"Ueber die Deformationen einer biegsamen unausdehnbahren Fläche," J. für reine u. angew. Math. **100** (1887), 296-310.

On the deformations of a flexible, inextensible surface

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In volume 22 of the Transactions of the Royal Irish Academy (1853), **Jellet** expressed his beautiful observation that a flexible and inextensible surface of everywhere positive curvature would not be deformable when a curve segment in it was fixed, while a surface of everywhere negative curvature would admit a deformation when one fixed one of its asymptotic lines. However, the argument by which **Jellet** developed those theorems was not convincing. The same thing was true of the conclusions by which **Lecornu** [Journal de l'École Polytechnique, Cahier 42, (1880)] later derived some geometric properties of the deformation of a flexible, inextensible surface in a treatise that dealt with one of the conditions for equilibrium of such a surface, which did not touch upon the crux of that treatise. Both authors took the starting points of their developments to be the linear partial differential equations upon which the *infinitely-small* deformations of a surface that is considered to be flexible and inextensible depend.

The concept of *infinitely-small* deformations of such a surface initially allows for two different way of looking at things whose intrinsic agreement is by no means obvious. On the one hand, one can consider a given surface whose points are determined by the values of two independent variables p, q to be a particular exemplar of a family of surfaces:

$$x = f_1 (p, q, t), \qquad x = f_1 (p, q, t), \qquad x = f_1 (p, q, t),$$

and think of the continuous functions x, y, z of the arguments p, q, t as being subject to the partial differential equations:

$$\left(\frac{\partial x}{\partial p}\right)^2 + \left(\frac{\partial y}{\partial p}\right)^2 + \left(\frac{\partial z}{\partial p}\right)^2 = E,$$

$$\frac{\partial x}{\partial p}\frac{\partial x}{\partial q} + \frac{\partial y}{\partial p}\frac{\partial y}{\partial q} + \frac{\partial z}{\partial p}\frac{\partial z}{\partial q} = F,$$

$$\left(\frac{\partial x}{\partial q}\right)^2 + \left(\frac{\partial y}{\partial q}\right)^2 + \left(\frac{\partial z}{\partial q}\right)^2 = G,$$

in which the functions E, F, G are independent of the parameter t. Thus, an exemplar of that family that corresponds to the parameter t should be developable to one that corresponds to an altered value of that parameter; that is, it can be brought from its original position to the altered one with a continuous change of form. When two exemplars of that family have parameters that differ by an infinitely-small value τ, one can refer to one of them as the *infinitely-small* deformation of the other one.

On the other hand, one can refer to a surface that is generated by an infinitely-small displacement of a given surface as an *infinitely-small* deformation of a given surface when one thinks of that displacement as being constrained by the condition that it must change the distance between two infinitely-close points on the given surface only by quantities that are infinitely-small of order two or higher in comparison to the infinitely-small displacements of those points themselves.

It is plausible that when a surface can be referred to as an infinitely-small deformation of a given surface on the basis of the first conception of the term, it will also deserve that designation on the basis of the second one. However, it is not clear that an infinitely-small deformation in the second sense of the term will also represent one in the first sense. The arguments of **Jellet** and **Lecornu** refer to the concept of an infinitely-small deformation that corresponds to the second conception of the idea and which is broader in scope than the one that belongs to the concept of an infinitely-small deformation as an intermediate step in the finite change of form of an inextensible surface. When one follows the thinking of those geometers further into the respective calculations of each in relation to the infinitely-small displacements of the point of a surface by infinitely-small quantities of second order with zero, and those quantities are identified with zero, the conclusion itself will also prove to be untenable that the vanishing of one of a number of infinitely-small variables of the same number of homogeneous linear functions of those variables would imply the vanishing of the determinant of the those functions. One can *only* infer that this determinant is infinitely-small of the same order as the infinitely-small variables themselves, and that they cannot be considered to vanish in their own right then.

Therefore, a development of **Jellet**'s theorems that chooses the other viewpoint on the concept of infinitely-small deformations does not seem pointless. In the process of doing that, one will also get the fundamental formulas in an article on the theory of infinitely-small deformations of flexible, inextensible surfaces that I submitted to the Kgl. Preussischen Akademie der Wissenschaften on 28 January 1886. (Sitzungsbericht der Kgl. Preussischen Akademie der Wissenschaften, VI, 1886.)

