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INTRODUCTION TO

LINE GEOMETRY AND KINEMATICS

BY

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Foreword

According to **Felix Klein**, line geometry is the geometry of a quadratic manifold in a five-dimensional space. According to **Eduard Study**, kinematics – viz., the geometry whose spatial element is a motion – is the geometry of a quadratic manifold in a seven-dimensional space, and as such, a natural generalization of line geometry. The geometry of multidimensional spaces is then connected most closely with the geometry of three-dimensional spaces in two different ways. The present guide gives an introduction to line geometry and kinematics on the basis of that coupling.

In the treatment of linear complexes in R_3, the line continuum is mapped to an M_4^2 in R_5. In that subject, the linear manifolds of complexes are examined, along with the loci of points and planes that are linked to them that lead to their analytic representation, with the help of **Weitzenböck's** complex symbolism. One application of the map gives **Lie**'s line-sphere transformation. Metric (Euclidian and non-Euclidian) line geometry will be treated, up to the axis surfaces that will appear once more in ray geometry as chains. The conversion principle of ray geometry admits the derivation of a parametric representation of motions from **Euler**'s rotation formulas, and thus exhibits the connection between line geometry and kinematics. The main theorem on motions and transfers will be derived by means of the elegant algebra of biquaternions.

Maps in the usual sense can be contained in this book only to a lesser degree, since it will treat geometry in complex or multidimensional spaces, for the most part. Symbolic figures have been avoided for the sake of saving space.

The main facts of our guide are indissolubly coupled with the name of **Eduard Study**. The treatment of the line-sphere transformation, ray geometry, and kinematics here goes back to **Study**. We shall refer to the original literature by citations at the relevant places. With those references, we would like to satisfy not only the duty of gratitude to our esteemed teacher, but also the point out the path into **Study**'s world of ideas to the reader.

I would like to thank **W. Brach**, **H. Peters**, and **H. Schröder** for their assistance in the correction, and the publisher for the eagerness that they showed in their decision to print this guide.

Bonn, in January 1935.

Ernst August Weiss

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Chapter One

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§ 1. Plückerian line coordinates.

1. Notations. The basic facts of the geometry of the line, the plane, and space will be assumed in what follows. We denote binary parameters by Greek symbols ξ, η. The parameter ξ has the coordinates $\xi_1 : \xi_2$. The condition for the binary parameters ξ, η to coincide reads:

$$(\xi\,\eta) = \xi_1\eta_2 - \xi_2\eta_1 = 0. \tag{1}$$

We denote the homogeneous coordinates of the points $x, y, \ldots, p, q, \ldots$ and the planes $u, v, \ldots$ by $x_i, y_i, \ldots, p_i, q_i, \ldots$ and $u_i, v_i, \ldots, a_i, b_i, \ldots$, resp. in the quaternary domain (i = 0, 1, 2, 3). Like the binary parameters, these coordinates are complex numbers, except for the cases in which we expressly restrict ourselves to the real domain. The equation:

$$(u\,x) = u_0\,x_0 + u_1\,x_1 + u_2\,x_2 + u_3\,x_3 = 0 \tag{2}$$

represents the condition for the point x to be incident with the plane u. (For fixed u, it is the equation of the plane u; for fixed x, it is the equation of the point.)

The vanishing of the determinant $(x^0\,x^1\,x^2\,x^3)$ of the coordinates of the four points x^k gives the condition for the four points to be linearly dependent, so they will lie in (at least) one plane. Dually, $(u^0\,u^1\,u^2\,u^3) = 0$ gives the condition for the four planes u^k to be linearly dependent; i.e., to run through (at least one) common point.

2. Definition of line coordinates. A line of R_3 can be represented by a parametric representation:

$$\xi_1\,x + \xi_2\,y \tag{3}$$

as the connecting line of two points x, y, and by a system of two equations:

$$(u\,x) = 0, \qquad (v\,x) = 0 \tag{4}$$

as the intersection of two planes u, v, and thus, as the carrier of the pencil of planes that is spanned by the planes:

$$\xi_1\,(u\,x) + \xi_2\,(v\,y) = 0 \tag{3}$$

The line will then appear to be a locus of points (viz., a *ray*) or planes (viz., an *axis*).

If we, with **J. Plücker** ([1]), would now like to introduce the line as a *spatial element* – i.e., regard geometric figures as the loci of straight lines – then we would need to represent the lines by coordinates.

We define the *Plücker coordinates* of the connecting line $\mathfrak{X} = \widehat{xy}$ by the six two-rowed determinants in the matrix:

$$\begin{Vmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{Vmatrix},$$

so

$$(6) \qquad \begin{cases} \mathfrak{X}_{01} = x_0 y_1 - x_1 y_0, & \mathfrak{X}_{02} = x_0 y_2 - x_2 y_0, & \mathfrak{X}_{03} = x_0 y_3 - x_3 y_0, \\ \mathfrak{X}_{23} = x_2 y_3 - x_3 y_2, & \mathfrak{X}_{31} = x_3 y_1 - x_1 y_3, & \mathfrak{X}_{12} = x_1 y_2 - x_2 y_1. \end{cases}$$

The quantities are determined by the line only up to a non-zero factor. Namely, if one replaces the points x, y by two other points of the lines:

$$(7) \qquad x' = \xi_1 x + \xi_2 y, \qquad y' = \eta_1 x + \eta_2 y, \qquad (\xi\,\eta) \neq 0$$

then one will get:

$$(8) \qquad \mathfrak{X}'_{ik} = (\xi\,\eta) \cdot \mathfrak{X}_{ik}$$

for the coordinates $\mathfrak{X}'_{ik}$ of $\widehat{x'y'}$. One then comes down to only the ratios of the line coordinates. *The line coordinates are homogeneous coordinates.*

It follows from homogeneity that at most five of the coordinates are essential ones. However, there are only ∞^4 lines in R_3 (∞^6 point-pairs, but every line contains ∞^2 point-pairs: $\infty^6 : \infty^2 = \infty^4$). The line coordinates cannot be independent of each other then; an identity relation must exist between them. One will obtain that relation from the equation:

$$(9) \qquad (\widehat{xy}\ \widehat{xy}) = 0.$$

The four-rowed determinant vanishes, because two times two of its rows are the same. If one (with **Laplace**) develops it in its first two rows then one will get:

$$(10) \qquad \mathfrak{X}_{01}\,\mathfrak{X}_{23} + \mathfrak{X}_{02}\,\mathfrak{X}_{31} + \mathfrak{X}_{03}\,\mathfrak{X}_{12} = 0.$$

The expression on the left-hand side is called the *Plücker expression*, while the identity itself is called the *Plücker identity.*

Theorem 1: *The (homogeneous) Plücker line coordinates satisfy the quadratic Plücker identity.*

([1]) **J. Plücker**, *Neue Geometrie des Raumes, gegründet auf die Betrachtung der geraden Linie als Raumelement,* Leipzig, 1868.

3. Line and point. Now, is it also true that conversely, any system of six quantities $\mathfrak{X}_{ik}$ that satisfy the Plücker identity will determine a line? In order to answer that question, we must know the condition for a point x to lie on a line $\mathfrak{X} = \widehat{yz}$.

Let x be an arbitrary point, and let $\mathfrak{X} = \widehat{yz}$ be an arbitrary line. The equation of their connecting line (in variables point coordinates t) will then read:

$$(11) \qquad (x\ \widehat{yz}\ t) = 0.$$

Its coordinates – viz., the coefficients of t_i – will be the three-rowed determinants of the matrix:

$$(12) \qquad \begin{Vmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \\ z_0 & z_1 & z_2 & z_3 \end{Vmatrix},$$

namely:

$$(18) \qquad \left\{ \begin{array}{llllll} u_0 = & * & -x_1\mathfrak{X}_{23} & -x_2\mathfrak{X}_{31} & -x_3\mathfrak{X}_{12}, \\ u_1 = & x_0\mathfrak{X}_{23} & * & -x_2\mathfrak{X}_{03} & +x_3\mathfrak{X}_{02}, \\ u_2 = & x_0\mathfrak{X}_{31} & +x_1\mathfrak{X}_{03} & * & -x_3\mathfrak{X}_{01}, \\ u_3 = & x_0\mathfrak{X}_{12} & -x_1\mathfrak{X}_{02} & +x_2\mathfrak{X}_{01} & * & . \end{array} \right.$$

Now, should the point x lie on the line $\mathfrak{X}$, the connecting line would be undetermined, so the u_i would have to vanish. Therefore, one would have the equations:

$$(14) \qquad \left\{ \begin{array}{llllll} 0 = & * & -x_1\mathfrak{X}_{23} & -x_2\mathfrak{X}_{31} & -x_3\mathfrak{X}_{12}, \\ 0 = & x_0\mathfrak{X}_{23} & * & -x_2\mathfrak{X}_{03} & +x_3\mathfrak{X}_{02}, \\ 0 = & x_0\mathfrak{X}_{31} & +x_1\mathfrak{X}_{03} & * & -x_3\mathfrak{X}_{01}, \\ 0 = & x_0\mathfrak{X}_{12} & -x_1\mathfrak{X}_{02} & +x_2\mathfrak{X}_{01} & * & , \end{array} \right.$$

together with the identity:

$$(15) \qquad (x\ \widehat{yz}\ t) \equiv 0 \quad \{t\} \qquad \text{(viz., “identically zero for all } t\text{”)},$$

as *the necessary and sufficient conditions for the point x to lie on the line $\mathfrak{X}$.* The requirement that a point should lie on a line actually represents only two conditions – viz., the point must lie on two distinct planes that run through the line – while no equation in equations (14) is dispensable, on the grounds of symmetry, and since one or the other equation can break down.

If one now regards the quantities $\mathfrak{X}_{ik}$ in the system (14) as arbitrary, but fixed, then one will have a system of four linear, homogeneous equations for the x_i. Non-trivial solutions will exist if and only if its determinant vanishes. However, one finds that the determinant of the skew-symmetric is the square of the Plücker expression. If one then assumes that the given $\mathfrak{X}_{ik}$ satisfies the Plücker identity without all of them vanishing then the determinant will vanish, and – as a simple calculation will show – the matrix will have rank 2. There will then be ∞^2 solutions, and correspondingly ∞^1 points:

(16) $$\xi_1\, y + \xi_2\, z$$

that satisfy the system of equations. If one connects any two of these points then one will obtain a line with the coordinates $\mathfrak{X}'_{ik}$. If one poses the systems of equations for these $\mathfrak{X}'_{ik}$ that is analogous to (14) then one will obtain, by construction, a system of equations with (16) for its solutions. The quantities $\mathfrak{X}_{ik}$ will then be identical with the coordinates of the line $\mathfrak{X}' = \widehat{yz}$ (up to a factor). Q. E. D.

Theorem 2: *A system of six quantities $\mathfrak{X}_{ik}$ that do not all vanish and that satisfy the Plücker identity is the system of line coordinates of a well-defined line.*

4. Ray coordinates and axial coordinates. In the definition of line coordinates, the line was regarded as a locus of points, up to now. The coordinates thus-defined will also be called *ray coordinates* then. The *axial coordinates* of a line are defined dually. One then understands the coordinates $\mathfrak{A}_{ik}$ of the line of intersection of the planes u, v to mean the two-rowed determinants of the matrix:

$$\left\| \begin{matrix} u_0 & u_1 & u_2 & u_3 \\ v_0 & v_1 & v_2 & v_3 \end{matrix} \right\|,$$

so

(17) $$\begin{cases} \mathfrak{A}_{01} = u_0 v_1 - u_1 v_0, & \mathfrak{A}_{02} = u_0 v_2 - u_2 v_0, \quad \mathfrak{A}_{03} = u_0 v_3 - u_3 v_0, \\ \mathfrak{A}_{23} = u_2 v_3 - u_3 v_2, & \mathfrak{A}_{31} = u_3 v_1 - u_1 v_3, \quad \mathfrak{A}_{12} = u_1 v_2 - u_2 v_1. \end{cases}$$

Everything that we said about ray coordinates up to now can be carried over to axial coordinates dually. Here, we will next be interested in *the connection between the ray and axial coordinates of one and the same line.*

Let x, y be two different points, and let u, v be two distinct planes through one and the same line, such that:

(18) $$(u\, x) = 0, \qquad (v\, x) = 0, \qquad (v\, x) = 0, \qquad (v\, y) = 0.$$

Furthermore, let z, z' be two variable points, and let w, w' be two variable planes. From the multiplication theorem, the four-rowed determinant is then:

$$(19)\quad \begin{cases} (x\,y\,z\,z')\cdot(u\,v\,w\,w') = \begin{vmatrix} (xu) & (xv) & (xw) & (xw') \\ (yu) & (yv) & (yw) & (yw') \\ (zu) & (zv) & (zw) & (zw') \\ (z'u) & (z'v) & (z'w) & (z'w') \end{vmatrix} \\ \text{so, from (18):} \\ = \{(x\,w)(y\,w') - (x\,w')(y\,w)\}\{(z\,u)(z'\,v) - (z\,v)(z'\,u)\}. \end{cases}$$

We set:

(20) $$(x\,w)\,(y\,w') - (x\,w')\,(y\,w) = (xy,\, ww'),$$

to abbreviate. (19) can then be written in the form:

(21) $$(x\,y\,z\,z') \cdot (u\,v\,w\,w') = (xy,\, ww') \cdot (uv,\, zz').$$

In this, we now fix ww', and indeed, in such a way that either $(u\ v\ w\ w')$ or $(xy,\ ww')$ will vanish. These two expressions will then be non-zero numerical factors. An identity in $\widehat{z\,z'}$ remains. The coefficients of $\mathfrak{Z}_{ik} = z_i z_k' - z_k z_i'$ in the expressions $(x\ y\ z\ z')$ and $(uv,\ zz')$ will thus be proportional to each other:

$$\begin{aligned} &\mathfrak{X}_{01} : \mathfrak{X}_{02} : \mathfrak{X}_{03} : \mathfrak{X}_{23} : \mathfrak{X}_{31} : \mathfrak{X}_{12} \\ = \; &\mathfrak{A}_{01} : \mathfrak{A}_{02} : \mathfrak{A}_{03} : \mathfrak{A}_{23} : \mathfrak{A}_{31} : \mathfrak{A}_{12}\,. \end{aligned}$$

On the grounds of this relationship, it is not generally necessary to use ray and axial coordinate together. In what follows, if we speak of line coordinates with no further qualifiers then we will mean ray coordinates.

5. Line and plane. With these preliminaries, we can easily dualize what was said in no. **3** about a line and a point. From (13) and Theorem 3, we find that *the point of intersection of a line* $\mathfrak{X}$ *with the plane* u is:

$$(22)\quad \begin{cases} x_0 = \quad * \quad - u_1\mathfrak{X}_{23} - u_2\mathfrak{X}_{31} - u_3\mathfrak{X}_{12}, \\ x_1 = u_0\mathfrak{X}_{23} \quad * \quad - u_2\mathfrak{X}_{03} + u_3\mathfrak{X}_{02}, \\ x_2 = u_0\mathfrak{X}_{31} + u_1\mathfrak{X}_{03} \quad * \quad - u_3\mathfrak{X}_{01}, \\ x_3 = u_0\mathfrak{X}_{12} - u_1\mathfrak{X}_{02} + u_2\mathfrak{X}_{01} \quad * \quad . \end{cases}$$

One will obtain *the necessary and sufficient condition for the plane u to run through the line* $\mathfrak{X}$ when one demands that the point of intersection x should remain undetermined, and thus, when one sets $x_i = 0$ in (22).

6. Line and line. It remains for us to exhibit the condition for two lines $\mathfrak{X}$ and $\mathfrak{Y}$ to be incident. Let $\mathfrak{X}$ be the connecting line of the points x, x', and let $\mathfrak{Y}$ be the connecting line of the points y, y'. Should the two lines intersect, the four points would have to lie in a plane:

(28) $$(\widehat{x\,x'}\,\widehat{y\,y'}) = 0.$$

However, if one develops the left-hand side according to **Laplace**'s rule along the first two rows then it will follow that:

(24) $$\mathfrak{X}_{01}\,\mathfrak{Y}_{23} + \mathfrak{X}_{02}\,\mathfrak{Y}_{31} + \mathfrak{X}_{03}\,\mathfrak{Y}_{12} + \mathfrak{X}_{23}\,\mathfrak{Y}_{01} + \mathfrak{X}_{31}\,\mathfrak{Y}_{02} + \mathfrak{X}_{12}\,\mathfrak{Y}_{03} = 0.$$

One denotes the left-hand side of this expression by $(\mathfrak{X}\,\mathfrak{Y})$.

Theorem 4: *The condition for two lines* $\mathfrak{X}$ *and* $\mathfrak{Y}$ *to intersect reads* $(\mathfrak{X}\,\mathfrak{Y}) = 0$.

The **Plücker** expression can be written in the form $\frac{1}{2}(\mathfrak{X}\mathfrak{X})$ with the recently-introduced notations.

§ 2. The linear complex.

7. Definition of the linear complex. If one fixes the line $\mathfrak{Y}$ in the equation:

(1) $$(\mathfrak{X}\,\mathfrak{Y}) = 0$$

then one will get the condition for a variable line $\mathfrak{X}$ to intersect the fixed line $\mathfrak{Y}$, viz., *the equation of the line* $\mathfrak{Y}$. This equation is linear in the line coordinates $\mathfrak{X}_{ik}$, but it is not the most general relationship that is linear in the line coordinates: Equation (1) must exist between the coefficients $\mathfrak{Y}_{ik}$, namely, the **Plücker** relation.

We shall now consider the most general linear relation between line coordinates $\mathfrak{X}_{ik}$:

(2) $$(\mathfrak{C}\,\mathfrak{X}) = \mathfrak{C}_{01}\,\mathfrak{X}_{23} + \mathfrak{C}_{02}\,\mathfrak{X}_{31} + \ldots + \mathfrak{C}_{12}\,\mathfrak{X}_{03} = 0.$$

We call the locus of all lines $\mathfrak{C}$ whose coordinates satisfy the equation (2) a *linear complex*. A linear complex contains ∞^3 lines. [The ∞^4 lines in R_3 are subject to one condition (2).] There are ∞^5 linear complexes, corresponding to the six homogeneous coordinates $\mathfrak{C}_{ik}$ of the complex.

