"Ueber eine Klasse auf einander abwickelbarer Flächen," J. reine angew. Math. **59** (1861), 382-393.

On a class of mutually-developable surfaces

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In the theory of curved surfaces, it would seem that the surfaces of centers of curvature have been given only limited attention. However, the remark that the developable surfaces that are constructed from the normals to a given surface along a line of curvature intersect corresponding surface of centers of curvature along a geodetic line, embodies the essential property of those surfaces, if we are not mistaken. When we recall some of the theorems that **Gauss** presented in "Disquisitiones generales circa superficies curvas," their further study will lead to the theorem that the surfaces of centers of curvature of those surface for which one radius of principal curvature is determined in terms of only the other one at each point define closed classes of mutually-developable surfaces. That theorem, which has a close relationship to the theory of surfaces that be developed to surfaces of revolution, is the one that defines the subject of the present communication.

If a curved surfaces i given in such a way that the coordinates x, y, z of a point of it are determined by the values of two independent variables p and q, and X, Y, Z denote the cosines of the direction of the normal at the point (x, y, z) then those quantities will satisfy the equations:

$$X\frac{\partial x}{\partial p}+Y\frac{\partial y}{\partial p}+Z\frac{\partial z}{\partial p}=0,$$

$$X\frac{\partial x}{\partial q}+Y\frac{\partial y}{\partial q}+Z\frac{\partial z}{\partial q}=0,$$

$$X^2+Y^2+Z^2=1.$$

It is preferable to put them into the following form:

$$\text{(1)}\qquad \left\{\begin{aligned}
\frac{\partial x}{\partial p}&=M\frac{\partial X}{\partial p}+N\frac{\partial X}{\partial q}, & \frac{\partial x}{\partial q}&=M'\frac{\partial X}{\partial p}+N'\frac{\partial X}{\partial q},\\
\frac{\partial y}{\partial p}&=M\frac{\partial Y}{\partial p}+N\frac{\partial Y}{\partial q}, & \frac{\partial y}{\partial q}&=M'\frac{\partial Y}{\partial p}+N'\frac{\partial Y}{\partial q},\\
\frac{\partial z}{\partial p}&=M\frac{\partial Z}{\partial p}+N\frac{\partial Z}{\partial q}, & \frac{\partial z}{\partial q}&=M'\frac{\partial X}{\partial p}+N'\frac{\partial X}{\partial q},
\end{aligned}\right.$$

and when one determines the quantities M, M', N, N' appropriately, those equations will allow one to solve any suitably-chosen four of them for the two remaining ones (*).

If (x, y, z) and $(x + dx, y + dy, z + dz)$ are two infinitely-close points on the curved surface that belong to the same line of curvature, and dX, dY, dZ measure the differences in the cosines of the direction of normals at both points then it is known that the following equations will be true:

$$dx = \rho\, dX, \qquad dy = \rho\, dY, \qquad dz = \rho\, dZ, \tag{2}$$

in which ρ denotes the radius of principal curvature that belongs to the normal section that goes through (x, y, z) and $(x + dx, y + dy, z + dz)$, and that those equations will represent the differential equations for the lines of curvature.

A comparison of those equations with (1) will show that the differential equations of the lines of curvature in terms of the variables p and q will be obtained by eliminating ρ from the equations:

$$\begin{aligned} M\,dp + M'\,dq &= \rho\, dp, \\ N\,dp + N'\,dq &= \rho\, dq, \end{aligned}$$

and that ρ will be determined from the quadratic equation:

$$\begin{vmatrix} M-\rho & M' \\ N & N'-\rho \end{vmatrix} = 0.$$

If one denotes the radii of principal curvature that belong to the point of the surface in question that is determined by the values of p and q by ρ and ρ' then one will have:

$$\begin{cases} \rho + \rho' = M + N', \\ \rho\rho' = MN' - M'N. \end{cases} \tag{3}$$

At the same time, it will emerge from these relations that the differential quotients of the coordinates of a surface that can be developed to a plane cannot be put into the form of equations (1), in which the quantities M and N are not generally assumed to be finite.

