"Über die Theorie der aufeinander abwickelbaren Oberflächen," Festschrift der Kön. Techn. Hochschule zu Berlin (1884), 1-43.

ON

THE THEORY

OF

MUTUALLY-DEVELOPABLE

SURFACES

BY

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SUPPLEMENT TO THE
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As is known, the determination of all surfaces that are partially developable to parts of a given surface requires one to ascertain the most general real-valued functions *x*, *y*, *z* that depend upon two real variables and satisfy the equation:

$$dx^2 + dy^2 + dz^2 = E\,dp^2 + 2F\,dp\,dq + G\,dq^2, \tag{a}$$

in which *E*, *F*, *G* are given functions of the variables *p*, *q*, and indeed they denote the coefficients of the square of the line element of the given surface.

Ascertaining those functions *x*, *y*, *z* then depends upon integrating three simultaneous partial differential equations:

$$\begin{aligned}
&\left(\frac{\partial x}{\partial p}\right)^2 + \left(\frac{\partial y}{\partial p}\right)^2 + \left(\frac{\partial z}{\partial p}\right)^2 = E,\\
&\frac{\partial x}{\partial p}\frac{\partial x}{\partial q} + \frac{\partial y}{\partial p}\frac{\partial y}{\partial q} + \frac{\partial z}{\partial p}\frac{\partial z}{\partial q} = F,\\
&\left(\frac{\partial x}{\partial q}\right)^2 + \left(\frac{\partial y}{\partial q}\right)^2 + \left(\frac{\partial z}{\partial q}\right)^2 = G.
\end{aligned} \tag{a$'$}$$

In order to perform that integration, one next appeals to the path that consists of eliminating two of the functions to be determined – say, *y*, *z* – that are derived from the foregoing three equations with the help of partial differentiations of a single partial differential equation that the function *x* must satisfy. Due to the symmetry that exists between the functions *x*, *y*, *z*, the two functions *y*, *z* would also be coupled by that equation.

That path is, in fact, the path that EDMOND BOUR followed in his response to the Paris Academy of Science for the prize problem that was posed in the year 1860, which was concerned with the theory of mutually-developable surfaces, albeit modified by specialized forms of the squares of the line elements of the given surface that he chose to serve as the starting point for the investigation. In our opinion, as opposed to BOUR's ("Théorie de la déformation des surfaces," Journal de l'École polytechnique, tome XII, page 5), that starting point would lead to ancillary difficulties in the problem, instead of avoiding them, even if by overlooking those difficulties, the introduction of new independent variables into a partial differential equation would generally seem to be coupled with a change in the domain of the original variables.

Over a long span of time, it has occurred to us that no one has ever remarked that the elimination of two of the functions *x*, *y*, *z* from equations (a′) by GAUSS in his "Disquisitiones generales circa superficies curvas" was accomplished in full generality to the extent that it can be referred to as something that GAUSS himself contributed, even though that elimination was not completed in the cited reference.

One finds the following equation in Section XI of *Disquisitiones:*

$$AD = \frac{\partial^2 x}{\partial p^2}\Delta + \frac{\partial x}{\partial p}(nF - mG) + \frac{\partial x}{\partial q}(mF - nE)\,,$$

(b) $$AD' = \frac{\partial^2 x}{\partial p\,\partial q}\Delta + \frac{\partial x}{\partial p}(n'F - m'G) + \frac{\partial x}{\partial q}(m'F - n'E)\,,$$

$$AD'' = \frac{\partial^2 x}{\partial q^2}\Delta + \frac{\partial x}{\partial p}(n''F - m''G) + \frac{\partial x}{\partial q}(m''F - n''E)\,,$$

$$\Delta = EG - F^2,$$

and in Section X, one finds the equation:

(c) $$\frac{DD'' - D'^2}{\Delta^2} = k\,,$$

in which k denotes the curvature of the given surface that was represented in Section XI in terms of the coefficients E, F, G.

If one introduces the notation:

$$\begin{aligned} m\,G - n\,F &= \Delta M, & n\,E - m\,F &= \Delta N, \\ m'G - n'F &= \Delta M, & n'E - m'F &= \Delta N', \\ m''\,G - n''\,F &= \Delta M, & n''\,E - m''\,F &= \Delta N'', \end{aligned}$$

to simplify, and one introduces the notations x_1, x_2, x_{11}, x_{12}, x_{22} for the first and second partial derivatives of a function x with respect to the variables p, q, for the same reason, then the six quantities M, N, … will be quantities that are determined in the same way from the given E, F, G, and one will get the following equation from equations (b) and (c):

$$A^2\,k = (x_{11} - M\,x_1 - N\,x_2)\,(x_{22} - M''x_1 - N''x_2) - (x_{12} - M'x_1 - N'x_2)^2,$$

and as a result of the simple relation:

$$A^2 = (y_1\,z_2 - z_1\,y_2)^2 = (E - x_1^2)(G - x_2^2) - (F - x_1\,x_2)^2,$$

one will get the equation:

(d) $$\Delta\,[1 - \Delta_1\,(x)]\,k = (x_{11} - M\,x_1 - N\,x_2)\,(x_{22} - M''x_1 - N''x_2) - (x_{12} - M'x_1 - N'x_2)^2,$$

in which $\Delta_1\,(x)$ refers to the differential parameter of the function x:

$$\frac{E\,x_2^2 - 2F\,x_1\,x_2 + G\,x_1^2}{\Delta}\,,$$

which is the second-order partial differential equation that the function x must satisfy, along with the other two y, z, as a consequence of equations (a′).

If one chooses $E = 0$, $F = 2\lambda$, $G = 0$ (under the assumption that one has introduced complex conjugate variables p, q) then one will get equation (III) that BOUR gave on page 15 of the cited treatise from the equation above.

The only opinion that BOUR expressed that we cannot share is that this equation [or more generally, equation (d)] represents the differential equation of the coordinates of those surfaces that are developable from with the given surface with the line element $\sqrt{E\,dp^2 + 2F\,dp\,dq + G\,dq^2}$.

In fact, no real-valued function x of the variables p, q that satisfies the partial differential equation (d) can be considered to be a function that is suitable to fulfill the fundamental equation:

$$dx^2 + dy^2 + dz^2 = E\,dp^2 + 2F\,dp\,dq + G\,dq^2,$$

when combined with two other functions y, z, which it would have to though if we were to adopt the opinion that we do not share.

One can easily convince oneself of the validity of that assertion when one derives the differential equation (d) in a different way that provides more satisfying information about its content.

Namely, if one puts equation (a) into the form:

$$dx^2 + dy^2 + dz^2 = [E - x_1^2]\,dp^2 + 2\,[F - x_1\,x_2]\,dp\,dq + [G - x_2^2]\,dq^2$$

then one will see that the quadratic form in the of the quantities dp, dq that the right-hand side of the foregoing equation represents must possess zero curvature.

When one forms that curvature using the formula that GAUSS gave in the cited reference, as is easy to see, one will be led to a second-order partial differential equation for the function x that proves to be identical to the differential equation (d).

The differential equation (d) can then be regarded as the expression of the condition that the quadratic form:

$$[E - x_1^2]\,dp^2 + 2\,[F - x_1\,x_2]\,dp\,dq + [G - x_2^2]\,dq^2$$

mist possess zero curvature, or (which is know to say the same thing) that this form can be converted into the product of the differentials of two functions α, β of the variables p, q.

Any function x that satisfies the differential equation (d) will then fulfill the equation:

$$[E - x_1^2]\,dp^2 + 2\,[F - x_1\,x_2]\,dp\,dq + [G - x_2^2]\,dq^2 = d\alpha\,d\beta,$$

in which the functions α, β are easy to determine.

Namely, if one decomposes the quadratic form on the left-hand side of that equation in the known way into two linear factors $a\,dp + b\,dq$, $a'\,dp + b'\,dq$ then there will always exist a function ϕ that simultaneously satisfies the two equations:

$$\text{(e)} \qquad \begin{aligned} e^{\phi}\,(a\;\;dp + b\;\;dq) &= d\alpha, \\ e^{-\phi}\,(a'dp + b'dq) &= d\beta. \end{aligned}$$

The integrability equations that follow from these equations:

$$\frac{\partial\,(e^{\phi}a)}{\partial q} = \frac{\partial\,(e^{\phi}b)}{\partial p},$$

$$\frac{\partial\,(e^{-\phi}a')}{\partial q} = \frac{\partial\,(e^{-\phi}b')}{\partial p},$$

obviously determine the differential quotients $\frac{\partial\phi}{\partial p}$, $\frac{\partial\phi}{\partial q}$ in terms of the given quantities a, b, a', b', which are given by the given quadratic form, and their differential quotients. After determining $\frac{\partial\phi}{\partial p}$, $\frac{\partial\phi}{\partial q}$, one can get the function ϕ by a quadrature, and after ascertaining that function, the functions α and β themselves can be determined from equations (e) by quadrature.

However, it is only in the special case in which the functions α, β thus-obtained assume complex-conjugate values $y + zi$, $y - zi$ for all values of the variables p, q that belong to the domain in question (or part of it) that the function x, which satisfies equation (d), will have the property that two real-valued functions y and z exist that make the equation:

$$dx^2 + dy^2 + dz^2 = E\,dp^2 + 2F\,dp\,dq + G\,dq^2$$

into an identity, together with the function x.

That special case will occur only when the function x possesses the property that it satisfies the inequality:

$$\left[E - \left(\frac{\partial x}{\partial p}\right)^2\right]\left[G - \left(\frac{\partial x}{\partial q}\right)^2\right] - \left[F - \frac{\partial x}{\partial p}\frac{\partial x}{\partial q}\right]^2 > 0,$$

in addition to the property that it satisfies the differential equation (d), and that inequality expresses a simple geometric property of those functions x that can be considered to the x coordinate of a curved surface with the line element $\sqrt{E\,dp^2 + 2F\,dp\,dq + G\,dq^2}$.

In the case where that inequality does not apply (i.e., in the case where the functions α, β prove to be real-valued for the domain in question of the variables p, q), the product $d\alpha\,d\beta$ can only be put into the form $dy^2 - dz^2$, and that is therefore the case in which the function x that satisfies the differential equation (d) can possibly fulfill the equation:

$$dx^2 + dy^2 - dz^2 = E\,dp^2 + 2F\,dp\,dq + G\,dq^2$$

jointly with the functions y and z, and that equation is foreign to the problem of the mutual developability of curved surfaces.

As a result of the foregoing juxtaposition, one can then regard the partial differential equation (d) as also being the result of eliminating the functions y, z from one or the other series of the following two times three equations:

$$\left(\frac{\partial x}{\partial p}\right)^2 + \left(\frac{\partial y}{\partial p}\right)^2 + \varepsilon\left(\frac{\partial z}{\partial p}\right)^2 = E,$$

$$\frac{\partial x}{\partial p}\frac{\partial x}{\partial q} + \frac{\partial y}{\partial p}\frac{\partial y}{\partial q} + \varepsilon\frac{\partial z}{\partial p}\frac{\partial z}{\partial q} = F,$$

$$\left(\frac{\partial x}{\partial q}\right)^2 + \left(\frac{\partial y}{\partial q}\right)^2 + \varepsilon\left(\frac{\partial z}{\partial q}\right)^2 = G$$

that one obtains when one first understands ε to be first positive unity and then negative unity.

One further remarks that as long as one understands the functions E, F, G to mean only ones that are suitable for appearing as the coefficients of the square of the line element of a curved surface, the differential equation (d) in question will suggest two transformation problems that are essentially different for real-valued functions x, y, z, namely, the ones that are distinguished by the equations:

(a) $$dx^2 + dy^2 + dz^2 = E\,dp^2 + 2F\,dp\,dq + G\,dq^2,$$

(a*) $$dx^2 + dy^2 - dz^2 = E\,dp^2 + 2F\,dp\,dq + G\,dq^2,$$

of which, only the former relates to the problem of finding those surfaces that can be developed onto a given one, while the latter, which is not symmetric in x, y, z, requires only that the functions x, y must fulfill equation (d), while the function z must fulfill the following one:

(d*) $$-\Delta\,[1 + \Delta_1(z)]\,k = (z_{11} - M\,z_1 - N\,z_2)(z_{22} - M''z_1 - N''z_2) - (z_{12} - M'z_1 - N'z_2)^2.$$

In contrast to the differential equation (d), this latter equation, which can be regarded as the condition for the form:

$$[E + z_1^2]\,dp^2 + 2\,[F + z_1\,z_2]\,dp\,dq + [G + z_2^2]\,dq^2$$

to possess zero curvature, is in fact the decisive differential equation for the transformation problem (a*). Since the foregoing quadratic form in the differentials dp, dq is always essentially positive, every real-valued function z that satisfies that equation will correspond to two real-valued functions x, y that satisfy the equation:

$$(a^*) \qquad dx^2 + dy^2 - dz^2 = E\,dp^2 + 2F\,dp\,dq + G\,dq^2,$$

jointly with z. *Those functions x, y satisfy the differential equation* (d) *without being related to the problem of developability.*

The differential equation (d) then possesses greater scope in relation to the functions x that satisfy it than the one that is prescribed by the problem of the deformation of curved surfaces. One will therefore forsake that differential equation for the treatment of that problem, or else the two transformation problems that it defines would have to split from each other.

In his "Théorie de la déformation des surfaces," BOUR gave yet another method for finding three functions x, y, z that might fulfill equation (a). That method is linked with the assumption that the square of the line element of the given surface is given in the form:

$$dp^2 + G\,dq^2,$$

or can be put into it. It is free of the reproach that when one complies with it, one can also arrive at functions x, y, z that are foreign to the problem that was posed, and in fact the transformation problems that are defined by equations (a) and (a^*) will split from each other, although BOUR avoided that eventuality since his second method encompassed the same scope as his first one. The method itself is based upon special geometric properties of the geodetic lines of curved surfaces and requires that before one can enter into it, one must have the results of integrating the differential equation for geodetic lines for a given surface. Once that integration was considered to have been performed, BOUR reduced the determination of the desired functions x, y, z to the determination of three functions H, T, H_1 of the variables p, q that are already coupled by a homogeneous equation of degree two from two more simultaneous linear partial differential equations that govern those functions, and then to the resulting integration of a system of nine simultaneous first-order ordinary differential equations in the variable p, which is an integration that must be performed in such a way that the nine functions that are to be determined will likewise fulfill a similar system of nine differential equations in the variable q. The differential equations for the functions H, T, H_1 are the necessary conditions for it to be possible for the latter eighteen differential equations to be valid simultaneously.

BOUR referred to the equations that he presented for the functions H, T, H_1 as the *fundamental equations* for the problem of finding the surfaces that can be developed onto a given one.

Following BOUR, various authors have specified differential equations for the quantities that GAUSS denoted by D, D', D'' (or the products of those quantities with a simple power of the determinant Δ) that did not use the specialized form $\sqrt{dp^2 + G\,dq^2}$, but the general one $\sqrt{E\,dp^2 + 2F\,dp\,dq + G\,dq^2}$, as the starting point for their geometric investigations, so those equations would not be suitable to replace BOUR's fundamental equations. BOUR's functions H, T, H_1 are themselves such products.

