[AA']$ is determined when one knows the three spears σ_1', σ_2', σ_3' that correspond to the three spears σ_1, σ_2, σ_3. If the translation $\mathfrak{T}$ that takes the point $[AA']$ to O converts the spears σ_i (σ_i', resp.) into the spears τ_i (τ_i', resp.) that go through O, and if $\mathfrak{R}'$ is the rotation around O that takes the τ_i to the τ_i', resp., then one will obtain the spear that corresponds to an arbitrary spear σ_4 under $\mathfrak{R}$ easily from the relation:

$$\mathfrak{R} = \mathfrak{T}\,\mathfrak{R}'\,\mathfrak{T}^{-1}.$$

27. There is one and only one complex motion $\mathfrak{B}$ that takes three given spears σ_i, no two of which are syntactic, to three other ones σ_i'. The proof of that, and likewise, the construction of the spear σ_4' that corresponds to an arbitrarily-given spear σ_4 under $\mathfrak{B}$, will be first achieved when one ascertains the arrows $[AB]$ and $[A'B']$ of the cycle that is defined by the triples σ_i and σ_i' (no. **16**). The spear σ_i will be converted into σ_i'' by means of the translation $\mathfrak{T}$ that takes the point $[AB]$ to $[A'B']$. The rotation $\mathfrak{R}$ that takes the triple σ_1'', σ_2'', σ_3'' to σ_1', σ_2', σ_3' (no. **26**) will then bring the spear σ_4'' to the desired position σ_4', and one will have $\mathfrak{B} = \mathfrak{T}\mathfrak{R}$.

28. Since any direct, real, circle conversion possesses two real fixed points on the sphere κ that can coincide in special cases, any complex rotation can, in general, leave two distinct spears σ, τ fixed, and the line (σ, τ) is the rotational axis. For any complex motion $\mathfrak{B}$ that is not a translation, there are then two and only two distinct bundles of syntactic spears, in general, that remain invariant under $\mathfrak{B}$; if those bundles coincide in one then we will call the motion $\mathfrak{B}$ *parabolic*.

29. The following facts that relate to the fixed spears of an arbitrary motion $\mathfrak{B}$, and thus, to the minimal planes that $\mathfrak{B}$ fixes, will flow easily from the known properties of the system of coefficients α_i, β_i, γ_i (no. **17**):

In order for the first motion $\mathfrak{B}$ to be parabolic, it is necessary and sufficient that one must have:

$$\alpha_1 + \beta_2 + \gamma_3 = 3, \tag{8}$$

without all three of the numbers α_1, β_2, γ_3 being equal to 1; the last assumption is characteristic of translations. Any parabolic motion will leave one and only one bundle of syntactic spears invariant, but generally no spear (viz., *parabolic motion of the first kind*), since it displaces the bundle β into itself by translation. There are ∞^3 such parabolic motions that transform a given bundle β into itself.

In particular, a parabolic motion can leave any spear of a syntactic bundle fixed (viz., *parabolic motion of the second kind*). Any bundle β remains unchanged under ∞^6 motions of that kind.

Conversely, any motion that displaces a bundle of syntactic spears into itself by translation is either a translation or a parabolic motion of the first kind, and furthermore, the motion that fixes the individual spears of a syntactic bundle is either a translation or a parabolic motion of the second kind.

30. Any motion $\mathfrak{B}$ that does not satisfy the condition (8) will possess two different invariant bundles of syntactic spears, and inside of each of them, a fixed spear σ (τ, resp.). The complex line (σ, τ) is the *screw axis* of the motion. The spears of each of the two bundles will be permuted with each other by a direct similarity transformation $\mathfrak{A}$ ($\mathfrak{A}'$, resp.). That is, if the spears σ_i that are syntactic to σ correspond to the spears σ_i' under $\mathfrak{B}$, and the spears τ_i that are syntactic to τ correspond to τ_i' then the parallel triangles $\sigma\sigma_i\sigma_i'$ will all be directly similar to each other, and similarly for the triangles $\tau\,\tau_i\,\tau_i'$. Moreover, the latter triangles will be directly similar to the former ones; i.e., the similarity transformations $\mathfrak{A}$, $\mathfrak{A}'$ will be *inverse.*

The construction of the fixed spear of the motion that we defined in no. **27** will follow easily from these remarks.