The relation $\frac{1}{2}(\mathfrak{C}\ \mathfrak{C}) = 0$ will not generally exist between the $\mathfrak{C}_{ik}$. If that relation exists then we will call the complex *singular*. One is then dealing with the *complex of lines that meet the line* $\mathfrak{C}$. In the other cases, we shall speak of a *regular* complex.

8. The null system. Our next problem is to gain an overview of the mutual positions of the ∞^3 lines of a regular linear complex. To that end, we think of the line $\mathfrak{X}_{ik}$ as being the connecting line of the points x and y in (2):

$$(3) \qquad (\mathfrak{C}\,\widehat{x\,y}) = \mathfrak{C}_{01}(x_2 y_3 - x_3 y_2) + \mathfrak{C}_{02}(x_3 y_1 - x_1 y_3) + \ldots + \mathfrak{C}_{12}(x_0 y_3 - x_3 y_0) = 0.$$

Here, one has an *alternating bilinear form* in x, y on the left-hand side (i.e., the form will change sign when one switches x and y). The equation then represents a skew-symmetric correlative relationship. If one fixes the point x then the locus of all points y that produce a line of the complex when linked with x (viz., the locus of all lines of the complex that run through x) will be the plane whose equation is (3) and whose coordinates are:

$$(4) \qquad \left\{\begin{array}{llllll} u_0 = & * & -\mathfrak{C}_{23}x_1 & -\mathfrak{C}_{31}x_2 & -\mathfrak{C}_{12}x_3, \\ u_1 = & \mathfrak{C}_{23}x_0 & * & -\mathfrak{C}_{03}x_2 & +\mathfrak{C}_{02}x_3, \\ u_2 = & \mathfrak{C}_{31}x_0 & +\mathfrak{C}_{03}x_1 & * & -\mathfrak{C}_{01}x_3, \\ u_3 = & \mathfrak{C}_{12}x_0 & -\mathfrak{C}_{02}x_1 & +\mathfrak{C}_{01}x_2 & * \quad . \end{array}\right.$$

This system of equations is identical with the system (13) in § 1, except that we now assume that the complex is regular, and therefore that the determinant $\left[\frac{1}{2}(\mathfrak{C}\mathfrak{C})\right]^2$ of the system is non-zero. The relationship (4) will then be regular and will be called a *null system.*

Theorem 5: *A null system belongs to a regular, linear complex that associates every point x with a plane u that goes through it (viz., the null plane) and is the locus of all lines of the complex (viz., null lines) that go through x.*

The null lines that go through a point then define a planar pencil, and thus the lines of the complex that lie in a plane will fill up a pencil (since the relationship is regular, every plane will be associated with a *null point*). In order to obtain the relationship that assigns an associated null point to every plane, one must merely regard $\mathfrak{X}$ as the line of intersection of two planes in (2):

$$(5) \qquad (\mathfrak{C}\,\widehat{x\,y}) = \mathfrak{C}_{01}(u_0 v_1 - u_1 v_0) + \mathfrak{C}_{02}(u_0 v_2 - u_2 v_0) + \ldots + \mathfrak{C}_{12}(u_1 v_2 - u_2 v_1) = 0.$$

It will follow immediately from the skew symmetry in equation (3) that:

Theorem 6: *If x lies on the null plane of y then y will lie on the null plane of x.*

Thus, if a point runs through a plane then its null plane will rotate about the null point of the plane. If a point runs along a line $\mathfrak{G}$ (which can be regarded as the intersection of two planes) then its null plane will rotate around a line $\mathfrak{G}'$ (which can be regarded as the connecting line of the null points of those planes). The lines $\mathfrak{G}$, $\mathfrak{G}'$ are called *null polar* to each other. *The null lines are the lines that are null polar to themselves.* Later on (no. **11**), we will learn how to determine the null polar to a line analytically (i.e., to describe the null system in line coordinates).

In the case $\frac{1}{2}(\mathfrak{C}\ \mathfrak{C}) = 0$, we would like to call (4) a *singular null system*. A singular null system is determined by a line $\mathfrak{C}$, namely, the *guiding line of the complex.* Any point that does not lie on $\mathfrak{C}$ will be associated with its connecting line with $\mathfrak{C}$. A point of $\mathfrak{C}$ itself will not correspond to any well-defined plane.

9. Möbius's pair of tetrahedra. A simple application of the (regular) null system is the derivation of **Möbius's** pair of tetrahedra ([1]). One makes a tetrahedron that consists of the points p^0, p^1, p^2, p^3 $[(p^0\ p^1\ p^2\ p^3) \neq 0]$ and the faces a^0, a^1, a^2, a^3 subordinate to a null system. A second tetrahedron will arise whose faces are b^0, b^1, b^2, b^3, and whose vertices are q^0, q^1, q^2, q^3. q_i will then lie in a_i (as the null point of a_i) and b_i will run through p_i (as the null plane of p_i).

Theorem 7: *When one applies a null system to a tetrahedron, a second tetrahedron will arise that is in Möbius position with respect to the first one: i.e., the two tetrahedra are inscribed and circumscribed equilaterally to each other.*

10. The pencil of complexes. Two different complexes $\mathfrak{A}$ and $\mathfrak{B}$ span a pencil of linear complexes:

$$(\mathfrak{C}\ \mathfrak{X}) \equiv \xi_1(\mathfrak{A}\ \mathfrak{X}) + \xi_2(\mathfrak{B}\ \mathfrak{X}) = 0. \tag{6}$$

We would like to look for the singular complexes of the pencil. To that end, we set the **Plücker** expression that is defined by the complex $\mathfrak{C}$ equal to zero:

$$(\mathfrak{C}\ \mathfrak{C}) \equiv \xi_1^2(\mathfrak{A}\ \mathfrak{A}) + 2\xi_1\xi_2(\mathfrak{A}\ \mathfrak{B}) + \xi_2^2(\mathfrak{B}\ \mathfrak{B}) = 0. \tag{7}$$

This equation shows:

Theorem 8: *The pencil of complexes that is spanned by the two distinct linear complexes $\mathfrak{A}$ and $\mathfrak{B}$ contains:*

([1]) **A. F. Möbius**, "Kann von zwei dreiseitigen Pyramiden eine jede in bezug auf die andere um- und eingeschrieben zugleich heißen?" Crelle's Journal **3** (1829), pp. 273.

1. *For* $(\mathfrak{A}\mathfrak{A})\,(\mathfrak{B}\mathfrak{B}) - (\mathfrak{A}\mathfrak{B})^2 \neq 0$: *Two distinct singular complexes (regular pencil)*

2. *For* $(\mathfrak{A}\mathfrak{A})\,(\mathfrak{B}\mathfrak{B}) - (\mathfrak{A}\mathfrak{B})^2 = 0$, *Two coincident singular complexes (singular pencil).*
but $(\mathfrak{C}\mathfrak{C}) \not\equiv 0\ \{\xi\}$:

3. *For* $(\mathfrak{C}\mathfrak{C}) \not\equiv 0\ \{\xi\}$: ∞^1 *singular complexes. It is the pencil of lines that is spanned by the incident lines* $\mathfrak{A}$ *and* $\mathfrak{B}$.

The *regular pencils* can be spanned by two of the singular complexes $\mathfrak{G}$ and $\mathfrak{G}'$ that are contained in them:

(8) $$(\mathfrak{C}\mathfrak{X}) \equiv \eta_1\,(\mathfrak{G}\mathfrak{X}) + \eta_2\,(\mathfrak{G}'\mathfrak{X}) = 0.$$

Thus, since $(\mathfrak{G}\mathfrak{G}) = (\mathfrak{G}'\mathfrak{G}') = 0$, one will have:

(9) $$(\mathfrak{C}\mathfrak{C}) = 2\,\eta_1\eta_2\,(\mathfrak{G}\mathfrak{G}'),$$

and it will follow from this that $(\mathfrak{G}\mathfrak{G}') \neq 0$: The lines $\mathfrak{G}$ and $\mathfrak{G}'$ do not intersect; they are *skew.*

From (8), one sees that a line $\mathfrak{X}$ that cuts $\mathfrak{G}$ and $\mathfrak{G}'$ will be common to all complexes of the pencil:

Theorem 9: *The complexes of a regular pencil generate a "regular linear congruence," namely, the manifold of* ∞^1 *common lines of intersection of two skew lines* $\mathfrak{G}$ *and* $\mathfrak{G}'$ *(the guiding lines of the congruence).*

The guiding lines themselves do not belong to the congruence.

The *singular pencils* can be spanned by a regular complex $\mathfrak{A}$ of the pencil and the double-counted singular complex $\mathfrak{G}$ of the pencil:

(10) $$(\mathfrak{C}\mathfrak{X}) \equiv \eta_1\,(\mathfrak{A}\mathfrak{X}) + \eta_2\,(\mathfrak{G}\mathfrak{X}) = 0.$$

One will then have:

(11) $$(\mathfrak{C}\,\mathfrak{C}) \equiv \eta_1^2(\mathfrak{A}\,\mathfrak{A}) + 2\eta_1\eta_2(\mathfrak{A}\,\mathfrak{G}) = \eta_1\{\,\eta_1(\mathfrak{A}\mathfrak{A}) + 2\eta_2(\mathfrak{A}\mathfrak{G})\}.$$

This time, the discriminant of the quadratic form will vanish. It will then follow that:

(12) $$(\mathfrak{A}\mathfrak{G}) = 0.$$

Theorem 10: *The complexes of a singular pencil generate a "singular linear congruence," namely, the manifold of all* ∞^2 *null lines of a regular complex that cut a line* $\mathfrak{G}$ *of this complex (guiding line of the congruence).*

The guiding line itself belongs to the congruence. Later on (no. **22**), we will learn about an intuitive construction of a singular congruence.

11. The null system in line coordinates. For a line $\mathfrak{G}$, one can find the null polar of $\mathfrak{G}$ relative to the null system that is coupled to a regular complex $\mathfrak{A}$. We consider the pencil that is spanned by $\mathfrak{A}$ and $\mathfrak{G}$. Of the two singular complexes that are contained in it, we know one of them $\mathfrak{G}$ from the outset. The second one can be determined from equation:

$$\eta_1(\mathfrak{A}\mathfrak{A}) + 2\eta_2\,(\mathfrak{A}\mathfrak{G}) = 0, \qquad \eta_1 : \eta_2 = 2\,(\mathfrak{A}\mathfrak{G}) : -\,(\mathfrak{A}\mathfrak{A}), \tag{13}$$

as:

$$(\mathfrak{G}'\mathfrak{X}) \equiv 2(\mathfrak{A}\mathfrak{G}) \cdot (\mathfrak{A}\mathfrak{X}) - (\mathfrak{A}\mathfrak{A}) \cdot (\mathfrak{G}\mathfrak{X}) = 0. \tag{14}$$

We now state:

Theorem 11: *The null polar* $\mathfrak{G}'$ *of a line* $\mathfrak{G}$ *relative to a null system that is coupled to a regular complex* $\mathfrak{A}$ *is the second guiding line in the linear congruence that is determined by* $\mathfrak{G}$ *and* $\mathfrak{A}$.

Proof: A common line of intersection of two null polar lines $\mathfrak{G}$ and $\mathfrak{G}'$ is a null line of the complex (since the null plane of a point of $\mathfrak{G}$ runs through $\mathfrak{G}'$). Conversely: A null line that meets $\mathfrak{G}$ will also meet $\mathfrak{G}'$ (since the null system switches $\mathfrak{G}$ with $\mathfrak{G}'$, while transforming the null line into itself, and – like any correlation – leaving incidence unchanged). Q. E. D.

Thus, (14) represents the *null system in line coordinates.*

12. Conjugate complexes. The expression $(\mathfrak{AB})$ that is defined by two regular complexes $\mathfrak{A}$, $\mathfrak{B}$ appears in (7) for the first time. Two complexes that satisfy the equation:

$$(\mathfrak{AB}) = 0 \tag{15}$$

are called *conjugate.* We have already interpreted the vanishing of $(\mathfrak{AB})$ in the case for which the two complexes $\mathfrak{A}$ and $\mathfrak{B}$ are singular: The lines $\mathfrak{A}$ and $\mathfrak{B}$ are then incident. We have also already interpreted the equation $(\mathfrak{AB}) = 0$ in the case where one of the complexes $\mathfrak{A}$ is regular and the other one $\mathfrak{B}$ is singular: The line $\mathfrak{B}$ belongs to the complex $\mathfrak{A}$. The equation shall now be interpreted in the case where the complexes $\mathfrak{A}$ and $\mathfrak{B}$ are both regular. The pencil that they span:

$$(\mathfrak{CX}) \equiv \xi_1\,(\mathfrak{AX}) + \xi_2\,(\mathfrak{BX}) = 0 \tag{16}$$

is then regular; the quadratic form:

$$(\mathfrak{CC}) = \xi_1^2\,(\mathfrak{AA}) + \xi_2^2\,(\mathfrak{BB}) \tag{17}$$

will then have a discriminant that is, by assumption, non-zero:

$$D = (\mathfrak{AA})\,(\mathfrak{BB}) \neq 0. \tag{18}$$

The pencil has two distinct, skew, guiding lines $\mathfrak{G}$ and $\mathfrak{G}'$. In the binary domain $\xi_1 : \xi_2$ of the pencil of complexes, the equation of the pair of complexes $\mathfrak{A}$, $\mathfrak{B}$ will now be achieved by setting the quadratic form:

$$2\xi_1\,\xi_2 \tag{19}$$

equal to zero. The harmonic invariant ([3]) of the quadratic forms (17) and (19) will then vanish.

Theorem 12: *Two conjugate, regular complexes lie in the (regular) pencil that they span harmonically to the pair of singular complexes of the pencil.*

([3]) The *harmonic invariant* of the two quadratic forms:

$$a_{11}\,\xi_2^2 - 2a_{12}\,\xi_2\,\xi_1 + a_{22}\,\xi_1^2\,,$$
$$b_{11}\,\xi_2^2 - 2b_{12}\,\xi_2\,\xi_1 + b_{22}\,\xi_1^2$$

reads $a_{11}\,b_{22} - 2\,a_{12}\,b_{12} + a_{22}\,b_{11}$. Its vanishing is the necessary and sufficient condition for the pair of zero loci of the two forms to be harmonic to each other.

We would like to illustrate this theorem by an immediate consequence of it. A pencil of null systems is coupled to the pencil of complexes:

$$\xi_1(\mathfrak{A}\widehat{x\,y}) + \xi_2(\mathfrak{B}\widehat{x\,y}) = 0, \tag{20}$$

by which, a point x in general position (viz., one that does not lie on a guiding line) is associated with a pencil of planes. The axis of the pencil is the common line of intersection of $\mathfrak{G}$, $\mathfrak{G}'$ that runs through x. Now, that pencil of planes will be related projectively to the pencil of complexes by the identity of the parameters:

Theorem 13: *Let two regular complexes $\mathfrak{A}$ and $\mathfrak{B}$ be conjugate. The two planes that are associated with a point x in general position by the null system that is linked to them will then lie harmonically to the connecting planes of the point x with the skew guiding line $\mathfrak{G}$, $\mathfrak{G}'$ of the intersection congruence of* $\mathfrak{A}$, $\mathfrak{B}$.

We will learn about two further geometric interpretations of the important relation $(\mathfrak{A}\mathfrak{B}) = 0$ later on (no. **13**, **19**).

Chapter Two

Line geometry as geometry in R_5

§ 3. M_4^2 as the image of the line continuum.

13. Plücker's M_4^2. The situation that was treated in the last paragraph will become much more intuitive when one appeals to the following map ([4]): One interprets the six coordinates $\mathfrak{C}_{ik}$ of a linear complex as homogeneous point coordinates in R_5. A linear complex in R_3 then corresponds to a point in R_5. The ∞^4 singular complexes then correspond to the ∞^4 distinguished points whose coordinates fulfill the **Plücker** identity:

$$\tfrac{1}{2}(\mathfrak{C}\mathfrak{C}) \equiv \mathfrak{C}_{01}\,\mathfrak{C}_{23} + \mathfrak{C}_{02}\,\mathfrak{C}_{31} + \mathfrak{C}_{03}\,\mathfrak{C}_{12} = 0. \tag{1}$$

These points then lie on a four-dimensional quadratic manifold (M_4^2) that we shall call the *Plücker manifold.* M_4^2 is regular; by the substitution:

$$\begin{cases} \mathfrak{C}_{01} = x_1 + ix_2, & \mathfrak{C}_{02} = x_3 + ix_4, & \mathfrak{C}_{03} = x_3 + ix_6, \\ \mathfrak{C}_{23} = x_1 - ix_2, & \mathfrak{C}_{31} = x_3 - ix_4, & \mathfrak{C}_{12} = x_5 - ix_6 \end{cases} \tag{2}$$

(i.e., by the introduction of the so-called *Klein coordinates*), one can then bring its equation into the form:

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 = 0. \tag{3}$$

Theorem 1: *If one interpret the coordinates of a linear complex as homogeneous coordinates in R_5 then the singular complexes (i.e., the lines) will be mapped to the points of a regular M_4^2 (viz., Plucker's M_4^2).*

The results of the previous paragraph can now be expressed in an especially simple form in the new language. A pencil of linear complexes will be mapped to a line in R_5. In place of Theorem 8 in Chapter I, one will now have:

Theorem 2: *A line in R_5 either cuts Plucker's M_4^2 at two distinct points, or it cuts it at two coincident ones (i.e., it contacts it), or it lies inside of M_4^2 entirely. (Line in M_4^2: Image of a pencil of lines in R_3 .)*

If one further observes that the relation $(\mathfrak{A}\mathfrak{B}) = 0$ arises from (1) by polarization then one will have:

([4]) The idea of the map goes back to **F. Klein**, "Über Liniengeometrie und metrische Geometrie," Math. Ann. **5** (1872) = Werke I, pp. 106: "Die Liniengeometrie is wie die Geometrie auf einer M_4^2 in R_5."

Theorem 3: *Two complexes $\mathfrak{A}$ and $\mathfrak{B}$ are conjugate when their image points are conjugate relative to M_4^2.*

14. Linear manifolds on M_4^2. We have just seen that M_4^2, includes lines as the image of pencils of lines. There are then just as many lines on M_4^2 as there are pencils of lines in R_3, viz., ∞^5.