To our knowledge, it was only the step that leads from one's knowledge of the quantities D, D', D'' in order to arrive at a definitive representation of the coordinates x, y, z for the desired surface that was not discussed. It was only in recent times that LIPSCHITZ [Sitzungsberichte der Königl. Preuss. Akademie der Wissenschaften zu

Berlin (1883), pps. 550 and 551], while preserving the quantities H, T, H_1 as the ones to be determined, transformed BOUR's fundamental equations into more general ones that would be true for any given form of the line element, and specified the step in question by introducing new geometrically-defined functions that one would need to find. In my opinion, those elegant investigations do not reduce that step to its simplest form, but only abandon the symmetry in the final result by introducing asymmetric relations.

In what follows, we will prove that after succeeding in finding three fundamental quantities that are defined by equations of the same type as BOUR's that would definitively represent the functions x, y, z, only the integration of two second-order linear ordinary differential equations would be required, and that proof will also imply that the same demands would also suffice in relation to the solution of the transformation problem:

$$dx^2 + dy^2 - dz^2 = E\, dp^2 + 2F\, dp\, dq + G\, dq^2,$$

independently of the fact that we have already presented the same partial differential equation (d*) for that problem, which governs it completely.

Geometric considerations, as well as applications to examples, will be avoided in the following developments. However, when one starts from the known finite equations for the class of surfaces that can be developed from the surface of revolution of the evolute of the catenary that I gave in CRELLE-BORCHARDT's Journal (bd. 59), as well as the other class of surfaces that can be wrapped around a surface of revolution that I presented in that same journal (Bd. 62), it will be easy to verify those developments conversely, and to give integrals to equations (a), as well as BOUR's fundamental equations, in special cases of the given line element, which will provide information about the nature of those equations.

When one overlooks the class of surfaces that can be developed to the plane, which has been known since the time of EULER, the aforementioned two classes of surfaces, which can be expanded by way of the class of surfaces whose line element is:

$$\sqrt{dp^2 + 2q\, dp\, dq + 2\, p\, dq^2}$$

(which are determined by the minimal surfaces), are the only classes of mutually-developable surfaces whose finite equations have been given up to now.

I.

One must thank GAUSS for the fundamental discovery of a special invariance under the transformation of quadratic binary differential forms.

If the quadratic form in the differentials dp, dq :

$$A = a_{11}\, dp^2 + 2a_{12}\, dp\, dq + a_{22}\, dq^2,$$

in which a_{11} , a_{12} , a_{22} denote three given functions of the real variables p, q, goes to another one:

$$A' = a'_{11}\, dp'^2 + 2a'_{12}\, dp'\, dq' + a'_{22}\, dq'^2\,,$$

in which a'_{11}, a'_{12}, a'_{22} now denote known functions of the variables p', q' under the substitutions $p' = f\,(p,\ q)$, $q' = f\,(p,\ q)$ then a function k that GAUSS gave that is constructed from the coefficients $a_{i,k}$ and their first and second partial derivatives will go to the same function of the coefficients $a'_{i,k}$ and their partial derivatives with respect to the variables p', q' under those substitutions.

The invariant k that GAUSS gave can be represented in the symmetric form:

$$k = \frac{-1}{2\sqrt{a}}\left\{\frac{\partial}{\partial p}\left[\frac{a_{12}\dfrac{\partial \log \frac{a_{22}}{a_{11}}}{2\,\partial q} + \dfrac{\partial a_{22}}{\partial p} - \dfrac{\partial a_{12}}{\partial q}}{\sqrt{a}}\right] + \frac{\partial}{\partial q}\left[\frac{a_{12}\dfrac{\partial \log \frac{a_{11}}{a_{22}}}{2\,\partial p} + \dfrac{\partial a_{11}}{\partial q} - \dfrac{\partial a_{12}}{\partial p}}{\sqrt{a}}\right]\right\},$$

$$a = a_{11}\, a_{22} - a_{12}^2\,.$$

Although GAUSS apparently restricted the derivation of that result to those forms:

$$a_{11}\, dp^2 + 2\, a_{12}\, dp\, dq + a_{11}\, dq^2$$

that are capable of representing the square of the line element of a curved surface, one easily observes that the same thing would be valid for all forms with a non-vanishing determinant $a_{11}\ a_{22} - a_{12}^2$. As long as we do not misunderstand the motto "Ab his via sternitur ad maiora (†)" that he himself quoted as a preface to his essay "Allgemeine Lösung der Aufgabe, die Theile einer gegebenen Fläche auf die Theile einer anderen gegebenen Fläche so abzubilden, dass die Abbildung dem Abgebildeten in the kleinsten Theilen ähnlich wird (††)," the properties of the decomposition of a given binary quadratic differential form into linear factors – i.e., the properties of the transformation:

$$\lambda\,(a_{11}\, dp^2 + 2\, a_{12}\, dp\, dq + a_{11}\, dq^2) = d\alpha\, d\beta,$$

would become the starting point for the discovery of that invariant relationship.

In fact, as we have known for some time, one easily derives a second-order partial differential equation for the quantity log (λ) from the foregoing equation by performing a calculation that was suggested in the introduction, and the term in that equation that is free of the differential quotients will be identical with the quantity k.

(†) Translator: "This path will lead to greater things."

(††) Translator: "General solution to the problem of mapping part of a given surface to part of another given surface in such a way that the image will be similar to the mapped surface in its smallest parts." Note that the word "Theil" is an archaic form of the word "Teil" for "part" or sometimes "subset."

If one starts from that derivation of the transformed form then one will next note the agreement of the terms in the two differential equations that one obtains for the quantity log (λ) that are free of the differential quotients and conclude from that the equality of the quantity k with the k' that is constructed from the coefficients of the transformed form in the same way as the quantity k was constructed from the coefficients of the original form. The invariance that was mentioned consists of that equality.

If one considers a curved surface to be a flexible, inextensible body with one vanishingly small dimension then the geometric properties of that surface in the neighborhood of any of its points will be partly connected with its special form and will partly remain unchanged, which will bring with it new forms that can be linked with the condition of the inextensibility of its parts.

If one thinks of the position of each point P of the surface as being determined by the values of two independently-varying parameters p, q then properties of the latter kind will be determined completely by knowing the differential form:

$$E\,dp^2 + 2F\,dp\,dq + G\,dq^2,$$

which gives the distance from any point (p, q) to the infinitely-close one ($p + dp$, $q + dq$), while determining the properties of the former kind will require knowing a second differential form:

$$\mathfrak{E}\,dp^2 + 2\mathfrak{F}\,dp\,dq + \mathfrak{G}\,dq^2$$

that will depend upon the one that was just given. That second form is the value of the differential quantity:

$$dX\,dx + dY\,dy + dZ\,dz,$$

which is independent of the choice of rectangular axis system to which one thinks of the position of the given surface as being referred, and in which x, y, z denote the coordinates of the point (p, q), and X, Y, Z denote the cosines of the angle that the normal that is raised at that point make with the coordinate axes, while the differential sign refers to the difference between the quantities in question at the point (p, q) and the ones at the point ($p + dp$, $q + dq$).

The simultaneous algebraic invariants:

$$h = \frac{E\,\mathfrak{G} - 2F\,\mathfrak{F} + G\,\mathfrak{E}}{EG - F^2},$$

$$k = \frac{\mathfrak{E}\,\mathfrak{G} - \mathfrak{F}^2}{EG - F^2}$$

of the two given quadratic forms determine the sum of the principal curvatures r, r' at the point (p, q) of the surface considered in the first case, and the product of those curvatures in the second case. The agreement of the second one with the invariant k of the square of the line element was likewise proved by GAUSS.

The better part of the infinitesimal-geometric investigations that geometers since EULER have addressed are connected with the question of finding those surfaces for which one of those invariants is constant or a function of the other one. Other investigations were concerned with those surfaces for which a second type of simultaneous invariant of the aforementioned differential forms possessed given properties. That second type of simultaneous invariant, which contains not just the coefficients of the forms in question, but also their partial derivatives, corresponding to the invariants that GAUSS presented for an individual form, does not seem to have been introduced expressly into the theory of curved surfaces and evaluated for them up to now.

We therefore believe that although we have easily drifted away from the part of the following developments that is connected with the theory of mutually-developable surfaces, and have been able to explain them independently of the consideration of those invariants, some discussion must be prefixed in regard to the simultaneous transformation of binary quadratic differential forms, and all the more because that will shed some light upon the path of investigation into the theory of the curvature of surfaces that was pursued up to now.

Let:

$$A = a_{11}\, dp^2 + 2\, a_{12}\, dp\, dq + a_{11}\, dq^2$$

be a quadratic form in the differentials dp, dq whose coefficients a_{11}, a_{12}, a_{22} are real-valued functions of the real variables p, q that are given inside a known region of those variables. The choice of those functions in what follows shall be subject to the restriction that they should be suitable for representing the coefficients of the squares of the line elements of a curved surface; i.e., that the functions a_{11}, a_{12}, a_{22} will fulfill the inequalities:

$$a_{11} > 0, \qquad a_{22} > 0, \qquad a_{11}\, a_{22} - a_{12}^2 > 0$$

for all values of the variables p, q that fall within the given region.

Furthermore, let:

$$C = c_{11}\, dp^2 + 2\, c_{12}\, dp\, dq + c_{11}\, dq^2$$

be a second quadratic form in the differentials dp, dq whose coefficients c_{11}, c_{12}, c_{22} are likewise real-valued functions of the variables p, q, but the choice of those coefficients is not subject to any further restriction.

Let the two absolute simultaneous invariants of the forms A and C be denoted by H and K, in such a way that:

$$H = \frac{a_{11}\, c_{22} - 2 a_{12}\, c_{12} + a_{22}\, c_{11}}{a_{11}\, a_{22} - a_{12}^2},$$

$$K = \frac{c_{11}\, c_{22} - c_{12}^2}{a_{11}\, a_{22} - a_{12}^2}.$$

The simultaneous transformation of two given forms A, C by the introduction of new variables p', q', in place of the original ones p, q, will always be accompanied by two other forms that are connected with them that we will denote by B and E in what follows.

The first of those forms:

$$B = b_{11}\, dp^2 + 2\, b_{12}\, dp\, dq + b_{22}\, dq^2$$

is given by the following determination of its coefficients:

$$b_{11} = H\, c_{11} - K\, a_{11}\,, \quad b_{12} = H\, c_{12} - K\, a_{12}\,, \quad b_{22} = H\, c_{22} - K\, a_{22}\,,$$

and its determinant $b_{11}\, b_{22} - b_{12}^2$ obviously satisfies the equation:

$$b_{11}\, b_{22} - b_{12}^2 = K^2\, (a_{11}\, a_{22} - a_{12}^2),$$

and when one considers the convention on the choice of coefficients $a_{i,k}$, that equation will show that the determinant of the form B will possess a positive value for the entire domain of the values of the variables p, q.

The other accompanying form:

$$E = e_{11}\, dp^2 + 2\, e_{12}\, dp\, dq + e_{22}\, dq^2$$

is given by the equation:

$$E = \frac{1}{\sqrt{a_{11}\, a_{22} - a_{12}^2}}\, [(a_{11}\, dp + a_{12}\, dq)(c_{11}\, dp + c_{12}\, dq) - (a_{12}\, dp + a_{22}\, dq)(c_{11}\, dp + c_{12}\, dq)],$$

which agrees with the following one:

$$E = \frac{1}{\sqrt{a_{11}\, a_{22} - a_{12}^2}}\, [(a_{11}\, c_{12} - a_{12}\, c_{11})dp^2 + (a_{11}\, c_{22} - a_{22}\, c_{11})dp\, dq + (a_{12}\, c_{22} - a_{22}\, c_{12})dq^2].$$

The coefficients of the form E are then determined from the equations:

$$e_{11} = \frac{a_{11}\, c_{12} - a_{12}\, c_{11}}{\sqrt{a_{11}\, a_{22} - a_{12}^2}}\,, \qquad 2e_{12} = \frac{a_{11}\, c_{22} - a_{22}\, c_{11}}{\sqrt{a_{11}\, a_{22} - a_{12}^2}}\,, \qquad e_{22} = \frac{a_{12}\, c_{22} - a_{22}\, c_{12}}{\sqrt{a_{11}\, a_{22} - a_{12}^2}}\,.$$

That determination of the coefficients is connected with the system of quantities that emerges as a result and is defined by the following equations:

$$A_{11} = \frac{a_{22}\, c_{11} - a_{12}\, c_{12}}{a_{11}\, a_{22} - a_{12}^2}\,, \qquad A_{12} = \frac{a_{11}\, c_{12} - a_{12}\, c_{11}}{a_{11}\, a_{22} - a_{12}^2}\,,$$

$$A_{21} = \frac{a_{22}\, c_{12} - a_{12}\, c_{22}}{a_{11}\, a_{22} - a_{12}^2}\,, \qquad A_{22} = \frac{a_{11}\, c_{22} - a_{12}\, c_{12}}{a_{11}\, a_{22} - a_{12}^2}\,,$$

from which one easily defines the relations:

$$A_{11} + A_{22} = H,$$
$$A_{11}\,A_{22} - A_{12}\,A_{21} = K.$$

With the help of that system of quantities, the form E will assume the form:

$$\text{(e)} \qquad E = \sqrt{a_{11}\,a_{22} - a_{12}^2}\;[A_{12}\,dp^2 + (A_{22} - A_{11})\,dp\,dq - A_{21}\,dq^2],$$

and when one now denotes the determinant $a_{11}\,a_{22} - a_{12}^2$ of a form A by a, its coefficients will take the forms:

$$e_{11} = A_{12}\sqrt{a}\,, \qquad 2e_{12} = (A_{22} - A_{11})\sqrt{a}\,, \qquad e_{22} = -A_{21}\sqrt{a}\,.$$

The transformation of the given form B that takes places at the same time as the transformation of the form C leads to a system of quantities $B_{i,k}$ that correspond to the system of quantities $A_{i,k}$ and are derived from the former when one exchanges the coefficients $a_{i,k}$ with the corresponding ones $b_{i,k}$. One effortlessly notes the following relations for that system:

$$B_{11} = \frac{1}{K}A_{22}\,, \qquad B_{12} = -\frac{1}{K}A_{12}\,,$$

$$B_{21} = -\frac{1}{K}A_{21}\,, \qquad B_{22} = \frac{1}{K}A_{11}\,,$$

and then the further representation of the form E in the form:

$$\text{(e}'\text{)} \qquad E = K\sqrt{a}\,[-B_{12}\,dp^2 + (B_{11} - B_{22})\,dp\,dq + B_{21}\,dq^2]\,.$$

As far as the determinant $e = e_{11}\,e_{22} - e_{12}^2$ of the form E is concerned, which can be put into the form:

$$e = -\tfrac{1}{4}[4\,A_{12}\,A_{21} + (A_{22} - A_{11})^2]\,a = -\tfrac{1}{4}[H^2 - 4K]\,a$$

by means of the representation of the coefficients of that form in terms of the quantities $A_{i,k}$, one recognizes that this determinant will always have a negative value in the entire domain of the quantities p, q, because the former of the foregoing values for e agrees with the following one:

$$e = -\tfrac{1}{4}\left[\left(A_{22} - A_{11} - 2\frac{a_{12}}{a_{11}}A_{12}\right)^2 + 4\frac{A_{12}^2}{a_{11}^2}a\right]a\,,$$

and that representation will illuminate the validity of the statement that was made when one recalls the always-positive value of the determinant a.

Under the assumptions that were made, the form E can always be decomposed into *two real* factors $\alpha\, dp + \alpha'\, dq$, $\beta\, dp + \beta'\, dq$ that are linear and homogeneous in the differentials dp, dq.