31. There are ∞^2 complex motions that transform two given bundles of syntactic spears into themselves by means of given direct similarity transformations $\mathfrak{A}$ ($\mathfrak{A}' = \mathfrak{A}^{-1}$, resp.). In other words: There exist ∞^2 motions that fix two given spears σ, τ and convert a given spear σ_1 that is syntactic to σ into another one σ_1'. Among those motions, one will find a rotation $\mathfrak{R}$, and all of the rest will have the form $\mathfrak{R}\mathfrak{T} = \mathfrak{T}\mathfrak{R}$, where $\mathfrak{T}$ means any of the ∞^2 complex translations that fix the spears of each bundle individually.

32. In order to construct the rotation $\mathfrak{R}$ from the data of the previous number, we represent $\mathfrak{R}$ in the form $\mathfrak{T}\mathfrak{R}'\mathfrak{T}^{-1}$, where $\mathfrak{T}$ is any translation that converts σ and τ into two spears that go through O, so $\mathfrak{R}'$ means a rotation about O. The problem is then reduced to the following one: One knows the two fixed spears σ, τ of a rotation $\mathfrak{R}$ that fixes the coordinate origin O, and the similarity transformations $\mathfrak{A}$ and $\mathfrak{A}' = \mathfrak{A}^{-1}$ by which the spears that are syntactic to σ and τ are permuted amongst themselves. One then looks for the direct circle conversion V that corresponds to the rotation $\mathfrak{R}$ on the spherical surface κ.

The following fact serves to solve that problem: There are always two and only two diametrically-opposite points P_1, P_2 on κ that are converted into two diametrically-opposite points Π_1, Π_2 by a given circle conversion V; P_1 and P_2 are the fixed points of the direct circle conversion:

$$CVCV^{-1},$$

in which C means the indirect circle conversion that converts any point of the sphere into their diametric opposites. The points P_1, P_2 are symmetric to the points Π_1, Π_2 relative to the middle plane e of the two spears σ, τ. When one now associates any spear that is syntactic to σ with its midpoint, the similarity transformation $\mathfrak{A}$ will correspond to a similarity transformation of the plane e into itself that was referred to. If g is the line that lies in e that goes through O, and cuts σ and τ at right angles then there will be infinitely many point-pairs PP' on e such that P' is associated with the point P by means of $\mathfrak{A}$, and the connecting line PP' is perpendicular to g. The locus of P (P', resp.) is line h (h', resp.) that goes through O. The planes (h, σ) and (h', τ) then intersect along the line $P_1 P_2$. Now, the circle conversion V is over-determined when one knows its fixed points (i.e., the exit points of the spears σ, τ) and the two pairs of corresponding points $P_i\,\Pi_i$.

33. If the spears σ, τ are opposite then the previous construction will break down, since the points P_1 and Π_1 would now coincide in the exit point of σ, while P_2 and Π_2 would coincide in the exist point of τ. One would then be dealing with a complex rotation $\mathfrak{R}$ with a real axis. From now on, e will mean the plane that goes through O perpendicular to σ and τ. When one lets any spear that is syntactic to either of σ or τ correspond to the point where it pierces e, the given similarity transformations $\mathfrak{A}$ and $\mathfrak{A}'$ will determine two similarity transformations $\mathfrak{A}$ and $\mathfrak{A}'$ of the plane e into itself. If the points P_i are converted into P_i' by $\mathfrak{A}$ and into P_i'' by $\mathfrak{A}'$ then the triangles $O\,P_i\,P_i'$ will all be directly similar, and the triangles $O\,P_i''\,P_i$ will all be inversely similar, such that point-pairs $P_i'\,P_i''$ lie on a line that goes through O. Under $\mathfrak{R}$, the real points P_i will correspond to the cycle whose equator cuts the plane e perpendicularly at the points P_i' and P_i'', and will thus be oriented such that its tangents at P_i' are syntactic to σ, while its tangents at P_i'' are syntactic to τ. Naturally, analogous things are true for any plane that is perpendicular to the rotational axis such that the complex point to which a given real point goes under $\mathfrak{R}$, can be constructed immediately.

Now, in order to construct the spear s' that corresponds to an arbitrary spear s under $\mathfrak{R}$, one chooses two real points on s and ascertains the cycles that correspond to them, as well their common spears s', t' (no. **14**); among them, the desired s' is made known by the fact that it also belong to the cycle that is determined by s, σ, τ.