We now ask about linear manifolds of higher dimension on M_4^2. They would correspond to line manifolds in R_3 of such a kind that any line of the manifold could be linked to any other line of the manifold by a pencil of lines. In other words: Any line of the manifold must cut any other line of the manifold. One sees directly that there are two types of manifolds of that kind in R_3: the *bundles of lines* (manifolds of all lines through a point) and *line fields* (manifolds of all lines in a plane). These two-dimensional linear line manifolds will then be mapped to planes on M_4^2:

Theorem 4: *Plücker's M_4^2 contains two families of ∞^3 planes of the "first" and "second" kind, which correspond to the points and planes in R_3 .*

We investigate the relative position of these planes to each other and assert:

Theorem 5: *Two distinct planes of the same kind always cut at one and only one point. Planes of different kinds will generally be skew. However, if they have a point in common then they will cut along a line.*

Proof: Two bundles of lines always have a line in common, namely, the connecting line of their vertices. Two line fields always have a line in common, namely, the line of intersection of their planes. By contrast, a bundle and a field have no line in common, in general. However, in the special case for which they have a line in common, the vertex of the bundle will lie in the plane of the field, and the bundle will have a pencil of lines in common with the field. Q. E. D.

15. Automorphic collineations of M_4^2. A collineation of R_3:

(4)
$$x'_k = \sum_{i=0}^{3} a_{ki} x_i \qquad (k = 0, 1, 2, 3) \qquad |a_{ki}| \neq 0$$

induces a transformation of straight lines. One will get the induced transformation when one associates the connecting line of two points x, y with the connecting line of the transformed points x', y'. One will then have, e.g.:

(5)
$$\begin{aligned}\mathfrak{X}'_{01} &= (x'_0 y'_1 - x'_1 y'_0) \\ &= (a_{00}\, x_0 + a_{01}\, x_1 + \ldots)(a_{10}\, y_0 + a_{11}\, y_1 + \ldots)\end{aligned}$$

$$- (a_{10}\, x_0 + a_{11}\, x_1 + \ldots)(a_{00}\, y_0 + a_{01}\, y_1 + \ldots)$$
$$= (a_{00}\, a_{11} - a_{01}\, a_{10}) \cdot (x_0\, y_1 - x_1\, y_1 + \ldots)$$
$$= (a_{00}\, a_{11} - a_{01}\, a_{10}) \cdot \mathfrak{X}_{01} + \ldots$$

A transformation of the regular complex $\mathfrak{C}_{ik}$ will also be induced by a transformation of the line $\mathfrak{X}_{ik}$: The totality of lines of a complex $\mathfrak{C}_{ik}$ will be transformed into the totality of lines of the associated complex $\mathfrak{C}'_{ik}$ by the collineation. One will obtain the induced transformation quite simply when one replaces the line coordinates $\mathfrak{X}_{ik}$ in (5) with complex coordinates $\mathfrak{C}_{ik}$.

It is essential for us that the transformation equations prove to be linear in these formulas. It will then follow from this that the image of a collineation in R_3 will be a collineation in R_5 . The same thing will be true for correlations of R_3 . They will also be mapped to certain collineations on R_5 .

Since a projective transformation of R_3 transforms lines to lines, its image in R_5 will fix the image manifold M_4^2 of lines: One will be dealing with an *automorphic collineation of* M_4^2:

Theorem 6: *Plücker's* M_4^2 *admits a laminated group of* $2 \cdot \infty^{15}$ *collineations. The main sheet consists of collineations that transform the planes of each family to themselves (i.e., the images of collineations). The other sheet consists of collineations that switch the planes of the first kind with planes of the second kind (i.e., the images of correlations).*

§ 4. Involutory, automorphic collineations of M_4^2

16. Involutory collineations in R_n . It follows from the last theorem that was proved that the involutory projectivities of R_3 will be mapped to the involutory, automorphic collineations of M_4^2. In order to arrive at all types of involutory, automorphic collineations of M_4^2, we shall start with the general problem of finding all involutory collineations of a space R_n of arbitrary dimension. The basis for that investigation is defined by the theorem:

Theorem 7: *The fixed points of an involutory collineation of* R_n *span all of* R_n .

Proof: A fixed line of the collineation runs through a point x that is not a fixed point, namely, the connecting line with the associated point x' (since the collineation transforms the line $\widehat{xx'}$ to the line $\widehat{x'x}$). However, an involution will be "cut out" from the fixed line by the involutory collineation; i.e., the points of the line will be permuted by the involutory collineation in such a way that an involution will arise in the binary domain of the line. It will always have two distinct fixed points. One will arrive at $n + 1$ linearly-independent fixed points by a suitable choice of the fixed line. Q. E. D.

If one now picks $n + 1$ linearly-independent fixed points of the involutory collineation to be the vertices of the *coordinate simplex* (which is how we refer to the figure that is determined by the points 1 : 0 : … : 0, 0: 0 : … : 0, … 0 : 0 : 0 … 1) then the equations of the involutory collineation will be:

$$(1) \qquad x_0' = a_{00}\, x_0 , \quad x_1' = a_{11}\, x_1 , \ldots, \; x_n' = a_{nn}\, x_n .$$

Since the collineation is involutory, one must have:

$$(2) \qquad a_{00}^2 = a_{11}^2 = \ldots = a_{nn}^2 .$$

One can set $a_{ii}^2 = 1$ and find that $a_{ii} = \pm 1$. One then finds that there are as many different types of involutory collineations as there are possible combinations of signs. For example, in the space R_3, the following cases are possible:

1. $x_0' = + x_0 , \quad x_1' = + x_1, \quad x_2' = + x_2 , \quad x_3' = + x_3 ,$
2. $x_0' = - x_0 , \quad x_1' = + x_1, \quad x_2' = + x_2 , \quad x_3' = + x_3 ,$
3. $x_0' = - x_0 , \quad x_1' = - x_1, \quad x_2' = + x_2 , \quad x_3' = + x_3 .$

Case 1 is omitted, since it gives the identity ([1]). In case 2, the plane $x_0 = 0$ is fixed point-wise, as well as the point $p = (1, 0, 0, 0)$. The fixed lines are the lines through the point p (except for the lines in the plane $x_0 = 0$). In order to transform a point x, we connect x to p, intersect the connecting line with the plane $x_0 = 0$ (point q), and look for the fourth harmonic point x' to x relative to the point-pair p, q *(viz., perspective involution).*

In case 3, the lines $\mathfrak{G} : x_0 = 0, x_1 = 0$ and $\mathfrak{G}' : x_2 = 0, x_3 = 0$ remain point-wise fixed. Fixed lines are, in addition, the line of intersection of $\mathfrak{G}$ and $\mathfrak{G}'$. In order to find the point x' that is associated with x, we draw the common line of intersection of $\mathfrak{G}$ and $\mathfrak{G}'$ through x (viz., the line of intersection of the connecting planes $x\mathfrak{G}$ and $x\mathfrak{G}'$) and look for the four harmonic points to x relative to the two points of intersection (i.e., *a skew involution*). We summarize the results in the theorem:

Theorem 8: *There are two types of involutory collineations in the space* R_3: *the perspective involutions and the skew involutions.*

More generally, one gets, in the same way:

Theorem 9: *There are just as many different types of involutory collineations in* R_n *as there are distinct possibilities for spanning* R_n *with two subspaces* R_i *and* R_{n-i-1} *that have no common point. A fixed line of the collineation runs through a point x that does not lie on the "incidence domains"* R_i *and* R_{n-i-1}*, namely, the common line of intersection of* R_i *and* R_{n-i-1} *(i.e., the line of intersection of the domains* R_{i+1} *and* R_{n-i} *that connect x to the*

([1]) We do not count the identity among the involutory transformations.

incidence domains). The collineation cuts out an involution along that line whose fixed points are the points of intersection with the two incidence manifolds.

With this theorem, we can give the types of involutory collineations in any space immediately. Here, we are especially interested in the theorem:

Theorem 10: *There are three distinct types of involutory collineations in R_5. This incidence domains are:*

1. *A point and R_4.*
2. *A line and R_3.*
3. *Two planes.*

17. Applications to the automorphic collineations of M_4^2. In order to derive all types of involutory, automorphic, collineations of M_4^2 from the result that we just found, imagine that an involutory collineation J is determined completely by its incidence domains. Now, should J fix M_4^2, J would also have to fix the polar system P that is coupled to M_4^2.

However, it follows from this that J and P commute, and in turn ([1]), that P must fix the collineation J; i.e., it must fix the figure that consists of the incidence domains of that collineation. That can happen either in such a way that P permutes the incidence domains or in such a way that P transforms each of those domains into itself. However, since P is a correlation, the latter case can occur only when the incidence domains have the same dimension numbers. From this and Theorem 10, we find that the following possibilities exist for the incidence domains:

1. Point and polar R_4.
2. Line and polar R_3.
3. Plane and polar R_2.
4. Two generating planes of M_4^2.

We will now treat these four cases in succession. At the same time, that examination will give us an opportunity to get to known about all the types of linear manifolds of linear complexes.

§ 5. Complex and forest of complexes.

18. Classification of forests of complexes. Let $\mathfrak{A}$ be the notation for a linear complex and, at the same time, for its image point in R_5. The locus:

$$(\mathfrak{A}\mathfrak{C}) = 0 \tag{1}$$

([1]) If follows from $P = J^{-1}PJ$ that $JP = PJ$ and from this that $J = P^{-1}JP$.

of all complexes $\mathfrak{C}$ that are conjugate to $\mathfrak{A}$ is a four-dimensional linear complex manifold; the image manifold is an R_4 that lies in R_5 , namely, the polar R_4 of the point $\mathfrak{A}$. Conversely, if an R_4 in R_5 is given then a well-defined pole $\mathfrak{A}$ will belong to it. We can then say:

Theorem 11: *A four-dimensional linear manifold of complexes consists of all linear complexes $\mathfrak{C}$ that are conjugate to a fixed complex $\mathfrak{A}$.*

Corresponding to the regular and singular complexes $\mathfrak{A}$, there are regular and singular forests of complexes.

A *singular* forest of complexes that is the locus of all complexes that contain a fixed line $\mathfrak{G}$ will be mapped to the tangential R_4 in M_4^2 at the point $\mathfrak{G}$. It will intersect M_4^2 in a M_3^2 for which $\mathfrak{G}$ is a singular point (since it is conjugate to all points of R_4). M_3^2 is therefore a cone of rank 4 that one can obtain by projecting an M_2^2 that lies regularly in R_3 from $\mathfrak{G}$.

A *regular forest of complexes* will be mapped to an R_4 that does not contact M_4^2, and will therefore intersect it in a regular M_3^2. The points of this M_3^2 are the images of the lines that belong to the regular complex. Therefore:

Theorem 12: *The lines of a regular, linear complex will be mapped to the points of a regular M_4^2. It contains ∞^3 lines (corresponding to the ∞^3 pencils of complexes). The lines (like the points of R_3, and unlike the planes in M_4^2) define a continuum, namely, a quaternary domain.*

The last assertion follows from the fact that the line manifold of M_3^2 can be mapped to the quaternary domain of the points of R_3 in a single-valued way by means of the pencil that is contained in the complex.

19. The image of the null system. From no. **17**, an involutory, automorphic collineation of M_4^2 is determined by a point $\mathfrak{C}$ that does not lie on M_4^2 and its polar R_4 (which does not go through $\mathfrak{C}$). It is the image of an involutory projectivity of R_3 . In order to establish what kind of projectivity we are dealing with, we examine its fixed elements in R_3 . In R_3, it fixes:

1. The regular complex $\mathfrak{C}$, as the image of the point $\mathfrak{C}$.

2. Its intersection M_3^2 with the linear complex that is conjugate to $\mathfrak{C}$, and in particular, the lines of the complex, as the image of the points of the polar R_4.

Thus:

Theorem 13: *The null system in R_3 will be mapped to the involutory collineation of M_4^2 with a point and its polar R_4 as the incidence domains.*

We now consider two mutually-conjugate regular complexes $\mathfrak{A}$ and $\mathfrak{B}$ and their image points. The one image point lies in the polar R_4 of the other one, and a point-polar R_4 pair is then fixed by the involutory collineation that is coupled with the other one. The two collineations then commute with each other. We then have arrived at a new interpretation of the relationship of conjugacy of two regular linear complexes:

Theorem 14: *Two regular linear complexes are conjugate if and only if the null systems that are coupled with them commute.* (For an analytical proof, see Chap. III, Theorem 2.)

The product of two commuting, involutory transformations is again involutory ([1]). The product of commuting null systems is also involutory then, and indeed, an involutory collineation, as the product of two correlations.

In R_5, the composition of the two involutory collineations will, in turn, give an involutory collineation, for which the intersection R_3 of the two polar R_4 and the connecting line of the points $\mathfrak{A}$ and $\mathfrak{B}$ will remain fixed. The image of this collineation will be a skew involution in R_3 with the lines that are contained in the pencil $\xi_1\mathfrak{A} + \xi_2\mathfrak{B}$ as its guiding lines. The second type of involutory collineation (in Theorem 8) that comes under scrutiny fixes a bundle of lines and a line field in R_3 line-wise, and therefore a pair of planes in R_5 of different types in M_4^2 will be fixed point-wise. The following theorems are then proved:

Theorem 15: *Under the map of the line continuum to M_4^2 in R_5, a skew involution will be mapped to an automorphic, involutory collineation of M_4^2 with a line and its polar R_3 as the incidence domains. A perspective involution will correspond to an automorphic, involutory collineation with two planes of different types in M_4^2 as their incidence domains.*

Theorem 16: *The product of two commuting null systems is a skew involution.*

§ 6. Pencil of complexes and bush of complexes.

20. Classification of the pencils of complexes. We already saw in no. **13** that the pencil of complexes of R_3 is mapped to lines in R_5 and that three different kinds of pencils of complexes correspond to three different kinds of lines R_5 according to their position relative to M_4^2. Under the polarity of M_4^2, a line will correspond to a space R_3 as the locus of all points that are conjugate to all points of the line relative to M_4^2.

([1]) It follows from $S^2 = 1$, $T^2 = 1$, and $ST = TS$ that $(ST)^2 = STTS = SS = 1$.

Correspondingly, a three-dimensional linear manifold of linear complexes – viz., a *bush of complexes* – will consist of all complexes that are conjugate to any two distinct complexes of a pencil (and therefore, to all of them). The three different kinds of pencils of complexes correspond to three types of pencils of complexes:

1. *The regular bush of complexes.* This kind corresponds to the regular pencil of complexes. The image R_3 cuts M_4^2 in an M_2^2, and thus, a second-order surface, which we would like to show is regular (of rank 4). In fact: the M_2^2 is the image of all lines that are common to all complexes of the regular pencil, so it will be the image of the lines of a regular, linear congruence. However, that congruence will consist of the common lines of intersection of two skew guiding lines $\mathfrak{G}$ and $\mathfrak{G}'$, and it will follow from an examination of the line pencils that are contained in the congruence that:

Theorem 17:

A regular, linear congruence contains two different kinds of pencils of lines: Pencils of lines that connect a point of $\mathfrak{G}$ with all the points of $\mathfrak{G}'$ and pencils of lines that connect a point of $\mathfrak{G}'$ with all points of $\mathfrak{G}$.	*The image M_2^2 contains two different families of ∞^1 generators.*
Two pencils of the same kind have no line in common. Two pencils of different kinds have one line in common.	*Two generators of the same kind are skew. Two generators of different kinds are incident.*

With that, we have shown:

Theorem 18: *A regular bush of complexes consists of all complexes that contain two fixed, skew lines. The guiding lines of the singular complexes of the bush fill up a linear congruence that is determined the two lines. That congruence will be mapped to a regular M_2^2.*

2. *The singular bush of complexes.* The singular bush of complexes consists of all complexes that are conjugate to the complexes of a singular pencil. The guiding lines of the singular complex of such a bush then fill up a singular linear congruence. We look for the image of such a congruence on M_4^2.

If a line contacts M_4^2 then the point of contact $\mathfrak{G}$ will be conjugate to all points of the line. The polar R_4 of the point will then include the line. One will then obtain the polar R_3 of the line when one intersects that polar R_4 with the polar R_4 of any other point of the line. It will run through the point of contact $\mathfrak{G}$, in such a way that the polar R_3 contains the tangent to the point of contact. That point of R_3 (as a point of the tangent) will then be conjugate to all points of R_3 (relative to M_4^2, and therefore) relative to the M_2^2 along which the R_3 intersects M_4^2. $\mathfrak{G}$ will then be a singular point of M_2^2, and indeed, the only singular point:

Theorem 19: *A singular linear complex is mapped to a cone M_2^2 (of rank 3) by the map of the line continuum to M_4^2.*

One will obtain a second-order cone when one projects a second-order curve from a point that does not lie on the plane of that curve. Starting from that remark, we shall give an intuitive construction of a singular linear congruence, as long as we know the image of a conic section of M_4^2 in R_3 (no. **22**).

3. *The most-singular bush of complexes.* This bush will be defined by all complexes that contain a given pencil of lines. The manifold of singular complexes that are contained in the bush decomposes into: The manifold of all singular complexes whose guiding lines run through the vertex of the pencil and the manifold of all singular complexes whose guiding lines lie in the plane of the pencil. The two manifolds of lines generate the given pencil. It will then follow that:

Theorem 20: *The image R_3 of a most-singular bush of complexes – viz., the polar R_4 of a line G that lies in M_4^2 – cuts M_4^2 in a pair of planes. That pair consists of a plane of the first kind and a plane of the second kind, and the line of intersection of the two planes will be the line G.*

§ 7. Bundle of complexes.

21. Classification of bundles of complexes. A bundle of complexes can be spanned by three linearly-independent complexes $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$:

$$x_1\,(\mathfrak{A}\mathfrak{X}) + x_2\,(\mathfrak{B}\mathfrak{X}) + x_3\,(\mathfrak{C}\mathfrak{X}) = 0, \tag{1}$$

and will therefore be mapped to a plane in R_5. A second plane is polar to it relative to M_4^2. The planes in R_5 will then be associated with pairs by M_4^2.

