"Über die unendlich kleinen Deformationen einer biegsamen, unausdehnbaren Fläche," Sitzungsber. Kön. Preuss. Akad. Wiss Berlin (1886), 83-91.

On the infinitely-small deformations of a flexible, inextensible surface.

By J. WEINGARTEN

Translated by D. H. Delphenich

The general theory of curved surfaces coincides with the theory of simultaneous transformations of binary quadratic forms that are constructed from the differentials of two independent variable quantities and whose coefficients are functions of those variables themselves. When one continuously maps the points of a curved surface to a sphere of unit radius using GAUSS's process, two quadratic forms of that sort will emerge from the simplest geometric considerations. The first of them is the square of the distance between two infinitely-close points of the surface considered. The second one is represented by the product that one obtains when one multiplies that distance times the value of its image on the sphere and cosine of the angle that the connecting line between two infinitely-close points on the surface makes with the line that connects their images. In place of the second form, one can also appeal to the one that specifies the square of the distance between the images of two infinitely-close points on the given surface. That third quadratic form is connected with the first two by a linear homogeneous equation.

If a point on a curved surface is specified by x, y, z, which are functions of two independently-varying quantities p, q, and the cosines of the angles that the normal to the surface at that point on the surface makes with the rectangular coordinate axes are specified by X, Y, Z, resp. (i.e., the coordinates of the Gaussian image of that point on the sphere of unit radius with its center at the coordinate origin) then the three aforementioned quadratic forms will be the following ones:

$$dx^2 + dy^2 + dz^2 = a_{11}\, dp^2 + 2a_{12}\, dp\, dq + a_{22}\, dq^2,$$

$$dX\, dx + dY\, dy + dZ\, dz = c_{11}\, dp^2 + 2c_{12}\, dp\, dq + c_{22}\, dq^2,$$

$$dX^2 + dY^2 + dZ^2 = b_{11}\, dp^2 + 2b_{12}\, dp\, dq + b_{22}\, dq^2,$$

and their coefficients a_{ik} , b_{ik} , c_{ik} will be given functions of the variables p, q for a given surface.

Investigations that I carried out on the properties of those forms, and which will be published in the next issue of the Journal für Mathematik, have led me to some results that relate to the theory of deformations of curved surfaces that I dare to submit below to the Königlichen Akademie der Wissenschaften.

If one gives each point (x, y, z) of a surface in question an infinitely-small displacement σ whose components along the coordinate axes are determined by the equations:

$$\delta x = u\, i, \qquad \delta y = v\, i, \qquad \delta z = w\, i,$$

where the u, v, w in those equations are finite and continuous functions of the variables p, q, including their second derivatives, and i denotes an infinitely-small constant, then the square of the distance between two infinitely-close points (x, y, z) and $(x + dx, y + dy, z + dz)$ on that surface will go to the value:

$$(dx + i\, du)^2 + (dy + i\, dv)^2 + (dz + i\, dw)^2$$

under the displacement that took place. Should the chosen displacement of the original distance between two infinitely-close points not have changed, so the surface in question would have also changed without any change in its line element, then when one neglects quantities that have second order relative to the infinitely-small displacements, the following equation must be valid:

$$du\, dx + dv\, dy + dw\, dz = 0.$$

Since the coefficients of the products dp^2, $dp\, dq$, and dq^2 in that equation must vanish separately, it will decompose into three other ones that represent those simultaneous partial differential equations that the functions u, v, w must satisfy when a displacement of the points of a curved surface can be mediated by them that leads to an infinitely-close surface that is developable onto the original one.

Now, it can be shown that ascertaining such functions u, v, w can be made to depend upon the discovery of a single function ϕ that satisfies the linear partial differential equation:

$$\text{(I)} \qquad \frac{1}{\sqrt{a}}\left\{\frac{\partial}{\partial p}\left(\frac{c_{22}\dfrac{\partial \phi}{\partial p} - c_{12}\dfrac{\partial \phi}{\partial q}}{k\sqrt{a}}\right) + \frac{\partial}{\partial q}\left(\frac{c_{11}\dfrac{\partial \phi}{\partial p} - c_{12}\dfrac{\partial \phi}{\partial q}}{k\sqrt{a}}\right)\right\} + \frac{c_{11}\, a_{22} - 2c_{12}\, a_{12} + c_{22}\, a_{11}}{a_{11}\, a_{22} - a_{12}^2}\,\phi = 0.$$

The a_{ik} , c_{ik} in it denote the coefficients of the aforementioned quadratic forms, and a is the determinant $a_{11}\, a_{22} - a_{12}^2$ of the first of them. Furthermore, k is the curvature of the point (p, q) on the surface in question, which will be assumed to vanish only at individual point or lines on the surface.