34. If two triples of spears σ_1, σ_2, σ_3 and σ_1', σ_2', σ_3' are given, such that σ_1 is syntactic to σ_2, and σ_1' is syntactic to σ_2' then there will exist ∞^2 motions $\mathfrak{B}$ that convert the σ_i into the σ_i', resp. Namely, if $\mathfrak{R}$ is any of the rotations that take the spears σ_i into three spears σ_1'', σ_2'', σ_3'' that are syntactic to σ_1', σ_2', σ_3', resp., then there will be one and only one direct similarity transformation that converts σ_1'' into σ_1' and σ_2'' into σ_2' inside of the spear bundle β that contains σ_1' and σ_2'. The similarity transformation that is inverse to it, which transforms the bundle γ that is syntactic to σ_3'', and thus brings σ_3'' to the position σ_3', is then determined completely. From no. **31**, there will then exist a rotation $\mathfrak{R}$, under which the σ_i'' will go to the σ_i', and the ∞^2 motions $\mathfrak{B}$ are all of the form $\mathfrak{R}\mathfrak{R}'\mathfrak{T}$, where $\mathfrak{T}$ means one of the ∞^2 translations that leave the spears of the bundles β and γ fixed individually. For the case in which the figure of intersection of the four spears σ_1'', σ_2'', σ_1', σ_2' defines a parallelogram with any plane, one must choose a complex translation that is easy to discern in place of $\mathfrak{R}'$.

If the three spears σ_1, σ_2, σ_3, and likewise σ_1', σ_2', σ_3', are syntactic then taking the first triple to the second one will possible if and only if the two triangles σ_i and σ_i' are directly similar; in that case, there will be ∞^6 motions that facilitate that transition.

§ 3. Invariants of quadruples.

35. A quadruple of spears σ_1, σ_2, σ_3, σ_4, no two of which are syntactic, obviously possesses four independent real or two independent complex (absolute) invariants under the Γ_{12} of all complex motions. If τ_k is the spear that goes through the coordinate origin syntactic to σ_k, where P_k is its exit point from the unit sphere κ; then one of the double ratios of the four spherical points P_k can be taken to be one of the complex invariants above. We denote the coordinates of the point P_k by ξ_k, η_k, ζ_k, and denote its first parameter by λ_k, such that one then has:

$$\lambda_k = \frac{\xi_k + i\eta_k}{1+\zeta_k} = \frac{1-\zeta_k}{\xi_k - i\eta_k} \qquad (k = 1, 2, 3, 4),$$

and then write:

$$(klpq) = \frac{\lambda_l - \lambda_k}{\lambda_l - \lambda_p} : \frac{\lambda_p - \lambda_k}{\lambda_q - \lambda_p},$$

$$\delta_0 = (1234), \quad \delta_1 = (2314), \quad \delta_2 = (3124).$$

We further set the coordinates u_k, v_k, w_k of the spear τ_k equal to the expressions:

$$\xi_k\,\zeta_k - i\,\eta_k\,, \quad i\xi_k + \eta_k\,\zeta_k\,, \quad \zeta_k^2 - 1,$$

and write:

$$[kl] = u_k\,u_l + v_k\,v_l + w_k\,w_l\,,$$

to abbreviate; the relation will then exist:

$$\sqrt{[kl]} = 2i\sqrt{2}\,\frac{\sqrt{\bar\lambda_k}\sqrt{\bar\lambda_l}\,(\lambda_l - \lambda_k)}{(1+\lambda_k\bar\lambda_k)(1+\lambda_l\bar\lambda_l)},$$

which likewise means that the symbol $\sqrt{[kl]}$ will be made unambiguous when one understands $\sqrt{\bar\lambda_k}$ on the right-hand side to mean a certain one of the two complex numbers whose square equals $\bar\lambda_k$. If one then sets:

$$p_0 = \sqrt{[13]}\sqrt{[42]}\,, \qquad p_1 = \sqrt{[12]}\sqrt{[34]}\,, \qquad p_2 = \sqrt{[23]}\sqrt{[14]}$$

then one will have the relations:

$$p_0 + p_1 + p_2 = 0,$$

$$-\delta_0 = \frac{p_1}{p_2}, \qquad -\delta_1 = \frac{p_2}{p_0}, \qquad -\delta_2 = \frac{p_0}{p_1}.$$

These three double ratios can then be expressed in a known way in terms of one of them. The absolute value of the number δ_k is equal to the corresponding double ratio of the absolutely-taken line segment $P_k\,P_l$, while its arc is equal to one of the MÖBIUS double angles of the spherical rectangle $P_1\,P_2\,P_3\,P_4$. We call the numbers δ_k, as well as the real double ratios and double angles that they define, the *direction invariants* of the spear-quadruple σ_k.