1.

One considers two curved surfaces S' and S'' in space that satisfy the condition that each point (x', y', z') of the first one corresponds to a point (x'', y'', z'') of the second one in such a way that the distance between any two infinitely-close points of the first surface is equal to the distance between the corresponding points of the other one. If one thinks of the coordinates x', y', z' of each point of the surface S' as being given as functions of two independent variable quantities p, q, and the coordinates of the corresponding point x'', y'', z'' of the surface S'' as functions of the same variables then the condition that was posed can be expressed by the equations:

$$dx'^2 + dy'^2 + dz'^2 = dx''^2 + dy''^2 + dz''^2 = E\,dp^2 + 2F\,dp\,dq + G\,dq^2, \tag{I}$$

in which E, F, G denote known functions of the variables p, q.

We assume that the functions x', y', z' and the other ones x'', y'', z'' are regular functions of those variables inside of a domain of the variables p, q under consideration, and that in that domain they possess finite and single-valued values, along with all of their derivatives.

If one couples any two corresponding points of the surfaces S' and S'' with lines then the geometric locus of the midpoints of those connecting lines will represent a third surface S, which we would like to refer to as the *middle surface* of the given ones S' and S''. The points of that surface are determined by the equations:

$$x = \tfrac{1}{2}(x'' + x'), \qquad y = \tfrac{1}{2}(y'' + y'), \qquad z = \tfrac{1}{2}(z'' + z'),$$

and the functions x, y, z that give the coordinates of each point on it are likewise finite and single-valued, along with their derivatives, as well as the functions x', x'', etc.

If one introduces the notations:

$$u = \tfrac{1}{2}(x'' - x'), \qquad v = \tfrac{1}{2}(y'' - y'), \qquad w = \tfrac{1}{2}(z'' - z')$$

then the same things will be true for the functions u, v, w.

When one introduces the functions x, y, z and u, v, w into equations (I), instead of the original ones, that will yield the equation:

$$\text{(II)} \qquad dx\,du + dy\,dv + dz\,dw = 0,$$

which decomposes into the following three:

$$(\text{II}^*) \quad \begin{cases} \dfrac{\partial x}{\partial p}\dfrac{\partial u}{\partial p} + \dfrac{\partial y}{\partial p}\dfrac{\partial v}{\partial p} + \dfrac{\partial z}{\partial p}\dfrac{\partial w}{\partial p} = \sum \dfrac{\partial x}{\partial p}\dfrac{\partial u}{\partial p} = 0, \\ \dfrac{\partial x}{\partial p}\dfrac{\partial u}{\partial q} + \dfrac{\partial y}{\partial p}\dfrac{\partial v}{\partial q} + \dfrac{\partial z}{\partial p}\dfrac{\partial w}{\partial q} + \dfrac{\partial x}{\partial q}\dfrac{\partial u}{\partial p} + \dfrac{\partial y}{\partial q}\dfrac{\partial v}{\partial p} + \dfrac{\partial z}{\partial q}\dfrac{\partial w}{\partial p} = \sum \dfrac{\partial x}{\partial p}\dfrac{\partial u}{\partial q} + \sum \dfrac{\partial x}{\partial q}\dfrac{\partial u}{\partial p} = 0, \\ \dfrac{\partial x}{\partial q}\dfrac{\partial u}{\partial q} + \dfrac{\partial y}{\partial q}\dfrac{\partial v}{\partial q} + \dfrac{\partial z}{\partial q}\dfrac{\partial w}{\partial q} = \sum \dfrac{\partial x}{\partial q}\dfrac{\partial u}{\partial q} = 0. \end{cases}$$

Furthermore, let the equation:

$$dx^2 + dy^2 + dz^2 = a_{11}\,dp^2 + 2a_{12}\,dp\,dq + a_{22}\,dq^2$$

exist for the square of the line element of the middle surface S, in which the functions a_{ik} are known and are constrained by the same continuity conditions as the functions x, y, z.