The plane (1) cuts M_4^2 (when it does not belong to it completely) in a second-order curve:

$$x_1^2\,(\mathfrak{A}\mathfrak{A}) + x_2^2\,(\mathfrak{B}\mathfrak{B}) + x_3^2\,(\mathfrak{C}\mathfrak{C}) + 2x_2\,x_3\,(\mathfrak{B}\mathfrak{C}) + 2x_3\,x_1\,(\mathfrak{C}\mathfrak{A}) + 2x_1\,x_2\,(\mathfrak{A}\mathfrak{B}) = 0. \tag{2}$$

The bundles can be distinguished by the rank of that curve, and thus, the rank of the matrix:

$$\begin{Vmatrix} (\mathfrak{A}\mathfrak{A}) & (\mathfrak{A}\mathfrak{B}) & (\mathfrak{A}\mathfrak{C}) \\ (\mathfrak{B}\mathfrak{A}) & (\mathfrak{B}\mathfrak{B}) & (\mathfrak{B}\mathfrak{C}) \\ (\mathfrak{C}\mathfrak{A}) & (\mathfrak{C}\mathfrak{B}) & (\mathfrak{C}\mathfrak{C}) \end{Vmatrix}. \tag{3}$$

Singular bundles of complexes. We will call a bundle for which the rank of (3) is $r < 3$ "singular." There are the following three cases to distinguish:

1. $r = 0$. The bundle is a bundle of lines or a field of lines, and will be mapped to a plane of the first or second kind in M_4^2, resp. Such a plane is polar to itself.

2. $r = 1$. The image plane E cuts M_4^2 in a doubly-counted line. A plane of the first kind and a plane of the second kind in M_4^2 runs through it (as the image line of a pencil of lines). One gets the plane E' that is polar to E as the fourth harmonic to E relative to the two planes in M_4^2.

3. $r = 2$. The image plane E cuts M_4^2 in a pair of distinct lines. The guiding lines of the singular complexes of the bundle then define two *restricted pencils of lines* (two pencils with a common lines) with two different vertices p, p' and two different planes a, a'. Any line of the pair of restricted pencils of lines that is determined by $p, a' : p', a$ will cut every line of the first pair. Thus, the plane E' that is polar to E will also cut M_4^2 in a pair of distinct lines. The point of intersection of the two pairs of lines will be common to the planes E and E'.

22. Regular bundles of complexes. In the case $r = 3$, the bundle of complexes will be mapped to a plane E that cuts M_4^2 in a regular second-order curve. The polar plane E' will also cut it in a regular, second-order curve then, since, from no. **21**, there is no other possibility. The ∞^1 lines that correspond to the points of such a curve are pair-wise skew. Two distinct points of a regular, second-order curve are never conjugate to each other then. Just as the plane E' can be spanned by three points of its intersecting conic section, the bundle of complexes E' can be spanned by three pair-wise skew lines, and therefore, the polar bundle of complexes E can be characterized as the bundle of all complexes that contain three pair-wise skew lines. The conic section that lies in E will then be the image of the manifold of all common lines of intersection of three pair-wise skew lines. We call the figure of these ∞^1 lines a *regulus.* Just as a conic section is the locus of its points, a regulus is a *binary domain*, as the locus of its lines. We summarize the results:

Theorem 21: *The regular bundles of complexes are paired together as pairs of polar bundles of complexes. Each of the bundles can be characterized as the locus of all complexes that contain three pair-wise skew lines of the other. The guiding lines of the singular complexes of each of the two bundles fill up a regulus. The two reguli define a "pair of polar reguli": Any line of one regulus will cut every line of the other one.*

Once we know the image of a conic section of M_4^2 in R_3, we can give the *construction of the singular congruence* that was suggested at the end of no. **20**:

Theorem 22: *Let a regulus be given, along with a line $\mathfrak{G}$ (of the polar regulus) that cuts all lines of that regulus. We connect $\mathfrak{G}$ with each generator of the given regulus by a pencil of lines: The locus of all lines of that pencil is a singular, linear complex.*

A complex cuts a regulus (when it does not contain it completely) in two lines, since the polar R_4 of its image points cuts the image conic section of the regulus in two points. In particular, a line in general position will also cut two generators of a regulus then. A pair of polar reguli, as the locus of points in question, will then define a surface that is cut by a line in general position at two points, namely, a second-order surface. The generators of the surface will then be fixed individually under the polarity. Therefore:

Theorem 23: *Under the map of the line continuum onto M_4^2, the polarity on a regular, second-order surface will be mapped to an involutory, automorphic collineation with two polar planes as its incidence domains.*

23. Orientation of a second-order surface. In no. **20**, we mapped an M_2^2 to a linear congruence. There, we saw that the difference between the two types of generators of M_2^2 corresponded to the difference between the two types of guiding line for the congruence. In R_5, that distinction leads to the difference between the two points of intersection of a line (viz., the image line pencil of lines that runs through the congruence) with the M_4^2, and thus, to the convention on the sign of the roots of the discriminant of a binary form. In geometry, the convention regarding the sign of a root is called an *orientation*. The distinction (separation) between the two families of generators of a regular M_2^2 is then an "orientation process." That process can also be interpreted as a splitting of the two image planes E, E' onto whose conic section the reguli of the second-order surface will be mapped.

In order to orient a regular M_2^2 that is given by its equation ([1]):

$$a_0 x_0^2 + a_1 x_1^2 + a_2 x_2^2 + a_3 x_3^2 = 0, \qquad D = a_0\, a_1\, a_2\, a_3 \neq 0, \tag{4}$$

we next seek the *line equation for the surface:* We set $x = \xi_1\, y + \xi_2\, z$ in (4), set the discriminant of the binary quadratic form that arises equal to zero, and introduce the line coordinates of the line $\widehat{y\,z} = \mathfrak{X}$. That will produce an equation that is quadratic in $\mathfrak{X}_{ik}$:

$$a_0 a_1 \mathfrak{X}_{01}^2 + a_0 a_2 \mathfrak{X}_{02}^2 + \ldots + a_1 a_2 \mathfrak{X}_{12}^2 = 0 \tag{5}$$

(viz., a special quadratic complex), which is the equation that the tangents to the surface will satisfy. If we polarize that equation then we will have the *equation of the polar*

([1]) We choose a special example, since we shall not go into the symbolic calculus that is requisite for a proper treatment of the general case.

system of the surface in line coordinates, and when we interpret the line coordinates as point coordinates in R_5, we will have the representation of the involutory, automorphic collineations of M_4^2 with two incidence planes E and E':

$$(6)\qquad a_0\, a_1\, \mathfrak{X}_{01}\, \mathfrak{Y}_{01} + a_0\, a_2\, \mathfrak{X}_{02}\, \mathfrak{Y}_{02} + \ldots + a_1\, a_2\, \mathfrak{X}_{12}\, \mathfrak{Y}_{12} = 0.$$

In order to find the planes E, E' analytically, we must look for the fixed points of that collineation. To that end, we must replace $\mathfrak{X}'_{ik}$ with $\rho\, \mathfrak{X}_{ik}$ in the equations:

$$(7)\qquad \begin{cases} \mathfrak{X}'_{01} = a_2 a_3 \mathfrak{X}_{23}, & \mathfrak{X}'_{02} = a_3 a_1 \mathfrak{X}_{31}, \quad \mathfrak{X}'_{03} = a_1 a_2 \mathfrak{X}_{12}, \\ \mathfrak{X}'_{23} = a_0 a_1 \mathfrak{X}_{01}, & \mathfrak{X}'_{31} = a_0 a_2 \mathfrak{X}_{02}, \quad \mathfrak{X}'_{12} = a_0 a_3 \mathfrak{X}_{03}. \end{cases}$$

It follows that:

$$(8)\qquad \rho^2\, \mathfrak{X}_{ik} = a_2\, a_3\, \mathfrak{X}_{23} = a_0\, a_1\, a_2\, a_3\, \mathfrak{X}_{01}\,,$$

and therefore:

$$(9)\qquad \rho = \sqrt{a_0 a_1 a_2 a_3} = \sqrt{D}\,.$$

Thus:

$$(10)\qquad \begin{cases} \sqrt{a_0 a_1}\,\mathfrak{X}_{01} + \sqrt{a_2 a_3}\,\mathfrak{X}_{23} = 0, & \sqrt{a_0 a_1}\,\mathfrak{X}_{01} - \sqrt{a_2 a_3}\,\mathfrak{X}_{23} = 0, \\ \sqrt{a_0 a_2}\,\mathfrak{X}_{02} + \sqrt{a_3 a_1}\,\mathfrak{X}_{31} = 0, & \sqrt{a_0 a_2}\,\mathfrak{X}_{02} - \sqrt{a_3 a_1}\,\mathfrak{X}_{31} = 0, \\ \sqrt{a_0 a_3}\,\mathfrak{X}_{03} + \sqrt{a_1 a_2}\,\mathfrak{X}_{12} = 0, & \sqrt{a_0 a_3}\,\mathfrak{X}_{03} - \sqrt{a_1 a_2}\,\mathfrak{X}_{12} = 0 \end{cases}$$

are the equations for the desired incidence domain:

Theorem 24: *The orientation of a regular, second-order surface – i.e., the separation of its families of generators – requires a convention on the sign of the root of the determinant of the surface* (cf., no. **26**).

In the special case of a sphere of radius-squared r^2:

$$(11)\qquad -r^2 + x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0,$$

one will have $D = -r^2$. *The orientation of a sphere requires a convention on the sign of its radius.*

§ 8. The second-order surface as a double-binary domain.

24. Parametric representation of a second-order surface. The fact that was derived in no. **22** – viz., that the regular, second-order surface, as the carrier of its two reguli, is a binary domain – leads to a parametric representation of that surface. The equations:

(1) $$x_0 = \xi_1\tau_1\,, \qquad x_1 = \xi_1\tau_2\,, \qquad x_2 = \xi_2\tau_1\,, \qquad x_3 = \xi_2\tau_2\,,$$

in which ξ and τ independently run through the points of two different binary domains, give a parametric representation in R_3 of the second-order surface of rank 4:

(2) $$(x\,x) = 2x_0\,x_1 - 2x_1\,x_2 = 0; \qquad D = +\,1 \neq 0.$$

One will obtain the most general parametric representation of an arbitrary, regular, second-order surface when one subjects (1) to a collineation, so when one sets the coordinates x_i equal to four linearly-independent bilinear forms in ξ and τ.

If one fixes τ in (1) then one will obtain the parametric representation of a line (i.e., the generator τ is of the first kind), while if one fixes ξ then one will likewise obtain a line (i.e., the generator ξ is of the second kind). *The generators of the one regulus cut those of the other in projective point sequences.*

25. Automorphic collineations of regular, second-order surfaces. A collineation transforms lines into lines, so an automorphic collineation of a second-order surface will transform generators into generators, and since incidence is preserved by collineations, the generators of one family will be transformed into the generators of another family. There are thus two cases to distinguish, according to whether the two families are transformed into themselves (viz., proper, automorphic collineations) or permuted with each other (viz., improper, automorphic collineation). One will obtain certain automorphic collineations when one subjects each of the binary domains ξ and τ to a projective transformation:

(3) $$\begin{aligned} x_0' &= \xi_1'\tau_1' = (a_{11}\,\xi_1 + a_{12}\,\xi_2)\,(b_{11}\tau_1 + b_{12}\,\tau_2) \\ &= a_{11}\,b_{11}\,\xi_1\tau_1 + a_{11}\,b_{12}\,\xi_1\tau_2 + a_{12}\,b_{11}\,\xi_2\tau_1 + a_{12}\,b_{12}\,\xi_2\tau_2, \\ &= a_{11}\,b_{11}\,x_0 + a_{11}\,b_{12}\,x_1 + a_{12}\,b_{11}\,x_2 + a_{12}\,b_{12}\,x_3\,,\ *,\ *,\ * \end{aligned}$$

(i.e., ∞^2 binary relationships) or transforms the one domain into the other one:

(4) $$x_0' = \xi_1'\tau_1' = (a_{11}\tau_1 + a_{12}\tau_2)\,(b_{11}\xi_1 + b_{12}\,\xi_2) = \ldots,\ *,\ *,\ *,$$

and one can show that one obtains all automorphic collineations in that way:

Theorem 25: *A regular, second-order surface admits a group of* $2\cdot\infty^6$ *automorphic collineations. They will be induced by the projective transformations of the binary domains of the two families of generators. Any family of generators will be transformed into itself by a proper, automorphic collineation, while the two families will be permuted by an improper, automorphic collineation.*

26. Projective equivalence of point-quadruples on a second-order surface. There is a *conversion principle* that is based upon the parametric representation (1) and is completely analogous to the **Hessian** of the plane ([1]). We will not treat that principle thoroughly, but only derive an equation that will be important in what follows. When are two points $x = (\xi, \tau)$, $y = (\eta, \sigma)$ of the surface conjugate to each other? We define:

$$(5)\qquad \begin{aligned}(x\,y) &= x_0\,y_3 + x_3\,y_0 - x_1\,y_2 - x_2\,y_1 \\ &= \xi_1\tau_1\eta_2\sigma_2 + \xi_2\tau_2\eta_1\sigma_1 - \xi_1\tau_2\eta_2\sigma_1 - \xi_2\tau_1\eta_1\sigma_2 \\ &= (\xi\,\eta)\,(\tau\,\sigma).\end{aligned}$$

That equation contains the theorem:

Theorem 26: *Two points of a second-order surface are conjugate if and only if they belong to the same generator (of the first or second kind).*

Equation (5) will play the role of an auxiliary formula in the following investigation.

We consider two point-quadruples 0, 1, 2, 3 and 0′, 1′, 2′, 3′ on a second-order surface. Should it be possible to take one point-quadruple to the other one by an automorphic collineation of the surface, then it would have to be possible to find two binary projectivities that take the four generators of the first and second kind that go through the points of the first quadruple to the four generators of the first and second kind (or second first, resp.) that go through the points of the second quadruple. However, a necessary and sufficient condition for this is the equality of the double ratios of the corresponding line quadruples of the surface. That will bring one to the problem of *determining the double ratios of the quadruples of generators that run through a quadruple of points* x_0, x_1, x_2, x_3 ([2]).

Let ξ^i, τ^i be the parameters of the points x^i. We set:

$$(6)\qquad \begin{cases} (u_1) = (\xi^0\xi^1)(\xi^2\xi^3), & (u_2) = (\xi^0\xi^2)(\xi^3\xi^1), \quad (u_3) = (\xi^0\xi^3)(\xi^1\xi^2), \\ (u_1') = (\tau^0\tau^1)(\tau^2\tau^3), & (u_2') = (\tau^0\tau^2)(\tau^3\tau^1), \quad (u_3') = (\tau^0\tau^3)(\tau^1\tau^2), \end{cases}$$

such that:

$$(7)\qquad (u_1) + (u_2) + (u_3) = 0, \qquad (u_1') + (u_2') + (u_3') = 0,$$

and demand that the expressions (6) must all be non-zero, so no two points of the quadruple can lie on one and the same generator. One is then dealing with the calculation of the binary, absolute invariants ([3]):

$$(8)\qquad \begin{cases} d_1 = \dfrac{(u_2)}{(u_1)}, \; d_2 = \dfrac{(u_3)}{(u_1)}, \; d_3 = \dfrac{(u_1)}{(u_2)}, \\ d_1' = \dfrac{(u_2')}{(u_1')}, \; d_2' = \dfrac{(u_3')}{(u_1')}, \; d_3' = \dfrac{(u_1')}{(u_2')}, \end{cases}$$

([1]) Cf., **L. Bieberbach**, *Einleitung in die höhere Geometrie*, Leipzig 1933, Chap. II.

([2]) **E. Study**, “Betrachtungen über Doppelverhältnisse,” Leipziger Berichte (1896), pp. 200.

([3]) The double ratios are the expressions: $-d_i$, $-d_i'$.

in terms of the simplest quaternary, absolute invariants, namely, the quotients that are defined with the help of the expressions:

(9) $$U_1 = (x^0\,x^1)(x^2\,x^3), \qquad U_2 = (x^0\,x^2)(x^3\,x^1), \qquad U_3 = (x^0\,x^3)(x^1\,x^2),$$

and take the form:

(10) $$D_1 = \frac{U_2}{U_3}, \qquad D_2 = \frac{U_3}{U_1}, \qquad D_3 = \frac{U_1}{U_2}.$$

Therefore, from (5), the following relations:

(11) $$U_i = (u_i) \cdot (u_i')$$

will exist between the U_i and u_i .

Now, the two invariants d_1 and d_1' are the roots of the equation:

(12) $$\{\lambda - d_1\}\{\lambda - d_1'\} = 0,$$

whose coefficients can be expressed in terms of the D_i; one will then have, first of all:

(13) $$d_i \cdot d_i' = \frac{(u_2)}{(u_3)} \cdot \frac{(u_2')}{(u_3')} = \frac{U_2}{U_3},$$

and secondly:

(14) $$d_1 + d_1' = \frac{(u_2)}{(u_3)} + \frac{(u_2')}{(u_3')} = \frac{(u_2)(u_3') + (u_3)(u_2')}{(u_3)(u_3')}.$$

However, due to (7), one will have:

(15) $$\begin{aligned} &+ |\,(u_1) + (u_2) + (u_3)\,|\,(u_2') \\ &+ |\,(u_1) + (u_2) + (u_3)\,|\,(u_3') \\ &+ |\,(u_1') + (u_2') + (u_3')\,|\,(u_1) = 0 \end{aligned}$$

here, so:

(16) $$(u_2)(u_3') + (u_3)(u_2') = (u_1)(u_1') - (u_2)\,(u_2') - (u_3)(u_3'),$$

and it will then follow that:

(17) $$d_1 + d_1' = \frac{U_1 - U_2 - U_3}{U_3}.$$

Equation (12) can now be written in the form:

(18) $$\lambda^2 - \lambda\,\frac{U_1 - U_2 - U_3}{U_3} + \frac{U_2}{U_3} = 0.$$

Its discriminant will be:

$$\Delta = -4U_2 U_3 + (U_1 - U_2 - U_3)^2 \tag{19}$$

$$= U_1^2 + U_2^2 + U_3^2 - 2U_2U_3 - 2U_3U_1 - 2U_1U_2$$

$$= \begin{vmatrix} 0 & (x_0x_1) & (x_0x_2) & (x_0x_3) \\ (x_1x_0) & 0 & (x_1x_2) & (x_1x_3) \\ (x_2x_0) & (x_2x_1) & 0 & (x_2x_3) \\ (x_3x_0) & (x_3x_1) & (x_3x_2) & 0 \end{vmatrix}.$$

One can change the form of this determinant. If a^0, a^1, a^2, a^3 are the polar planes of the points x^0, x^1, x^2, x^3 then one will have:

$$(x^i\,x^k) = (x^i\,a^k), \tag{20}$$

and from the multiplication theorem, the determinant (19) can be written as a product $(x^0\,x^1\,x^2\,x^3) \cdot (a^0\,a^1\,a^2\,a^3)$. However, one has:

$$(a^0\,a^1\,a^2\,a^3) = D \cdot (x^0\,x^1\,x^2\,x^3), \tag{21}$$

in which D denotes the determinant of the surface. One will then have:

$$\Delta = D \cdot (x^0\,x^1\,x^2\,x^3)^2. \tag{22}$$

We assume that Δ is non-zero ([1]), and thus demand that the four starting points do not lie in a plane. However, a choice of sign for $\sqrt{\Delta}$ will, at the same time, differ from the choice of the sign for $\sqrt{D}$. Thus, d_1 will differ from d_1'; i.e., the two families of generators of the surface will be separate. (The surface will be "oriented." Cf., no. **23**).