One will then have:

$$E = (\alpha\, dp + \alpha'\, dq)(\beta\, dp + \beta'\, dq),$$

in which $\alpha, \alpha', \beta, \beta'$ are real-valued function of the variables p, q.

If λ and μ denote the integrating factors of the first and second, resp., of the linear factors of E (which always exist) then one will have the equations:

$$du = \lambda\,(\alpha\, dp + \alpha'\, dq),$$
$$dv = \mu\,(\beta\, dp + \beta'\, dq),$$

in which the quantities u and v denote real-valued functions of the variables p, q, and the form E can be put into the form:

$$E = \frac{1}{\lambda\mu}\, du\, dv\,.$$

If one now introduces the functions u and v as new variables in the four forms A, B, C, E ([1]), in place of the original variables p, q, and adds an asterisk to the coefficients of the original forms in order to denote the coefficients of the transformed ones then one will get the equations:

$$A = a_{11}^*\, du^2 + 2a_{11}^*\, du\, dv + a_{22}^*\, dv^2\,,$$
$$B = b_{11}^*\, du^2 + 2b_{11}^*\, du\, dv + b_{22}^*\, dv^2\,,$$
$$C = c_{11}^*\, du^2 + 2c_{11}^*\, du\, dv + c_{22}^*\, dv^2\,,$$

$$E = \frac{1}{\sqrt{a^*}}[(a_{11}^*\, c_{12}^* - a_{12}^*\, c_{11}^*)\, du^2 + (a_{11}^*\, c_{22}^* - a_{22}^*\, c_{11}^*)\, du\, dv + (a_{12}^*\, c_{22}^* - a_{22}^*\, c_{12}^*)\, dv^2]\,,$$

the last of which must coincide with the representation that was just given for the form E.

Therefore, the relations:

$$a_{11}^*\, c_{12}^* - a_{12}^*\, c_{11}^* = 0,$$
$$a_{12}^*\, c_{22}^* - a_{22}^*\, c_{12}^* = 0,$$
$$a_{11}^*\, c_{22}^* - a_{22}^*\, c_{11}^* = \frac{\sqrt{a^*}}{\lambda\mu}$$

are necessary. When one recalls the third of them, the first two will imply that:

([1]) The functions u and v will not be mutually-independent only in the case where the determinant e of the form E vanishes identically. That case, which is characterized by the invariant relation:

$$H^2 = 4K,$$

shall be excluded from what follows.

$$a_{12}^{*} = 0, \qquad c_{12}^{*} = 0.$$

When one introduces the variables u, v, the forms A and C will then take on the forms:

$$A = a_{11}^{*}\,du^2 + a_{22}^{*}\,dv^2,$$
$$C = c_{11}^{*}\,du^2 + c_{22}^{*}\,dv^2.$$

If one constructs the two simultaneous invariants H and K from those transformed forms then one will obtain them from the equations:

$$H = \frac{c_{11}^{*}}{a_{11}^{*}} + \frac{c_{22}^{*}}{a_{22}^{*}},$$

$$K = \frac{c_{11}^{*}}{a_{11}^{*}} \cdot \frac{c_{22}^{*}}{a_{22}^{*}}.$$

After introducing the relations:

$$\frac{c_{11}^{*}}{a_{11}^{*}} = w, \qquad \frac{c_{22}^{*}}{a_{22}^{*}} = w',$$

those invariants can be represented in the forms:

$$H = w + w',$$
$$K = ww',$$

and the quantities w and w' prove to the two roots (which are always real) of the quadratic equation:

$$w^2 - H\,w + K = 0,$$

which is an equation that can be constructed from the known simultaneous invariants of the quadratic forms A, C with no further analysis when they are also given special forms.

When one uses the variables u, v that were just defined instead of the original variables p, q and incorporates the irrational invariants w, w' into the calculations instead of the invariants H, K, the four jointly-considered forms A, B, C, E will go to the transformed forms that are given by the following system of equations:

$$(1) \qquad \begin{aligned} a_{11}\,dp^2 + 2a_{12}\,dp\,dq + a_{22}\,dq^2 &= a_{11}^{*}\,du^2 + a_{22}^{*}\,dv^2, \\ c_{11}\,dp^2 + 2c_{12}\,dp\,dq + c_{22}\,dq^2 &= w\,a_{11}^{*}\,du^2 + w'\,a_{22}^{*}\,dv^2, \\ b_{11}\,dp^2 + 2b_{12}\,dp\,dq + b_{22}\,dq^2 &= w^2\,a_{11}^{*}\,du^2 + w'^2\,a_{22}^{*}\,dv^2, \end{aligned}$$

$$e_{11}\,dp^2 + 2e_{12}\,dp\,dq + e_{22}\,dq^2 = \sqrt{a_{11}^*\,a_{22}^*}\,(w - w')\,du\,dv\,.$$

If δp, δq, δu, δv denote the variations of the variables p, q, u, v, resp., – i.e., quantities between which the same linear relations exist as the ones that exist between the differentials dp, dq, du, dv, namely, the following ones:

$$\delta p = \frac{\partial p}{\partial u}\delta u + \frac{\partial p}{\partial v}\delta v\,, \qquad \delta q = \frac{\partial q}{\partial u}\delta u + \frac{\partial q}{\partial v}\delta v\,,$$

then along with the first set (1) of transformations of the forms A, B, C, E, it is known that one will simultaneously have the second set:

$$(2)\quad \begin{aligned}
&a_{11}\,\delta p\,dq + a_{12}\,(\delta q\,dp + \delta p\,dq) + a_{22}\,\delta p\,dq = a_{11}^*\,\delta u\,du + a_{22}^*\,\delta v\,dv\,,\\
&c_{11}\,\delta p\,dq + c_{12}\,(\delta q\,dp + \delta p\,dq) + c_{22}\,\delta p\,dq = w\,a_{11}^*\,\delta u\,du + w'\,a_{22}^*\,\delta v\,dv\,,\\
&b_{11}\,\delta p\,dq + b_{12}\,(\delta q\,dp + \delta p\,dq) + b_{22}\,\delta p\,dq = w^2\,a_{11}^*\,\delta u\,du + w'^2\,a_{22}^*\,\delta v\,dv\,,\\
&e_{11}\,\delta p\,dq + e_{12}\,(\delta q\,dp + \delta p\,dq) + e_{22}\,\delta p\,dq = \tfrac{1}{2}\sqrt{a_{11}^*\,a_{22}^*}\,(w' - w)(\delta v\,du + \delta u\,dv)\,.
\end{aligned}$$

A set of variations δp, δq, δu, δv will obviously be given by the equations:

$$(\delta)\quad \begin{aligned}
&\delta p = -\frac{1}{\sqrt{a}}\frac{\partial\phi}{\partial q}\,, \qquad && \delta q = \frac{1}{\sqrt{a}}\frac{\partial\phi}{\partial p}\,,\\
&\delta u = -\frac{1}{\sqrt{a^*}}\frac{\partial\phi}{\partial v}\,, \qquad && \delta v = \frac{1}{\sqrt{a^*}}\frac{\partial\phi}{\partial u}\,,
\end{aligned}$$

in which ϕ denotes an arbitrary function of the variables p, q. They will emerge immediately from the equations that express the differential quotients of a function ϕ with respect to the new variables u, v in terms of the differential quotients of the original variables, when one considers the equation:

$$a = a^*\left(\frac{\partial u}{\partial p}\frac{\partial v}{\partial q} - \frac{\partial u}{\partial q}\frac{\partial v}{\partial p}\right)^2.$$

A second set of variations will follow directly from the one above. If the transformation relation exists that:

$$P\,dp + Q\,dq = U\,du + V\,dv$$

then a system of quantities will also be given by the equations:

$$\delta p = -\frac{1}{\sqrt{a}} Q, \qquad \delta q = \frac{1}{\sqrt{a}} P,$$

$$\delta u = -\frac{1}{\sqrt{a^*}} V, \qquad \delta v = \frac{1}{\sqrt{a^*}} U$$

that possesses the property of a system of variations.

We will make use of the introduction of the system of variations δ in order to exhibit an equation that corresponds to the equations (2), in which we think of the coefficients e_{ik} as being determined by the equation (e′).

When one considers the relation:

$$K (B_{11} + B_{22}) = H$$

and appeals to the relations ϕ_1 , ϕ_2 for the partial derivatives of ϕ, the introduction of the variations δ will convert the last of equations (2) into the following one:

$$K [(B_{11} \phi_1 + B_{12} \phi_2) dp + (B_{21} \phi_1 + B_{22} \phi_2) dq] - \tfrac{1}{2} H \, d\phi = \tfrac{1}{2}(w' - w) [\phi_1 \, du - \phi_2 \, dv],$$

and when one adds $\frac{1}{2} H \, d\phi$ to each side of the equality and divides by $K = w \, w'$, the following transformation relation, which is true for any function ϕ, will seem to emerge from that:

$$\left(B_{11}\frac{\partial\phi}{\partial p} + B_{12}\frac{\partial\phi}{\partial q}\right)dp + \left(B_{21}\frac{\partial\phi}{\partial p} + B_{22}\frac{\partial\phi}{\partial q}\right)dq = \frac{1}{w}\frac{\partial\phi}{\partial u}du + \frac{1}{w'}\frac{\partial\phi}{\partial v}dv. \tag{3}$$

Use will likewise be made from now on of the transformation:

$$\left(A_{11}\frac{\partial\phi}{\partial p} + A_{12}\frac{\partial\phi}{\partial q}\right)dp + \left(A_{21}\frac{\partial\phi}{\partial p} + A_{22}\frac{\partial\phi}{\partial q}\right)dq = w\frac{\partial\phi}{\partial u}du + w'\frac{\partial\phi}{\partial v}dv, \tag{3*}$$

which is developed in the same way with the help of equation (e).

It hardly needs to be mentioned that each of the transformation relations that were presented up to now, as well as each of the other ones that are developed by introducing the system δ of variations or the second system that was cited in place of the differentials *dp*, *dq*, *du*, *dv*, will be converted into a new transformation relation that is likewise true for every function ϕ. Those conversions often perform a welcome service in the transformation of partial differential equations that are closely connected with the foregoing investigations.

II.

In order to represent such simultaneous invariants of the forms A, C, as well as those of the accompanying forms, which are composed from the coefficients a_{ik} , c_{ik} and their partial derivatives analogously to the way GAUSS constructed the invariant k from a given individual form, the GAUSSian curvature next provides the family of forms $A + \lambda C$; i.e., the ones with the form:

$$(a_{11} + \lambda\, c_{11})\, dp^2 + 2\,(a_{12} + \lambda\, c_{12})\, dp\, dq + (a_{22} + \lambda\, c_{22})\, dq^2,$$

in which λ is understood to mean an arbitrary constant.

The coefficients of the powers of the constant λ that appear in the development of that curvature in powers of λ were represented for a simultaneous invariant of the forms A and B.

It is only in that way that the invariants that one obtains would include differential quotients of the coefficients $a_{i,k}$, $c_{i,k}$ up to and including order two, while different considerations might bring about the appearance of simultaneous invariants that would lead only to the first-order differential quotients of those coefficients.

In the simultaneous transformation of two forms A, C, those couplings of the coefficients and their differential quotients that already bring to prominence one of those forms – e.g., the form A – will, in turn, appear to be obvious couplings. Those couplings are the quantities that already appeared in the introduction and were denoted by M, M', M'', N, N', N'' there.

Once we drop the GAUSSian notations E, F, G for the coefficients of the binary differential forms of degree two that was applied in the introduction and adopt the notations a_{11}, a_{12}, a_{22}, which will be more appropriate in what follows, the quantities m, m', m'', n, n', n'', which were likewise introduced by GAUSS, will be given by the following equations:

$$m = \tfrac{1}{2}\frac{\partial a_{11}}{\partial p}, \qquad m' = \tfrac{1}{2}\frac{\partial a_{11}}{\partial q}, \qquad m'' = \frac{\partial a_{12}}{\partial q} - \tfrac{1}{2}\frac{\partial a_{22}}{\partial p},$$

$$n = \frac{\partial a_{12}}{\partial p} - \tfrac{1}{2}\frac{\partial a_{11}}{\partial q}, \qquad n' = \tfrac{1}{2}\frac{\partial a_{22}}{\partial p}, \qquad n'' = \tfrac{1}{2}\frac{\partial a_{22}}{\partial q}.$$

We will choose a notation that goes back to CHRISTOFFEL for the quantities that were previously denoted by M, M', M'', N, N', N'' and set:

$$\frac{m\, a_{22} - n\, a_{12}}{a_{11}\, a_{22} - a_{12}^2} = \begin{Bmatrix} 11 \\ 1 \end{Bmatrix}_a, \qquad \frac{n\, a_{11} - m\, a_{12}}{a_{11}\, a_{22} - a_{12}^2} = \begin{Bmatrix} 11 \\ 2 \end{Bmatrix}_a,$$

$$\frac{m'\, a_{22} - n'\, a_{12}}{a_{11}\, a_{22} - a_{12}^2} = \begin{Bmatrix} 12 \\ 1 \end{Bmatrix}_a, \qquad \frac{n'\, a_{11} - m'\, a_{12}}{a_{11}\, a_{22} - a_{12}^2} = \begin{Bmatrix} 12 \\ 2 \end{Bmatrix}_a,$$

$$\frac{m'' a_{22} - n'' a_{12}}{a_{11}\, a_{22} - a_{12}^2} = \begin{Bmatrix} 2\,2 \\ 1 \end{Bmatrix}_a , \qquad \frac{n'' a_{11} - m'' a_{12}}{a_{11}\, a_{22} - a_{12}^2} = \begin{Bmatrix} 2\,2 \\ 2 \end{Bmatrix}_a ,$$

in what follows.

The index a that is introduced into the CHRISTOFFEL notation suggests the quadratic form A, from which one infers the characteristic system of quantities $\begin{Bmatrix} i\,k \\ h \end{Bmatrix}_a$, and when one infers that system from another form G, it will be replaced with the corresponding index g.

The simultaneous invariants of the forms A and C that we speak of, which only lead to first-order differential quotients of the coefficients of those forms, are obtained from considering two couplings $c_a\,(p)$, $c_a\,(q)$ of those coefficients and their derivatives that are linear in the quantities $\begin{Bmatrix} i\,h \\ k \end{Bmatrix}_a$ of the first form and the coefficients $c_{i,k}$ of the other form and its derivatives.