The use of equations (II^*) will be lightened essentially with the aid of a new function φ, which is defined by the function:

$$2\varphi\sqrt{a_{11}\,a_{22}-a_{12}^2} = \sum\frac{\partial x}{\partial p}\frac{\partial u}{\partial q}-\sum\frac{\partial x}{\partial q}\frac{\partial u}{\partial p},$$

and in which one easily knows one invariant of the differential expression:

$$u\,dx + v\,dy + w\,dz\,.$$

Equations (II*) can then be replaced with the following four:

$$\text{(III)}\qquad \begin{cases} \sum\dfrac{\partial x}{\partial p}\dfrac{\partial u}{\partial p}=0, & \sum\dfrac{\partial x}{\partial p}\dfrac{\partial u}{\partial q}=\varphi\sqrt{a_{11}\,a_{22}-a_{12}^2}\,, \\ \sum\dfrac{\partial x}{\partial q}\dfrac{\partial u}{\partial p}=-\varphi\sqrt{a_{11}\,a_{22}-a_{12}^2}\,, & \sum\dfrac{\partial x}{\partial q}\dfrac{\partial u}{\partial q}=0. \end{cases}$$

The variables p, q that enter into the foregoing equations depend upon not only the function φ, but the coordinates of a point on the middle surface S and the finite displacements u, v, w through which one has displaced that point in the directions of the coordinates, or its opposite, in order to make the point (x'', y'', z'') on the surface S'' or the point (x', y', z') on the surface S' coincide with it. The equations:

$$x'' = x + u, \qquad y'' = y + v, \qquad z'' = z + w,$$

along with the other ones:

$$x' = x - u, \qquad y' = y - v, \qquad z' = z - w,$$

then determine the surfaces S' and S''. One remarks that both surfaces can be regarded as exemplars of the family of surfaces:

$$x''' = x + t\,u, \quad y''' = y + t\,v, \quad z''' = z + t\,w,$$

for the values + 1 and – 1 of the parameter t, and that it will follow from equation (II) that any two exemplars of that family that correspond to equal and opposite values of those parameters will possess equally large line elements, while the surface S will still remain the middle surface for them. For infinitely-small values τ of t, the exemplars of that family will define infinitely-small deformations of the surface S itself.

Now let X, Y, Z denote the cosines of the angles that the normal to the surface S that is raised at the point (x, y, z) that corresponds to the values p, q of the variables defines with the coordinate axes, and the associated differential form:

$$dX\,dx + dY\,dy + dZ\,dz$$

might imply the equation.

$$dX\,dx + dY\,dy + dZ\,dz = c_{11}\,dp^2 + 2\,c_{12}\,dp\,dq + c_{22}\,dq^2\,.$$

The curvature k of the surface S at the point (x, y, z) is determined by the formula:

$$k = \frac{c_{11}\,c_{22} - c_{12}^2}{a_{11}\,a_{22} - a_{12}^2}.$$

As far as the functions c_{11}, c_{12}, c_{22} are concerned, as long as the determinant $a_{11}\,a_{22} - a_{12}^2$ does not vanish for individual points or lines in the surface S inside of the domain of the independent variables under consideration, they will be finite and single-valued functions of those variables, along with their derivatives, as a result of the assumption that was made on the domain in question, or at least a finite part of it.

In Chap. XI of his "Disquisitiones generales circa superficies curvas," **Gauss** (*Werke*, v. IV, page 235) presented formulas by which the second derivatives of any of the coordinates x, y, z of a point of a surface S could be expressed in terms of the first derivatives of that same coordinate, and indeed in terms of certain couplings that are composed from the coefficients of the line element and its first derivatives, as well as by means of the quantities X, Y, Z, and the functions c_{11}, c_{12}, c_{22} in question. With the notation that we have assumed, and under the assumption of a notation that **Christoffel** introduced for the aforementioned couplings, those formulas are the following ones:

$$\text{(IV)} \qquad \left\{\begin{aligned} \frac{\partial^2 x}{\partial p^2} &= \begin{Bmatrix}11\\1\end{Bmatrix}\frac{\partial x}{\partial p} + \begin{Bmatrix}11\\2\end{Bmatrix}\frac{\partial x}{\partial q} - c_{11}X, \\ \frac{\partial^2 x}{\partial p\,\partial q} &= \begin{Bmatrix}12\\1\end{Bmatrix}\frac{\partial x}{\partial p} + \begin{Bmatrix}12\\2\end{Bmatrix}\frac{\partial x}{\partial q} - c_{12}X, \\ \frac{\partial^2 x}{\partial q^2} &= \begin{Bmatrix}22\\1\end{Bmatrix}\frac{\partial x}{\partial p} + \begin{Bmatrix}22\\2\end{Bmatrix}\frac{\partial x}{\partial q} - c_{22}X. \end{aligned}\right.$$