36. If one lets M_0, M_1, M_2 denote the moments of the three complex line-pairs (no. **15**):

$$(\sigma_1, \sigma_3), (\sigma_4, \sigma_2); \qquad (\sigma_1, \sigma_2), (\sigma_3, \sigma_4); \qquad (\sigma_1, \sigma_4), (\sigma_2, \sigma_3),$$

resp., and denotes the four-rowed determinant that is defined by the coordinates u_k, v_k, w_k, ϖ_k of the four spears by Δ then one will find that:

$$M_h = \frac{\Delta}{p_h^2}.$$

Obviously, each of the numbers δ_k , together with any of the quantities M_k , defines the complete system of invariants of the spear-quadruple σ_1, σ_2, σ_3, σ_4 under our Γ_{12} . As before, we come to the problem of replacing the δ_k , as well as the M_k , by real, constructible, geometric quantities.

37. To that end, we let P_h^k denote the base point on σ_h of the shortest distance between the spears σ_h and σ_k ; if those spears have a point in common then the points P_h^k and P_k^h will coincide in the point of intersection. Moreover, we understand d_h^{kl} to mean the line segment $P_h^k P_k^l$, and let $d_h^{kl} > 0$ or $d_h^{kl} < 0$ according to whether direction that points from P_h^k to P_h^l does or does not coincide with that of the spear σ_h , resp. Finally, we set:

$$\{hklm\} = d_k^{hl} - d_l^{km} + d_m^{lh} - d_h^{mk} .$$

Since that expression, which we would like to call a *double difference* of our spear-quadruple, obviously changes sign when one permutes the indices *hklm* cyclically or writes them down in the opposite order, the 24 double differences will reduce to the following three that are distinct, up to sign:

$$D_0 = \{1234\},\quad D_1 = \{2314\},\quad D_2 = \{3124\},$$

which are coupled by the relation:

$$D_0 + D_1 + D_2 = 0,$$

in their own right.

The geometric definition of the double difference is also applicable when the quadruple contains one or two pairs of anti-tactic spears. If, e.g., σ_1 is anti-tactic to σ_2 then one will understand P_1^2 to mean any point of σ_1, and P_2^1 to mean its vertical projection onto σ_2 ; the expression D_0 will obviously be entirely independent of the choice of point P_1^2 then.

38. The double differences of the spear-quadruples are connected with the complex moments M_h by the following relations:

$$(1)\qquad \left\{\begin{aligned} D_h &= -\frac{i}{2}\left(\frac{p_h^2}{p_0 p_1 p_2} M_h - \frac{\bar{p}_h^2}{\bar{p}_0 \bar{p}_1 \bar{p}_2} \bar{M}_h\right) \\ &= -\frac{i}{2}\left(\frac{\Delta p_h}{p_0 p_1 p_2} - \frac{\bar{\Delta}\,\bar{p}_h}{\bar{p}_0 \bar{p}_1 \bar{p}_2}\right) \end{aligned}\right. ,\qquad (h = 0, 1, 2).$$

These relations can be solved for Δ and $\bar{\Delta}$; i.e., the moment M_h can be represented as a linear, homogeneous expression in the quantities D_0, D_1, D_2 whose coefficients are direction invariants. An exception exists only in the case where the equations:

$$\frac{p_0}{\bar{p}_0} = \frac{p_1}{\bar{p}_1} = \frac{p_2}{\bar{p}_2}$$

are valid, each of which has the other ones as a consequence, and which express the fact that one of the three double ratios δ_h is real, and as a result, all three of them, so the four exit points P_k on the unit sphere κ of the spear τ_k that is drawn through O syntactic to σ_k will belong to the same spherical circle. If we refer to the quadruple σ_k as *special* in the latter case then we can say:

For a general quadruple of spears, the complete system of absolute invariants under the group Γ_{12} *of all complex motions will be represented by any two of the MÖBIUS double ratios and double angles of the spherical rectangle* P_1 P_2 P_3 P_4 *and by any two of the double differences* D_0, D_1, D_2 .