Those couplings are defined by the following equations:

$$(4)\qquad \begin{aligned} c_a\,(p) &= \frac{1}{\sqrt{a}}\left[\frac{\partial}{\partial p}\left(\frac{c_{22}}{\sqrt{a}}\right) - \frac{\partial}{\partial q}\left(\frac{c_{12}}{\sqrt{a}}\right) + \frac{c_{11}}{\sqrt{a}}\begin{Bmatrix} 2\,2 \\ 1 \end{Bmatrix}_a - 2\frac{c_{12}}{\sqrt{a}}\begin{Bmatrix} 1\,2 \\ 1 \end{Bmatrix}_a + \frac{c_{22}}{\sqrt{a}}\begin{Bmatrix} 1\,1 \\ 1 \end{Bmatrix}_a\right], \\ c_a\,(q) &= \frac{1}{\sqrt{a}}\left[\frac{\partial}{\partial q}\left(\frac{c_{11}}{\sqrt{a}}\right) - \frac{\partial}{\partial p}\left(\frac{c_{12}}{\sqrt{a}}\right) + \frac{c_{11}}{\sqrt{a}}\begin{Bmatrix} 2\,2 \\ 2 \end{Bmatrix}_a - 2\frac{c_{12}}{\sqrt{a}}\begin{Bmatrix} 1\,2 \\ 2 \end{Bmatrix}_a + \frac{c_{22}}{\sqrt{a}}\begin{Bmatrix} 1\,1 \\ 2 \end{Bmatrix}_a\right], \end{aligned}$$

which can also be represented in the following form:

$$(4')\qquad \begin{aligned} c_a\,(p) &= \frac{1}{a}\left[\frac{\partial c_{22}}{\partial p} - \frac{\partial c_{12}}{\partial q} + c_{11}\begin{Bmatrix} 2\,2 \\ 1 \end{Bmatrix}_a + c_{12}\left(\begin{Bmatrix} 2\,2 \\ 2 \end{Bmatrix}_a - \begin{Bmatrix} 2\,1 \\ 1 \end{Bmatrix}_a\right) - c_{22}\begin{Bmatrix} 1\,2 \\ 2 \end{Bmatrix}_a\right], \\ c_a\,(q) &= \frac{1}{a}\left[\frac{\partial c_{11}}{\partial q} - \frac{\partial c_{12}}{\partial p} - c_{11}\begin{Bmatrix} 1\,2 \\ 1 \end{Bmatrix}_a + c_{12}\left(\begin{Bmatrix} 1\,1 \\ 1 \end{Bmatrix}_a - \begin{Bmatrix} 1\,2 \\ 2 \end{Bmatrix}_a\right) + c_{22}\begin{Bmatrix} 1\,1 \\ 2 \end{Bmatrix}_a\right]. \end{aligned}$$

The couplings $c_a\,(p)$, $c_a\,(q)$ themselves ([1]) do not have the character of invariants. However, between them and the corresponding couplings that one infers from the transforms of the form A, C, namely:

$$c_a\,(p') = \frac{1}{\sqrt{a'}}\left[\frac{\partial}{\partial p'}\left(\frac{c'_{22}}{\sqrt{a'}}\right) - \frac{\partial}{\partial q'}\left(\frac{c'_{12}}{\sqrt{a'}}\right) + \frac{c'_{11}}{\sqrt{a'}}\begin{Bmatrix} 2\,2 \\ 1 \end{Bmatrix}_{a'} - 2\frac{c'_{12}}{\sqrt{a'}}\begin{Bmatrix} 1\,2 \\ 1 \end{Bmatrix}_{a'} + \frac{c'_{22}}{\sqrt{a'}}\begin{Bmatrix} 1\,1 \\ 1 \end{Bmatrix}_{a'}\right],$$

([1]) One immediately notes the validity of the equations:

$$a_a\,(p) = 0, \qquad a_a\,(q) = 0.$$

$$c_a(q') = \frac{1}{\sqrt{a'}}\left[\frac{\partial}{\partial q'}\left(\frac{c'_{11}}{\sqrt{a'}}\right) - \frac{\partial}{\partial p'}\left(\frac{c'_{12}}{\sqrt{a'}}\right) + \frac{c'_{11}}{\sqrt{a'}}\begin{Bmatrix}22\\2\end{Bmatrix}_{a'} - 2\frac{c'_{12}}{\sqrt{a'}}\begin{Bmatrix}12\\2\end{Bmatrix}_{a'} + \frac{c'_{22}}{\sqrt{a'}}\begin{Bmatrix}11\\2\end{Bmatrix}_{a'}\right],$$

the following equations exist:

$$(6)\qquad \begin{aligned} c_a(p') &= \frac{\partial p'}{\partial p} c_a(p) + \frac{\partial p'}{\partial q} c_a(q), \\ c_a(q') &= \frac{\partial q'}{\partial p} c_a(p) + \frac{\partial q'}{\partial q} c_a(q). \end{aligned}$$

One proves that in a way that we believe we should only suggest, for the sake of brevity. It is known that every identity:

$$P\,dp + Q\,dp = P'dp' + Q'dq'$$

between two linear differential expressions that can be transformed into each other will lead, under the assumption of the second identity:

$$a_{11}\,dp^2 + a_{12}\,dp\,dq + a_{22}\,dq^2 = a'_{11}\,dp'^2 + 2a'_{12}\,dp'\,dq' + a'_{22}\,dq'^2,$$

to the equation:

$$(\varepsilon)\qquad \frac{1}{\sqrt{a}}\left(\frac{\partial P}{\partial q} - \frac{\partial Q}{\partial p}\right) = \frac{1}{\sqrt{a'}}\left(\frac{\partial P'}{\partial q'} - \frac{\partial Q'}{\partial p'}\right).$$

If one applies that remark twice to the identity:

$$\frac{1}{\sqrt{a}}\left[\left(c_{11}\frac{\partial\phi}{\partial q} - c_{12}\frac{\partial\phi}{\partial p}\right)dp - \left(c_{22}\frac{\partial\phi}{\partial p} - c_{12}\frac{\partial\phi}{\partial q}\right)dq\right] = \frac{1}{\sqrt{a'}}\left[\left(c'_{11}\frac{\partial\phi}{\partial q'} - c'_{12}\frac{\partial\phi}{\partial p'}\right)dp' - \left(c'_{22}\frac{\partial\phi}{\partial p'} - c'_{12}\frac{\partial\phi}{\partial q'}\right)dq'\right],$$

in which one understands the function ϕ to first mean the function p' of the variables p, q and then the function q', then one will get two equations that correspond to equation (ε). One can eliminate the second differential quotients of the functions p', q' from them with the help of the equations (9) that CHRISTOFFEL gave in his treatise “Über die Transformation der homogenen Differentialausdrücke zweiten Grades” (CRELLE-BORCHARDT’s Journal, Bd. 70) and then obtain equations (6).

It follows from equations (6) that the system of quantities $c_a(p)$, $c_a(q)$, $c_a(p')$, $c_a(q')$ possesses the properties of a system of variations δp, δq, $\delta p'$, $\delta q'$ (a system of differentials, respectively), and is therefore suitable for converting each of the transformation relations that is true for the latter system into a new one that represents an identity relation between expressions that are composed in the same way, on the one hand, from the coefficients a_{ik}, c_{ik}, and on the other hand, from the coefficients a'_{ik}, c'_{ik}, and their corresponding derivatives.

By employing that property, one can then derive a wealth of simultaneous invariants of two simultaneously-transformed forms *A* and *C*, by which, one understands that to mean forms that play a preeminent role in the investigations of infinitesimal geometry.

For example, if one replaces the variations in the transformation relation:

$$g_{11}\,\delta p\,dp + g_{12}\,(\delta q\,dp + \delta p\,dq) + g_{22}\,\delta q\,dq = g'_{11}\,\delta p'dp' + g'_{12}(\delta q'dp' + \delta p'dq') + g'_{22}\,\delta q'dq',$$

which is valid for any form *G*, with the quantities $h_a(p)$, $h_a(q)$, $h_a(p')$, $h_a(q')$, which are inferred from a second form *H* in conjunction with the form *A*, then one will be led to the equation:

$$(7)\qquad [g_{11}\,h_a(p) + g_{12}\,h_a(q)]\,dp + [g_{12}\,h_a(p) + g_{22}\,h_a(q)]\,dq$$

$$= [g'_{11}\,h_a(p') + g'_{12}\,h_a(q')]\,dp' + [g'_{12}\,h_a(p') + g'_{22}\,h_a(q')]\,dq',$$

which can be applied to the question whose analogue in the theory of curved surfaces is the question: Under what conditions on a surface will the lines of curvature be suitable for dividing that surface into infinitely-small squares?

We would like to specify the conditions under which a form *C* :

$$c_{11}\,dp^2 + 2c_{12}\,dp\,dq + c_{22}\,dq^2$$

will be suitable for transforming the *given* form *A* :

$$a_{11}\,dp^2 + 2a_{12}\,dp\,dq + a_{22}\,dq^2$$

into the form:

$$f(u, v)\,[du^2 + dv^2]$$

by introducing the variables *u*, *v* that were defined in Section I.

If one chooses the form *G* in equation (7) to be the form *E* that was defined in the same place, and likewise for the form *H*, and uses the variables *u*, *v* in place of the arbitrary variables p', q' then that equation will be converted into:

$$(8)\qquad [e_{11}\,e_a(p) + e_{12}\,e_a(q)]\,dp + [e_{12}\,e_a(p) + e_{22}\,e_a(q)]\,dq = e^*_{12}\,[e_a(v)\,du + e_a(u)\,dv],$$

in which the coefficients of the form *E* that was transformed by the introduction of the variables *u*, *v* are denoted by adding an asterisk, as we have done up to now, once one considers that $e^*_{11} = 0$, $e^*_{22} = 0$.

Now, as a result of the definitions (4), one will have:

$$e_a(u) = -\frac{1}{\sqrt{a^*}}\left[\frac{\partial}{\partial v}\left(\frac{e^*_{12}}{\sqrt{a^*}}\right) + 2\frac{e^*_{12}}{\sqrt{a^*}}\begin{Bmatrix}12\\1\end{Bmatrix}_{a^*}\right],$$

$$e_a(v) = -\frac{1}{\sqrt{a^*}}\left[\frac{\partial}{\partial u}\left(\frac{e_{12}^*}{\sqrt{a^*}}\right) + 2\frac{e_{12}^*}{\sqrt{a^*}}\begin{Bmatrix}12\\2\end{Bmatrix}_{a^*}\right],$$

and as a result of the meaning of the notations $\begin{Bmatrix}ik\\h\end{Bmatrix}_{a^*}$:

$$\begin{Bmatrix}12\\1\end{Bmatrix}_{a^*} = \frac{1}{2}\frac{1}{a_{11}^*}\frac{\partial a_{11}^*}{\partial v}, \qquad \begin{Bmatrix}12\\2\end{Bmatrix}_{a^*} = \frac{1}{2}\frac{1}{a_{22}^*}\frac{\partial a_{22}^*}{\partial u},$$

and furthermore:

$$e_{12}^* = \tfrac{1}{2}\sqrt{a^*}\,(w' - w),$$

so one will also have:

$$e_a(u) = \tfrac{1}{2}\frac{w - w'}{\sqrt{a^*}}\frac{\partial \log(w - w')}{\partial v},$$

$$e_a(v) = \tfrac{1}{2}\frac{w - w'}{\sqrt{a^*}}\frac{\partial \log(w - w')}{\partial u}.$$

With the use of those equations, equation (8) will be converted into the following one:

$$[e_{11}\, e_a(p) + e_{12}\, e_a(q)]\, dp + [e_{12}\, e_a(p) + e_{22}\, e_a(q)]\, dq$$

$$= -\tfrac{1}{4}(w - w')^2\left[\frac{\partial \log(w - w')\, a_{22}^*}{\partial u}du + \frac{\partial \log(w - w')\, a_{11}^*}{\partial v}dv\right].$$

Now, should the form C possess the property that introducing the variables u, v will generate the coefficients a_{11}^*, a_{22}^* of the transformed form A and the quantity λ in the same way as before, then the differential expression:

$$\{[e_{11}\, e_a(p) + e_{12}\, e_a(q)]\, dp + [e_{12}\, e_a(p) + e_{22}\, e_a(q)]\, dq\}(w - w')^{-2} = \Omega$$

would obviously have to go to the total differential:

$$-\tfrac{1}{4}d\,[\log((w - w')\,\lambda)].$$

Conversely, if the differential expression Ω is the total differential of a function of the variables p, q then, as is easy to see, the introduction of the new variables $u = f(u)$, $v = \phi(v)$ will suffice to transform the form A into the form:

$$\lambda\,(du'^2 + dv'^2).$$

The condition for the possibility that a form C is suitable for putting the form:

$$A = a_{11}\, dp^2 + 2a_{12}\, dp\, dq + a_{22}\, dq^2$$

into the form:

$$A = \lambda\, (du^2 + dv^2)$$

by introducing the variables u, v is then expressed by the partial differential equation:

$$\frac{\partial}{\partial p}\left[\frac{e_{12}\, e_a(p) + e_{22}\, e_a(q)}{H^2 - 4K}\right] = \frac{\partial}{\partial q}\left[\frac{e_{11}\, e_a(p) + e_{12}\, e_a(q)}{H^2 - 4K}\right].$$

That equation, which admits a multitude of conversions and simplifications, is the source of the article that was presented to the Königlich Preussischen Akademie der Wissenschaften zu Berlin on 8 November 1883: "Über die Differentialgleichung der Oberflächen, welche durch ihre Krümmungslinien in unendlich kleine Quadrate getheilt werden können." Insofar as the three coefficients c_{11}, c_{12}, c_{22} of the form C only need to be coupled with each other by *one* equation in order for that form to effect the desired conversion of the given form A, it will still remain that those coefficients can be subjected to two more arbitrary conditions.

That suggests the remark that when one has succeeded in determining the three coefficients c_{ik} in accordance with that one equation of condition that was posed, the variables u, v themselves will be arrived at by quadratures. In fact, the equation:

$$\frac{e_{11}\, e_a(p) + e_{12}\, e_a(q)}{H^2 - 4K}\, dp + \frac{e_{12}\, e_a(p) + e_{22}\, e_a(q)}{H^2 - 4K}\, dq = -\tfrac{1}{4}\, d \log\, [(w - w')\, \lambda],$$

which is equivalent to the equation of condition, allows one to determine the quantity $\lambda(w - w')$ by a quadrature.

Knowing that quantity will suffice for one to represent a given quadratic form by means of the equation:

$$\frac{1}{(w - w')\lambda}\, [e_{11}\, dp^2 + 2\, e_{12}\, dp\, dq + e_{22}\, dq^2] = du\, dv\,.$$

u and v can be obtained from that equation by means of quadratures using the process that was suggested in the introduction. Carrying out the process in question will show that for those required quadratures, it is not the value of the quantity $\lambda\, (w - w')$ itself that is given directly by the condition equation, but only the differential quotients of its natural logarithm, in such a way that the quadrature that is required first can be skipped, and that will yield some simple and elegant expressions for the variables u, v after one makes some conversions that are close at hand. That last remark is also obviously true when one ascertains the lines of curvature in the corresponding geometric problem.

We do not believe that this method for converting a given quadratic form:

$$a_{11}\, dp^2 + 2a_{12}\, dp\, dq + a_{22}\, dq^2$$

into the form:

$$\lambda\,(du^2 + dv^2)$$

by consulting a second one that contains three arbitrary coefficients is unworthy of attention.

III.

We would now like to choose the form $c_{11}\,dp^2 + 2c_{12}\,dp\,dq + c_{22}\,dq^2$ that is to be transformed along with $a_{11}\,dp^2 + 2a_{12}\,dp\,dq + a_{22}\,dq^2$ in such a way that its *three* coefficients will be subject to the *two* condition equations:

$$c_a(p) = 0, \qquad c_a(q) = 0. \tag{10}$$

When those two conditions are fulfilled for any two original variables p, q, as a result of equations (6), the corresponding conditions:

$$c_a(p') = 0, \qquad c_a(q') = 0 \tag{10*}$$

will also be fulfilled for any new arbitrary variables p', q' that one introduces, and conversely.