Finally, the solution of equation (18) will yield:

$$d_1 \cdot d_1' = \frac{U_1 - U_2 - U_3 \pm \sqrt{\Delta}}{2U_1}. \tag{23}$$

This result shows that the $d_i \cdot d_i'$ are determined completely from the U_i :

([1]) If Δ vanishes then one will have $d_1 = d_1'$: The two quadruples of lines that are determined by a quadruple of points are projective (in the sequence that is determined by the points) if and only if the four points of the quadruple belong to a plane. $[(x^0\,x^1\,x^2\,x^3)] = 0.]$ The plane mediates a projective relationship between the two families of generators, under which, generators will correspond when they intersect on the plane. – From Theorem 17, a conversion of this theorem into line space will yield: *The product of two projective point sequences on skew carriers is a regulus.*

Theorem 27: *A quadruple of linearly-independent points of a regular, second-order surface, no two of which belong to the same generator, is characterized, relative to the automorphic collineations of that surface, by the values of the expressions:*

$$D_1 = \frac{(x^0x^2)(x^3x^1)}{(x^0x^3)(x^1x^2)}, \quad D_2 = \frac{(x^0x^3)(x^1x^2)}{(x^0x^1)(x^2x^3)}, \quad D_3 = \frac{(x^0x^1)(x^2x^3)}{(x^0x^2)(x^3x^1)} \tag{24}$$

(i.e., the "double ratios of four points on the second-order surface"). The (binary) double ratios of the generators that run through the quadruple of points can be calculated from these quaternary, absolute invariants.

The proof of that theorem, in particular, the derivation of the equation (5) is linked with the special parametric representation (1). However, since the result is present in an invariant form, one will have $\sum_{i,k=0}^{3} a_{ik}x_ix_k = 0$ for any regular, second-order surface. One merely has to set:

$$(x\,y) = \sum_{i,k=0}^{3} a_{ik}x_iy_k\,. \tag{25}$$

§ 9. Classification of quadruples of lines.

27. The common lines of intersection of four lines. Four lines $\mathfrak{G}_0$, $\mathfrak{G}_1$, $\mathfrak{G}_2$, $\mathfrak{G}_3$ will first be classified by their *rank*; viz., the rank of the matrix that their coordinates define. Four lines of rank 4 span a bush of complexes. There will then be three different kinds of bushes (since there are three different possibilities for the conjugate pencil), so corresponding to them, the following cases can occur:

Theorem 28: *Four lines of rank 4 can have two distinct, two coincident, or a pencil of common lines of intersection.*

If one omits two opposite edges of a tetrahedron then the four remaining edges will give example of a quadruple of lines of the first kind. The omitted edges are the two common lines of intersection. The generator of the regulus of the first kind of a second-order surface and a tangent to the surface will determine a quadruple of the second kind. The generator of the second kind that goes through the point of contact is the doubly-counted common line of intersection. Finally, three linearly-independent lines of a bundle, together with a line that does not belong to the bundle, will define a quadruple of the third kind. The pencil of lines of intersection is the pencil of lines of the four lines that are contained in the bundle.

In order to *distinguish the three cases analytically*, from § 6, we must examine the intersection M_2^2 of the image points of the R_3 that is spanned by the four lines:

$$x_0\,\mathfrak{G}_0 + x_1\,\mathfrak{G}_1 + x_2\,\mathfrak{G}_2 + x_3\,\mathfrak{G}_3 \tag{1}$$

with the M_4^2:

(2) $$(x\,x) \equiv 2x_0\,x_1(\mathfrak{G}_0\;\mathfrak{G}_0) + 2x_0\,x_2(\mathfrak{G}_0\;\mathfrak{G}_2) + \ldots + 2x_2\,x_3(\mathfrak{G}_2\;\mathfrak{G}_3) = 0$$

in regard to its rank. That rank will be the rank of the determinant:

(4) $$\begin{vmatrix} * & (\mathfrak{G}_0\mathfrak{G}_1) & (\mathfrak{G}_0\mathfrak{G}_2) & (\mathfrak{G}_0\mathfrak{G}_3) \\ (\mathfrak{G}_1\mathfrak{G}_0) & * & (\mathfrak{G}_1\mathfrak{G}_2) & (\mathfrak{G}_1\mathfrak{G}_3) \\ (\mathfrak{G}_2\mathfrak{G}_0) & (\mathfrak{G}_2\mathfrak{G}_1) & * & (\mathfrak{G}_2\mathfrak{G}_3) \\ (\mathfrak{G}_3\mathfrak{G}_0) & (\mathfrak{G}_3\mathfrak{G}_1) & (\mathfrak{G}_3\mathfrak{G}_2) & * \end{vmatrix}.$$

Theorem 29: *Four lines $\mathfrak{G}_i$ of rank 4 have two different, two coincident, or a pencil of common lines of intersection, according to whether the determinant (4) has rank 4, 3, or 2, respectively.*

28. Projective equivalence of quadruples of lines. Two quadruples of lines that belong to a class in the classification that was just expounded do not need to be *projectively* equivalent, for that very reason. Here, we would like to address only the projective equivalence of two quadruples of lines that both possess two different common lines of intersection. Since the linear congruences that are determined by these lines of intersection are certainly projectively equivalent, the one of them can then be taken to the other one by a collineation or correlation, so it will suffice to investigate the projective equivalence of quadruples of lines of one and the same linear congruence. Under our map, that problem will be converted into the problem that we solved already in no. **26** of classifying the quadruples of points of a regular, second-order surface projectively.

If $\mathfrak{X}$ and $\mathfrak{Y}$ are two complexes of the bush (1), and x, y are its image points in the image-R_3 then, from (2), one will have:

(5) $$(x\,y) = (\mathfrak{X}\;\mathfrak{Y}).$$

From the quantities in (24) in § 8, one will then have:

(6) $$D_1 = \frac{(\mathfrak{G}_0\mathfrak{G}_2)(\mathfrak{G}_3\mathfrak{G}_1)}{(\mathfrak{G}_0\mathfrak{G}_3)(\mathfrak{G}_1\mathfrak{G}_2)}, \qquad D_2 = \frac{(\mathfrak{G}_0\mathfrak{G}_3)(\mathfrak{G}_1\mathfrak{G}_2)}{(\mathfrak{G}_0\mathfrak{G}_1)(\mathfrak{G}_2\mathfrak{G}_3)}, \qquad D_3 = \frac{(\mathfrak{G}_0\mathfrak{G}_1)(\mathfrak{G}_2\mathfrak{G}_3)}{(\mathfrak{G}_0\mathfrak{G}_2)(\mathfrak{G}_3\mathfrak{G}_1)}.$$

These absolute invariants of the four lines are called *Grassmannian double ratios of the lines.* The translation of Theorem 27 yields:

Theorem 30: *A quadruple of pair-wise skew, linearly-independent lines of a regular, linear congruence will be characterized completely in regard to projective transformations by its Grassmannian double ratios.*

Moreover, the double ratios $-d_i$, $-d_i'$ of the generators of the first and second kind, resp., that run through the four points of the second-order surface will be mapped to the double ratios that the four lines single out by their common lines of intersection. From § 8, (23), one will then have:

(7) $$-d_i,\ -d_i' = \frac{-U_1 + U_2 + U_3 \pm \sqrt{\Delta}}{2U_3}, \text{ resp.,}$$

in which one sets:

(8) $$U_1 = (\mathfrak{G}_0\ \mathfrak{G}_1)\ (\mathfrak{G}_2\ \mathfrak{G}_3), \qquad U_2 = (\mathfrak{G}_0\ \mathfrak{G}_2)\ (\mathfrak{G}_3\ \mathfrak{G}_1), \qquad U_3 = (\mathfrak{G}_0\ \mathfrak{G}_3)\ (\mathfrak{G}_1\ \mathfrak{G}_2),$$

as one of these pairs of double ratios. In it, Δ denotes the determinant (4). In the limiting case of the singular congruence, (7) yields a double ratio of the four points at which $\mathfrak{G}_0$, $\mathfrak{G}_1$, $\mathfrak{G}_2$, $\mathfrak{G}_3$ cut their single common line of intersection.

Four lines determine six **Grassmannian** double ratios, in all: viz., the expressions (6) and their reciprocals. As in the binary case, one can examine the question of what situation will lead to less than six different double ratios ([1]).

29. A spatial analogue of Desargues's theorem. From **Desargues**'s theorem, two associated *triangles* in the plane (Triangle = figure composed of a three-angle of rank 3 and the trilateral of rank 3 that is coupled with it) that are perspective as three-angles (i.e., the connecting lines of corresponding points run through a point) will also be perspective as trilaterals (i.e., the points of intersection of corresponding sides lie along a line). We would like to prove an analogous theorem for two *tetrahedra* whose points and planes are x^i, a^i and y^i, b^i, resp.:

Theorem 31 ([2]): *Let the connecting lines of corresponding points of two associated tetrahedra be pair-wise skew and of rank 4. The lines of intersection of corresponding planes of the two tetrahedra will also be pair-wise skew and of rank 4, and the quadruple of lines of intersection is projective to the quadruple of the connecting lines. In particular, if the common lines of intersection of the connecting lines of the points coincide then the lines of intersection of the lines of intersection of the planes will also coincide.*

In order to prove this, we shall need an auxiliary formula: If one sets the four-rowed determinant equal to:

([1]) **H. Mohrmann**, "Über die Graßmannschen Doppelverhältnisse von vier geraden Linien im Raume," Math. Ann. **79** (1919).

([2]) **E. Study**, "Beweis und Erweiterung eines von **E. Heß** angegebenen Satzes," Ber. d. Oberhess. Naturf. Ges. (1900).

(9) $$(x^0\; x^1\; x^2\; x^3)(u^0\; u^1\; u^2\; u^3) = \begin{vmatrix} (x^0u^0) & (x^0u^1) & (x^0u^2) & (x^0u^3) \\ (x^1u^0) & (x^1u^1) & (x^1u^2) & (x^1u^3) \\ (x^2u^0) & (x^2u^1) & (x^2u^2) & (x^2u^3) \\ (x^3u^0) & (x^3u^1) & (x^3u^2) & (x^3u^3) \end{vmatrix}$$

(10) $$u^0 = \overbrace{x^1x^2x^3}$$

in the equation for the multiplication theorem then it will follow that:

(11) $$(x^0\; u^0) \cdot (x^1\; x^2\; x^3,\; u^1\; u^2\; u^3) = (x^0\; u^0) \begin{vmatrix} (x^1u^1) & (x^1u^2) & (x^1u^3) \\ (x^2u^1) & (x^2u^2) & (x^2u^3) \\ (x^3u^1) & (x^3u^2) & (x^3u^3) \end{vmatrix},$$

and therefore, since that identity is true for $(x^0\; u^0) \neq 0$, in particular:

(12) $$(x^1\; x^2\; x^3,\; u^1\; u^2\; u^3) = \begin{vmatrix} (x^1u^1) & (x^1u^2) & (x^1u^3) \\ (x^2u^1) & (x^2u^2) & (x^2u^3) \\ (x^3u^1) & (x^3u^2) & (x^3u^3) \end{vmatrix}.$$

Now, let x^i be the points of a tetrahedron, $(x^0\; x^1\; x^2\; x^3) \neq 0$, and let a^i be the planes of a tetrahedron, such that:

(13) $$a^0 = \overbrace{x^1x^2x^3}, \quad a^1 = \overbrace{x^0x^2x^3}, \quad a^2 = \overbrace{x^0x^3x^1}, \quad a^3 = \overbrace{x^0x^1x^2}.$$

From (12), one will then have:

(14) $$\begin{aligned}(a^0\; a^1\; u\; v) &= (x^1\; x^2\; x^3,\; x^0\; x^2\; u^3,\; u,\; v) \\ &= \begin{vmatrix} (x^1x^0x^2x^3) & (x^1u) & (x^1v) \\ (x^2x^0x^2x^3) & (x^2u) & (x^2v) \\ (x^3x^0x^2x^3) & (x^3u) & (x^3v) \end{vmatrix} \\ &= -(x^2\; x^3,\; u\; v) \cdot (x^0\; x^1\; x^2\; u^3), \qquad \text{etc.}\end{aligned}$$

If one now sets:

(15) $$\mathfrak{X}_i = \widehat{x^iy^i}, \qquad \mathfrak{U}_i = \widehat{a^ib^i},$$

in which y^i, b^i represents a second tetrahedron, then one will have:

$$(16) \qquad (\mathfrak{U}_0\,\mathfrak{U}_1) = (a^0\,b^0\,a^1\,b^1) = -\,(a^0\,a^1\,b^0\,b^1)$$
$$= -\,(x^2\,x^3\,y^2\,y^3)\cdot(x^0\,x^1\,y^2\,y^3)\cdot(y^0\,y^1\,y^2\,y^3)$$
$$= +\,(x^2\,y^2\,x^3\,y^3)\cdot(x^0\,x^1\,y^2\,y^3)\cdot(y^0\,y^1\,y^2\,y^3)$$
$$= (\mathfrak{X}_2\,\mathfrak{X}_3)\cdot(x^0\,x^1\,y^2\,y^3)\cdot(y^0\,y^1\,y^2\,y^3).$$

Finally, if one introduces the quantities:

$$(17) \qquad (\mathfrak{U}_0\,\mathfrak{U}_1)\,(\mathfrak{U}_2\,\mathfrak{U}_3) = (\mathfrak{X}_0\,\mathfrak{X}_1)\,(\mathfrak{X}_2\,\mathfrak{X}_3)\,(x^0\,x^1\,y^2\,y^3)^2\cdot(y^0\,y^1\,y^2\,y^3)^2, *, *$$

then it will follow for the quantities that are defined analogously to (6) with the $\mathfrak{U}_i$ ($\mathfrak{X}_i$, resp.) that:

$$(18) \qquad D_i^* = D_i\,.$$

From nos. **27**, **28**, that will give the theorem that was to be proved.

20. Lines in hyperboloidal position. Four different lines of rank 3 are said to be in *hyperboloidal position* (since they will belong to a one-sheeted hyperboloid in the case of pair-wise skew, real lines). One proves, in a manner that is similar to what was just done:

Theorem 32: *If the connecting lines of corresponding vertices of two associated tetrahedra are found to be in hyperboloidal position then the lines of intersection of corresponding planes will also be in hyperboloidal position, and conversely* ([1]).

The two tetrahedra are then said to be in "hyperboloidal position." A theorem of **Chasles** that is analogous to one of **Plücker** (no. **89**) states:

Theorem 33: *Two tetrahedra that are polar relative to a second-order surface are found to be in hyperboloidal position* ([1]).

If the surface class of the surface degenerates into a conic section (which will be chosen to be the absolute conic section of the Euclidian metric) then the polar tetrahedron will degenerate into a rectangle in the imaginary plane, and the theorem of **J. Steiner** will then arise: *The altitudes of a tetrahedron are found to be in hyperboloidal position.* (They do not go through a point, in general.)

§ 10. Generating linear complexes,

31. Chasles's method of generation. Under the map of the linear complexes to the points of R_5, a regular, linear complex will be mapped to a point that does not belong to M_4^2. Its polar R_4 will cut M_4^2 in a regular M_3^2 whose points are the images of the lines

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([1]) Cf., the papers of **L. Berzolari** and **L. Brusotti** in Palermo Rendiconti **20** (1905). Furthermore, **E. A. Weiss**, Math. Zeit. **33** (1931).

of the complex. In order to generate the complex, we single out a pencil of R_3 in the polar R_4 . That pencil will fill up R_4 completely: i.e., a well-defined R_3 of the pencil will run through every point of R_4 that is in general position. The points of the base plane of the pencil will contain all R_3 in the pencil.

For our purposes, we can now choose that base plane in two different ways relative to M_3^2: as a plane E that cuts M_3^2 in a regular, second-order curve or one that cuts it in a pair of lines. We first consider the former case. The polar plane E' to E relative to M_4^2 will also cut M_4^2 in a regular, second-order curve, and the pencil of R_3 will be polar to a pencil of planes in the planes E'. That pencil will cut out an involution on the second-order curve. Let $\mathfrak{P}$ and $\mathfrak{P}'$ be two distinct associated points of that involution. The polar R_4 will then intersect in an R_3 of our initial pencil. If we let $\widehat{\mathfrak{P}\mathfrak{P}'}$ run through the lines of our pencil then the polar R_3 to the pencil of R_3 will run through E.

If we now translate this construction into the language of R_3 then the involution of the point pair $\mathfrak{P}$, $\mathfrak{P}'$ will be a regulus for the involution of the line pair $\mathfrak{P}$, $\mathfrak{P}'$. Just as the M_2^2 that the R_3 of the pencil cut out of M_3^2 fill up M_3^2 completely, the linear complexes will generate linear congruences whose guiding lines are the line-pairs of the image involution.

Theorem 34: *Let an involution be given in a regulus. The common lines of intersection of associated lines of the involutions will then generate a regular linear complex.*

The lines of intersection of the fixed lines of the involution (which indeed do not all belong to the linear complex) merit a special investigation. A fixed line will be the guiding line of a singular congruence of complexes, and indeed the congruence that one obtains (from Theorem 22) when one links the guiding lines with the generators of the polar regulus by pencils. In fact: An R_3 is polar to a tangent to a conic section in the plane E' that cuts M_3^2 in a cone M_2^2 that contains the conic section of the plane E. One then obtains the cone when one connects the vertex – viz., the point of contact of the tangent – with every point of that conic section with a line. Q. E. D.