For the sake of further conclusions that one can infer from equations (III), one defines the following equation from the first two of those equations, with the introduction of the notation $a = a_{11}\,a_{22} - a_{12}^2$:

$$\frac{\partial \varphi\sqrt{a}}{\partial p} = \sum \frac{\partial u}{\partial q}\frac{\partial^2 x}{\partial p^2} - \sum \frac{\partial u}{\partial p}\frac{\partial^2 x}{\partial p\,\partial q},$$

defines another one from the last two:

$$\frac{\partial \varphi\sqrt{a}}{\partial q} = \sum \frac{\partial u}{\partial q}\frac{\partial^2 x}{\partial p\,\partial q} - \sum \frac{\partial u}{\partial p}\frac{\partial^2 x}{\partial q^2},$$

and one then replaces the second differential quotients of the coordinates x, y, z in the equations thus-obtained by means of equations (IV). When one recalls equations (III), and appeals to the relations:

$$\frac{1}{\sqrt{a}} \cdot \frac{\partial \varphi \sqrt{a}}{\partial p} = \begin{Bmatrix} 11 \\ 1 \end{Bmatrix} + \begin{Bmatrix} 12 \\ 2 \end{Bmatrix}, \qquad \frac{1}{\sqrt{a}} \cdot \frac{\partial \varphi \sqrt{a}}{\partial q} = \begin{Bmatrix} 12 \\ 1 \end{Bmatrix} + \begin{Bmatrix} 22 \\ 2 \end{Bmatrix},$$

which are easy to see, one will then be led to the following determinations:

$$\text{(V)} \qquad \left\{ \begin{aligned} \frac{\partial \varphi}{\partial p} &= \frac{-c_{11} \sum X \frac{\partial u}{\partial q} + c_{12} \sum X \frac{\partial u}{\partial p}}{\sqrt{a}}, \\ \frac{\partial \varphi}{\partial q} &= \frac{-c_{11} \sum X \frac{\partial u}{\partial q} + c_{22} \sum X \frac{\partial u}{\partial p}}{\sqrt{a}}. \end{aligned} \right.$$

In the case where the curvature k vanishes for every point of the surface S, so for all values of p, q, the equation:

$$c_{11}\, c_{22} - c_{12}^2 = 0$$

will be true, and that will yield the equations:

$$\text{(VI)} \qquad c_{22} \frac{\partial \varphi}{\partial p} - c_{12} \frac{\partial \varphi}{\partial q} = 0$$

as the first-order linear partial differential equation that the function φ is subject to.

By contrast, when the determinant $c_{11}\, c_{22} - c_{12}^2$ vanishes at only isolated points or lines on the surface S, the following equations will be valid:

$$\text{(VII)} \qquad \left\{ \begin{aligned} \sum X \frac{\partial u}{\partial p} &= X \frac{\partial u}{\partial p} + Y \frac{\partial v}{\partial p} + Z \frac{\partial w}{\partial p} = \frac{c_{11} \frac{\partial \varphi}{\partial q} - c_{12} \frac{\partial \varphi}{\partial p}}{k \sqrt{a}}, \\ \sum X \frac{\partial u}{\partial q} &= X \frac{\partial u}{\partial q} + Y \frac{\partial v}{\partial q} + Z \frac{\partial w}{\partial q} = -\frac{c_{22} \frac{\partial \varphi}{\partial p} - c_{12} \frac{\partial \varphi}{\partial q}}{k \sqrt{a}} \end{aligned} \right.$$

in the domain of the variables p, q, with the exception of the locations in question. When one differentiates the first of them with respect to q and the second one with respect to p and subtracts the results obtained from each other, that will yield:

$$\frac{\partial \dfrac{c_{22}\dfrac{\partial\varphi}{\partial p} - c_{12}\dfrac{\partial\varphi}{\partial q}}{k\sqrt{a}}}{\partial p} + \frac{\partial \dfrac{c_{11}\dfrac{\partial\varphi}{\partial q} - c_{12}\dfrac{\partial\varphi}{\partial p}}{k\sqrt{a}}}{\partial q} = \sum \frac{\partial X}{\partial q}\frac{\partial u}{\partial p} - \sum \frac{\partial X}{\partial p}\frac{\partial u}{\partial q}.$$

