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A contribution to the theory of elastic stability

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There exist essentially two different derivations of stability criteria for elastic equilibrium, in principle. One of them is the *Biezeno-Grammel-Hencky* derivation, which is based upon dynamical arguments, and which one finds presented thoroughly in the standard text on dynamics in Section 5 of Chapter 1. The other one is based upon energetic considerations and was developed by *Trefftz* ([1,2]) and *Kreutzer* ([3]).

Starting from the fact that the process of overshooting the stability limit can be regarded kinematically as something that corresponds to a temporal passage through a discontinuity that is similar to when one passes through a wave surface, it will be shown that the so-called "dangerous" system of displacements, with the terminology of *Trefftz*, is exactly the one that will lead to a well-defined "wave surface" in the classical sense of continuum mechanics. Allow me to give a brief overview of that here in order to pursue that line of inquiry in more detail with examples in a paper that will appear later in this journal.

Let $\mathfrak{K}$ be a simply-connected, finite continuum that is piecewise bounded by surfaces Γ_i that are continuously differentiable at least four times. Let x^1, x^2, x^3 be the rectangular coordinates of a point in the initial state at time $t = 0$. Let X_1, X_2, X_3 be the components of the continuously-distributed volume forces per unit volume. $\rho(x^1, x^2, x^3)$ will denote the continuous density in the ground state, and e will denote the internal energy per unit volume. The functions:

$$\xi^\alpha = x^\alpha + u^\alpha(x^1, x^2, x^3; t),$$

which are continuously differentiable in all variables at least at least four times, determine the position of the point x^1, x^2, x^3 at time t, which is subject to volume forces and surface tractions per unit area with the components Ξ_1, Ξ_2, Ξ_3. (We shall appeal to the notation of *Trefftz* and *Kreutzer*.)

We use the deformation matrix:

([1]) *E. Trefftz*, "Über die Ableitung der Stabilitätskriterien des elastischen Gleichgewichts aus der Elastizitätstheorie endliche deformations," Internationaler Kongreß für technische Mechanik 1930, Stockholm, Teil III, pp. 44.

([2]) *E. Trefftz*, "Zur Theorie der Stabilität des elastischen Gleichgewichts," ZAMM **13** (1933), Heft 2, pp. 160, *et seq*.

([3]) *K. Kreutzer*, "Die Stabilität des gedrückten Stabes," ZAMM **12** (1932), Heft 6, pp. 351, *et seq.*

$$T = E + \left(\frac{\partial u^\alpha}{\partial x^\beta}\right) = E + (u_{\alpha\beta})$$

to define the positive-definite matrix:

$$T' \cdot T = E + (\gamma_{\alpha\beta}) ,$$

in which T' denotes the transpose of the matrix *T*. Now, should the classical *Newtonian* law of impulse be also true for our continuum mechanics then, as is known, *e* can depend upon only the $\gamma_{\alpha\beta}$, and in particular upon only the three orthogonal invariants when one deals with an isotropic medium, which *does not* need to be the case here. Thus, we have:

$$\gamma_{\alpha\beta} = \frac{\partial u^\alpha}{\partial x^\beta} + \frac{\partial u^\beta}{\partial x^\alpha} + \sum_{\kappa=1}^{3} \frac{\partial u^\kappa}{\partial x^\alpha} \cdot \frac{\partial u^\kappa}{\partial x^\beta}$$

here. The *Lagrangian* stress matrix will then be:

$$S = \left(\frac{\partial e}{\partial u_{\alpha\beta}}\right) = (\sigma^{\alpha\beta}) ,$$

which is *not* symmetric in the general case. *Hamilton's principle* will then say that for $\mathfrak{K}$, one has:

$$\text{I.} \qquad \rho \frac{\partial^2 \xi^\alpha}{\partial t^2} = X_\alpha + \sum_{\beta=1}^{3} \frac{\partial \sigma^{\alpha\beta}}{\partial x^\beta} ; \qquad \alpha = 1, 2, 3.$$