32. Sylvester's method of generation. We now start with a base plane E in R_4 that cuts M_3^2 in two different lines. The plane E' that is polar to it relative to M_4^2 will cut M_4^2 in two distinct lines (no. **21**), and as before, a pencil of lines in E' will correspond to the pencil of R_3 . The two lines will be related to each other projectively by the pencil of lines in E'. Let $\mathfrak{P}$ and $\mathfrak{P}'$ be the associated points of the two lines. Their polar R_4 intersect in an R_3 of our pencil. If $\widehat{\mathfrak{P}\mathfrak{P}'}$ runs through the lines of the pencil in the plane E then the polar spaces will run through the pencil of R_3 in E':

Theorem 35: *Let two restricted pencils of lines be related to each other projectively in such a way that their common line corresponds to itself. The linear congruences that* *have corresponding lines of the two pencils for their guiding lines will then generate a linear complex.* (The lines of intersection of the lines that are common to the two pencils deserve a special investigation.)

33. Constructing a complex from five lines. Just as an R_4 in R_5 is determined by five linearly-independent points, a linear complex is determined by five linearly-independent lines. If five lines are given then one can easily give a **Chasles** or **Sylvester** generator for the complex. In the former case, three lines will determine a regulus, and the remaining two will establish the involution on the polar regulus. In the latter case, one can then begin to construct two null polar lines of the complex as the common line of intersection of four of the given lines of the complex. One of those lines will then determine the two pencils with the pair that was found. The five lines will cut out associated lines from the two pencils and will thus establish the projective relationship between them.

Once the product of the two projective pencils of lines that are found in special position is known, one can ask what the product of two projective pencils of lines in general position would be. The product will be a so-called *tetrahedral complex*, which is the locus of all lines that cut the planes of a tetrahedron with a constant double ratio. Due to the aforementioned special position of the two pencils, that complex (which is quadratic; cf., no. **54**) will decompose into a regular complex and the complex of lines of intersection of the common lines to the pencils.

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Chapter Three

Weitzenböck's complex symbolism.

§ 11. The product of two null systems.

34. Weitzenböck chains. In the previous chapter, we learned about the relationships, point loci, and plane loci that are linked with linear systems of linear complexes. In order to be able to represent these figures properly, we shall require some new analytical tools.

We write the equation of a linear complex in the form:

(1) $$(p\,\bar{p}\,x\,y) = 0.$$

In this, p and $\bar{p}$ shall be symbols that possess no real meaning in their own right. However, $p_i\bar{p}_k$ shall represent a coefficient, and indeed, one shall have:

(2) $$p_i\bar{p}_k = -p_k\bar{p}_i = \tfrac{1}{2}\mathfrak{P}_{ik}\,.$$

Due to the convention (2), the symbols are said to be *alternating*. One will then have:

(3) $$(p\,\bar{p}\,x\,y) = (p_0\bar{p}_1 - p_1\bar{p}_0)(x_2\,y_3 - x_3\,y_2) + \ldots = \mathfrak{P}_{01}\,\mathfrak{X}_{23} + \ldots$$

On the basis of (20) in § 1, one will now have:

(4) $$(p\,\bar{p}\,,\,uv) = (pu)(\bar{p}\,v) - (pv)(\bar{p}\,u) = 2\,(up)(\bar{p}\,v)\,.$$

We now set:

(5) $$(up)(\bar{p}\,v) = [u\,\mathfrak{P}\,\mathrm{v}]$$

to abbreviate, and call the expression a *Weitzenböck chain* ([1]). When the chain (5) is set equal to zero, it will represent the linear complex $\mathfrak{P}$ that is linked with the null system.

In order to be able to write down the dual equation, it proves to be *necessary to distinguish between ray coordinates and axial coordinates.* We denote the axial coordinates of the complex $\mathfrak{P}$ by $\mathfrak{P}'$, such that:

(6) $$(xp')(\bar{p}'\,y) = [x\,\mathfrak{P}\,y] = 0$$

will be the new representation of the null system that is coupled to the complex in point coordinates.

([1]) **R. Weitzenböck**, "Zum Formensystem von linearen Komplexen in R_3," Jahresbericht der Deutschen. Mathematiker-Vereinigung **19** (1910).

With the use of the chain symbolism, the invariant of two linear complexes $\mathfrak{A}$ and $\mathfrak{P}$ can also be written in the new form:

$$(7)\qquad (\mathfrak{A}\ \mathfrak{P}) = (a\bar{a},\, p'\,\bar{p}') = (ap')(\bar{a}\,\bar{p}') - (a\bar{p}')(\bar{a}\,p') = 2\,(\bar{a}p')(\bar{p}'a) = -\,2\,[\mathfrak{A}\ \mathfrak{P}'].$$

35. The skew involution. Now, let two linear complexes $\mathfrak{A}$ and $\mathfrak{B}$ be given that are associated with the null relationships:

$$(8)\qquad \begin{cases} [x\,\mathfrak{A}'y] = 0, & [x\,\mathfrak{B}'y] = 0, \\ [u\,\mathfrak{A}\,v] = 0, & [u\,\mathfrak{B}\,v] = 0. \end{cases}$$

These can be combined in two different sequences. Should the relationships commute, one of the following two identities must exist:

$$(9)\qquad [x\,\mathfrak{A}'\mathfrak{B}\,u] \pm [x\,\mathfrak{B}'\mathfrak{A}\,u] \equiv 0, \quad \{x, u\}.$$

We first treat the case of the lower sign. One will have:

$$(10)\qquad \begin{aligned}
[x\,\mathfrak{A}'\mathfrak{B}\,u] = {} & x_0\,a_0'\,\bar{a}_1'\,b_1\,\bar{b}_0\,u_0 = \tfrac{1}{4}\{-\,x_0\,\mathfrak{A}_{23}\,\mathfrak{B}_{01}\,u_0 \\
& + \quad \bar{b}_2\,u_2 \qquad + x_0\,\mathfrak{A}_{23}\,\mathfrak{B}_{12}\,u_2 \\
& + \quad \bar{b}_3\,u_3 \qquad - x_0\,\mathfrak{A}_{23}\,\mathfrak{B}_{31}\,u_3 \\
& + \quad \bar{a}_2'\,b_2\,\bar{b}_0\,u_0 \qquad - x_0\,\mathfrak{A}_{31}\,\mathfrak{B}_{02}\,u_0 \\
& + \quad \bar{b}_1\,u_1 \qquad - x_0\,\mathfrak{A}_{31}\,\mathfrak{B}_{12}\,u_1 \\
& + \quad \bar{b}_3\,u_3 \qquad + x_0\,\mathfrak{A}_{31}\,\mathfrak{B}_{23}\,u_3 \\
& + \quad \bar{a}_3'\,b_3\,\bar{b}_0\,u_0 \qquad - x_0\,\mathfrak{A}_{12}\,\mathfrak{B}_{03}\,u_0 \\
& + \quad \bar{b}_1\,u_1 \qquad + x_0\,\mathfrak{A}_{12}\,\mathfrak{B}_{31}\,u_1 \\
& + \quad \bar{b}_2\,u_2 \qquad - x_0\,\mathfrak{A}_{12}\,\mathfrak{B}_{23}\,u_2 \\
& + \quad \ldots \qquad + \ldots\}
\end{aligned}$$

$$\begin{aligned}
= -\tfrac{1}{4}\{ & x_0\,u_0\,[\mathfrak{A}_{23}\,\mathfrak{B}_{01} + \mathfrak{A}_{31}\,\mathfrak{B}_{02} + \mathfrak{A}_{12}\,\mathfrak{B}_{03}] \\
& + x_0\,u_1\,[\mathfrak{A}_{31}\,\mathfrak{B}_{12} - \mathfrak{A}_{12}\,\mathfrak{B}_{31}] + x_0\,u_2\,[\mathfrak{A}_{12}\,\mathfrak{B}_{23} - \mathfrak{A}_{23}\,\mathfrak{B}_{12}] \\
& + x_0\,u_3\,[\mathfrak{A}_{23}\,\mathfrak{B}_{31} - \mathfrak{A}_{31}\,\mathfrak{B}_{23}] + \ldots\}.
\end{aligned}$$

The expansion of:

$$(11)\qquad [x\,\mathfrak{A}'\ \mathfrak{B}\,u] - [x\,\mathfrak{B}'\ \mathfrak{A}\,u]$$

then shows that the coefficients of $x_i\, u_k$ $(i \neq k)$ are the twelve two-rowed determinants of the matrix:

$$\left\| \begin{matrix} \mathfrak{A}_{01} & \mathfrak{A}_{02} & \mathfrak{A}_{03} & \mathfrak{A}_{23} & \mathfrak{A}_{31} & \mathfrak{A}_{12} \\ \mathfrak{B}_{01} & \mathfrak{B}_{02} & \mathfrak{B}_{03} & \mathfrak{B}_{23} & \mathfrak{B}_{31} & \mathfrak{B}_{12} \end{matrix} \right\|, \tag{12}$$

namely, all possible determinants with the exception of $\mathfrak{A}_{01}\mathfrak{B}_{23} - \mathfrak{A}_{23}\mathfrak{B}_{01}$, *, *, while the coefficients of $x_i\, u_i$ are linear combinations of these three missing determinants:

Theorem 1: *The necessary and sufficient condition that two complexes* $\mathfrak{A}$ *and* $\mathfrak{B}$ *should be identical is (in quaternary form) the identical vanishing of their covariant* (11).

We now assume that $\mathfrak{A}$ and $\mathfrak{B}$ are different, and treat the case of the identity (9), except with the upper sign.

With consideration given to (7), equation (10) will give:

$$[x\, \mathfrak{A}'\mathfrak{B}\, u] + [x\mathfrak{B}'\mathfrak{A}\, u] = [\mathfrak{A}'\mathfrak{B}] \cdot (x\, u). \tag{13}$$

Theorem 2: *The null systems of two different complexes commute if and only if the complexes are conjugate.* (Cf., Chap. II, Theorem 14)

From Chap. II, Theorem 16, and under the assumption that $[\mathfrak{A}'\mathfrak{B}] = 0$, the equation:

$$[x\, \mathfrak{A}'\mathfrak{B}\, u] = 0 \tag{14}$$

represents the skew involution that is coupled with the complexes $\mathfrak{A}$ and $\mathfrak{B}$. It follows from this that:

Theorem 3: *If* $\mathfrak{A}$ *and* $\mathfrak{B}$ *are two distinct complexes of a regular pencil of complexes then the equation:*

$$[x\, \mathfrak{A}'\mathfrak{B}\, u] - [x\, \mathfrak{B}'\mathfrak{A}\, u] = 0 \tag{15}$$

will represent the skew involution that is coupled to the two guiding lines of the pencil.

Proof: The expression on the left-hand side of (15) possesses the *combination property:* viz., it will not change when one replaces $\mathfrak{A}$ and $\mathfrak{B}$ with two other complexes of the pencil. If one replaces $\mathfrak{A}$ and $\mathfrak{B}$ with – e.g., $\tau_1\,\mathfrak{A} + \tau_2\mathfrak{B}$ – then the left-hand side of (15) will become:

$$\begin{aligned} \tau_1\, [x\, \mathfrak{A}'\mathfrak{B}\, u] + \tau_2\, [x\mathfrak{B}'\mathfrak{A}\, u] - \tau_1\, [x\mathfrak{B}'\mathfrak{A}\, u] - \tau_2\, [x\, \mathfrak{A}'\mathfrak{B}\, u] \\ = \tau_1\{[x\, \mathfrak{A}'\mathfrak{B}\, u] - [x\, \mathfrak{B}'\mathfrak{A}\, u]\}. \end{aligned} \tag{16}$$

If we now assume that $\mathfrak{A}$ and $\mathfrak{B}$ are conjugate [which is always possible, since the pencil always contains two distinct conjugate complexes (in fact, ∞^1 pairs of them)] then, from (13), we will have:

$$(17) \qquad -[x\mathfrak{B}'\mathfrak{A}\,u] = [x\,\mathfrak{A}'\mathfrak{B}\,u],$$

and from (15), we will have:

$$(18) \qquad 2\,[x\,\mathfrak{A}'\mathfrak{B}\,u] = 0.$$

However, this equation represents [cf., (14)] the skew involution that is coupled with $\mathfrak{A}$ and $\mathfrak{B}$, and therefore with the guiding lines of the pencil. Q. E. D.

36. Point of intersection of two lines. From Theorem 3, equation (15) is implied in the case of a regular pencil of complexes. In the case of a *singular* pencil of complexes, the skew involution will degenerate. The case of the *pencil of lines* is particularly noteworthy:

Theorem 4: *If $\mathfrak{A}$ and $\mathfrak{B}$ are two distinct, incident lines then the left-hand side of* (15) *will split into the product of two linear forms:*

$$(19) \qquad [x\,\mathfrak{A}'\mathfrak{B}\,u] - [x\,\mathfrak{B}'\mathfrak{A}\,u] = (v\,x)\cdot(s\,u)$$

that will yield the connecting line and point of intersection of the two lines when they are set to zero.

We remark that the fact that the null system is involutory leads to the equation:

$$(20) \qquad [x\,\mathfrak{A}'\mathfrak{B}\,u] = \tfrac{1}{4}[\mathfrak{A}'\mathfrak{A}]\cdot(x\,u),$$

which is implied by (13).

§ 12. The product of three null systems.

37. Equation of the second-order surface that is determined by three lines. Let a regulus be determined by three pair-wise skew lines $\mathfrak{G}_1$, $\mathfrak{G}_2$, $\mathfrak{G}_3$. If x is a point of the surface that is determined by the regulus then there will be a generator of the second kind $\mathfrak{G}^*$ though x that cuts $\mathfrak{G}_1$, $\mathfrak{G}_2$, $\mathfrak{G}_3$. We now connect x to $\mathfrak{G}_1$, intersect the connecting plane with $\mathfrak{G}_2$, and then connect the point of intersection thus-obtained with $\mathfrak{G}_3$. The connecting plane contains the lines $\mathfrak{G}^*$, and thus the point x, in particular. The point x then satisfies the equation:

$$(1) \qquad [x\,\mathfrak{G}_1'\,\mathfrak{G}_2\,\mathfrak{G}_3'\,x] = 0.$$

The chain on the left-hand side of (1) is a combination of the bundle of complexes that is spanned by $\mathfrak{G}_1$, $\mathfrak{G}_2$, $\mathfrak{G}_3$. In fact, if one replaces $\mathfrak{G}_1$ with $y_1\mathfrak{G}_1 + y_2\mathfrak{G}_2 + y_3\mathfrak{G}_3$ then one will get:

(2) $$y_1[x\mathfrak{G}_1'\,\mathfrak{G}_2\,\mathfrak{G}_3'\,x] + y_2[x\mathfrak{G}_2'\,\mathfrak{G}_2\,\mathfrak{G}_3'\,x] + y_3[x\mathfrak{G}_3'\,\mathfrak{G}_2\,\mathfrak{G}_3'\,x] = y_1[x\mathfrak{G}_1'\,\mathfrak{G}_2\,\mathfrak{G}_3'\,x];$$

one will then have:

(3) $$[x\mathfrak{G}_2'\,\mathfrak{G}_2\,\mathfrak{G}_3'\,x] = \tfrac{1}{4}[\mathfrak{G}_2\mathfrak{G}_2']\cdot[x\mathfrak{G}_3'x] \equiv 0\ \{x\}$$ [§ 11, (20)]

and

(4) $$\begin{cases} [x\mathfrak{G}_2'\mathfrak{G}_2\mathfrak{G}_3'x] & = -[x\mathfrak{G}_2'\mathfrak{G}_3\mathfrak{G}_3'x] + \tfrac{1}{2}[\mathfrak{G}_2\mathfrak{G}_3']\cdot[x\mathfrak{G}_3'x] \\ & = -\tfrac{1}{4}[\mathfrak{G}_3\mathfrak{G}_3']\cdot[x\mathfrak{G}_2'x] + \tfrac{1}{2}[\mathfrak{G}_2\mathfrak{G}_3']\cdot[x\mathfrak{G}_3'x] \equiv 0\{x\}. \end{cases}$$ [§ 11, (13)]

The three lines can then be replaced with any three linearly-independent complexes of the bundle that they span.

Theorem 5: *Let* $\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{A}_3$ *be three linearly-independent complexes of a regular bundle. The second-order surface and the surface of class two that are determined by the regulus of the bundle will then have the equations:*

(5) $$[x\mathfrak{A}_1'\,\mathfrak{A}_2\,\mathfrak{A}_3'\,x] = 0, \qquad [u\,\mathfrak{A}_1\,\mathfrak{A}_2'\,\mathfrak{A}_3\,u] = 0,$$

resp.

§ 13. The Kummer configuration.

38. Polar hexatope of M_4^2. Just as one can construct ∞^6 tetrahedra that are polar to a second-order surface, there are ∞^{15} hexatopes in R_5 that are polar to M_4^2 – i.e., systems of six points that are pair-wise conjugate relative to M_4^2. In R_3, such a hexatope (of rank 6) will go to a system of six linear complexes (of rank 6) that are pair-wise conjugate. Such a system of six pair-wise conjugate complexes will then depend upon 15 constants.

Let $\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{A}_3$, $\mathfrak{A}_4$, $\mathfrak{A}_5$, $\mathfrak{A}_6$ be one such system. The bundle that is spanned by $\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{A}_3$ is polar to the bundle that is spanned by $\mathfrak{A}_4$, $\mathfrak{A}_5$, $\mathfrak{A}_6$. The surfaces of order two and class two that are spanned by the bundles, namely:

(1) $$[x\mathfrak{A}_1'\,\mathfrak{A}_2\,\mathfrak{A}_3'\,x] = 0, \quad [u\,\mathfrak{A}_1\,\mathfrak{A}_2'\,\mathfrak{A}_3\,u] = 0,$$

(2) $$[x\mathfrak{A}_4'\,\mathfrak{A}_5\,\mathfrak{A}_6'\,x] = 0, \quad [u\,\mathfrak{A}_4\,\mathfrak{A}_5'\,\mathfrak{A}_6\,u] = 0$$

will then be identical, and one will then get the identity when one composes the polarities that they determine. For that reason, there exists an identity of the form:

(3) $$[x\mathfrak{A}_1'\,\mathfrak{A}_2\,\mathfrak{A}_3'\,\mathfrak{A}_4'\,\mathfrak{A}_5\,\mathfrak{A}_6'\,u] = \rho\cdot(xu),$$

in which ρ denotes a factor that depends upon only the complexes $\mathfrak{A}_i$, and which we shall not elaborate upon.