When one appeals to the known formulas:

$$\frac{\partial X}{\partial p} = \frac{a_{22}\,c_{11} - a_{12}\,c_{12}}{a}\frac{\partial x}{\partial p} + \frac{a_{11}\,c_{12} - a_{12}\,c_{11}}{a}\frac{\partial x}{\partial q},$$

$$\frac{\partial X}{\partial q} = \frac{a_{22}\,c_{12} - a_{12}\,c_{22}}{a}\frac{\partial x}{\partial p} + \frac{a_{11}\,c_{12} - a_{12}\,c_{12}}{a}\frac{\partial x}{\partial q},$$

the last equation that was developed will be converted into the following second-order partial differential equation:

$$\text{(VIII)}\quad \frac{1}{\sqrt{a}}\left\{\frac{\partial \dfrac{c_{22}\dfrac{\partial\varphi}{\partial p} - c_{12}\dfrac{\partial\varphi}{\partial q}}{k\sqrt{a}}}{\partial p} + \frac{\partial \dfrac{c_{11}\dfrac{\partial\varphi}{\partial q} - c_{12}\dfrac{\partial\varphi}{\partial p}}{k\sqrt{a}}}{\partial q}\right\} + \frac{c_{11}\,a_{22} - 2c_{12}\,a_{12} + c_{22}\,a_{11}}{a}\,\varphi = 0$$

that the function φ satisfies.

Equations (VII) and (III) further imply expressions for the six first derivatives of the functions u, v, w in terms of the values of the function φ and its first derivatives by means of the equations:

$$\text{(IX)}\quad \left\{\begin{aligned} \frac{\partial u}{\partial p} &= \frac{c_{11}\left(X\dfrac{\partial\varphi}{\partial q} - \varphi\dfrac{\partial X}{\partial q}\right) - c_{12}\left(X\dfrac{\partial\varphi}{\partial p} - \varphi\dfrac{\partial X}{\partial p}\right)}{k\sqrt{a}}, \\ \frac{\partial u}{\partial q} &= \frac{c_{22}\left(X\dfrac{\partial\varphi}{\partial p} - \varphi\dfrac{\partial X}{\partial p}\right) - c_{12}\left(X\dfrac{\partial\varphi}{\partial q} - \varphi\dfrac{\partial X}{\partial q}\right)}{k\sqrt{a}}, \end{aligned}\right.$$

from which, the equations that relate to the derivatives of the functions v, w will emerge when one permutes X with Y, Z.

The foregoing equations that were developed for the geometry of the middle surface of two surfaces with equal line elements are, in fact, the ones that one obtains for the determination of the infinitely-small deformations of an arbitrary curved surface S, since even the latter represent only the same consequences of the condition equation for the infinitely-small displacements without extension, namely, the equation:

(II) $$dx\,du + dy\,dv + dz\,dw = 0.$$

2.

The surfaces S' and S'' that were considered in the previous section relate to each other in such a way that every well-defined point of one surface corresponds to a well-defined point of the other one, such that the distance between two infinitely-close neighboring points on the surface S' would be equal to the distance between the corresponding points on the surface S''. Every line in the first surface will then correspond to a well-defined line in the second one.