39. We now consider a special quadruple of spears σ_1, σ_2, σ_3, σ_4, and initially assume that the spherical circle K that contains the four points P_k is not a great circle; i.e., that the four spears σ_k do not run parallel to one and the same real plane. P means the spherical center of the circle K that lies on the smaller cap on κ, ζ means the cosine of the spherical radius of K, σ means the spear OP, e means any real plane that is perpendicular to σ, Q_k is the point at which it is pierced by the spear σ_k, σ'_k is the vertical projection of σ_k onto the plane e, and finally, σ''_k means the spear that lies in e, goes through Q_k, and is perpendicular to σ'_k, whose direction is chosen so that the sequence of three spears σ'_k, σ''_k, σ appears to be oriented analogously to the coordinate cross X, Y, Z, respectively. If one then decomposes the complex moment M_h into its real and imaginary parts:

$$M_h = M''_h + i\,M'_h ,$$

and lets D'_h (D''_h, resp.) denote the double differences of the two spear quadruples σ'_1, σ'_2, σ'_3, σ'_4 and σ''_1, σ''_2, σ''_3, σ''_4, resp., which are defined analogously to D_h, then one will easily find the relations:

$$M'_h = (1-\zeta^2)^{-2}\,\frac{p_0\,p_1\,p_2}{p_h^2}\,D'_h = \frac{p_0\,p_1\,p_2}{p_h^2}\,D_h , \tag{2}$$

$$M''_h = \zeta\,(1-\zeta^2)^{-2}\,\frac{p_0\,p_1\,p_2}{p_h^2}\,D''_h , \tag{3}$$

$$D_0 : D_1 : D_0 = D_0' : D_1' : D_2' = D_0'' : D_1'' : D_2'' = p_0 : p_1 : p_2 \,.$$

For a special quadruple of spears that are not parallel to the same real planes, the complete system of absolute invariants will then consist of any one of the three real double ratios δ_h and one of the three number-pairs:

$$\zeta(1-\zeta^2)^{-2} D_h'', \qquad D_h = (1-\zeta^2)^{-2} D_h' \,.$$

40. This real-geometric interpretation of the invariants of a special quadruple breaks down if and only if the four spears σ_k are parallel to the same real plane e. Now, if ρ_k is the plane that goes through σ_k parallel to e, and Q_k is the point where it is pierced by any spear σ that is perpendicular to e then $D_h' = D_h$, and in place of (3), one will have the equation:

$$M_0'' = \frac{H}{2\sin^2\frac{(13)}{2}\cdot\sin^2\frac{(24)}{2}},$$

as well as the relations for M_1'' and M_2'' that emerge from it by cyclic permutation of 1, 2, 3. In this, H means the expression:

$$H = \sum (-1)^{k+l+1} Q_k Q_l \sin(pq)\,;$$

$Q_k\, Q_l$ is a positive or negative line segment according to whether the direction that points from Q_k to Q_l does or does not coincide with that of σ, resp. Furthermore, sin (pq) shall mean the positive or negative sine of the angle between the spears σ_p and σ_q according to whether the three spears σ_p, σ_q, σ are oriented analogously to the coordinate system or not, resp. The indices k, l, p, q represent a permutation of the numerals 1, 2, 3, 4, and the sum is taken over the six terms that arise when each of the pairs kl and pq run through the six choices:

12, 13, 14, 23, 34, 42.

With that, for four spears σ_k, no two of which are syntactic, the absolute invariants of the group Γ_{12} are represented in all cases by quantities that have a geometrically simple meaning.

41. Incidentally, it follows from these results that if four spears belong to the same cycle then all of its double differences will vanish. That theorem is invertible when the quadruple is not special. In the latter case, the four spears belong to the same cycle if and only if the double differences D_h'' that were defined above vanish (the expression H, resp., in the event that the σ_k are parallel to the same plane), in addition to the double differences D_h (and D_h'). It further follows that in order for four spears σ_k to be able to go to four spears in the same real plane by complex motions, it is necessary and sufficient that the quadruple must be special and that the double differences D_h'' (the expression H, resp.) must vanish ([1]).