If one introduces the functions u, v as the new variables, which possess the property that the form E that accompanies the forms A and C will go to the product of their differentials multiplied by a function of u, v, then equations (4′) will next imply the following ones:

$$c_a(u) = \frac{1}{a^*}\left[\frac{\partial w' a_{22}^*}{\partial u} + w\,a_{11}^*\begin{Bmatrix}22\\1\end{Bmatrix}_{a^*} - w'\,a_{22}^*\begin{Bmatrix}12\\2\end{Bmatrix}_{a^*}\right],$$

$$c_a(v) = \frac{1}{a^*}\left[\frac{\partial w' a_{11}^*}{\partial v} - w\,a_{11}^*\begin{Bmatrix}12\\1\end{Bmatrix}_{a^*} + w'\,a_{22}^*\begin{Bmatrix}11\\2\end{Bmatrix}_{a^*}\right],$$

and as a result of the equations:

$$\begin{Bmatrix}22\\1\end{Bmatrix}_{a^*} = -\frac{1}{2}\frac{1}{a_{11}^*}\frac{\partial a_{22}^*}{\partial u}, \qquad \begin{Bmatrix}12\\2\end{Bmatrix}_{a^*} = -\frac{1}{2}\frac{1}{a_{22}^*}\frac{\partial a_{22}^*}{\partial u},$$

$$\begin{Bmatrix}12\\1\end{Bmatrix}_{a^*} = \frac{1}{2}\frac{1}{a_{11}^*}\frac{\partial a_{11}^*}{\partial v}, \qquad \begin{Bmatrix}11\\2\end{Bmatrix}_{a^*} = -\frac{1}{2}\frac{1}{a_{22}^*}\frac{\partial a_{11}^*}{\partial v},$$

one will obtain:

$$c_a(u) = \frac{1}{a^*}\left[\frac{\partial w'}{\partial u} - \tfrac{1}{2}(w - w')\frac{\partial \log a_{22}^*}{\partial u}\right],$$

(11)
$$c_a(v) = \frac{1}{a^*}\left[\frac{\partial w}{\partial v} - \tfrac{1}{2}(w' - w)\frac{\partial \log a_{11}^*}{\partial v}\right].$$

Under the assumption that equations (10) [equations (10*), resp.] are true, the differential equations will be true:

(12)
$$\tfrac{1}{2}\frac{\partial \log a_{22}^*}{\partial u} = \frac{\dfrac{\partial w'}{\partial u}}{w - w'},$$
$$\tfrac{1}{2}\frac{\partial \log a_{11}^*}{\partial u} = \frac{\dfrac{\partial w'}{\partial v}}{w - w'}.$$

Under the assumption that is expressed by equations (10) in regard to the coefficients of the form C, the form B that accompanies the forms A and C will possess a remarkable property, whose derivation we shall move on to. It is that:

$$B = b_{11}\,dp^2 + 2b_{12}\,dp\,dq + b_{22}\,dq^2 = w^2\,a_{11}^*\,du^2 + w'^2\,a_{22}^*\,dv^2.$$

When one constructs the invariant k_b (or curvature) that GAUSS gave from that quadratic form, and indeed in its representation in terms of the variables u, v, with the representation that was given in Section I, that will yield the same thing that one determines from the equation:

$$k_b = \frac{-1}{2ww'\sqrt{a_{11}^*\,a_{22}^*}}\left[\frac{\partial}{\partial u}\left(\frac{1}{ww'\sqrt{a_{11}^*\,a_{22}^*}}\right)\frac{\partial\, w'^2 a_{22}^*}{\partial u} + \frac{\partial}{\partial v}\left(\frac{1}{ww'\sqrt{a_{11}^*\,a_{22}^*}}\right)\frac{\partial\, w^2 a_{11}^*}{\partial v}\right].$$

After introducing the values of the differential quotients $\dfrac{\partial w'}{\partial u}$, $\dfrac{\partial w}{\partial v}$ in equations (12), one will note the relations:

$$\frac{\partial\, w'^2 a_{22}^*}{\partial u} = ww'\frac{\partial\, a_{22}^*}{\partial u}, \qquad \frac{\partial\, w^2 a_{11}^*}{\partial v} = ww'\frac{\partial\, a_{11}^*}{\partial v},$$

and with their help, one will get the following representation for k_b :

$$k_b = -\frac{1}{2}\frac{1}{ww'\sqrt{a_{11}^*\,a_{22}^*}}\left[\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{a_{11}^*\,a_{22}^*}}\frac{\partial\, a_{22}^*}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{1}{\sqrt{a_{11}^*\,a_{22}^*}}\frac{\partial\, a_{11}^*}{\partial v}\right)\right].$$

One recognizes the curvature k_a of the given form $a_{11}\, dp^2 + 2a_{12}\, dp\, dq + a_{22}\, dq^2$ in the quantity that is found in the factor $w\, w'$ in the reciprocal product on the right-hand side of the foregoing equation.

Under the assumption that equations (10) are true, the very important equation:

(13) $$k_a = \frac{1}{w\,w'} k_a$$

will also be valid. The form B is to the form C what the form A is to the form C, which emerges from equations (1) in the same algebraic relations. If one takes the form B itself to be the starting form then that, in conjunction with the form C, will imply the form A in the same way that the form B is obtained from A and C when one simply exchanges the quantities w, w' with their reciprocals.

The quantities $c_b\,(p)$, $c_b\,(q)$ are also closely related to the quantities $c_a\,(p)$, $c_a\,(q)$, which would emerge from the simple relations between the quantities $c_b\,(u)$, $c_b\,(v)$ and $c_a(u)$, $c_a\,(v)$.

Namely, if one constructs the latter quantities from the equations that serve to define them then one will find, after a brief calculation, that:

$$c_b\,(u) = -\frac{1}{w^2}\frac{1}{w\,w'}\,c_a\,(u),$$

$$c_b\,(v) = -\frac{1}{w'^2}\frac{1}{w\,w'}\,c_a\,(v)\,.$$

Those equations also imply that under the assumption that was made about equations (10^*) being true, the following equations:

$$c_b\,(u) = 0, \qquad c_b\,(v) = 0$$

will also be fulfilled, and as a result of them, the equations:

(14) $$c_b\,(p) = 0, \qquad c_b\,(q) = 0$$

will also be true.

We would further like to determine the three coefficients c_{11}, c_{12}, c_{22} of the form C, which are already subject to the conditions (10), upon further assuming that the form B, which is obviously always positive, represents the square of the line element of a surface of *constant* curvature ε, where we understand ε to mean one of the *two* square roots of unity.

As a result of equation (13), that assumption can be expressed by the equation:

$$\varepsilon = \frac{1}{w\,w'} k_a\,,$$

which is an equation that will take the form:

$$\frac{c_{11}\,c_{22}-c_{12}^2}{a_{11}\,a_{22}-a_{12}^2} = \varepsilon\,k_a$$

after one introduces the value of the invariant ww' that is expressed in terms of the coefficients of the forms A, C.

The three conditions that the coefficients c_{11}, c_{12}, c_{22} of the form C will be subject to as a result are now given by the following equations:

$$\text{(I)}\qquad \begin{aligned} &\frac{\partial c_{22}}{\partial p}-\frac{\partial c_{12}}{\partial q}+c_{11}\begin{Bmatrix}22\\1\end{Bmatrix}_a+c_{12}\left(\begin{Bmatrix}22\\2\end{Bmatrix}_a-\begin{Bmatrix}21\\1\end{Bmatrix}_a\right)-c_{22}\begin{Bmatrix}12\\2\end{Bmatrix}_a = 0,\\ &\frac{\partial c_{11}}{\partial q}-\frac{\partial c_{12}}{\partial p}-c_{11}\begin{Bmatrix}12\\1\end{Bmatrix}_a+c_{12}\left(\begin{Bmatrix}11\\1\end{Bmatrix}_a-\begin{Bmatrix}12\\2\end{Bmatrix}_a\right)+c_{22}\begin{Bmatrix}11\\2\end{Bmatrix}_a = 0,\\ &\frac{c_{11}\,c_{22}-c_{12}^2}{a_{11}\,a_{22}-a_{12}^2} = \varepsilon\,k_a\,. \end{aligned}$$

Under the assumption that the equations are fulfilled for any system of quantities c_{11}, c_{12}, c_{22}, from the composition that exists in them, the form:

$$b_{11}\,dp^2 + 2b_{12}\,dp\,dq + b_{22}\,dq^2 = w^2\,a_{11}^*\,du^2 + w'^2\,a_{22}^*\,dv^2$$

will represent the square of the line element of a surface of constant curvature ε.

In volume 94 of CRELLE-BORCHARDT's Journal, pp. 201, as well as in volume 95, pp. 326-329, we have carried out the proof of that for the case in which the form:

$$b_{11}\,dp^2 + 2b_{12}\,dp\,dq + b_{22}\,dq^2$$

represents the square of the line element of a surface of constant curvature ε by integrating two second-order linear ordinary differential equations in the real-valued functions X, Y, Z, which are always real and which fulfill the equations:

$$\begin{aligned} dX^2 + dY^2 + \varepsilon\,dZ^2 &= b_{11}\,dp^2 + 2b_{12}\,dp\,dq + b_{22}\,dq^2,\\ X^2 + Y^2 + \varepsilon\,Z^2 &= \varepsilon. \end{aligned}$$

It would seem superfluous to once more exhibit the differential equations in question in the notations that were chosen here.

The functions X, Y, Z themselves, which were denoted by x, y, z in the first-mentioned article, satisfy the equation that were denoted by (XII) there, and which can be represented by the following ones in the notations that were chosen here:

$$
\text{(II)}\qquad
\begin{aligned}
&\frac{\partial^2\xi}{\partial p^2}-\begin{Bmatrix}11\\1\end{Bmatrix}_b\frac{\partial\xi}{\partial p}-\begin{Bmatrix}11\\2\end{Bmatrix}_b\frac{\partial\xi}{\partial q}+\varepsilon\, b_{11}\,\xi = 0,\\
&\frac{\partial^2\xi}{\partial p\,\partial q}-\begin{Bmatrix}12\\1\end{Bmatrix}_b\frac{\partial\xi}{\partial p}-\begin{Bmatrix}12\\2\end{Bmatrix}_b\frac{\partial\xi}{\partial q}+\varepsilon\, b_{12}\,\xi = 0,\\
&\frac{\partial^2\xi}{\partial q^2}-\begin{Bmatrix}22\\1\end{Bmatrix}_b\frac{\partial\xi}{\partial p}-\begin{Bmatrix}22\\2\end{Bmatrix}_b\frac{\partial\xi}{\partial q}+\varepsilon\, b_{22}\,\xi = 0.
\end{aligned}
$$

By introducing the functions *u*, *v* as the new variables, as a result of their invariance that was pointed out in *loc. cit.*, those equations will go to the following ones:

$$
\text{(II}^*\text{)}\qquad
\begin{aligned}
&\frac{\partial^2\xi}{\partial u^2}-\begin{Bmatrix}11\\1\end{Bmatrix}_{b^*}\frac{\partial\xi}{\partial u}-\begin{Bmatrix}11\\2\end{Bmatrix}_{b^*}\frac{\partial\xi}{\partial v}+\varepsilon\, w^2\, a_{11}^*\,\xi = 0,\\
&\frac{\partial^2\xi}{\partial u\,\partial v}-\begin{Bmatrix}12\\1\end{Bmatrix}_{b^*}\frac{\partial\xi}{\partial u}-\begin{Bmatrix}12\\2\end{Bmatrix}_{b^*}\frac{\partial\xi}{\partial v} = 0,\\
&\frac{\partial^2\xi}{\partial v^2}-\begin{Bmatrix}22\\1\end{Bmatrix}_{b^*}\frac{\partial\xi}{\partial u}-\begin{Bmatrix}22\\2\end{Bmatrix}_{b^*}\frac{\partial\xi}{\partial v}+\varepsilon\, w'^2\, a_{22}^*\,\xi = 0.
\end{aligned}
$$

As a result of the relations:

$$
\begin{Bmatrix}12\\1\end{Bmatrix}_{b^*} = \frac{\dfrac{\partial(1/w)}{\partial v}}{\dfrac{1}{w'}-\dfrac{1}{w}}, \qquad
\begin{Bmatrix}12\\2\end{Bmatrix}_{b^*} = \frac{\dfrac{\partial(1/w')}{\partial u}}{\dfrac{1}{w}-\dfrac{1}{w'}},
$$

which are obtained effortlessly from the equations that define the quantities $\begin{Bmatrix}i\,k\\h\end{Bmatrix}_{b^*}$ with the help of equations (12), the middle of those equations, the consideration of which is all that is required next, can be put into the form:

$$
\left(\frac{1}{w}-\frac{1}{w'}\right)\frac{\partial^2\xi}{\partial u\,\partial v}+\frac{\partial(1/w)}{\partial v}\frac{\partial\xi}{\partial u}-\frac{\partial(1/w')}{\partial u}\frac{\partial\xi}{\partial v} = 0,
$$

from which, the equation:

$$
\frac{\partial}{\partial v}\left(\frac{1}{w}\frac{\partial\xi}{\partial u}\right) = \frac{\partial}{\partial u}\left(\frac{1}{w'}\frac{\partial\xi}{\partial v}\right)
$$

will emerge.

The property of a function ξ that satisfies that equation can be expressed by saying that the differential expression:

$$\frac{1}{w}\frac{\partial\xi}{\partial u}du+\frac{1}{w'}\frac{\partial\xi}{\partial v}dv$$

is equivalent to a total differential of a function of the variables u, v is therefore a property of each of the three functions X, Y, Z.

As a result of that remark, one will have the three equations:

$$\text{(III)}\qquad \begin{aligned} dx &= \frac{1}{w}\frac{\partial X}{\partial u}du+\frac{1}{w'}\frac{\partial X}{\partial v}dv, \\ dy &= \frac{1}{w}\frac{\partial Y}{\partial u}du+\frac{1}{w'}\frac{\partial Y}{\partial v}dv, \\ dz &= \frac{1}{w}\frac{\partial Z}{\partial u}du+\frac{1}{w'}\frac{\partial Z}{\partial v}dv, \end{aligned}$$

in which x, y, z denote three new functions of the variable u, v.