Theorem 6: *The product of six linearly-independent, pair-wise commuting null systems is the identity.*

39. The group $\mathfrak{G}_{16}$. We shall denote the six null systems by [1], [2], [3], [4], [5], [6], to abbreviate, as one does in the theory of theta functions ([1]). We will obtain 15 skew involutions $[x \mathfrak{A}'_i \mathfrak{A}_k u] = 0$ by composing these null systems in pairs. We would like denote the skew involution that arises upon composing the null systems [i] and [k] by (ik).

The 15 involutions (ik), together with the identity [for which, we introduce the notation (0)], define a group G_{16} of 16 two-sided ([2]), commuting collineations. In fact: The equation (3) shows directly that one will obtain, perhaps by composing:

(12) and (34), the skew involution (56),

while naturally, the composition of:

(12) and (23) will yield the involution (13), trivially.

Theorem 7: *One will obtain a group G_{16} of two-sided, commuting collineations by composing any two null systems of a system of six linearly-independent, pair-wise conjugate linear complexes.*

40. Commuting, skew involutions. The latter argument shows that we have to distinguish between two kinds of commuting, skew involutions. In the case for which the symbols of the two involutions have no index in common, the guiding lines of the one involution will intersect the guiding lines of the other one. Any involution will leave the guiding lines of the other one individually fixed; we then call such involutions +*commuting*. In the case for which the symbols have a common index [e.g., (12) and (23)], the guiding lines will belong to a regulus, and (since the involutions commute) they will define two harmonic pairs on that regulus. One involution will permute the guiding lines of the other one; we then call such involutions –*commuting*.

Theorem 8: *Any skew involution of the group G_{16} is +commuting with six other ones and –commuting with eight other ones.*

([1]) **H. Weber**, Crelle's Journal **84** (1877).

([2]) Since we did not call the identity an involution, we need a term that will combine the identity transformation with the involutory transformations.

41. The group G_{32} . Twenty polar systems [*i j k*] will arise from composing the three null systems [*i*], [*j*], *k*]. We already showed above that these polar systems are pair-wise identical. For instance, the polar systems [123] and [456] are equal to each other.

If one composes a polar system with a null system or two polar systems then one will obtain a transformation of G_{16} – e.g.:

(4) [123] · [1] = (23), [123] · [4] = (56), [123] · [124] = (34),

and when one composes a polar system or a null system with a collineation of G_{16}, one of these polar or null systems:

(5) [123] · (12) = [3], [123] · (34) = [124], [1] · (23) = [123]

will again arise. Therefore:

Theorem 9: *A group* G_{32} *of two-sided, pair-wise commuting, projective transformations belongs to a system of six linearly-independent, pair-wise conjugate, linear complexes that contains six null systems* [*i*], *16 collineations* (0) *and* (*ik*), *and 10 polar systems* [*ikl*] = [*mno*].

42. The $(16_6, 16_6)$ **configuration**. We now subject a point to all 32 transformations of the group and thus obtain a system of 16 points and 16 planes, which we denote with the symbols of the transformation that produced them. The 16 points are then:

(6) (0), (12), (13), (14), (15), (16), (23), (24), (25), (26), (34),
(35), (36), (45), (46), (56),

while the 16 planes are:

(7) [1], [2], [3], [4], [5], [6], [123], [124], [125], [126], [134],
[135], [136], [145], [146], [156].

This shows that:

Theorem 10: *The 16 points* (6) *and the 16 planes* (7) *define a* $(16_6, 16_6)$ *configuration*: *Any point contains six of the planes, and any plane contains six of the point. This "Kummer configuration" will be transformed into itself by the group* G_{32} .

In fact: It follows from the existence of the point and plane systems that the figure is transformed into itself by G_{32} . It is then clear that the point:

(0) lies on its null planes [1], [2], [3], [4], [5], [6].

However, if one applies the null system [1] to this figure then it will follow that the plane:

[1] contains the points (0), (12), (13), (14), (15), (16),

and furthermore, when one applies the null system [2] to this figure:

(12) will lie on the planes [2], [1], [123], [124], [125], [126],

and finally, that the plane:

[123] = [456] will contains the points (23), (31), (12), (56), (64), (45).

In regard to the relative positions of these points and planes, one has the theorem (which is easily proved by means of **Pascal**'s theorem) that the six points that lie in a plane of the configuration belong to a conic section, and the six planes that run through a point of the configuration belong to a second-order cone.

Since the system of six pair-wise conjugate, linear complexes depends upon 15 constants, and a point in space depends upon 3 constants, there will be ∞^{18} *Kummer configurations*. Since a system of six points also depends upon 18 constants, one can assume that the six points determine a finite number of **Kummer** configurations. **H. Weber** has determined that number: Six points in general position can be extended to a **Kummer** configuration in 12 different ways.

43. Möbius's tetrahedra. We mention that one can regard the **Kummer** configuration in different ways as a *system of four tetrahedra that are found to be pair-wise in Möbius position.* (Cf., no. **9**).

In order to prove this, we start with the tetrahedra:

I.	(12), (13), (14), (56),	[134], [124], [123], [1],

and subject them to the null systems [1], [5], [6], in succession. We will obtain the tetrahedra:

II.	(34), (24), (23), (0),	[2], [3], [4], [156],
III.	(26), (36), (46), (15),	[125], [135], [145], [6].
IV.	(25), (35), (45), (16),	[126], [136], [146], [5].

The four tetrahedra subsume all points and planes of the configuration. By construction, I is found to be in Möbius position with II, III, IV. II goes to III by [6] and to IV by [5], and III goes to IV by means of [1]. Q. E. D.

§ 14. Multi-term chains.

44. The four-term chain. We would like to treat only two types of multi-term chains. We subject the four complexes of the chain:

(1) $$[x\, \mathfrak{A}'\, \mathfrak{B}\, \mathfrak{C}'\, \mathfrak{D}\, u]$$

to all permutations and endow them with a positive or negative sign according to whether one is dealing with an even or odd permutation, respectively. The expression:

(2) $$\sum \pm [x\,\mathfrak{A}'\,\mathfrak{B}\,\mathfrak{C}'\,\mathfrak{D}\,u],$$

which is similar to a determinant, will arise. This expression is a combination of the bush that is spanned by $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ (in case they are linearly-independent). If we replace it with four, pair-wise conjugate, linear complexes then (2) will reduce to a single term, due to (13) in § 11.

The geometric interpretation of the equation that arises by setting (2) equal to zero is obtained from Theorem 6. According to it, in the general case, one will be dealing with skew involutions of the two lines in which the four complexes intersect. In the case where the four complexes are linearly-dependent, these lines will be undetermined, and with them, the skew involution that they determine.

Theorem 11: *The existence of the identity:*

(3) $$\sum \pm [x\,\mathfrak{A}'\,\mathfrak{B}\,\mathfrak{C}'\,\mathfrak{D}\,u] \equiv 0 \qquad \{x, u\}$$

gives the necessary and sufficient condition (in quaternary form) for the four complexes $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ *to be linearly-independent.*

If the identity (3) *does not exist, and the four complexes span a regular bush then the equation:*

(4) $$\sum \pm [x\,\mathfrak{A}'\,\mathfrak{B}\,\mathfrak{C}'\,\mathfrak{D}\,u] \equiv 0$$

will represent the skew involution on both lines that have the four complexes in common. This relationship will degenerate when the two lines coincide.

In the case where the complexes have a pencil of lines in common, the left-hand side will split into a product of two linear forms that represent the vertex and plane of the pencil of lines when they are set equal to zero. (Analogue of Theorem 4).

45. The six-chain. The condition for six complexes $\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{A}_3$, $\mathfrak{A}_4$, $\mathfrak{A}_5$, $\mathfrak{A}_6$ to be linearly-dependent is the vanishing of the six-rowed determinant $(\mathfrak{A}_1\,\mathfrak{A}_2\,\mathfrak{A}_3\,\mathfrak{A}_4\,\mathfrak{A}_5\,\mathfrak{A}_6)$ of its coordinates. The equations of the complexes that are conjugate to five linearly-independent complexes $\mathfrak{A}_i$ – in particular, the complexes that run through five lines $\mathfrak{A}_i$ – then reads:

(5) $$(\mathfrak{A}_1\,\mathfrak{A}_2\,\mathfrak{A}_3\,\mathfrak{A}_4\,\mathfrak{A}_5\,\mathfrak{A}_6) = 0.$$

The form of this equation is taken from the geometry of R_5. The invariant property of the expression on the left-hand side under quaternary projective transformations will not be evident by this notation. That will first come about when the determinant is expressed with the help of chains. We give the result without proof ([1]):

$$(\mathfrak{A}_1\,\mathfrak{A}_2\,\mathfrak{A}_3\,\mathfrak{A}_4\,\mathfrak{A}_5\,\mathfrak{A}_6) = -\frac{2}{6!}\sum \pm [\mathfrak{A}_1\,\mathfrak{A}_2'\,\mathfrak{A}_3\,\mathfrak{A}_4'\,\mathfrak{A}_5\,\mathfrak{A}_6'] \tag{6}$$

$$= [\mathfrak{A}_1'\,\mathfrak{A}_2\,\mathfrak{A}_3'\,\mathfrak{A}_4\,\mathfrak{A}_5'\,\mathfrak{A}_6] - [\mathfrak{A}_1\,\mathfrak{A}_2'\,\mathfrak{A}_3\,\mathfrak{A}_4'\,\mathfrak{A}_5\,\mathfrak{A}_6'] .$$

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([1]) **B. L. van der Waerden**, "Über Determinanten aus Formenkoefficienten," K. Ak. d. Wet. t. Amsterdam, Proc. XXV (1922).

Chapter Four

The line-sphere transformation

§ 15. Map of a line-pair in the space of a linear complex to a second-order surface on an M_3^2.

46. Trace of a point and a plane. One application of the map of the projective line continuum to the M_4^2 in R_5 is *Lie's line-sphere transformation* ([1]).

We single out a regular, linear complex $\mathfrak{C}$ in R_3 (i.e., line space) as the *basic complex.* It will correspond to an R_4 in R_5 that cuts M_4^2 in a regular M_3^2. The pencils of lines of the complex $\mathfrak{C}$ will be mapped to the lines of M_3^2 in a one-to-one and invertible way.

Now, an arbitrary plane in R_3 contains a pencil of lines of $\mathfrak{C}$, just as an arbitrary point of R_3 does. In both cases, we would like to call the pencil of lines the *trace* of the plane or the point in the linear complex. Points and planes will be mapped to generating planes of the first and second kind, resp., in M_4^2, and both of them will cut out one line from M_3^2 in R_4, namely, their *trace* on M_3^2.

Points and planes determine unique traces in a linear complex. However, the initial element will not, conversely, be determined uniquely by the trace.

A pencil of lines of $\mathfrak{C}$ determines a point and a plane.	A plane of the first kind and a plane of the second kind in M_4^2 run through a line of M_3^2.

In order to make the map uniquely invertible, *we cover the trace* (and therefore, the pencil of lines of $\mathfrak{C}$ and the line of M_3^2) *with two sheets.* The first sheet shall be the points that correspond to the second of the planes in R_3. One will then have, without exception, a single-valued, invertible correspondence:

Point of R_3	Line of the first sheet of M_3^2
Plane of R_3	Line of the second sheet of M_3^2
Application of the null system that is linked with $\mathfrak{C}$	Switching of lines of both sheets that "overlap" each other
Points and planes are incident (i.e., their traces have a common line)	Lines of the first sheet and lines of the second sheet are incident.

([1]) The presentation that follows here goes back to **E. Study**, "Vereinfachte Begründung von S. Lies Geraden-Kugeltransformation," Sitzungsber. Preuß. Ak. d. Wiss. (1926).

In this, we also refer to two lines on the right-hand side as "incident" when they "overlap," which is a case that will occur when the points and planes on the left-hand side are related as null points and null planes.

47. The image of a line. We now consider an arbitrary line $\mathfrak{G}$ in R_3 that does not belong to the complex $\mathfrak{C}$ to be the locus of its points and planes. Its ∞^1 points correspond to a system of ∞^1 lines of the first sheet in M_3^2, and its ∞^1 planes will correspond to ∞^1 lines of the second sheet. Since every point of the line $\mathfrak{G}$ is incident with every plane of the line $\mathfrak{G}$, every line of the first sheet will intersect every line of the second one. The two families of lines then define the two families of generators of a regular, second-order surface, and indeed, an oriented surface, since the two families of generators can be put into a well-defined sequence, namely, in the well-defined way that the two sheets of lines are distributed on M_3^2 (no. **23**).

A line that is not a null line	Regular, oriented, second-order surface on M_3^2
Its null polar	Oppositely-oriented second-order surface

By contrast, a null line corresponds to a point on M_3^2. The null lines that it intersects will define a singular linear congruence, which, as we know (Chap. II, Theorem 19), will be mapped to a second-order cone. Any generator of that cone must be regarded as a line of the first sheet and a line of the second sheet:

Null line (as the locus of points and planes)	Second-order cone on M_3^2 (simultaneously the locus of lines of the first and second sheets)

§ 10. The relationship as a contact transformation.

48. Surface element and leaf. Up to now, under our map, lines and pencils of lines (as spatial elements in line space) appeared on M_3^2 as points and lines, resp. We shall now consider the map to be an association of spatial elements of a different kind.

With **S. Lie**, we call the figure that consists of a point and a plane in united position in R_3 a *surface element*. Under our map, such a surface element will correspond to the figure that consists of two incident lines on M_3^2, namely, a line of the first sheet and a line of the second one (which can also overlap), which **E. Study** called a *leaf*. Our map will then give a single-valued and invertible correspondence between the ∞^5 surface elements of R_3 and the ∞^5 leaves on M_3^2. The application of the null system to a surface element corresponds to a re-orientation of the associated leaf (i.e., the two lines of the leaf will be simultaneously subjected to a change of sheet).

Surface element	Leaf
Surface element of the null system (Point and plane are related as null points and null plane.)	Singular leaf (The lines of the leaf overlap.)

49. The line as a locus of surface elements. Just as a line is the locus of ∞^2 surface elements, an oriented second-order surface on M_3^2 will be the locus of ∞^2 leaves. Any tangential plane to the surface will contain such a leaf. Therefore, one will have the correspondence:

Line that does not belong to the basic complex $\mathfrak{C}$, as the locus of its surface elements	Oriented, second-order surface, as the locus of its leaves

If one maps the ∞^2 surface elements of a null line to M_3^2 then one will get a leaf that corresponds to that surface element, and that will contain any arbitrary generator of the image cone as a line of the first and second kind. It will produce the manifold of all ∞^2 leaves that go through the vertex of the cone.

Null line, as the locus of surface elements	Point, as the locus of leaves

The only way that one can speak of a well-defined degeneracy of a geometric figure is when one is given the figure that it will degenerate into as a locus of spatial elements. For instance, a conic section will degenerate into a pair of lines as a locus of points and a pair of points as a locus of lines. Similarly, we have here: If a line in general position goes to a null line in line space then the surface that corresponds to the lines on M_3^2, as a locus of points, will go to a second-order cone, as a locus of surface elements at a point.

50. Unions. Let the point coordinates *x*, as well as the plane coordinates *u*, be analytically dependent upon a certain number of parameters, and indeed, in such a way that the equation:

(1) $$(u\,x) = 0$$

will be fulfilled identically. One will then have the equation:

(2) $$(u\,dx) = 0,$$

and since (1) makes this equivalent to the equation:

(3) $$(x\,du) = 0,$$

one will call the manifold of elements that is defined by equations (1)-(3) in the domain of existence of the functions a *union* (**S. Lie**). We say that every element of a union is "united" with each *consecutive* one.

One easily sees that only one and two-dimensional unions can exist in R_3. If the locus of points is an analytical surface patch then the union will consist of the tangential element to that surface (i.e., tangential planes and their contact points). If the locus of points is an analytical curve segment then a surface element of the union will consist of a point of the curve segment, together with a tangential plane to the curve segment at that point. If the locus of points is a point then the union will consist of elements through that point. Along with the unions that were enumerated, there are also the unions that are dual to them.

51. Contact transformations. A transformation of surface elements that takes unions to unions is called a *contact transformation:* It will take unions that *contact each other* – i.e., have an element in common – to unions that contact each other. Collineations and correlations are very special examples of contact transformations, as well as the "extended point transformations"; i.e., the transformations of surface elements that are induced by point transformations.

Now, a contact transformation is also the relationship between surfaces elements in line space and leaves (viz., "oriented surface elements") in M_3^2 that we have considered. It associates unions of surface elements with unions of leaves. We communicate that result without proof [**E. Study**, Math. Ann. **91** (1924), 106-107], since we shall not need it in that generality for the following special case:

Theorem 1: *Two incident null-planes will be mapped to two contacting, oriented, second-order surfaces by the line-sphere transformation.*

In fact: The line, as the locus of its surface elements, is a union (viz., any element of the line is united with any other one), and the oriented, second-order surface, as the locus of its leaves, is a union (viz., consecutive leaves are united). Furthermore: Two incident lines, when regarded as unions of elements, will have a common surface element, and it will be associated with a common leaf of the two corresponding, oriented, second-order surface by the relationship. We shall pass over any examination of the special cases.

§ 17. Stereographic projection of M_3^2 to R_3.

52. Stereographic projection of the cone. The relationship that we have been dealing with up to now mediates the connection between figures in a certain R_3 (viz., line space) and other figures on a M_3^2 in R_4. It is possible to go from this M_3^2 to a second R_3 by "stereographic projection," and in that way, establish a relationship between two R_3.