The function ϕ itself is the invariant of the differential expression:

$$u\, dx + v\, dy + w\, dz = P\, dp + Q\, dq\ ;$$

i.e., it is the quantity that is determined by the equation:

$$\phi = \frac{1}{2}\frac{\dfrac{\partial P}{\partial q} - \dfrac{\partial Q}{\partial p}}{\sqrt{a_{11}\,a_{22} - a_{12}^2}}\,.$$

If any real-valued integral ϕ of the differential equation (I) is known then it will correspond to a displacement without extension of the given curved surface in such a way that the displacement *ri* of a point on it along a fixed direction *r* can be ascertain from the equations:

$$\frac{\partial r}{\partial p} = R^2\,\frac{c_{11}\dfrac{\partial \phi R^{-1}}{\partial q} - c_{12}\dfrac{\partial \phi R^{-1}}{\partial p}}{k\sqrt{a}}\,,$$

$$\frac{\partial r}{\partial q} = -R^2\,\frac{c_{22}\dfrac{\partial \phi R^{-1}}{\partial p} - c_{12}\dfrac{\partial \phi R^{-1}}{\partial q}}{k\sqrt{a}}$$

by quadrature. In those equations, *R* denotes the cosine of the angle that the fixed direction *r* makes with the normal that is raised at the point (*p*, *q*).

Any well-defined real-valued function ϕ that satisfies equation (I) will then correspond (up to an additive constant) to a well-defined displacement without extension of the surface that we speak of, and conversely, every well-defined displacement without extension of it will correspond to a well-defined function ϕ that satisfies the differential equation.

In what follows, such a function ϕ that is intrinsically linked with an infinitely-small displacement without extension of a surface will be given the name of *displacement function*, and the differential equation (I) will be referred to as the differential equation of the *displacement function.*

The following theorems will be true then:

Any linear homogeneous function:

$$a\,X + b\,Y + c\,Z$$

of the cosines X, Y, Z represents a displacement function, and the displacement of a curved surface that is connected with that function consists of an infinitely-small rotation of the fixed surface around a line whose direction cosines are proportional to the constants a, b, c and an infinitely-small advance of that surface in an arbitrary direction.

Conversely:

Any infinitely-small possible motion of the fixed surface corresponds to a displacement function ϕ *of the form:*

$$\phi = a\,X + b\,Y + c\,Z.$$

Any integral ϕ *of the differential equation* (I) *that cannot be represented as a homogeneous linear function of the quantities X, Y, Z corresponds to an infinitely-small deformation of the surface under consideration.*

The foregoing theorems are easily obtained by considering the changes that the values of the given coefficients c_{ik} will have suffered at the corresponding point of the second surface that arises from an infinitely-small displacement. If one denotes the variations of those coefficient that correspond to the variations:

$$\delta x = u\,i, \qquad \delta y = v\,i, \qquad \delta z = w\,i$$

by δc_{ik} then after introducing the displacement function ϕ, those variations can be obtained in a simple form that is closely linked with the forms that appear in the theory of the transformations of quadratic differential expressions.