42. Four spears σ_k, two of which (say, σ_1 and σ_2) are syntactic, possess merely two independent real invariants under the group Γ_{12}, and therefore, one complex one, for which one can choose the expression:

$$M = M_0 = M_2 = \frac{\Delta}{[13][24]},$$

while the complex moment M_1 is illusory. If we further set:

$$M = M' + i\, M''$$

then the double difference D_1 will be equal to $-M''$, while D_0 and D_2 will be indeterminate. As before, we let P_1, P_3, P_4 be the exit points on the sphere k of the spears that are drawn through O and syntactic to $\sigma_1, \sigma_3, \sigma_4$, resp., and further let P be a real point on σ_1, while P' is its vertical projection onto σ_2. If one draws a spear through the sphere point P_1 that is syntactic to the spear PP' then it will contact the sphere κ at P_1, and thus determine a point P_2 on the spherical surface κ that is infinitely close to the point P_1. The arc of the complex invariant M will then be one of the MÖBIUS double angles of the spherical rectangle $P_1\, P_2\, P_3\, P_4$.

Furthermore, we consider the case in which σ_1 is syntactic to σ_2, and likewise, σ_3 is syntactic to σ_4. Since there are ∞^{12} such quadruples, and each of them admits ∞^2 complex motions (translations), there will also exist two independent real invariants in this case, and therefore one complex invariant, which we can choose to have the expression:

$$J = \frac{(\varpi_2 - \varpi_1)(\varpi_4 - \varpi_3)}{[13]},$$

([1]) The subgroup Γ_6 of our Γ_{12} that permute the spears of a real plane e amongst themselves is also a subgroup of LIE's group Γ_{10} of all real contact transformations that take circles to circles. É. LAGUERRE (*Recherches sur la Géométrie de Direction*, Paris 1885) has studied these spear transformations in the plane e without, however, recognizing the connection above. [On that, cf., C. STEPHANOS, in C. R. **92** (1881), pp. 1195.] The generating LAGUERRE transformation is simply the reflection in the complex plane:

$$\alpha x + \beta y + i\, \gamma z + \delta = 0,$$

in which $\alpha, \beta, \gamma, \delta$ mean real numbers, and the plane e is chosen to be the xy-plane. Four spears σ_k of the plane e possess two invariants under this LAGUERRE group Γ_6, namely, one of the three real number-pairs (δ_h, D_h). LAGUERRE referred to the invariant D_0 as the *longitude* of the quadruple $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ [Bull. Soc. Math. **3** (1880), 200].

under the assumption that the numbers u_1, v_1, w_1 are *equal* to u_2, v_2, w_2, resp., and u_3, v_3, w_3 are equal to u_4, v_4, w_4, resp. If d denotes the distance between the spears σ_1, σ_2, and d' denotes the distance between the spears σ_3, σ_4, while ω is the angle between the two directions σ_1 and σ_3 then one will find the value:

$$| J | = \sqrt{\frac{dd'}{2}} \cdot \frac{1}{1 - \cos \omega}$$

for the absolute value of J. If the points P_1, P_3 that lie on the sphere k have the meaning that was explained above, and if P, Q are arbitrary points on σ_1 (σ_3, resp.), while P', Q' are their vertical projections onto σ_2 (σ_4, resp.) then one can further determine the points P_2 (P_4, resp.) on the spherical surface κ that are infinitely close to P_1 (P_3, resp.) such that the spears P_1P_2 and P_3P_4 are syntactic to the spears PP' (QQ', resp.), so the arc of the complex invariant J will be equal to one of the MÖBIUS double angles of the spherical rectangle $P_1\,P_2\,P_3\,P_4$; i.e., up to sign, they will be equal to the angles that the oriented spherical circle that goes through P_1 and contacts the spear $P_2\,P_4$ makes with the spear $P_1\,P_2$.

In order for a spear quadruple σ_1, σ_2, σ_3, σ_4, in which, the first three (all four, resp.) spears are syntactic to be able to go to an analogous quadruple σ'_k by a complex motion, from no. **34**, it is necessary and sufficient that the parallel triangles $\sigma_1\,\sigma_2\,\sigma_3$ and $\sigma'_1\sigma'_2\sigma'_3$ (the rectangles in question, resp.) must be directly similar. Thus, the surface angles that appear in the triangles (rectangles, resp.) in question prove to be the invariants of a quadruple of that type.

From the considerations of this number, the quadruple invariants are also characterized geometrically for all cases in which syntactic spears enter among the four given ones.

Munich, in October 1903.