As is known, for Γ_i, one will have:

$$\text{II.} \qquad \sum_\beta \sigma^{\alpha\beta} \cos(n, \beta) = \Xi_\alpha ; \qquad \alpha = 1, 2, 3.$$

When I is written out, it will read:

$$\rho \frac{\partial^2 \xi^\alpha}{\partial t^2} = X_\alpha + \sum_{\beta,\mu,\nu=1}^{3} \frac{\partial^2 e}{\partial u_{\alpha\beta}\, \partial u_{\mu\nu}} \cdot \frac{\partial^2 e}{\partial x^\nu\, \partial x^\beta} + \sum_{\nu,\beta=1}^{3} \frac{\partial}{\partial x^\nu} \left(\frac{\partial e}{\partial u_{\alpha\beta}}\right) ; \quad \alpha = 1, 2, 3.$$

(∂' means differentiation with respect to the x^ν that appear explicitly in *e*.) The nonlinear *Euler* equations and matrices will not be needed in this Note. Let:

$$t = \varphi(x^1, x^2, x^3)$$

be a wave surface with the property that the velocities of its particles do not change upon crossing the wave surface, but their accelerations experience a jump:

$$Sp\left(\frac{\partial^2 \xi^\alpha}{\partial t^2}\right) = s^\alpha .$$

One will then have:

$$Sp\left(\frac{\partial^2 u^\alpha}{\partial x^\nu \, \partial x^\beta}\right) = Sp\left(\frac{\partial^2 \xi^\alpha}{\partial t^2}\right) \cdot \frac{\partial \varphi}{\partial x^\nu} \cdot \frac{\partial \varphi}{\partial x^\beta} .$$

It then follows from I that:

$$\rho \, s^\alpha = \sum_{\beta,\mu,\nu=1}^{3} \frac{\partial^2 e}{\partial u_{\alpha\beta} \, \partial u_{\mu\nu}} \cdot s^\mu \, \frac{\partial \varphi}{\partial x^\nu} \frac{\partial \varphi}{\partial x^\beta} = \sum_{\mu=1}^{3} s^\mu \cdot h_{\alpha\mu} ; \qquad \alpha = 1, 2, 3.$$

With that, one has:

$$\begin{vmatrix} h_{11} - \rho & h_{12} & h_{13} \\ h_{21} & h_{22} - \rho & h_{23} \\ h_{31} & h_{32} & h_{33} - \rho \end{vmatrix} = 0 ,$$

in the event that all s^α do not vanish simultaneously, which can possibly happen only at points or along lines. For a "stationary" wave surface, one will then have:

$$| h_{ik} | = 0 .$$

We shall now show that the linear system of differential equations that determines the so-called "dangerous" displacements and possesses precisely the manifold above for its characteristic manifold, will suffice completely to determine the stability values. To that end, we form:

$$\delta^2 E = \iiint_{\mathfrak{K}} \delta^2 e \cdot dx^1 \cdot dx^2 \cdot dx^3 ,$$

in which one has:

$$\delta^2 e = \sum_{\mu,\nu,\alpha,\beta} \frac{\partial^2 e}{\partial u_{\alpha\beta} \, \partial u_{\mu\nu}} \cdot \frac{\partial \, \delta u^\mu}{\partial x^\nu} \frac{\partial \, \delta u^\alpha}{\partial x^\beta} .$$

[As *Trefftz* showed, cf., ([1]), pp. 47, only those quantities come into question when assessing stability.] If we set $\delta u^\alpha = v^\alpha$ then the "dangerous" system of displacements will be determined by the *Jacobi* criterion from:

$$\delta J = \delta \iiint_{\mathfrak{K}} \sum_{\mu,\nu,\alpha,\beta} \frac{\partial^2 e}{\partial u_{\alpha\beta} \, \partial u_{\mu\nu}} \cdot \frac{\partial \, v^\mu}{\partial x^\nu} \frac{\partial \, v^\alpha}{\partial x^\beta} \cdot dx^1 \cdot dx^2 \cdot dx^3 = 0 .$$