When one appeals to a notation that CHRISTOFFEL introduced into the theory of the transformations of quadratic differential expressions (E. B. CHRISTOFFEL, "Über die Transformation der homogenen Differentialausdrücke zweiten Grades," Borchardt's Journal, Bd. **70**) and in relation to the quadratic form:

$$dX^2 + dY^2 + dZ^2 = b_{11}\,dp^2 + 2\,b_{12}\,dp\,dq + b_{22}\,dq^2,$$

one introduces the notation:

$$\Delta^{11} = \frac{\partial^2\phi}{\partial p^2} - \begin{Bmatrix}11\\1\end{Bmatrix}\frac{\partial\phi}{\partial p} - \begin{Bmatrix}11\\2\end{Bmatrix}\frac{\partial\phi}{\partial q} + b_{11}\,\phi,$$

$$\Delta^{12} = \frac{\partial^2\phi}{\partial p\,\partial q} - \begin{Bmatrix}12\\1\end{Bmatrix}\frac{\partial\phi}{\partial p} - \begin{Bmatrix}12\\2\end{Bmatrix}\frac{\partial\phi}{\partial q} + b_{12}\,\phi,$$

$$\Delta^{22} = \frac{\partial^2\phi}{\partial q^2} - \begin{Bmatrix}22\\1\end{Bmatrix}\frac{\partial\phi}{\partial p} - \begin{Bmatrix}22\\2\end{Bmatrix}\frac{\partial\phi}{\partial q} + b_{22}\,\phi,$$

to abbreviate, then one will find the following values for the variations of the coefficients c_{11}, c_{12}, c_{22} of the original surface under a displacement of it that is mediated by a displacement function ϕ:

$$\delta c_{11} = \frac{1}{\sqrt{b}}(-\,c_{11}\,\Delta^{12} + c_{12}\,\Delta^{11})\,i,$$

$$\delta c_{12} = \frac{1}{\sqrt{b}}(-\,c_{11}\,\Delta^{22} + c_{12}\,\Delta^{12})\,i,$$

$$\delta c_{12} = \frac{1}{\sqrt{b}} (+ c_{22}\,\Delta^{11} - c_{12}\,\Delta^{12})\, i,$$

$$\delta c_{22} = \frac{1}{\sqrt{b}} (+ c_{22}\,\Delta^{12} - c_{12}\,\Delta^{22})\, i,$$

in which b means the determinant $b_{11}\,b_{22} - b_{12}^2$.

If one chooses the parameters u, v of the curvature lines of the given surface to be the variables p, q, resp., then since $c_{12} = 0$, $b_{12} = 0$ under that assumption, the foregoing equations will be converted into the following ones:

$$\delta c_{11} = -\frac{1}{\sqrt{b}}\, c_{11}\,\Delta^{12}\, i, \qquad \delta c_{12} = -\frac{1}{\sqrt{b}}\, c_{11}\,\Delta^{22}\, i,$$

$$\delta c_{12} = \frac{1}{\sqrt{b}}\, c_{22}\,\Delta^{11}\, i, \qquad \delta c_{22} = \frac{1}{\sqrt{b}}\, c_{22}\,\Delta^{12}\, i\,.$$

With their help, one can resolve the question that has been touched upon many times of the conditions under which a given surface will admit an infinitely-small deformation without extension under which the lines of curvature will, in turn, go to lines of curvature.

In order to go further into those conditions, it is obviously necessary that $c_{12} + \delta c_{12}$ should vanish for all points of the deformed surface, and since one already has $c_{12} = 0$, that δc_{12} itself should be equal to zero.

A displacement function ϕ must then exist for the given surface that simultaneously satisfies the equations:

$$\Delta^{11} = 0, \qquad \Delta^{22} = 0,$$

while Δ^{12} remains non-zero. If Δ^{12} were likewise zero then that would yield a linear homogeneous function of the cosines X, Y, Z for the displacement function in question, so its corresponding displacement would be one without deformation, and the conservation of the lines of curvature would be obvious.

Since the quantities Δ^{11}, Δ^{12}, Δ^{22} fulfill the equations:

$$0 = \frac{\partial}{\partial p}\left(\frac{\Delta^{22}}{\sqrt{b}}\right) - \frac{\partial}{\partial q}\left(\frac{\Delta^{12}}{\sqrt{b}}\right) + \frac{\Delta^{11}}{\sqrt{b}}\begin{Bmatrix}22\\1\end{Bmatrix} - 2\frac{\Delta^{12}}{\sqrt{b}}\begin{Bmatrix}12\\1\end{Bmatrix} + \frac{\Delta^{22}}{\sqrt{b}}\begin{Bmatrix}11\\1\end{Bmatrix},$$