We would like to assume the further property of both surfaces S' and S'' that a *single* line on one of them will correspond to a line on the other that is *congruent* to it, and indeed, in such a way that the points of both lines that overlap as a result of that congruence will also be corresponding points of both surfaces. One can then put both surfaces into a position in which the two congruent lines are made to overlap from the outset. One can also assume, with no loss of generality, that one of the independent variables that determine the points of the surfaces – e.g., p – has a constant value p_0 along the common line to the two surfaces in question, while the other one q varies along that line. Our assumption will then demand that the functions x'', y'', z'' of the variables p, q that were defined in the first section, as well as the other ones x', y', z', will coincide for the value p_0 of p and every value of q. The functions u, v, w, which are subject to equations (II*) and the consequences that they imply will also satisfy the equations:

$$u_{p=p_0} = 0, \qquad v_{p=p_0} = 0, \qquad w_{p=p_0} = 0,$$

for every value of q, and the ones that follow from them directly:

(*a*) $$\left(\frac{\partial u}{\partial q}\right)_{p=p_0} = 0, \qquad \left(\frac{\partial v}{\partial q}\right)_{p=p_0} = 0, \qquad \left(\frac{\partial w}{\partial q}\right)_{p=p_0} = 0.$$

For further considerations, we shall exclude an especially interesting singularity that is defined when the determinant $a_{11}\,a_{22} - a_{12}^2$ vanishes for $p = p_0$, and then exclude the case in which the middle surface S proves to be developable from the surfaces S' and S'', for the sake of brevity. As a result of excluding the singularity, in a domain of the independent variables that includes a finite value of $p = p_0$, the function φ itself will be single-valued and finite, along with its derivatives, even though the existence of a line for which $a_{11}\,a_{22} - a_{12}^2$ vanishes will imply that the original domain of the variables p, q for which only x'', y'', z'' and x', y', z', along with their derivatives, are single-valued and finite might have suffered a reduction.

Moreover, as a consequence of equations (*a*), the second of equations (III) will imply that:

(*b*) $$\varphi_{p=p_0} = 0,$$

and therefore also:

$$(c) \qquad \left(\frac{\partial\varphi}{\partial q}\right)_{p=p_0} = 0, \qquad \left(\frac{\partial^2\varphi}{\partial q^2}\right)_{p=p_0} = 0, \qquad \left(\frac{\partial^3\varphi}{\partial q^3}\right)_{p=p_0} = 0, \qquad \text{etc.}$$

When one recalls equations (*a*), the second of equations (VII), or the required case of equation (VI), will then imply, in turn, the consequence that:

$$(d) \qquad (c_{22})_{p=p_0}\left(\frac{\partial\varphi}{\partial p}\right)_{p=p_0} = 0.$$

Under the assumption that will established next that for the value $p = p_0$, the function c_{22} does not vanish for any value of q, the foregoing equation will imply:

$$(e) \qquad \left(\frac{\partial\varphi}{\partial p}\right)_{p=p_0} = 0$$

and furthermore:

$$(e') \qquad \left(\frac{\partial^2\varphi}{\partial p\,\partial q}\right)_{p=p_0} = 0, \qquad \left(\frac{\partial^3\varphi}{\partial p\,\partial q^2}\right)_{p=p_0} = 0, \qquad \text{etc.}$$

If one then thinks of the partial differential equation (VIII) that the function φ is subjected to for all values of p, q in the domain that we speak of as being arranged in terms of the derivatives of that function, and one then sets $p = p_0$ then the foregoing equations will imply the vanishing of all terms of that differential equation, up to the following one, whose vanishing it required by the existence of the differential equation, namely:

$$\left[(c_{22})\left(\frac{\partial^2\varphi}{\partial p^2}\right)\right]_{p=p_0} = 0.$$

The further equations will then be true:

$$\left(\frac{\partial^3\varphi}{\partial p^2\,\partial q}\right)_{p=p_0} = 0, \qquad \text{etc.}$$

Upon differentiating the differential equation (VIII), when ordered in the aforementioned way, with respect to p and setting the value of p equal to to p_0 , that will further yield:

$$\left[(c_{22})\left(\frac{\partial^3\varphi}{\partial p^3}\right)\right]_{p=p_0} = 0.$$

and with repeated differentiation with respect to that variable and continual use of the equations that exist already, one will arrive at the result that the function φ itself will vanish along the line $p = p_0$, along with all of its derivatives with respect to p . When one recalls the assumptions that were made for the function φ, that result will bring one to the conclusion that the this function will vanish in the domain of the variables p, q that finitely includes the line $p = p_0$.

As a result of equations (IX), the functions u, v, w will also vanish in that domain of the independent variables, and the surfaces S' and S'' will coincide at all points of that domain.