If one constructs the sum:

$$dx^2 + dy^2 + \varepsilon\, dz^2$$

from the foregoing equations then when one considers the equation that is fulfilled by the functions X, Y, Z :

$$dX^2 + dY^2 + \varepsilon\, dZ^2 = w^2\, a_{11}^*\, du^2 + w'^2\, a_{22}^*\, dv^2,$$

one will find the following equation:

$$dx^2 + dy^2 + \varepsilon\, dz^2 = a_{11}^*\, du^2 + a_{22}^*\, dv^2 = A.$$

The functions x, y, z, which are given by equations (III) by quadratures, are therefore ones that correspond to the transformation problem that was spoken of in the introduction and is expressed by the equation:

$$dx^2 + dy^2 + \varepsilon\, dz^2 = a_{11}\, dp^2 + 2a_{12}\, dp\, dq + a_{22}\, dq^2 = A.$$

With the help of formula (3) of Section I, the differentials dx, dy, dz of those functions will be transformed directly into expressions that are represented in terms of the original variables p, q, and their differentials dp, dq, namely, the following ones:

$$dx = \left(B_{11}\frac{\partial X}{\partial p}+B_{12}\frac{\partial X}{\partial q}\right)dp+\left(B_{21}\frac{\partial X}{\partial p}+B_{22}\frac{\partial X}{\partial q}\right)dq,$$

(IV)
$$dy = \left(B_{11}\frac{\partial Y}{\partial p} + B_{12}\frac{\partial Y}{\partial q} \right) dp + \left(B_{21}\frac{\partial Y}{\partial p} + B_{22}\frac{\partial Y}{\partial q} \right) dq ,$$

$$dz = \left(B_{11}\frac{\partial Z}{\partial p} + B_{12}\frac{\partial Z}{\partial q} \right) dp + \left(B_{21}\frac{\partial Z}{\partial p} + B_{22}\frac{\partial Z}{\partial q} \right) dq ,$$

in which one understands the quantities B_{ik} to have the following values:

$$B_{11} = \frac{1}{k_a}\frac{a_{11}\,c_{22} - a_{12}\,c_{12}}{a_{11}\,a_{22} - a_{12}^2}, \qquad B_{12} = \frac{1}{k_a}\frac{a_{12}\,c_{11} - a_{11}\,c_{12}}{a_{11}\,a_{22} - a_{12}^2},$$

$$B_{21} = \frac{1}{k_a}\frac{a_{12}\,c_{22} - a_{22}\,c_{12}}{a_{11}\,a_{22} - a_{12}^2}, \qquad B_{22} = \frac{1}{k_a}\frac{a_{22}\,c_{11} - a_{12}\,c_{12}}{a_{11}\,a_{22} - a_{12}^2},$$

from the formulas that were cited in Section I. As long as the quantities c_{11}, c_{12}, c_{22} in these equations refer to three functions that satisfy equations (I), the former equations will include a solution to the problem: Determine three real-valued functions x, y, z of the variables p, q in such a way that the equation:

$$dx^2 + dy^2 + \varepsilon\, dz^2 = a_{11}\, dp^2 + 2a_{12}\, dp\, dq + a_{22}\, dq^2$$

will become an identity.

In the case where ε denotes positive unity, those functions will give the so-called general solution to the problem of deforming a curved surface in the way that BOUR himself presented it, under the assumption that one has succeeded in integrating the differential equation for the geodetic lines for any given surface. When one makes that assumption, the equation:

$$X^2 + Y^2 + Z^2 = 1$$

will exist between the functions X, Y, Z, and after equations (IV) have been multiplied by X, Y, Z, resp., and one has added together the products that one obtains, that will give the further equation:

$$X\, dx + Y\, dy + Z\, dz = 0,$$

from which it will follow that those functions X, Y, Z represent the cosines of the angles that the normal that is raised at the point (p, q) of the surface that is represented by equations (IV) makes with the x, y, z coordinate axes, resp.

If one further multiplies equations (IV), or equations (III), which are identical to them, by dX, dY, dZ, resp., and adds together the resulting products then one will be led to the equation:

$$dX\, dx + dY\, dy + dZ\, dz = w\, a_{11}^*\, du^2 + w'\, a_{22}^*\, dv^2 ;$$

i.e., to the following one:

$$dX\,dx + dY\,dy + dZ\,dz = c_{11}\,dp^2 + 2c_{12}\,dp\,dq + c_{22}\,dq^2,$$

which will yield the geometric meaning of the coefficients c_{11}, c_{12}, c_{22} that was mentioned already in the foregoing.

Finally, the equation:

$$dX^2 + dY^2 + dZ^2 = b_{11}\,dp^2 + 2b_{12}\,dp\,dq + b_{22}\,dq^2$$

shows that the form B gives the square of the line element for the image of the map of the surface that is given by equations (IV) to the GAUSSian sphere.

From the known formulas for the theory of the curvature of curved surfaces, the simultaneous invariants H, K of the forms A and C, resp., prove to be identical with the sum of the principal curvatures at the point (p, q) of the surface considered, in the first case, and their product, in the second case. If one denotes those two principal curvatures, which are now the values of the invariants w and w', by r and r', resp., then the linear relationship:

$$r\,r'(dx^2 + dy^2 + dz^2) - (r + r')(dX\,dx + dY\,dy + dZ\,dz) + dX^2 + dY^2 + dZ^2 = 0$$

will exist between the three forms A, B, C, which is a relationship that allows one to recognize immediately the known theorem of the similarity of infinitely-small parts of those surfaces for which $r + r' = 0$ with the corresponding parts under their map to the GAUSSian sphere.

One effortlessly observes that equations (IV) are nothing but the ones that I introduced before into the theory of curved surfaces in the year 1861 in volume 59 of CRELLE-BORCHARDT's Journal ("Über eine besondere Klasse von aufeineander abwickelbaren Oberflächen").

As far as the determination of the functions X, Y, Z by integrating two second-order linear ordinary differential equations is concerned, that determination can lead to other mutually-distinct systems of those functions depending upon what constants one assumes for the integration. However, as was shown already in the aforementioned article (CRELLE-BORCHARDT's Journal, Bd. 94), the relations:

$$\begin{aligned} X' &= a\;X + b\;\;Y + c\;\;Z, \\ Y' &= a_1\,X + b_1\,Y + c_1\,Z, \\ Z' &= a_2\,X + b_2\,Y + c_2\,Z \end{aligned}$$

exist between two systems (X, Y, Z), (X', Y', Z'), in which the coefficients a_i, b_i, c_i are *constant* coefficients, and each of the systems represents a system of linearly-independent integrals of equations (II). Moreover, due to the equations:

$$\begin{aligned} X^2 + Y^2 + Z^2 &= 1, \\ X'^2 + Y'^2 + Z'^2 &= 1, \end{aligned}$$

the coefficients represent an orthogonal substitution.

When one understands the ε in equations (IV) to mean positive unity and c_{11}, c_{12}, c_{22} to mean a triple of quantities that satisfy equations (I), after one determines a system of quantities X, Y, Z, those equations will always lead to *only one* well-defined surface, although it can be obtained by rotating the axis system into different positions or reflecting it in a plane.

With the assumption $\varepsilon = -1$ in the equations that were spoken of, that will yield a class of real-valued functions x, y, z that fulfill the equation:

$$dx^2 + dy^2 - dz^2 = a_{11}\, dp^2 + 2a_{12}\, dp\, dq + a_{22}\, dq^2.$$

The functions x, y of that class satisfy the second-order partial differential that was denoted by (d) in the introduction, which does not relate to the problem of the developability of curved surfaces.

IV.

The developments that were presented in the previous section reduced the determination of three functions x, y, z that satisfied the equation:

$$dx^2 + dy^2 + \varepsilon\, dz^2 = a_{11}\, dp^2 + 2a_{12}\, dp\, dq + a_{22}\, dq^2$$

not to the integration of three simultaneous differential equations:

$$\left(\frac{\partial x}{\partial p}\right)^2 + \left(\frac{\partial y}{\partial p}\right)^2 + \left(\frac{\partial z}{\partial p}\right)^2 = a_{11},$$

$$\frac{\partial x}{\partial p}\frac{\partial x}{\partial q} + \frac{\partial y}{\partial p}\frac{\partial y}{\partial q} + \frac{\partial z}{\partial p}\frac{\partial z}{\partial q} = a_{12},$$

$$\left(\frac{\partial x}{\partial q}\right)^2 + \left(\frac{\partial y}{\partial q}\right)^2 + \left(\frac{\partial z}{\partial q}\right)^2 = a_{22},$$

but to the integration of the system of equations (I). The latter system, which likewise includes three simultaneous equation, further requires only the fulfillment of two partial differential equations in three functions c_{11}, c_{12}, c_{22}, which are themselves coupled to each other by a second-degree algebraic equation. That will allow one to eliminate one of the functions $c_{i,k}$, so that two partial differential equations in the remaining two functions will exist that are *linear* in regard to the differential quotients of the functions that are included in them. The question of ascertaining the three functions x, y, z will then be reduced to the integration of those two equations.

Although one can see that this reduction represents a simplification of the problem in question, it must still not conceal the fact that exhibiting a *single* differential equation whose integration will resolve the problem under the assumption that $\varepsilon = 1$ is not just *necessary*, but also sufficient, and that this was *not* achieved by that reduction, which has not be observed anywhere, up to now. One will indeed be led to a single partial differential equation for the last of the two remaining functions $c_{i,k}$ by further eliminating one of them from the two aforementioned partial differential equations. That partial differential equation was a necessary consequence of equations (I), although it possesses integrals that go beyond the integrals of equations (I). The subset of those integrals that are foreign to the solution of the problem must then be separated from the ones that are not, and that separation is tantamount to a new problem whose difficulty cannot be overlooked, and all the more so because that problem will lack a precise formulation before that integration has been accomplished.

Now it actually proves to be easy to express each of the three functions $c_{i,k}$ that appear in equations (I) in terms of a single function ϕ in such a way that those equations happen to be sufficient when the function ϕ satisfies a second-order partial differential equation. However, any real-valued integral ϕ of the latter will always correspond to real-valued $c_{i,k}$ in the case where $\varepsilon = -1$, whereas in the case of $\varepsilon = 1$, those integrals can also correspond to purely-imaginary values of the $c_{i,k}$, which are excluded due to the nature of the problem that was posed.

From a detailed consideration of the formulas for the products AD, AD', AD'' that GAUSS gave and were reproduced in the introduction [and are easily confirmed by carrying out a simple calculation that will not be communicated here, since it requires the introduction of some further formal devices in regard to the differential quotients of the quantities $\begin{Bmatrix} i\,k \\ h \end{Bmatrix}_a$ and the quotients $\frac{a_{11}}{a}$, $\frac{a_{12}}{a}$, $\frac{a_{22}}{a}$], one will see that quantities c_{11}, c_{12}, c_{22} will be given by the equations:

$$c_{11} = \frac{\dfrac{\partial^2\phi}{\partial p^2} - \begin{Bmatrix} 11 \\ 1 \end{Bmatrix}_a \dfrac{\partial\phi}{\partial p} - \begin{Bmatrix} 11 \\ 2 \end{Bmatrix}_a \dfrac{\partial\phi}{\partial q}}{\sqrt{\alpha - \varepsilon\dfrac{1}{a}\left[a_{11}\left(\dfrac{\partial\phi}{\partial q}\right)^2 - 2a_{12}\dfrac{\partial\phi}{\partial p}\dfrac{\partial\phi}{\partial q} + a_{22}\left(\dfrac{\partial\phi}{\partial p}\right)^2\right]}},$$

$$c_{12} = \frac{\dfrac{\partial^2\phi}{\partial p\,\partial q} - \begin{Bmatrix} 12 \\ 1 \end{Bmatrix}_a \dfrac{\partial\phi}{\partial p} - \begin{Bmatrix} 12 \\ 2 \end{Bmatrix}_a \dfrac{\partial\phi}{\partial q}}{\sqrt{\alpha - \varepsilon\dfrac{1}{a}\left[a_{11}\left(\dfrac{\partial\phi}{\partial q}\right)^2 - 2a_{12}\dfrac{\partial\phi}{\partial p}\dfrac{\partial\phi}{\partial q} + a_{22}\left(\dfrac{\partial\phi}{\partial p}\right)^2\right]}},$$

$$c_{22} = \frac{\dfrac{\partial^2\phi}{\partial q^2} - \begin{Bmatrix} 22 \\ 1 \end{Bmatrix}_a \dfrac{\partial\phi}{\partial p} - \begin{Bmatrix} 22 \\ 2 \end{Bmatrix}_a \dfrac{\partial\phi}{\partial q}}{\sqrt{\alpha - \varepsilon\dfrac{1}{a}\left[a_{11}\left(\dfrac{\partial\phi}{\partial q}\right)^2 - 2a_{12}\dfrac{\partial\phi}{\partial p}\dfrac{\partial\phi}{\partial q} + a_{22}\left(\dfrac{\partial\phi}{\partial p}\right)^2\right]}}$$

that satisfy equations (I), as long as one understands the function ϕ to mean an integral of the partial differential equation:

$$(\phi) \quad \varepsilon\, k_a \{\alpha - \varepsilon\, \Delta_1(\phi)\} a = \left[\phi_{22} - \begin{Bmatrix} 11 \\ 1 \end{Bmatrix}_a \phi_1 - \begin{Bmatrix} 11 \\ 2 \end{Bmatrix}_a \phi_2\right]\left[\phi_{22} - \begin{Bmatrix} 22 \\ 1 \end{Bmatrix}_a \phi_1 - \begin{Bmatrix} 22 \\ 2 \end{Bmatrix}_a \phi_2\right] - \left[\phi_{12} - \begin{Bmatrix} 12 \\ 1 \end{Bmatrix}_a \phi_1 - \begin{Bmatrix} 12 \\ 2 \end{Bmatrix}_a \phi_2\right]^2,$$

in which the partial derivatives of the function ϕ are suggested by indices.

In that differential equation, $\Delta_1(\phi)$ denotes the differential parameter of the function ϕ:

$$\frac{a_{11}\left(\dfrac{\partial\phi}{\partial q}\right)^2 - 2a_{12}\dfrac{\partial\phi}{\partial p}\dfrac{\partial\phi}{\partial q} + a_{22}\left(\dfrac{\partial\phi}{\partial p}\right)^2}{a_{11}\,a_{22} - a_{12}^2},$$

for the sake of brevity, and α refers to an arbitrary constant that might have real positive values, including zero.

In the case where one assumes that $\varepsilon = -1$, the functions c_{11}, c_{12}, c_{22} that are given by the foregoing equations will always be real-valued when ϕ is chosen to be a real-valued solution of the partial differential equation (ϕ). If the arbitrary constant α is not assumed to be equal to zero then that partial differential equation will coincide with equation (d^*) in the introduction, and determining x, y, z by integrating it will require only quadratures, but no longer the integration of equations (IV).

When one understands the ε in equation (ϕ) to mean negative unity and the α to be mean a constant that is positive or zero, one can then regard that equation as the partial differential equation whose integration would be necessary and sufficient for resolution of the transformation problem:

$$dx^2 + dy^2 - dz^2 = a_{11}\, dp^2 + 2a_{12}\, dp\, dq + a_{22}\, dq^2;$$

i.e., as the differential equation whose set of real-valued integrals would not be greater than what is required to resolve the problem.

Things are not the same when one understands the ε in equations (I) to mean positive unity. If one assumes that the constant α has the value zero then the three quantities c_{11}, c_{12}, c_{22} will be pure imaginary ([1]) for any real-valued integral of equation (ϕ), and will suffice to solve the problem of the mutual developability of surfaces, whereas when one

([1]) The equation (9) that OSSIAN BONNET presented in his "Mémoire sur la théorie des surfaces applicables sur une surface donnée" [Journal de l'École impériale polytechnique, **25** (1867), pp. 3] for the solution of the problem of the developability of a surface onto a given one will coincide with equation (ϕ) above when one chooses $\alpha = 0$ in it and introduces the line element $\sqrt{4\psi^2\, dx\, dy}$ that BONNET chose.

To our way of looking at things, the equation that the distinguished geometer gave will not satisfy the conditions of the problem in question; however, its treatment would be sufficient for the accompanying problem in pseudo-geometry.

assumes that α is equal to a positive constant, the quantities c_{11}, c_{12}, c_{22} will prove to be real-valued only when:

$$\frac{a_{11}\left(\frac{\partial \phi}{\partial q}\right)^2 - 2a_{12}\frac{\partial \phi}{\partial p}\frac{\partial \phi}{\partial q} + a_{22}\left(\frac{\partial \phi}{\partial p}\right)^2}{a_{11}\,a_{22} - a_{12}^2} < \alpha;$$

i.e., only for those real-valued functions ϕ that satisfy the differential equation (ϕ) itself, as well as the foregoing condition, which comes from the condition that was discussed in the introduction, as is easy to see. The differential equation (ϕ) will have more real-valued integrals than the ones that are connected with the problem of developability, and a comment that was already expressed before will still apply in regard to the separation of those solutions.