$$0 = \frac{\partial}{\partial q}\left(\frac{\Delta^{11}}{\sqrt{b}}\right) - \frac{\partial}{\partial p}\left(\frac{\Delta^{12}}{\sqrt{b}}\right) + \frac{\Delta^{11}}{\sqrt{b}}\begin{Bmatrix}22\\2\end{Bmatrix} - 2\frac{\Delta^{12}}{\sqrt{b}}\begin{Bmatrix}12\\2\end{Bmatrix} + \frac{\Delta^{22}}{\sqrt{b}}\begin{Bmatrix}11\\2\end{Bmatrix},$$

identically for every arbitrary function ϕ and any variables p, q (Cf., "Über die Theorie der aufeinander abwickelbaren Oberflächen," Festschrift der Königlichen Technischen Hochschule zu Berlin, 1884, pp. 35), the simultaneous vanishing of Δ^{11}, Δ^{22} for non-vanishing Δ^{12} would require the fulfillment of the equations:

$$\frac{\partial}{\partial v}\left(\frac{\Delta^{12}}{\sqrt{b}}\right)+2\frac{\Delta^{12}}{\sqrt{b}}\begin{Bmatrix}12\\1\end{Bmatrix}=0, \qquad \frac{\partial}{\partial u}\left(\frac{\Delta^{12}}{\sqrt{b}}\right)+2\frac{\Delta^{12}}{\sqrt{b}}\begin{Bmatrix}12\\2\end{Bmatrix}=0,$$

which would go directly to the following ones:

$$\frac{\partial}{\partial v}\log\left(\frac{\Delta^{12}b_{11}}{\sqrt{b}}\right)=0, \qquad \frac{\partial}{\partial u}\log\left(\frac{\Delta^{12}b_{22}}{\sqrt{b}}\right)=0,$$

corresponding to the definition of the $\begin{Bmatrix}i\,k\\h\end{Bmatrix}$. It follows from them that:

$$\frac{b_{22}}{b_{11}}=\frac{V}{U}, \qquad \Delta^{12}=\sqrt{UV},$$

in which one understands U to mean a function of only the variable u, and V to mean a function of only the variable v. One must then be able to put the equation:

$$dX^2 + dY^2 + dZ^2 = b_{11}\,du^2 + b_{22}\,dv^2$$

into the form:

$$dX^2 + dY^2 + dZ^2 = \frac{b_{11}}{U}(U\,du^2 + V\,dv^2),$$

or the following one:

$$dX^2 + dY^2 + dZ^2 = \lambda\,(du'^2 + dv'^2),$$

in the latter of which, one defines:

$$du' = \sqrt{U}\,du, \qquad dv' = \sqrt{V}\,dv, \qquad \lambda = b_{11}\,U^{-1}.$$

Under the assumption that the functions u', v' (which can, in turn, be referred to as the parameters of the lines of curvature of the surface considered) were introduced as the variables u, v, the necessary conditions for the surface to admit an infinitely-small deformation that preserves the lines of curvature will now read:

$$\lambda = b_{11} = b_{22}, \qquad \Delta^{12} = 1.$$

However, as a result of the linear partial differential equations that the quantities X, Y, Z are subject to, those necessary conditions will also immediately prove to be sufficient, since one infers from the equations:

$$dX^2 + dY^2 + dZ^2 = \lambda\,(du^2 + dv^2),$$
$$X^2 + Y^2 + Z^2 = 1$$

the conclusion that the following differential expressions:

(II)
$$\frac{1}{\lambda}\left(\frac{\partial X}{\partial v}du + \frac{\partial X}{\partial u}dv\right) = d\alpha,$$
$$\frac{1}{\lambda}\left(\frac{\partial Y}{\partial v}du + \frac{\partial Y}{\partial u}dv\right) = d\beta,$$
$$\frac{1}{\lambda}\left(\frac{\partial Z}{\partial v}du + \frac{\partial Z}{\partial u}dv\right) = d\gamma$$

represent the total differentials of three functions α, β, γ of the variables u, v, from which those functions can be obtained by quadratures. After doing that, the function:

$$\phi = \alpha X + \beta Y + \gamma Z$$

will represent a function of the quantities u, v that does, in fact, satisfy the conditions:

$$\Delta^{11} = 0, \qquad \Delta^{12} = 0, \qquad \Delta^{22} = 0\,.$$

It will then follow that:

In order for a surface to admit an infinitely-small deformation that is not coupled with any extension, and under which its lines of curvature will go to lines of curvature in the deformed surface, it is necessary and sufficient that under the map of its lines of curvature to the GAUSSian sphere, that sphere is suitable for being associated with infinitely-small squares.