The variables p and q determine two systems of curves on the given surface, in such a way that for one of them, only p varies, while only q varies for the other one. For what follows, we will choose those variables in such a way that the first system corresponds to a family of lines of curvature, while the second one corresponds to the family of all curves for which the radius of principal curvature that belongs to the chosen system of lines of curvature will remain unchanged. In relation to the variable q, it remains for us

(*) The actual representation of the quantities M, M', N, N' will be unnecessary in what follows. When one appeals to the notations that **Gauss** introduced, an easy calculation will yield:

$$M = \frac{D''E - D'F}{DD'' - D'D'}\sqrt{EG - FF}\,, \qquad N = \frac{-D'E + DF}{DD'' - D'D'}\sqrt{EG - FF}\,,$$

$$M' = \frac{-D'E + D''F}{DD'' - D'D'}\sqrt{EG - FF}\,, \qquad N' = \frac{DG - D'F}{DD'' - D'D'}\sqrt{EG - FF}\,.$$

to introduce an arbitrary function of it as a new variable in such a way that the conditions that were posed will not be harmed in any way.

In general, the introduction of the variables p, q that were just defined as the independent variables is possible in *two* ways for any surface. The plane and the sphere define obvious exceptions to that statement, for which there are *no possible ways*, along with those surfaces for which one radius of principal curvature remains unchanged along the corresponding line of curvature, such as the developables and the channel surfaces for which that introduction is possible in *only one* way.

If ξ, η, ζ are the coordinates of the center of curvature that corresponds to the radius of curvature $\rho = p$ then:

$$x - \xi = p\,X, \qquad y - \eta = p\,Y, \qquad z - \zeta = p\,Z. \tag{4}$$

In relation to the point $\left(x + \frac{\partial x}{\partial p}\,dp,\ y + \frac{\partial y}{\partial p}\,dp,\ z + \frac{\partial z}{\partial p}\,p\right)$, one will have these equations:

$$\left\{\begin{aligned} \left(\frac{\partial x}{\partial p} - \frac{\partial \xi}{\partial p}\right) dp &= p\,\frac{\partial X}{\partial p}\,dp + X\,dp, \\ \left(\frac{\partial y}{\partial p} - \frac{\partial \eta}{\partial p}\right) dp &= p\,\frac{\partial Y}{\partial p}\,dp + Y\,dp, \\ \left(\frac{\partial z}{\partial p} - \frac{\partial \zeta}{\partial p}\right) dp &= p\,\frac{\partial Z}{\partial p}\,dp + Z\,dp. \end{aligned}\right. \tag{5}$$

Due to the definitions of the variables p, q, a variation of p alone will correspond to an advance of the point (x, y, z) to the point $(x + dx, y + dy, z + dz)$ along the line of curvature q = const.

As a result of that, from equations (2), one will have:

$$\frac{\partial x}{\partial p}\,dp = p\,\frac{\partial X}{\partial p}\,dp, \qquad \frac{\partial y}{\partial p}\,dp = p\,\frac{\partial Y}{\partial p}\,dp, \qquad \frac{\partial z}{\partial p}\,dp = p\,\frac{\partial Z}{\partial p}\,dp,$$

and equations (5) will yield:

$$\left\{\begin{aligned} \frac{\partial x}{\partial p} &= p\,\frac{\partial X}{\partial p}, \quad & X &= -\frac{\partial \xi}{\partial p}, \\ \frac{\partial y}{\partial p} &= p\,\frac{\partial Y}{\partial p}, \quad & Y &= -\frac{\partial \eta}{\partial p}, \\ \frac{\partial z}{\partial p} &= p\,\frac{\partial Z}{\partial p}, \quad & Z &= -\frac{\partial \zeta}{\partial p}. \end{aligned}\right. \tag{6}$$

The coordinates x, y, z are then determined by the coordinates of the corresponding centers of curvature as follows:

(7) $$x = \xi - p\frac{\partial \xi}{\partial p}, \qquad y = \eta - p\frac{\partial \eta}{\partial p}, \qquad z = \zeta - p\frac{\partial \zeta}{\partial p}.$$