One can succeed in fulfilling the equation:

$$dx^2 + dy^2 + dz^2 = a_{11}\,dp^2 + 2a_{12}\,dp\,dq + a_{22}\,dq^2 \tag{a}$$

in *only one* case that is linked with a second-order partial differential equation whose set of all real-valued integrals does not exceed the set of all integrals that fulfill the foregoing equation. That is the case in which the quadratic form:

$$a_{11}\,dp^2 + 2a_{12}\,dp\,dq + a_{22}\,dq^2$$

represents the square of the line element of a surface of constant curvature k. As is easily verified ([1]), the equations:

$$c_{11} = \frac{\partial^2 \phi}{\partial p^2} - \begin{Bmatrix} 11 \\ 1 \end{Bmatrix}_a \frac{\partial \phi}{\partial p} - \begin{Bmatrix} 11 \\ 2 \end{Bmatrix}_a \frac{\partial \phi}{\partial q} + k\,a_{11}\phi,$$

$$c_{12} = \frac{\partial^2 \phi}{\partial p\,\partial q} - \begin{Bmatrix} 12 \\ 1 \end{Bmatrix}_a \frac{\partial \phi}{\partial p} - \begin{Bmatrix} 12 \\ 2 \end{Bmatrix}_a \frac{\partial \phi}{\partial q} + k\,a_{12}\,\phi,$$

$$c_{22} = \frac{\partial^2 \phi}{\partial q^2} - \begin{Bmatrix} 22 \\ 1 \end{Bmatrix}_a \frac{\partial \phi}{\partial p} - \begin{Bmatrix} 22 \\ 2 \end{Bmatrix}_a \frac{\partial \phi}{\partial q} + k\,a_{22}\,\phi,$$

in which ϕ denotes an arbitrary real-valued function of the variables p, q, will determine three functions c_{11}, c_{12}, c_{22} of the variables p, q that satisfy the first two of equations (I) identically. In order for them to also satisfy the third of those equations, the function ϕ must be determined from the following second-order partial differential equation:

([1]) That is done most easily by using the invariance properties of equations (I), which can also take into account the foregoing remarks.

$$\frac{\left[\phi_{11}-\begin{Bmatrix}11\\1\end{Bmatrix}_a\phi_1-\begin{Bmatrix}11\\1\end{Bmatrix}_a\phi_2+k\,a_{11}\,\phi\right]\left[\phi_{22}-\begin{Bmatrix}22\\1\end{Bmatrix}_a\phi_1-\begin{Bmatrix}22\\2\end{Bmatrix}_a\phi_2+k\,a_{22}\,\phi\right]-\left[\phi_{12}-\begin{Bmatrix}12\\1\end{Bmatrix}_a\phi_1-\begin{Bmatrix}12\\2\end{Bmatrix}_a\phi_2+k\,a_{12}\,\phi\right]^2}{a_{11}\,a_{22}-a_{12}^2}=k\,.$$

Any real-valued integral of that differential equation will lead to three functions c_{11} , c_{12} , c_{22} that satisfy equations (I), and then to three functions *x*, *y*, *z* that satisfy equation (a), just as conversely three functions *x*, *y*, *z* that satisfy the latter equation will determine a real-valued integral of the foregoing differential equation.

The proof of the converse that was just spoken of can be obtained from the following consideration:

Let *x*, *y*, *z* be three given functions for which the sum of the squares of their differentials yields a quadratic form $a_{11}\,dp^2 + 2a_{12}\,dp\,dq + a_{22}\,dq^2$ whose curvature possesses the constant value ε, where ε is understood to mean a square root of unity, with no loss of generality. When one regards those functions as the rectangular coordinates of a point in space, a well-defined curved will then be given by those functions whose curvature at every point will be equal to the number ε. If one calculates the cosines *X*, *Y*, *Z* of the angles that the normal that is raised at the point (*p*, *q*) on that surface makes with the axes of the chosen coordinates then those cosines will be given functions of the variables *p*, *q*. One will obtain the following three quadratic forms with the help of the six functions *x*, *y*, *z*, *X*, *Y*, *Z* :

$$dx^2 + dy^2 + dz^2 = a_{11}\,dp^2 + 2a_{12}\,dp\,dq + a_{22}\,dq^2 = A,$$

$$dX\,dx + dY\,dy + dZ\,dz = c_{11}\,dp^2 + 2c_{12}\,dp\,dq + c_{22}\,dq^2 = C,$$

$$dX^2 + dY^2 + dZ^2 = b_{11}\,dp^2 + 2b_{12}\,dp\,dq + b_{22}\,dq^2 = B,$$

whose coefficients are *known* functions of the variables *p*, *q*. The parameters of the lines of curvature of the surface in question are the quantities that were previously denoted by *u*, *v*, and the invariants *w*, *w′* of the first two of the foregoing forms are the principal curvature of that surface at the point *p*, *q*.

Since the form $a_{11}\,dp^2 + 2a_{12}\,dp\,dq + a_{22}\,dq^2$ has constant curvature ε, it will determine three functions that satisfy the equations:

$$\mathfrak{X}^2 + \mathfrak{Y}^2 + \varepsilon\,\mathfrak{Z}^2 = \varepsilon,$$
$$d\,\mathfrak{X}^2 + d\,\mathfrak{Y}^2 + \varepsilon\,d\,\mathfrak{Z}^2 = a_{11}\,dp^2 + 2a_{12}\,dp\,dq + a_{22}\,dq^2.$$

Each of those functions then satisfies the three linear partial differential equations:

$$\frac{\partial^2\eta}{\partial p^2}-\begin{Bmatrix}11\\1\end{Bmatrix}_a\frac{\partial\eta}{\partial p}-\begin{Bmatrix}11\\2\end{Bmatrix}_a\frac{\partial\eta}{\partial q}+\varepsilon\,a_{11}\,\eta=0,$$

$$\frac{\partial^2\eta}{\partial p\,\partial q}-\begin{Bmatrix}12\\1\end{Bmatrix}_a\frac{\partial\eta}{\partial p}-\begin{Bmatrix}12\\2\end{Bmatrix}_a\frac{\partial\eta}{\partial q}+\varepsilon\,a_{12}\,\eta=0,$$

$$\frac{\partial^2\eta}{\partial q^2} - \begin{Bmatrix} 22 \\ 1 \end{Bmatrix}_a \frac{\partial \eta}{\partial p} - \begin{Bmatrix} 22 \\ 2 \end{Bmatrix}_a \frac{\partial \eta}{\partial q} + \varepsilon\, a_{22}\, \eta = 0,$$

which will coincide with the following three equations when one introduces the parameters u, v of the lines of curvature in place of the variables p, q :

$$\frac{\partial^2\eta}{\partial u^2} - \begin{Bmatrix} 11 \\ 1 \end{Bmatrix}_{a^*} \frac{\partial \eta}{\partial u} - \begin{Bmatrix} 11 \\ 2 \end{Bmatrix}_{a^*} \frac{\partial \eta}{\partial v} + \varepsilon\, a_{11}^*\, \eta = 0,$$

$$\frac{\partial^2\eta}{\partial u\,\partial v} - \begin{Bmatrix} 12 \\ 1 \end{Bmatrix}_{a^*} \frac{\partial \eta}{\partial u} - \begin{Bmatrix} 12 \\ 2 \end{Bmatrix}_{a^*} \frac{\partial \eta}{\partial v} = 0,$$

$$\frac{\partial^2\eta}{\partial u^2} - \begin{Bmatrix} 22 \\ 1 \end{Bmatrix}_{a^*} \frac{\partial \eta}{\partial u} - \begin{Bmatrix} 22 \\ 2 \end{Bmatrix}_{a^*} \frac{\partial \eta}{\partial v} + \varepsilon\, a_{22}^*\, \eta = 0,$$

the middle of which, as a result of the relations:

$$\begin{Bmatrix} 12 \\ 1 \end{Bmatrix}_{a^*} = \frac{1}{2}\frac{\partial \log a_{11}^*}{\partial v} = \frac{\dfrac{\partial r}{\partial v}}{r' - r},$$

$$\begin{Bmatrix} 12 \\ 2 \end{Bmatrix}_{a^*} = \frac{1}{2}\frac{\partial \log a_{22}^*}{\partial u} = \frac{\dfrac{\partial r'}{\partial u}}{r - r'},$$

can be put into the form:

$$\frac{\partial}{\partial v}\left(r\frac{\partial \eta}{\partial u}\right) = \frac{\partial}{\partial u}\left(r'\frac{\partial \eta}{\partial v}\right).$$

Since each of the three functions $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$ satisfy the foregoing equation, there will exist three other functions $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$ that fulfill the equations ([1]):

$$d\,\mathfrak{x} = r\frac{\partial \mathfrak{X}}{\partial u}du + r'\frac{\partial \mathfrak{X}}{\partial v}dv,$$

(V) $$d\,\mathfrak{y} = r\frac{\partial \mathfrak{Y}}{\partial u}du + r'\frac{\partial \mathfrak{Y}}{\partial v}dv,$$

([1]) One can express $d\,\mathfrak{x}$, $d\,\mathfrak{y}$, $d\,\mathfrak{z}$ directly in terms of the original variables p, q with the help of formula (3^*) in Section I.

$$d\,\mathfrak{z} = r\frac{\partial \mathfrak{Z}}{\partial u}du + r'\frac{\partial \mathfrak{Z}}{\partial v}dv\,.$$

The functions $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$ will then be determined in such a way that:

$$\text{(VI)} \qquad d\,\mathfrak{X}^2 + d\,\mathfrak{Y}^2 + \varepsilon\,d\,\mathfrak{Z}^2 = a_{11}^* du^2 + a_{22}^* dv^2 = a_{11}\,dp^2 + 2a_{12}\,dp\,dq + a_{22}\,dq^2.$$

If one forms the sum $d\,\mathfrak{x}^2 + d\,\mathfrak{y}^2 + \varepsilon\,d\,\mathfrak{z}^2$ while recalling the foregoing equations then one will obviously find that:

$$d\,\mathfrak{x}^2 + d\,\mathfrak{y}^2 + \varepsilon\,d\,\mathfrak{z}^2 = r^2 a_{11}^* du^2 + r'^2 a_{22}^* dv^2 = b_{11}\,dp^2 + 2b_{12}\,dp\,dq + b_{22}\,dq^2,$$

whereas the differential form:

$$d\,\mathfrak{X}\,d\mathfrak{x} + d\,\mathfrak{Y}\,d\mathfrak{y} + \varepsilon\,d\mathfrak{Z}\,d\mathfrak{z}$$

will then have the equation:

$$\text{(VII)} \quad d\,\mathfrak{X}\,d\mathfrak{x} + d\,\mathfrak{Y}\,d\mathfrak{y} + \varepsilon\,d\mathfrak{Z}\,d\mathfrak{z} = r^2 a_{11}^* du^2 + r'^2 a_{22}^* dv^2 = b_{11}\,dp^2 + 2b_{12}\,dp\,dq + b_{22}\,dq^2.$$

Finally, one notes that due to the relation:

$$\mathfrak{X}^2 + \mathfrak{Y}^2 + \varepsilon\,\mathfrak{Z}^2 = \varepsilon,$$

one will have the equation:

$$\text{(VII)} \qquad \mathfrak{X}\,d\mathfrak{x} + \mathfrak{Y}\,d\mathfrak{y} + \varepsilon\,\mathfrak{Z}\,d\mathfrak{z} = 0.$$

If one now considers the sum $\mathfrak{X}\,\mathfrak{x} + \mathfrak{Y}\,\mathfrak{y} + \varepsilon\,\mathfrak{Z}\,\mathfrak{z}$ (which might be denoted by Q) the equations (VII) and (VIII) will imply the following ones:

$$Q = \mathfrak{X}\,\mathfrak{x} + \mathfrak{Y}\,\mathfrak{y} + \varepsilon\,\mathfrak{Z}\,\mathfrak{z}\,,$$

$$\frac{\partial Q}{\partial p} = \frac{\partial \mathfrak{X}}{\partial p}\mathfrak{x} + \frac{\partial \mathfrak{Y}}{\partial p}\mathfrak{y} + \varepsilon\frac{\partial \mathfrak{Z}}{\partial p}\mathfrak{z}\,,$$

$$\frac{\partial Q}{\partial q} = \frac{\partial \mathfrak{X}}{\partial q}\mathfrak{x} + \frac{\partial \mathfrak{Y}}{\partial q}\mathfrak{y} + \varepsilon\frac{\partial \mathfrak{Z}}{\partial q}\mathfrak{z}\,,$$

$$\frac{\partial^2 Q}{\partial p^2} = \frac{\partial^2 \mathfrak{X}}{\partial p^2}\mathfrak{x} + \frac{\partial^2 \mathfrak{Y}}{\partial p^2}\mathfrak{y} + \varepsilon\frac{\partial^2 \mathfrak{Z}}{\partial p^2}\mathfrak{z} + c_{11}\,,$$

$$\frac{\partial^2 Q}{\partial p\,\partial q} = \frac{\partial^2 \mathfrak{X}}{\partial p\,\partial q}\mathfrak{x} + \frac{\partial^2 \mathfrak{Y}}{\partial p\,\partial q}\mathfrak{y} + \varepsilon\frac{\partial^2 \mathfrak{Z}}{\partial p\,\partial q}\mathfrak{z} + c_{12}\,,$$

$$\frac{\partial^2 Q}{\partial q^2} = \frac{\partial^2 \mathfrak{X}}{\partial q^2}\mathfrak{x} + \frac{\partial^2 \mathfrak{Y}}{\partial q^2}\mathfrak{y} + \varepsilon\frac{\partial^2 \mathfrak{Z}}{\partial q^2}\mathfrak{z} + c_{22}\,,$$

the last three of which will go to the following ones:

$$\frac{\partial^2 Q}{\partial p^2} = \begin{Bmatrix}11\\1\end{Bmatrix}_a \frac{\partial Q}{\partial p} + \begin{Bmatrix}11\\2\end{Bmatrix}_a \frac{\partial Q}{\partial q} - \varepsilon\, a_{11}\, Q + c_{11}\,,$$

$$\frac{\partial^2 Q}{\partial p\,\partial q} = \begin{Bmatrix}12\\1\end{Bmatrix}_a \frac{\partial Q}{\partial p} + \begin{Bmatrix}12\\2\end{Bmatrix}_a \frac{\partial Q}{\partial q} - \varepsilon\, a_{12}\, Q + c_{12}\,,$$

$$\frac{\partial^2 Q}{\partial q^2} = \begin{Bmatrix}22\\1\end{Bmatrix}_a \frac{\partial Q}{\partial p} + \begin{Bmatrix}22\\2\end{Bmatrix}_a \frac{\partial Q}{\partial q} - \varepsilon\, a_{22}\, Q + c_{22}$$

when one uses the linear partial differential equations that govern the functions $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$.