The family of surfaces that is characterized by the property that was expressed in the foregoing theorem possesses a feature that is analogous to the one that I exhibited in a publication that I presented to the Königlichen Akademie on 8 November 1883 for those surfaces that can be associated with infinitely-small squares by way of their lines of curvature.

When one understands ρ and ρ' to mean the radii of principal curvature of a curved surface at the point (x, y, z), that feature can be expressed in the following form:

In order for the map of the lines of curvature of a surface to the GAUSSian sphere to be capable of being assigned squares in the infinitely small, it is necessary and sufficient that the differential equation:

$$\left\{-\rho\rho'\left(\frac{\partial(\rho+\rho')}{\partial x}dX + \frac{\partial(\rho+\rho')}{\partial y}dY + \frac{\partial(\rho+\rho')}{\partial z}dZ\right) + d(\rho\rho')\right\}(\rho-\rho')^{-2}$$

should be the total differential of a function of position on that surface.

Proving that theorem and exhibiting of the *fourth*-order partial differential equation that the family of surfaces that we spoke of must satisfy emerge from an almost direct reproduction of the conclusions that were reached in the aforementioned article on 8 November 1883.

If Ω denotes the foregoing differential expression and one assumes that Ω proves to be the total differential of a function of position for a given surface then the sum:

$$dX^2 + dY^2 + dZ^2 = b_{11}\,dp^2 + 2b_{12}\,dp\,dq + b_{22}\,dq^2$$

will always take on the form:

$$\lambda\,(du^2 + dv^2)$$

by the introduction of suitable parameters for the lines of curvature of that surface, and one will determine the function λ as a function of the original variables p, q from the equation:

$$\Omega = \tfrac{1}{2}\,d\log\lambda$$

by quadrature without it being necessary to know the parameters u, v as functions of the p, q.

Equations (II) then give rise to a new consequence. One can easily convert the first one into the form:

$$d\alpha = \frac{1}{\lambda(\rho - \rho')}\{2\,(Y\,dz - Z\,dy) - (\rho + \rho')\,(Z\,dY - Y\,dZ)\},$$

from which the other two can be derived by cyclically permuting X, Y, Z. The function α, and as a result, β and γ, as well, can be obtained from that equation by quadrature without having to ascertain the parameters u, v. The same thing will therefore be true for the displacement function in question:

$$\phi = \alpha X + \beta Y + \gamma Z\,.$$

In order to ascertain the displacement function that mediates the deformation of a surface that preserves the lines of curvature, when the surface admits such a deformation, it is only necessary to perform quadratures of differential expressions that are given in terms of the original variables and their differentials.

Finally, the equation:

$$d\alpha\,dX + d\beta\,dY + d\gamma\,dZ = 2\,du\,dv$$

also shows that the parameters of the lines of curvature in the surface that we speak of can be determined by quadratures in their own right.

When a surface admits an infinitely-small bending that takes its lines of curvature to other ones on the bent surface, nothing will follow from that fact alone besides the fact that the second surface will, in turn, possess that same property. Should the property in question also remain true for that surface, and also for further deformations, which would then follow, then it would not be sufficient for the square of the map of its line elements to the GAUSSian sphere to be capable of being put into the form $\lambda\,(du^2 + dv^2)$ in terms of the parameters of the lines of curvature, but it would be necessary, in addition, that λ should be a function of only the parameter u or the other one v.

The type of surface that corresponds to the latter condition, which belongs to the general class that was defined above, can be given completely and proves to belong to MONGE's *moulure* (milling) surfaces, to which one can also count surfaces of revolution. That remark agrees with a study by CODAZZI, who showed in volume VII of series 1 of TORTOLONI's Annali (1856) that the surfaces in question and the developables were the only ones that could be subjected to finite deformations that preserved their lines of curvature.

The developable surfaces were expressly excluded from our analysis.

____