The conclusion that one infers from equation (d) will break down when the function c_{22} vanishes for the value p_0 of p and every value of q. In that case, a non-zero function φ can exist that satisfies the differential equation (VIII) and equation (b).

The validity of the equation:

$$(c_{22})_{p=p_0} = 0,$$

which can also be written in the form:

$$\frac{\partial X}{\partial q}\frac{\partial x}{\partial q} + \frac{\partial Y}{\partial q}\frac{\partial y}{\partial q} + \frac{\partial Z}{\partial q}\frac{\partial z}{\partial q} = 0,$$

as a result of the definition of the quantities c_{ik} , demands that the differential equation:

$$dX\,dx + dY\,dy + dZ\,dz = 0$$

must be true along the curve $p = p_0$. As is known, the foregoing differential equation, in conjunction with the one:

$$X\,dx + Y\,dy + Z\,dz = 0,$$

which is likewise fulfilled on the aforementioned curve, says that the normals to the surface S along that curve coincide with the binormals of the latter, as long as one continues to exclude the assumption that the curve is a straight line.

As one knows, a double family of curves with that property will exist in all cases of everywhere negative curvature. However, surfaces with everywhere non-negative curvature can also possess individual lines with that property, namely, any planar line of zero curvature that they might contain.

Therefore, when the situation arises that the curve that one assumes to be common to the surfaces S' and S'' proves to be a curve with the aforementioned character on the middle surface, the coincidence of the surfaces S' and S'' in a domain that finitely contains that curve cannot be inferred from equations (III).

Finally, as far as the other case that was excluded up to now is concerned – namely, that the surface S is a developable surface – some simple considerations that are similar to the foregoing ones in regard to equations (III), (V), and (VI) will show that even in that case, when c_{22} does not vanish for the value p_0 of p, the functions u, v, w must vanish in a domain that finitely contains the curve $p = p_0$, while that conclusion cannot be inferred when $c_{22} = 0$ for $p = p_0$. The curve $p = p_0$ will then be an edge of regression for S, or a planar curve that it contains.

As a result of the foregoing developments, the following theorem can then be stated:

When two mutually-developable surfaces have a corresponding line that is not straight and have the same common points along it, those surfaces will either coincide on a domain that finitely contains that line or the normal family of the middle surface to both surfaces will define the family of binormals to that common line along that line.

The foregoing theorem makes it possible to resolve the question of what the necessary conditions would be for a flexible, inextensible surface to admit continuous, finite deformations that would preserve the position of a given curve in it.

If one imagines that such a deformation is possible for a given surface S_0 , and one lets S_t denote the position of that surface after a continuous deformation that is performed over a time t then, as a result of the foregoing theorem, the curve that is assumed to be fixed in S_0 must possess the property at every time t that the system of its normals on the middle surface of the surfaces S_0 and S_t , which likewise varies with t, must coincide with the unvarying system of its binormals. If one lets each of the limits of t decrease then S_t and the common middle surface will continuously approach the surface S_0 without altering the position of that system of binormals. The given curve must then already possess the aforementioned property for the surface S_0 , and preserve it under all continuous deformations of S_0 , moreover.

Surfaces of everywhere-positive curvature do not possess any curves with that character. Such surfaces are not deformable then when one fixes a finite, but still quite small, curve segment in them. That is because if two such mutually-developable surfaces exist that coincide along that curve segment then, from the foregoing, they would have to coincide along a finite domain that includes that curve segment, and could separate only in a boundary of that domain that is common to them, but that boundary would, in turn, need to possess the character of an asymptotic line. Such a boundary would not exist in the further extent of both surfaces, with the exception of the case in which one would be led to have zero curvature along a planar line, but that curvature could not be ascribed the property of being positive.

For surfaces of everywhere-negative curvature, the foregoing argument will not suffice to prove undeformability when one fixes a small, but finite, curve segment that is included in it that does not belong to an asymptotic line. That is because an enlargement of the necessarily-common finite domain of two mutually-developable surfaces that possess that curve segment in common would lead to limits at which a separation would seem possible, namely, to the asymptotic lines of the surface that go through the endpoints of the curve segment.