Due to the fact that:

$$\frac{c_{11}\,c_{22} - c_{12}^2}{a_{11}\,a_{22} - a_{12}^2} = \varepsilon,$$

those equations will imply the second-order differential equation:

$$\frac{\left[Q_{11} - \begin{Bmatrix}11\\1\end{Bmatrix}_a Q_1 - \begin{Bmatrix}11\\2\end{Bmatrix}_a Q_2 + \varepsilon\, a_{11} Q\right]\left[Q_{22} - \begin{Bmatrix}22\\1\end{Bmatrix}_a Q_1 - \begin{Bmatrix}22\\2\end{Bmatrix}_a Q_2 + \varepsilon\, a_{22} Q\right] - \left[Q_{12} - \begin{Bmatrix}12\\1\end{Bmatrix}_a Q_1 - \begin{Bmatrix}12\\2\end{Bmatrix}_a Q_2 + \varepsilon\, a_{12} Q\right]^2}{a_{11}a_{22} - a_{12}^2} = \varepsilon$$

for the function Q, which coincides with the one that was presented above for the function ϕ ([1]).

Any real-valued solution ϕ of that differential equation will, in fact, correspond to a surface of constant curvature ε, and conversely, any such surface will correspond to a real-valued solution of that differential equation.

In the case where ε is set equal to positive unity, the latter developments will include an interesting result that relates to surfaces of unity curvature. In that case, as a result of the equation:

$$\mathfrak{X}\; d\mathfrak{x} + \mathfrak{Y}\; d\mathfrak{y} + \mathfrak{Z}\; d\mathfrak{z} = 0,$$

equations (V) will determine a surface for which the functions $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$ give the cosines of the angles that the normal to it that is raised at the point (u, v) make with the

([1]) The introduction of the intermediate variables u, v, which will not appear in the final result of all of the foregoing developments, is not necessary. The properties of the couplings $c_a\,(p)$, $c_a\,(q)$ of the original variables suffices completely to develop the final result free of all variables u, v. However, avoiding those variables would lead to a greater expenditure of calculation and would make the geometric relationships to the results harder to see.

coordinate axes $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$, resp. That is because the curvature of the form *B*, by which the square of the line element:

$$d\mathfrak{x}^2 + d\mathfrak{y}^2 + d\mathfrak{z}^2$$

of that surface is represented, is equal to unity, since that form yields the square of the line element of the map to the GAUSS sphere for the original surface (*x*, *y*, *z*).

Equations (V) further show that the variables *u*, *v* also represent the parameters of the lines of curvature for that second surface, and that *r* and *r′* are its radii of principal curvature at the point (*u*, *v*), while the same quantities will represent the parameters *u*, *v* of the corresponding principal curvatures for the original surface. The relations:

$$dx^2 + dy^2 + dz^2 = a_{11}\,dp^2 + 2a_{12}\,dp\,dq + a_{22}\,dq^2,$$

$$dX^2 + dY^2 + dZ^2 = b_{11}\,dp^2 + 2b_{12}\,dp\,dq + b_{22}\,dq^2,$$

$$d\mathfrak{x}^2 + d\mathfrak{y}^2 + d\mathfrak{z}^2 = b_{11}\,dp^2 + 2b_{12}\,dp\,dq + b_{22}\,dq^2,$$

$$d\mathfrak{X}^2 + d\mathfrak{Y}^2 + d\mathfrak{Z}^2 = a_{11}\,dp^2 + 2a_{12}\,dp\,dq + a_{22}\,dq^2,$$

which exist for the given surface [*x*, *y*, *z*] and the one that is derived from it [$\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$], now contain the following theorem:

The points of a given surface of unity curvature always correspond to the points of a second surface of equal curvature that those points determine in such a way that the line element of the first one is equal to the line element of the image of the second one under the map to the GAUSSian sphere, and conversely. Under that correspondence, the lines of curvature of both surfaces will correspond to each other, and the principal curvatures will be equal at corresponding points, although the associated lines of curvature will be switched with each other.

This would be a good place to mention the connection between the developments that were just employed and some other investigations into the theory of surfaces.

Instead of examining those properties of curved surfaces that are linked with a given form for the square of its line element, one can also shift one's attention to the properties of surface that emerge from the map of a given form for the square of the line element onto the GAUSSian sphere. OSSIAN BONNET published that genre of investigations in his treatise "Mémoire sur l'emploi d'un nouveau système de variables dans l'étude des propriétes des surfaces courbes" [LIOUVILLE's Journal **5** (1860)], and indeed under the assumption that the square of the line element for the map to the GAUSSian sphere possessed the form:

$$d\phi^2 + \sin^2\phi\,dv^2.$$

CHRISTOFFEL presented his beautiful study "Über die Bestimmung der Gestalt einer krummen Oberfläche durch locale Messungen auf derselben" (CRELLE-BORCHARDT's Journal, Bd. 64) from the same standpoint.

If one imagines that the point (x, y, z) of a curved surface is determined by the values of two variables p, q, and one lets X, Y, Z denote the cosines of the angles that the normal that is raised at the point (x, y, z) of that surface makes with the coordinate axes – i.e., the coordinates of the point (X, Y, Z) on the GAUSSian sphere that is the image of the point (x, y, z) – and one lets the quadratic form:

$$b_{11}\,dp^2 + 2b_{12}\,dp\,dq + b_{22}\,dq^2$$

of the *given* representation be the sum:

$$dX^2 + dY^2 + dZ^2$$

then the cosines X, Y, Z will individually satisfy the three simultaneous linear partial differential equations:

$$\frac{\partial^2 U}{\partial p^2} - \begin{Bmatrix} 11 \\ 1 \end{Bmatrix}_b \frac{\partial U}{\partial p} + \begin{Bmatrix} 11 \\ 2 \end{Bmatrix}_b \frac{\partial U}{\partial q} + b_{11}\,U = 0,$$

$$\frac{\partial^2 U}{\partial p\,\partial q} - \begin{Bmatrix} 12 \\ 1 \end{Bmatrix}_b \frac{\partial U}{\partial p} + \begin{Bmatrix} 12 \\ 2 \end{Bmatrix}_b \frac{\partial U}{\partial q} + b_{12}\,U = 0,$$

$$\frac{\partial^2 U}{\partial q^2} - \begin{Bmatrix} 22 \\ 1 \end{Bmatrix}_b \frac{\partial U}{\partial p} + \begin{Bmatrix} 22 \\ 2 \end{Bmatrix}_b \frac{\partial U}{\partial q} + b_{22}\,U = 0.$$

If one considers the function P, which is defined by the equation:

$$P = Xx + Yy + Zz, \tag{15}$$

or the algebraic value of the normal that is raised to the tangent plane at the point (x, y, z) of the given surface when the origin of the coordinates is *arbitrary*, then that will yield the following equations for its first differential quotients:

$$\text{(16)} \qquad \begin{aligned} \frac{\partial P}{\partial p} &= \frac{\partial X}{\partial p} x + \frac{\partial Y}{\partial p} y + \frac{\partial Z}{\partial p} z, \\ \frac{\partial P}{\partial q} &= \frac{\partial X}{\partial q} x + \frac{\partial Y}{\partial q} y + \frac{\partial Z}{\partial q} z, \end{aligned}$$

while the second differential quotients of that function will be given by the further equations:

$$\frac{\partial^2 P}{\partial p^2} = \begin{Bmatrix} 11 \\ 1 \end{Bmatrix}_b \frac{\partial P}{\partial p} + \begin{Bmatrix} 11 \\ 2 \end{Bmatrix}_b \frac{\partial P}{\partial q} - b_{11} P + \left[\frac{\partial X}{\partial p}\frac{\partial x}{\partial p} + \frac{\partial Y}{\partial p}\frac{\partial y}{\partial p} + \frac{\partial Z}{\partial p}\frac{\partial z}{\partial p} \right],$$

$$\frac{\partial^2 P}{\partial p\,\partial q} = \begin{Bmatrix}12\\1\end{Bmatrix}_b \frac{\partial P}{\partial p} + \begin{Bmatrix}12\\2\end{Bmatrix}_b \frac{\partial P}{\partial q} - b_{12} P + \left[\frac{\partial X}{\partial p}\frac{\partial x}{\partial q} + \frac{\partial Y}{\partial p}\frac{\partial y}{\partial q} + \frac{\partial Z}{\partial p}\frac{\partial z}{\partial q}\right],$$

$$\frac{\partial^2 P}{\partial p\,\partial q} = \begin{Bmatrix}12\\1\end{Bmatrix}_b \frac{\partial P}{\partial p} + \begin{Bmatrix}12\\2\end{Bmatrix}_b \frac{\partial P}{\partial q} - b_{12} P + \left[\frac{\partial X}{\partial q}\frac{\partial x}{\partial p} + \frac{\partial Y}{\partial q}\frac{\partial y}{\partial p} + \frac{\partial Z}{\partial q}\frac{\partial z}{\partial p}\right],$$

$$\frac{\partial^2 P}{\partial q^2} = \begin{Bmatrix}22\\1\end{Bmatrix}_b \frac{\partial P}{\partial p} + \begin{Bmatrix}22\\2\end{Bmatrix}_b \frac{\partial P}{\partial q} - b_{22} P + \left[\frac{\partial X}{\partial q}\frac{\partial x}{\partial q} + \frac{\partial Y}{\partial q}\frac{\partial y}{\partial q} + \frac{\partial Z}{\partial q}\frac{\partial z}{\partial q}\right]$$

when one uses the linear partial differential equations that the cosines X, Y, Z satisfy.

After introducing the relations:

$$c_{11} = \frac{\partial^2 P}{\partial p^2} - \begin{Bmatrix}11\\1\end{Bmatrix}_b \frac{\partial P}{\partial p} - \begin{Bmatrix}11\\2\end{Bmatrix}_b \frac{\partial P}{\partial q} + b_{11} P\,,$$

$$c_{12} = \frac{\partial^2 P}{\partial p\,\partial q} - \begin{Bmatrix}12\\1\end{Bmatrix}_b \frac{\partial P}{\partial p} - \begin{Bmatrix}12\\2\end{Bmatrix}_b \frac{\partial P}{\partial q} + b_{12} P\,,$$

$$c_{22} = \frac{\partial^2 P}{\partial q^2} - \begin{Bmatrix}22\\1\end{Bmatrix}_b \frac{\partial P}{\partial p} - \begin{Bmatrix}22\\2\end{Bmatrix}_b \frac{\partial P}{\partial q} + b_{22} P\,,$$

one will recognize the validity of the equation:

$$c_{11}\,dp^2 + 2c_{12}\,dp\,dq + c_{22}\,dq^2 = dX\,dx + dY\,dy + dZ\,dz\,.$$

Now, as one easily convinces oneself, the simultaneous absolute invariants of the two forms $dX^2 + dY^2 + dZ^2$, $dX\,dx + dY\,dy + dZ\,dz$ for any surface; i.e., the following values:

$$\frac{b_{11}\,c_{22} - 2\,b_{12}\,c_{12} + b_{22}\,c_{11}}{b_{11}\,b_{22} - b_{12}^2}\,,$$

$$\frac{c_{11}\,c_{22} - c_{12}^2}{b_{11}\,b_{22} - b_{12}^2}\,,$$

are identical to the sum of the radii of principal curvature ρ, ρ' at the point (p, q) of the surface in question in the former case and their product in the latter, and the following equations will be true:

(17) $$\frac{b_{11}\,c_{22} - 2\,b_{12}\,c_{12} + b_{22}\,c_{11}}{b_{11}\,b_{22} - b_{12}^2} = \rho + \rho',$$

$$\frac{c_{11}\,c_{22}-c_{12}^2}{b_{11}\,b_{22}-b_{12}^2}=\rho\,\rho'=\frac{1}{k}. \tag{18}$$

When one considers the definitions that were given for the quantities c_{ik} , equation (17) can be easily brought into the form:

$$\frac{1}{\sqrt{b_{11}\,b_{22}-b_{12}^2}}\left[\frac{\partial}{\partial p}\left(\frac{b_{22}\frac{\partial P}{\partial p}-b_{12}\frac{\partial P}{\partial q}}{\sqrt{b_{11}\,b_{22}-b_{12}^2}}\right)+\frac{\partial}{\partial p}\left(\frac{b_{11}\frac{\partial P}{\partial q}-b_{12}\frac{\partial P}{\partial p}}{\sqrt{b_{11}\,b_{22}-b_{12}^2}}\right)\right]+2P=\rho+\rho'. \tag{19}$$

The coordinates x, y, z can be determined using equations (15) and (16) :

$$\begin{aligned}
x &= PX+\frac{b_{11}\frac{\partial X}{\partial q}\frac{\partial P}{\partial q}-b_{12}\left(\frac{\partial X}{\partial p}\frac{\partial P}{\partial q}+\frac{\partial X}{\partial q}\frac{\partial P}{\partial p}\right)+b_{22}\frac{\partial X}{\partial p}\frac{\partial P}{\partial p}}{b_{11}\,b_{22}-b_{12}^2},\\
y &= PY+\frac{b_{11}\frac{\partial Y}{\partial q}\frac{\partial P}{\partial q}-b_{12}\left(\frac{\partial Y}{\partial p}\frac{\partial P}{\partial q}+\frac{\partial Y}{\partial q}\frac{\partial P}{\partial p}\right)+b_{22}\frac{\partial Y}{\partial p}\frac{\partial P}{\partial p}}{b_{11}\,b_{22}-b_{12}^2},\\
z &= PZ+\frac{b_{11}\frac{\partial Z}{\partial q}\frac{\partial P}{\partial q}-b_{12}\left(\frac{\partial Z}{\partial p}\frac{\partial P}{\partial q}+\frac{\partial Z}{\partial q}\frac{\partial P}{\partial p}\right)+b_{22}\frac{\partial Z}{\partial p}\frac{\partial P}{\partial p}}{b_{11}\,b_{22}-b_{12}^2}.
\end{aligned} \tag{20}$$

The foregoing equations contain the elements of a theory of the curvature of surfaces that corresponds to the assumption that the cosines of the angles that the normal that is raised at point (p, q) of a surface makes with the coordinate axes are given as functions of the variables p, q.

They show that when the sum of the radii of principal curvature of a surface is supposed to be a known function of the variables p, q under that assumption, which depends upon the determination of that surface by integrating the second-order *linear* differential equation (19), the coordinates of the points of that surface can be determined by integration from the function P and its derivatives alone.

By contrast, if a function k of the variables p, q were given each point of a surface that was supposed to represent its curvature then one would have to integrate the complicated differential equation (18), and the determination of the coordinates would result from that integration, as before.

Finally, if the cosines X, Y, Z themselves are not given, but only the sum of the squares of their differentials:

$$b_{11}\,dp^2+2b_{12}\,dp\,dq+b_{22}\,dq^2,$$

then the integration of two second-order linear ordinary differential equations that has been mentioned many times by now would always allow one to determine a system of functions X, Y, Z that could represent those cosines. In place of that system, one could also choose other ones that are connected with it by the relationship of an orthogonal substitution.

In the context of that remark, equations (20) obviously prove the following theorem:

If the map of the points of a curved surface to the GAUSSian sphere (the celestial sphere, respectively) is given and one knows the distance from a point in the tangent plane to that surface at a fixed point of the surface to each point of the surface then the form of the surface will be determined completely, while its position is space will not be fixed by that determination.

Obviously, that theorem is linked with the assumption that the distance from points that are coupled with the given surface to the corresponding tangent plane, which varies from point to point, will always admit first and second partial differential quotients with well-defined values.

Berlin 1884.