The substitution of those values of *x*, *y*, *z*, *X*, *Y*, *Z* in the equation:

$$X\frac{\partial x}{\partial q} + Y\frac{\partial y}{\partial q} + Z\frac{\partial z}{\partial q} = 0$$

will lead to the following one:

$$\frac{p}{2}\frac{\partial}{\partial q}\left[\left(\frac{\partial \xi}{\partial p}\right)^2 + \left(\frac{\partial \eta}{\partial p}\right)^2 + \left(\frac{\partial \zeta}{\partial p}\right)^2\right] - \left(\frac{\partial \xi}{\partial p}\frac{\partial \xi}{\partial q} + \frac{\partial \eta}{\partial p}\frac{\partial \eta}{\partial q} + \frac{\partial \zeta}{\partial p}\frac{\partial \zeta}{\partial q}\right) = \frac{p}{2}\frac{\partial E}{\partial q} - F = 0,$$

and due to the fact that:

$$E = X^2 + Y^2 + Z^2 = 1,$$

the former equation will be converted into:

(8) $$0 = \frac{\partial \xi}{\partial p}\frac{\partial \xi}{\partial q} + \frac{\partial \eta}{\partial p}\frac{\partial \eta}{\partial q} + \frac{\partial \zeta}{\partial p}\frac{\partial \zeta}{\partial q} = F.$$

It will follow from this that the variables *p* and *q* correspond to two systems of curves on the surface of the centers of curvature that intersect at right angles at each point. As is known, the system for which only *p* varies is a system of geodetic lines.

If one compares the first row of equations in (6) with equations (1) then one will note that one will have:

$$M = p, \qquad N = 0$$

for the chosen variables. The determinations of the radii of principal curvature that is given in (3) then shows that *N* is identical with the second principal curvature ρ', which will be p', by analogy.

When one considers those remarks and substitutes the values of *x*, *y*, *z*, *X*, *Y*, *Z* in ξ, η, ζ, the second row of equations can be represented in the form:

$$\frac{\partial \xi}{\partial q} - p\frac{\partial^2 \xi}{\partial p\,\partial q} = -M'\frac{\partial^2 \xi}{\partial p^2} - p'\frac{\partial^2 \xi}{\partial p\,\partial q},$$

$$\frac{\partial \eta}{\partial q} - p\frac{\partial^2 \eta}{\partial p\,\partial q} = -M'\frac{\partial^2 \eta}{\partial p^2} - p'\frac{\partial^2 \eta}{\partial p\,\partial q},$$

$$\frac{\partial \zeta}{\partial q} - p\frac{\partial^2 \zeta}{\partial p\,\partial q} = -M'\frac{\partial^2 \zeta}{\partial p^2} - p'\frac{\partial^2 \zeta}{\partial p\,\partial q},$$

and once one has multiplied those equations by $\frac{\partial\xi}{\partial q}$, $\frac{\partial\eta}{\partial q}$, $\frac{\partial\zeta}{\partial q}$, in turn, and added them that will give the relation:

$$\left(\frac{\partial\xi}{\partial q}\right)^2+\left(\frac{\partial\eta}{\partial q}\right)^2+\left(\frac{\partial\zeta}{\partial q}\right)^2=-M'\left(\frac{\partial F}{\partial p}-\frac{1}{2}\frac{\partial E}{\partial q}\right)+\frac{p-p'}{2}\frac{\partial}{\partial p}\left[\left(\frac{\partial\xi}{\partial q}\right)^2+\left(\frac{\partial\eta}{\partial q}\right)^2+\left(\frac{\partial\zeta}{\partial q}\right)^2\right].$$

If one denotes the sum:

$$\left(\frac{\partial\xi}{\partial q}\right)^2+\left(\frac{\partial\eta}{\partial q}\right)^2+\left(\frac{\partial\zeta}{\partial q}\right)^2$$

by G then since $F = 0$ and $\partial E / \partial q = 0$, when one suppresses the vanishing part, one will have:

$$G = \frac{p-p'}{2}\frac{\partial G}{\partial p},$$

and as a result:

$$G = Q\exp\left(2\int\frac{dp}{p-p'}\right),$$

in which Q denotes a function of only q.