The conclusions that were just inferred pertained to the *necessary* condition for the existence of a continuous, finite deformation of a surface that fixes a curve segment in it. However, it was not proved that this condition for performing the deformation was also *sufficient*. Strictly speaking, that proof was only completed by deriving the equation for the family of mutually-developable surfaces of negative curvature that included an asymptotic line that was common to all of its individual exemplars. By itself, that derivation did not seem to be practicable given the present state of the theory. We shall satisfy ourselves then with the proof that infinitely-small deformations of a surface S_0 of everywhere-negative curvature are possible for which a given asymptotic line on it will remain an asymptotic line.

Let x, y, z be the coordinates of a point of that surface, when they are expressed in terms of the independent variables p, q, and let $p = p_0$ be the equation of a well-defined asymptotic line on it. While preserving the notations that were introduced, a surface S_τ that is infinitely-close to it, where τ denotes an infinitely-small constant, can be represented by the equations:

$$x' = x + \tau u, \quad y' = y + \tau v, \quad z' = z + \tau w,$$

and both surfaces will possess the same line element when the functions u, v, w are subjected to the condition:

$$\text{(II)} \qquad dx\,du + dy\,dv + dz\,dw = 0$$

for all p, q. That will imply all of the consequences in regard to the functions u, v, w that were inferred in section 1.

Formulas (IX) of this section will then show that when a function φ can be ascertained from the differential equation (VIII), under the condition that $\varphi_{p=p_0} = 0$, which is a condition that implies the equation:

$$\left(\frac{\partial \varphi}{\partial q}\right)_{p=p_0} = 0,$$

as a result of the assumption:

$$(c_{22})_{p=p_0} = 0,$$

the derivatives of the functions u, v, w that are determined by those formulas, moreover, will satisfy the equations:

$$\left(\frac{\partial u}{\partial q}\right)_{p=p_0} = 0, \qquad \left(\frac{\partial v}{\partial q}\right)_{p=p_0} = 0, \qquad \left(\frac{\partial w}{\partial q}\right)_{p=p_0} = 0,$$

and that for a suitable determination of the constants, the values of u, v, w will vanish along the curve $p = p_0$.

The surface S_τ that corresponds to such a function φ would then represent an infinitely-small deformation of the surface and would include the line $p = p_0$. However, the surface S_τ would not satisfy the condition that this line is also one of its asymptotic lines with no further assumptions.

One would effortlessly determine the equation:

$$X' - X = \frac{\dfrac{\partial X}{\partial q}\dfrac{\partial \varphi}{\partial p} - \dfrac{\partial X}{\partial p}\dfrac{\partial \varphi}{\partial q}}{k\sqrt{a_{11}\,a_{22} - a_{12}^2}} \cdot \tau$$

for the difference between the cosines X', Y', Z' of the angles between the normal at a point of the surface S_τ that corresponds to the values p, q of the independent variables and the cosines X, Y, Z at the corresponding point of S, in which the a_{ik} denote the coefficients of the line element of S. One obtains the differences $Y' - Y$, $Z' - Z$ from it by permuting the X with Y, Z, resp.

The necessary coincidence of the normals to the surfaces S and S_τ along the line $p = p_0$ will demand the vanishing of the derivatives $\frac{\partial \varphi}{\partial p}$ for $p = p_0$, as long as that line is not planar. The function φ that mediates a possible infinitely-small deformation of the surface S is then linked with the *two* conditions:

$$\varphi_{p=p_0} = 0, \qquad \left(\frac{\partial \varphi}{\partial p}\right)_{p=p_0} = 0,$$

along with the differential equation (VIII). Suppressing the second one would lead to an infinitely-small deformation of S that could not be considered to be an intermediate stage of a finite deformation of that surface.

That remark confirms what we said in the introduction about the concept of an infinitely-small deformation of a surface using the second approach to defining that concept that was mentioned there being broader in scope than what would be required for the concept of the continuous deformation of a surface.

We believe that the sufficiently-known deformations of developable surfaces should be excluded from out developments.

Finally, permit us to remark that the foregoing considerations are closely connected with the study of those domain boundaries in an n-dimensional manifold on which the integrals of the second-order partial differential equations with n independent variables that are single-valued, finite, and continuous, along with their first derivatives, can branch, but we shall postpone an examination of that situation to a later publication.

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