With the boundary conditions:

$$\text{IV.} \qquad \sum_{\beta} \frac{\partial e}{\partial u_{\alpha\beta}} \cos(n, \beta) = 0 \, ; \qquad \alpha = 1, 2, 3,$$

i.e., one must have:

$$\text{III.} \qquad \sum_{\alpha,\beta,\nu} \frac{\partial}{\partial x_\nu} \left(\frac{\partial^2 e}{\partial u_{\alpha\beta}\, \partial u_{\mu\nu}} \cdot \frac{\partial v^\alpha}{\partial x^\beta} \right) = 0 \, ; \qquad m = 1, 2, 3.$$

For the buckled rod, as one calculates directly from the *Trefftz* formula [cf., ([1]), pp. 48, formula (28)], and with the special assumption on the dependency of the internal energy on the γ_{ik}, one has:

$$h_{11} = G \frac{m-1}{m-2} \left(\frac{\partial \varphi}{\partial x^1} \right)^2 + \frac{G}{2} \left(\frac{\partial \varphi}{\partial x^2} \right)^2 + \frac{G}{2} \left(\frac{\partial \varphi}{\partial x^3} \right)^2 - \frac{p}{2(1+\varepsilon_{x^1})^2} \left(\frac{\partial \varphi}{\partial x^3} \right)^2,$$

$$h_{22} = \frac{G}{2} \left(\frac{\partial \varphi}{\partial x^1} \right)^2 + G \frac{m-1}{m-2} \left(\frac{\partial \varphi}{\partial x^2} \right)^2 + \frac{G}{2} \left(\frac{\partial \varphi}{\partial x^3} \right)^2 - \frac{p}{2(1+\varepsilon_{x^2})^2} \left(\frac{\partial \varphi}{\partial x^3} \right)^2,$$

$$h_{33} = \frac{G}{2} \left(\frac{\partial \varphi}{\partial x^1} \right)^2 + \frac{G}{2} \left(\frac{\partial \varphi}{\partial x^2} \right)^2 + G \frac{m-1}{m-2} \left(\frac{\partial \varphi}{\partial x^3} \right)^2 - \frac{p}{2(1+\varepsilon_{x^3})^2} \left(\frac{\partial \varphi}{\partial x^3} \right)^2,$$

$$h_{12} = h_{21} = \frac{G}{m-2} \frac{\partial \varphi}{\partial x^1} \cdot \frac{\partial \varphi}{\partial x^2},$$

$$h_{23} = h_{32} = \frac{G}{m-2} \frac{\partial \varphi}{\partial x^2} \cdot \frac{\partial \varphi}{\partial x^3},$$

$$h_{13} = h_{31} = \frac{G}{m-2} \frac{\partial \varphi}{\partial x^1} \cdot \frac{\partial \varphi}{\partial x^3}.$$

$| h_{ik} | = 0$ is then the characteristic manifold of the system:

$$G \left(\Delta v^2 + \frac{m}{m-2} \frac{\partial \Theta}{\partial x^\nu} \right) = \frac{p}{(1+\varepsilon_{x^\nu})^2} \frac{\partial^2 v^\nu}{(\partial x^3)^2} \, ; \qquad \nu = 1, 2, 3.$$

However, those are precisely the *Jacobi* equations for the "dangerous" system of displacements v^1, v^2, v^3, which is given by III.

The boundary conditions IV remain the same. However, *Kreutzer* [cf., ([3]), pp. 362] has calculated the *Euler* buckling loads from that system very logically.

In this brief overview, only the methods were suggested for addressing the stability cases for an elastic continuum and associating them with the known theory of wave surfaces, and one can draw some very interesting parallels between the wave surfaces of gas dynamics, on the one hand, and the "dangerous" systems of displacements in anisotropic elastic media, on the other. Even when one drops the classical law of impulse, that will still give interesting parallels between crystal optics and the "stability criterion" in an ether medium. All of that will be treated in a more detailed investigation.

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