The functions ξ, η, ζ then satisfy the differential equations:

$$E = \left(\frac{\partial\xi}{\partial p}\right)^2+\left(\frac{\partial\eta}{\partial p}\right)^2+\left(\frac{\partial\zeta}{\partial p}\right)^2 = 1,$$

$$F = \frac{\partial\xi}{\partial p}\frac{\partial\xi}{\partial q}+\frac{\partial\eta}{\partial p}\frac{\partial\eta}{\partial q}+\frac{\partial\zeta}{\partial p}\frac{\partial\zeta}{\partial q} = 0,$$

$$G = \left(\frac{\partial\xi}{\partial q}\right)^2+\left(\frac{\partial\eta}{\partial q}\right)^2+\left(\frac{\partial\zeta}{\partial q}\right)^2 = Q\exp\left(2\int\frac{dp}{p-p'}\right),$$

in which one can set $Q = 1$ with no loss of generality, which would be equivalent to introducing:

$$\int\sqrt{Q}\,dq$$

as the new variable.

Those remarkable relations lead to the assumption that p' (i.e., the second radius of principal curvature for the given surface) depends upon only p, so it will satisfy the condition:

$$p' = \lambda(p),$$

which can be replaced with a second-order partial differential equation with no further discussion, with the aforementioned consequences in connection with the theory of developable surfaces.

Under that assumption, one will get:

$$E = 1, \qquad F = 0, \qquad G = \exp\left[2\int \frac{dp}{p-\lambda(p)}\right] = \varphi(p).$$

The line element $d\sigma$ of the surface of centers of curvature, which corresponds to the radius p, will then be:

$$d\sigma^2 = d\xi^2 + d\eta^2 + d\zeta^2 = dp^2 + \varphi(p)\, dq^2,$$

which is an expression that can be given by the choice of variables p and q for the line element of the surface of centers of curvature of any curved surface that satisfies the differential equation:

$$p' = \lambda(p) .$$

One then has the following theorem:

The surfaces of centers of curvature of all surfaces for which one of the radii of principal curvature can be determined from the other one alone in the same way at each point can be developed to each other.

We need to examine whether the surfaces that are given by the aforementioned theorem, whose line elements can be put into the form:

$$dp^2 + \varphi(p)\, dq^2,$$

are the only ones that possess that property. It is known that this form is the only one that can give the line element of any surface of revolution for a suitable determination of the function $\varphi(p)$. Our question is then identical with this one: Is every curved surface that can be developed to a given surface of revolution contained in the system of surfaces of centers of curvature for a family of surfaces that is characterized by the equation $p' = \lambda(p)$?

Let:

$$u = r \cos q, \quad v = r \sin q, \quad w = F(r)$$

be the equation of a surface of rotation. That will imply:

$$du^2 + dv^2 + dw^2 = [1 + F'(r)^2]\, dr^2 + r^2\, dq^2$$

for the square of the line element. The substitution:

$$p = \int \sqrt{1 + F'(r)^2}\, dr\,,$$

from which it follows that:

$$r^2 = \varphi(p),$$

will give it the desired form:

$$du^2 + dv^2 + dw^2 = dr^2 + \varphi(p)\, dq^2.$$

Let ξ, η, ζ denote the coordinates of the point on the surface S that can be developed to a surface of revolution that corresponds to the point (u, v, w) under that development. Three well-defined values u, v, w will then correspond to three associated ξ, η, ζ. The former are determined by choosing values of p and q, and as a result, the latter, as well. ξ, η, ζ are then functions of p and q that must verify the condition:

$$du^2 + dv^2 + dw^2 = d\xi^2 + d\eta^2 + d\zeta^2,$$

due to the nature of developability. They will then fulfill the partial differential equations:

$$\text{(9)} \qquad \left\{ \begin{aligned} 1 &= \left(\frac{\partial \xi}{\partial p}\right)^2 + \left(\frac{\partial \eta}{\partial p}\right)^2 + \left(\frac{\partial \zeta}{\partial p}\right)^2, \\ 0 &= \frac{\partial \xi}{\partial p}\frac{\partial \xi}{\partial q} + \frac{\partial \eta}{\partial p}\frac{\partial \eta}{\partial q} + \frac{\partial \zeta}{\partial p}\frac{\partial \zeta}{\partial q}, \\ \varphi(p) &= \left(\frac{\partial \xi}{\partial q}\right)^2 + \left(\frac{\partial \eta}{\partial q}\right)^2 + \left(\frac{\partial \zeta}{\partial q}\right)^2. \end{aligned} \right.$$

One now considers the quantities:

$$\text{(10)} \qquad \left\{ \begin{aligned} x &= \xi - p\frac{\partial \xi}{\partial p}, \\ y &= \eta - p\frac{\partial \eta}{\partial p}, \\ z &= \zeta - p\frac{\partial \eta}{\partial p} \end{aligned} \right.$$

to be the coordinates of a point on a third surface T, which is possible, generally speaking, and one lets X, Y, Z denote the differential quotients $-\frac{\partial \xi}{\partial p}$, $-\frac{\partial \eta}{\partial p}$, $-\frac{\partial \zeta}{\partial p}$, for brevity. The equations:

$$X\, dx + Y\, dy + Z\, dz = 0,$$

$$X^2 + Y^2 + Z^2 = 1$$

will then be a consequence of equations (9), which are fulfilled by assumption, and they will say that X, Y, Z are the cosines of the direction of the normal at the point (x, y, z) of the surface T. It is easy to see the geometric meaning of the variables p, q as it relates to that surface. One needs only to introduce the values of the coordinates x, y, z and the cosines X, Y, Z into equations (1) and to remark that due to the equations:

$$\frac{\partial x}{\partial p} = p\frac{\partial X}{\partial p}, \quad \frac{\partial y}{\partial p} = p\frac{\partial Y}{\partial p}, \quad \frac{\partial z}{\partial p} = p\frac{\partial Z}{\partial p},$$

which are fulfilled identically, one will have the relations:

$$M = p, \qquad N = 0.$$

As a result of that, one will get the equations:

$$\rho + \rho' = p + N',$$

$$\rho\,\rho' = p\,N'.$$

From them and the foregoing system of equations, one will see that the quantities q, p, N' are, in turn, identical with the parameters of a family of lines of curvature on the surface T with the radius of principal curvature that corresponds to it and the other one p'.

A treatment of the second row of equations in (1) that was carried out before will once more lead to the equation:

$$\left(\frac{\partial \xi}{\partial q}\right)^2 + \left(\frac{\partial \eta}{\partial q}\right)^2 + \left(\frac{\partial \zeta}{\partial q}\right)^2 = \frac{p-p'}{2}\frac{\partial}{\partial p}\left[\left(\frac{\partial \xi}{\partial q}\right)^2 + \left(\frac{\partial \eta}{\partial q}\right)^2 + \left(\frac{\partial \zeta}{\partial q}\right)^2\right],$$

which can be converted into:

$$\varphi(p) = \frac{p-p'}{2}\varphi'(p)$$

by using the last of equations (9). That will imply that:

$$p' = p - \sqrt{\varphi(p)}\,\frac{dp}{d\sqrt{\varphi(p)}} = \lambda(p)\,. \tag{11}$$

The surface T will then belong to the class of surfaces for which one of the radii of principal curvature at any point is determined by the other one alone. As equations (10) will show, when one gives them the form:

$$x - \xi = p\,X, \qquad y - \eta = p\,Y, \qquad z - \zeta = p\,Z,$$

it will have the surface S for its surface of centers of curvature.

The result of our investigation would then be that the class of those surfaces whose line elements can be given the form:

$$dp^2 + \varphi(p)\, dq^2$$

is, in fact, congruent to the system of shells of surfaces of centers of curvature that correspond to the radius of principal curvature ρ, which is a family of surfaces that is characterized by the partial differential equation:

$$\rho' = \rho - \sqrt{\varphi(p)}\,\frac{d\rho}{d\sqrt{\varphi(p)}} = \lambda(p),$$

which is a fact that could not be considered to impair that result, despite it being inessential. The consequences that were just inferred from equations (10) are, in fact, illusory in the case where those equations do not determine a surface. However, the quantities x, y, z will then be determined by the values of a single variable τ that is itself a function of both variables p, q, or one of them. Under that assumption, the equations:

$$\frac{\partial x}{\partial p}\frac{\partial \xi}{\partial q} + \frac{\partial y}{\partial p}\frac{\partial \eta}{\partial q} + \frac{\partial z}{\partial p}\frac{\partial \zeta}{\partial q} = 0,$$

$$\frac{\partial x}{\partial q}\frac{\partial \xi}{\partial q} + \frac{\partial y}{\partial q}\frac{\partial \eta}{\partial q} + \frac{\partial z}{\partial q}\frac{\partial \zeta}{\partial q} = \varphi(p) - \tfrac{1}{2}p\,\varphi'(p),$$

which are fulfilled identically as a result of equations (9), will become:

$$(12)\qquad \left\{\begin{aligned} &\left(\frac{\partial x}{\partial \tau}\frac{\partial \xi}{\partial q} + \frac{\partial y}{\partial \tau}\frac{\partial \eta}{\partial q} + \frac{\partial z}{\partial \tau}\frac{\partial \zeta}{\partial q}\right)\frac{\partial \tau}{\partial p} = 0 \\ &\left(\frac{\partial x}{\partial \tau}\frac{\partial \xi}{\partial q} + \frac{\partial y}{\partial \tau}\frac{\partial \eta}{\partial q} + \frac{\partial z}{\partial \tau}\frac{\partial \zeta}{\partial q}\right)\frac{\partial \tau}{\partial q} = \varphi(p) - \tfrac{1}{2}p\,\varphi'(p). \end{aligned}\right.$$

The validity of the first of those equations demands that either:

$$\frac{\partial x}{\partial \tau}\frac{\partial \xi}{\partial q} + \frac{\partial y}{\partial \tau}\frac{\partial \eta}{\partial q} + \frac{\partial z}{\partial \tau}\frac{\partial \zeta}{\partial q} = 0$$

or

$$\frac{\partial \tau}{\partial p} = 0.$$

If the first of these two conditions is fulfilled then the second of equations (12) will demand that:

$$(13)\qquad \varphi(p) = \tfrac{1}{2}p\,\varphi'(p), \qquad \text{so} \qquad \varphi(p) = c\,p^2,$$

and as a result:

$$d\xi^2 + d\eta^2 + d\zeta^2 = dp^2 + c\, p^2\, dq^2,$$

which is a form for the line element that only a surface that is developable to the plane possesses, which is a property that should not be assumed for the surface *S*.

Fulfilling the other condition $\partial\tau / \partial p = 0$ will reduce the three quantities *x*, *y*, *z* to three functions *V*, *V′*, *V″* of *q* alone, which means that ξ, η, ζ must have the form:

$$(14) \qquad \begin{cases} \xi = pU + V, \\ \eta = pU' + V', \\ \zeta = pU'' + V'', \end{cases}$$

in which *U*, *U′*, *U″* likewise suggest quantities that are determined by only *q*. Since the coefficient $\varphi(p)$ of dq^2 in the line element of the surfaces that are represented by those equations should be a function of only *p*, the case to be treated will occur only when:

$$\varphi(p) = \alpha p^2 + 2\beta p + \gamma,$$

in which α, β, γ denote constants, or when:

$$(15) \qquad \varphi(p) = \alpha p^2 + \gamma,$$

when one shifts the origin, which is no loss of generality. That form for $\varphi(p)$, which includes the one that was given in (13), is therefore the only one that can be linked with the occurrence of the aforementioned situation. The complete determination of the surfaces whose line element can be put into the form:

$$d\sigma^2 = dp^2 + (\alpha p^2 + \gamma)\, dq^2$$

is not sufficient, which results from knowing the surfaces that are defined by the partial differential equation:

$$\rho\,\rho' = -\frac{\gamma}{\alpha},$$

which emerges from (11), but is associated with the corresponding surfaces of centers of curvature of the family of surfaces (14) that are generated by straight lines, under the condition that the line element must assume the required form. It is not difficult to determine the six functions *U* and *V* in the equations (14) of that condition accordingly from any one of the remaining ones. Among the surfaces that are determined in that way, one will find, e.g., the helicoid:

$$\xi = p \cos q, \qquad \eta = p \sin q, \qquad \zeta = a\, q,$$

whose line element is equal to $dp^2 + (p^2 + a^2)\, dq^2$, and which can be regarded as the surface of the centers of curvature of a surface of constant curvature $-1 / a^2$, unlike the surface of revolution of the catenary:

$$y = \frac{a}{2}(e^{x/a} + e^{-x/a}),$$

which has the same line element.

What was just proved will imply the following theorem that relates to the theory of surfaces that can be developed to surfaces of revolution, which can be expressed without having to refer to them, when one resolves the exceptional case that relates to the partial differential equation $\rho\,\rho' = c$:

The system of the shells of surfaces of centers of curvature of a family of surfaces that is characterized by the partial differential equation:

$$\rho' = \lambda(\rho)$$

that correspond to the radius of principal curvature ρ constitutes a closed class of mutually-developable surfaces that can be regarded as the representative of a surface of revolution that is determined by the function λ,

and conversely:

The class of surfaces that can be developed to a given surface of revolution is congruent to the system of surfaces of centers of curvature of a family of surfaces that is characterized by the partial differential equation:

$$\rho' = \lambda(\rho)$$

and corresponds to the radius ρ, in which the function λ is determined from the given surface of revolution accordingly.

That theorem, which characterizes a necessary and sufficient condition for a surface to be wrapped around a surface of revolution, leads to a finite representation of a complete class of surfaces that can be developed to a special surface of revolution in at least *one* case. That is the case in which the equation that exists between ρ and ρ' is that of a surface of minimal area:

$$\rho + \rho' = 0,$$

whose integration is known. Under that condition, the common expression for the line element of all surfaces of centers of curvature for this family of surfaces will be:

$$d\sigma^2 = dp^2 + p\, dq^2.$$

The surface of revolution whose line element assumes that form is found to be given by the equations:

$$\zeta = -\frac{r}{2}\sqrt{4r^2 - 1} + \tfrac{1}{4}\log(2r + \sqrt{4r^2 - 1}),$$

$$r = \sqrt{\xi^2 + \eta^2},$$

by an easy calculation. In fact, if one sets:

$$\xi = \sqrt{p}\cos q,$$

$$\eta = \sqrt{p}\sin q,$$

$$z = -\tfrac{1}{2}\sqrt{p(4p-1)} + \tfrac{1}{4}\log(2\sqrt{p} + \sqrt{4p-1})$$

then that will yield:

$$d\xi^2 + d\eta^2 + d\zeta^2 = dp^2 + p\,dq^2,$$

as required. As one knows, the meridian curve of that surface of revolution is the evolute of the catenary:

$$y = \tfrac{1}{8}(e^{4x} + e^{-4x}),$$

which is a result that will be obvious with no calculation when one recalls that the surface of revolution of any catenary (around the corresponding axis) is a surface that satisfies the partial differential equation:

$$\rho + \rho' = 0,$$

and as a result, the surface of revolution of its evolute will then be the corresponding surface of centers of curvature.

The surfaces of centers of curvature of the surfaces of least area then define the class of the surfaces that can be developed to the surface of revolution of one, or more correctly, any catenary evolute.

Finally, as far as the surfaces that can be developed to a plane are concerned, which have been excluded from our considerations due to the form to which they will lead, one easily convinces oneself, and in various ways, that the surfaces of their centers of curvature (when they exist) are, in turn, surfaces that can be developed to a plane.

Berlin, June 1861.