Before we treat the projection of M_3^2 onto R_3, it will be convenient to examine the corresponding relationship in one less dimension: We would like to project a real, regular cone stereographically – i.e., from one of its real points p – onto a plane. As long as one restricts oneself to real points, that projection will be uniquely invertible, and as long as one establishes that the line G along which the tangential plane to p cuts the image plane shall be referred to as an *accessory point.* The real projective plane B will then be the *Gaussian plane.*

In the complex domain, however, the one-to-one character can no longer be produced by such a convention on the terminology. In fact, the following elements will correspond:

Center of projection p	All points of the line G
Arbitrary points of the generator of the first (second, resp.) kind through p that are different from p	"Absolute points" of the first (second, resp.) kind

A non-decomposable section of the sphere – and thus, a regular circle – will be mapped to a circle or a line in the image plane according to whether it does or does not include the center of projection p, resp., and which one can refer to as the "circle through the accessory point." The projection of M_3^2 onto R_3 behaves similarly.

53. Stereographic projection of M_3^2**.** We next arrive, by a (necessarily imaginary) collineation, at the fact that the M_3^2 that is cut out of the **Plückerian** M_4^2 by R_4, namely:

$$\mathfrak{X}_{01}\,\mathfrak{X}_{23} + \mathfrak{X}_{02}\,\mathfrak{X}_{31} + \mathfrak{X}_{03}\,\mathfrak{X}_{12} = 0, \tag{1}$$

is a spherical manifold (viz., one whose the equation is $-x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 = 0$). The R_3-sections of that M_3^2 will then be spheres, and those spheres will once more go to spheres or planes under the stereographic projection M_3^2 onto R_3. However, we now obtain a map of lines in line space to oriented spheres in Euclidian R_3 that is no longer free of singularities.

The tangential R_3's at the center of projection cut M_3^2 in a second-order cone that meets the image R_3 in a conic section, in its own right. That conic section will be the *absolute conic section* that is common to all spheres in Euclidian R_3. Its lines of intersection (viz., *minimal lines*), which are projections of lines in M_3^2, will be images of the points in line space. The points of Euclidian R_3, which are projections of points of M_3^2**,** will be images of the null lines in line space.

We now go on to a more precise treatment of the singularities that might appear: It will then be clear that the nucleus of the line-sphere transformation is contained in the relationship between line space and spherical M_3^2 that we have presented, and that the singularities will first come about when we wish to convert that relationship artificially into a relationship between two R_3 (but not considered from the viewpoint that was assumed here, since it was anthropocentrically influenced).

§ 18. The Dupin cyclide as the image of a second-order surface.

54. Definition of the cyclide. One will get the equation of a *quadratic line complex* by setting a quadratic form in line space equal to zero. As was pointed out on pp. ?, the lines of such a complex that lie in a plane will envelope a curve of class 2, while the complex lines through a point will define a second-order cone.

If one now intersects a quadratic complex in line space with the basic complex then one will obtain a congruence (2, 2), namely, a system of ∞^2 lines such that two lines of the system will run through any point in general position, while two lines of the system will lie in any plane in general position. The image of that congruence in R_4 is the (general) M_4^2-section of two quadratic manifolds. The intersection manifolds that are obtained in that way, and likewise, their projections onto R_3, will be surfaces of order 4 or 3 that contain the absolute conic section, when counted twice or once, resp., and that will be called *cyclides.*

If one starts with the tangent complex to a second-order surface in line space, in particular, then one will obtain a *Dupin cyclide.*

55. Map of a second-order surface in general position. We apply the line-sphere transformation to a second-order surface F^2 that is in general position with respect to the basic complex; i.e., the basic complex shall cut each of the two families of generators of F^2 in two distinct generators $\mathfrak{N}_1$, $\mathfrak{N}_1'$ and $\mathfrak{N}_2$, $\mathfrak{N}_2'$ (cf., no. **75**).

The lines of the basic complex that cut two skew lines that are not null polar to each other define a regulus (no. **22**). In the limiting case, the null lines that contact F^2 along a generator will also define a regulus. Such a circle will be mapped to a conic section on M_3^2, and thus, to a circle, when we assume that M_3^2 is spherical. The image of F^2 – viz., a *Dupin cyclide* – will go to two families of ∞^1 circles that correspond to two families of generators, and just as any surface tangent cuts each generator of both families, and otherwise no generator, two circles of the same family will be disjoint, while any circle of the one family will cut any circle of the other family at a point.

Among the circles, one finds, in particular, ones that correspond to the null lines $\mathfrak{N}_1$, $\mathfrak{N}_1'$ of the first family of generators and $\mathfrak{N}_2$, $\mathfrak{N}_2'$, of the second one. Since $\mathfrak{N}_1$ cuts the null lines $\mathfrak{N}_2$, $\mathfrak{N}_2'$, the regulus will decompose into the null lines that are tangent to $\mathfrak{N}_1$ in the pair of restricted pencils and $\mathfrak{N}_1\,\mathfrak{N}_2$ and $\mathfrak{N}_1\,\mathfrak{N}_2'$. Correspondingly, the cyclide will contain two circles in each of its two families of circles that decompose into pairs of lines (in R_3: pairs of minimal lines). They collectively define a spatial quadrilateral. The cyclide does not contain more than those four lines, since F^2 possesses no more than four pencils of tangent null lines.

The vertices of the spatial quadrilateral $\mathfrak{N}_1$, $\mathfrak{N}'_1$; $\mathfrak{N}_2$, $\mathfrak{N}'_2$ (viz., the double points of the cyclide) correspond to the four null generators of F^2. Since any generator of the second kind cuts the lines $\mathfrak{N}_1$, $\mathfrak{N}'_1$, each of the circles of second kind will run through the point $\mathfrak{N}_1$, $\mathfrak{N}'_1$, and likewise, each of the circles of the first kind will run through the points $\mathfrak{N}_2$, $\mathfrak{N}'_2$.

56. Construction of the cyclide. The two families of generators of F^2 will be mapped to two simply-infinite families of oriented spheres. From Theorem 1, every sphere of the one family will contact every sphere of the family. Just as F^2, as the locus of the common lines of intersection of three generators, can contain one and same family, the *Dupin cyclide* can be obtained as the envelope of all spheres that contact the three spheres of one of the two families.

Theorem 2: *Under the line-sphere transformation, a Dupin cyclide on M_3^2 will be the image of an F^2 that cuts the basic complex in a spatial quadrilateral of generators. That cyclide will be a fourth-order surface with four double points. The four double points are the vertices of a spatial quadrilateral whose four sides are the only lines in the cyclide. The cyclide is the envelope of two simply-infinite families of spheres, and will be covered by two families of ∞^1 circles that run through two opposite points of the four double points.*

57. Lines of curvature and asymptotic lines. "Consecutive" spheres from one of the two families intersect in circles of the corresponding family of circles. The surface normals along one such circle will (obviously) define a cone. The circles are thus the lines of curvature of the cyclide. Under the map, they will correspond to the generators of F^2, and thus to the asymptotic lines of F^2. We then have the simplest example of **Lie**'s theorem, which says that *principal tangent curves in line space and lines of curvature in sphere space will correspond under the line-sphere transformation.*

Lines of curvature and asymptotic lines cannot be regarded as loci of points for a rigorous formulation of that theorem, but as loci of surface elements.

58. Special cases. If F^2 assumes a special position with respect to the basic complex then a degenerate case will appear in place of the general *Dupin cyclide.* In the case where the complex cuts out a pair of coincident lines from one of the two families of generators, the points of a pair of double points will coincide. In the case where the basic complex contains the two generating reguli of F^2 completely, the cyclide will split into two second-order cones as a locus of points (in R_3, they will be *minimal cones*; i.e., cones that contain the absolute conic section) and the circle of intersection of those two cones as a locus of leaves.

With that, we have exhausted the cases in which a regular, second-order surface will appear in line space. In the case of a cone, one will get a fourth-order ruled surface ([1]).

§ 19. Study's double-five.

59. Heuristic process. As a further application of the line-sphere transformation, we would like to derive (this time, starting in sphere space) a special configuration of two-times-five lines:

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 1' & 2' & 3' & 4' & 5' \end{matrix}$$

with the property that every line cuts four other ones, namely, the ones that are not in the same row or column; that configuration is a so-called *double-five.*

It is not difficult to construct a double-five: One starts with five lines 1, 2, 3, 4, 5 in general position and constructs one of the two common lines of intersection to any four of them. The existence of a double-five, upon which one imposes no other demands, is in no way remarkable then. (By contrast, the existence of a *double-six* will rest upon the validity of a closing theorem.)

However, the existence of a double-five whose pairs of lines are pairs of polar lines relative to a null system is remarkable. Such a double-five shall be called a *Study double-five*, since **Study** ([2]) found it in the following way:

If one draws three smaller spheres in a large one then one will see that one can find yet a fifth one (upon which, the three smaller spheres in the larger one will lie), in such a way that the spheres thus-obtained will contact pair-wise. If one orients these spheres then one will obtain a double-five of oriented spheres:

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 1' & 2' & 3' & 4' & 5' \end{matrix}$$

in such a way that every sphere will contact the other ones that are not in the same row and column.

If one now subjects that figure to the line-sphere transformation then a pair of null-polar lines will arise from a pair of associated, oriented spheres. Hence:

([1]) **E. A. Weiss**, "Zykliden als Bilder von Flächen 2. Ordnung in der Geradenkugeltransformation," Mathematica **8** (1933).

([2]) **E. Study**, in the beginning of the paper that was cited in no. **46**.

Theorem 3: *If one subjects a quintuple of non-oriented spheres that contact pair-wise to the line-sphere transformation then a double-five will arise whose line-pairs are pairs of null-polar lines in a regular null system*

60. Projective derivation of the double-five. One can pose the problem of constructing such a double-five in projective space (i.e., independently of the line-sphere transformation) ([1]). For that, it is convenient to consider the figure of 5 · 2 points on **Plücker**'s M_4^2 that correspond to the 5 · 2 lines of the double-five.

Since the line-pairs are pairs of lines that are polar relative to a fundamental complex $\mathfrak{C}$, a point $\mathfrak{C}$ in R_5 that does not lie on M_4^2 is singled out in such a way that the connecting lines of the image points of associated lines 11′, …, 55′ run through $\mathfrak{C}$ (no. **19**). We consider two of these lines 11′ and 22′. They span a plane E that cuts M_4^2 in a second-order curve. However, that curve will decompose into a pair of lines.

We now consider the points p_1 and p_2 that the lines 11′ and 22′ cut out of the R_4 that is polar to the point $\mathfrak{C}$. Its line of intersection will be the line of intersection of the plane E with the polar R_4 and (as the polar to the point $\mathfrak{C}$ relative to the pair of lines) will run through the vertex of the pair of lines. The line $p_1\, p_2$ is then a tangent to the M_3^2 that the polar R_4 cuts out of **Plücker**'s M_4^2 .

Theorem 4: *The connecting lines of the image points of associated lines of a Study double-five cut out a tangent pentatope in M_3^2 from the image R_4 of the fundamental complex.*

61. Tangent pentatope to M_3^2. Up to now, we have inferred the fact that a tangent pentatope to an M_3^2 exists only intuitively: We started with five spheres that contacted pair-wise. The existence of a tangent pentatope, and therefore, a **Study** double-five, shall now be proved. To that end, we seek to inscribe the coordinate simplex in R_4 in an M_3^2:

(1) $$a_{11}\, x_1^2 + \ldots + a_{55}\, x_5^2 + 2a_{12}\, x_1\, x_2 + \ldots + 2a_{45}\, x_4\, x_5 = 0,$$

in such a way that the ten edge lines will be tangents to M_3^2. For example, should the connecting line:

(2) $$\xi_1\,(1, 0, 0, 0, 0) + \xi_2\,(0, 1, 0, 0, 0)$$

be tangent to M_3^2, then the discriminant of the form:

(3) $$a_{11}\,\xi_1^2 + 2a_{12}\;\xi_1\;\xi_2 + a_{22}\,\xi_2^2$$

([1]) **A. Maurer**, "Doppelvieren und Dopplefünften," Diss. Bonn, 1929.

would have to vanish, and thus the two-rowed determinant:

(4) $$a_{11}\, a_{22} - a_{12}^{\ 2}.$$

One sees in the same way that all ten two-rowed determinants that one can select from the matrix of the M_3^2 that are symmetric about the main diagonal must vanish. One can then calculate the remaining ones from the given values of the diagonal elements of that matrix. If one sets:

(5) $$a_{ii} = a_i^2$$

then one will have:

(6) $$a_{ii}\, a_{kk} - a_{ik}^{\ 2} = 0, \qquad \text{thus: } a_{ii}\, a_{kk} = a_{ik}^{\ 2}, \qquad (i \neq k)$$

(7) $$a_{ik} = \pm\, a_i \cdot a_k\,.$$

It is clear that one cannot take all of the signs in the determinant to be positive, since the M_3^2 would then be a doubly-counted R_3. One can then ask how many ways that the signs in the matrix can be assigned if one is to obtain a regular M_3^2. We shall pass over that question, and establish only that there are regular M_3^2. One of them corresponds to the following sign arrangement:

$$\begin{matrix} - & + & + & + & + \\ + & - & + & + & + \\ + & + & - & + & + \\ + & + & + & - & + \\ + & + & + & + & - \end{matrix}$$

and the simplest M_3^2 of that type is:

(8) $$-\, x_1^{\ 2} - \ldots - x_5^{\ 2} + 2x_1\, x_2 + \ldots + 2x_4\, x_5 = 0.$$

At the same time, we see that the M_3^2 that one can inscribe in a given simplex such that it becomes a tangent pentatope will depend upon four essential constants. A pentatope in R_4 depends upon $5 \cdot 4 = 20$ constants, so the figure that consists of an M_3^2 with a tangent pentatope will depend upon 24 constants, and since there are ∞^{14} M_3^2 in R_4, one can circumscribe ∞^{10} tangent pentatopes to a given M_3^2.

That fact defines the foundation for the enumeration of the **Study** double-fives: Since the R_4, and therefore the complex $\mathfrak{C}$, can be chosen in ∞^5 different ways, one will have:

Theorem 5: *There are* ∞^{15} *Study double-fives.*

In conclusion, we mention yet another remarkable double-five, namely, the double-five of **B. Segre**. Any line of such a double-five is the single line of intersection of the associated quadruple. The lines of the double-five belong to a linear complex, and will be mapped to a pentatope that is, at the same time, inscribed and circumscribed in the image M_3^2 of the linear complex ([1]).

([1]) **B. Segre**, "Le piramidi inscritte e circoscritte alle quadriche di S_4 e una notevole configurazione di rette dello spazio ordinario," Memorie della R. Acc. dei Lincei (6) **2** (1927).

Chapter Five

Metric line geometry

§ 20. Basic formulas of non-Euclidian geometry.

62. The three kinds of non-Euclidian geometry in R_3 . The geometry of the group of automorphic collineations of a regular quadratic manifold in R_n is called *non-Euclidian geometry* in R_n . The automorphic collineations themselves will also be called "non-Euclidian motions" and "transfers" in connection with that. From the standpoint of real, projective geometry, one distinguishes just as many types of real, non-Euclidian geometries as there are different types of real, non-Euclidian manifolds. Thus, in the plane, the real conic sections with and without real points will correspond to two kinds of non-Euclidian geometry, namely, hyperbolic and elliptic, respectively, but in space, there will be three kinds: the real, regular, second-order surfaces without real points, the ones with real points, but no real lines, and finally, the ones that also have real lines. Therefore, whenever we are concerned with questions of reality at all in what follows, we shall treat elliptic (absolute surface without real points) or hyperbolic (absolute surface with real points, but no real lines) geometry exclusively ([1]).

63. Distance between two points. A so-called **Cayley** metric will be defined by a basic second-order surface in space that assigns the distance between two points and dually the angle between two planes to two points and two planes, respectively.

One defines the distance between two points x, y with the help of the double ratio that the two points determine, along with two other ones at which their connecting line:

$$z = \xi_1\, x + \xi_2\, y \tag{1}$$

cuts the absolute surface $(x\,x) = 0$:

$$(\xi \mid \xi) = \xi_1^2\,(x\,x) + 2\,\xi_1\,\xi_2\,(x\,y) + \xi_2^2\,(y\,y) = 0. \tag{2}$$

64. Double ratio of two points and a point-pair. We next pose the problem of determining the double ratio that two points ξ_1 and ξ_2 determine with the zero locus of a quadratic form:

$$(\xi \mid \xi) = (\alpha^1\,\xi) \cdot (\alpha^2\,\xi). \tag{3}$$

It follows immediately that the discriminant of that form is:

$$D = C \cdot (\alpha^1\,\alpha^2)^2, \tag{4}$$

([1]) For a thorough presentation of non-Euclidian geometry, we refer to **J. L. Coolidge**, *The elements of non-euclidian geometry*, Oxford, 1909.

in which C denotes a yet-to-be-determined constant. The discriminant will then be quadratic in the coefficients of the form, and will vanish if and only if the zero locus of the form is a point. However, if one sets:

(5) $$\alpha_1^1 : \alpha_2^1 = 1 : 0, \qquad \alpha_1^2 : \alpha_2^2 = 0 : 1,$$

(6) $$(\alpha^1\, \xi) \cdot (\alpha^2\, \xi) = -\, \xi_1 \cdot \xi_2\,,$$

in particular, then one will have:

(7) $$D = -\tfrac{1}{4}, \qquad (\alpha^1 \alpha^2) = 1, \quad \text{so} \quad c = -\tfrac{1}{4},$$

and one will ultimately have:

(8) $$D = -\tfrac{1}{4}\,(\alpha^1 \alpha^2)^2.$$

Furthermore, one will have:

(9) $$(\xi^1 \mid \xi^2) = \tfrac{1}{2}\{(\alpha^1 \xi^1)\,(\alpha^2 \xi^2) + (\alpha^1 \xi^2)(\alpha^2 \xi^1)\}.$$

Equations (8) and (9) suffice to derive the desired double ratio:

(10) $$\begin{aligned} DV\,(\alpha^1 \alpha^2\, \xi^1 \xi^2) &= \frac{(\alpha^1\xi^1)(\alpha^2\xi^2)}{(\alpha^1\xi^2)(\alpha^2\xi^1)} \\ &= \frac{\frac{1}{2}\{(\alpha^1\xi^1)(\alpha^2\xi^2)+(\alpha^1\xi^2)(\alpha^2\xi^1)\}+\frac{1}{2}(\alpha^1\alpha^2)\cdot(\xi^1\xi^2)}{\frac{1}{2}\{(\alpha^1\xi^1)(\alpha^2\xi^2)+(\alpha^1\xi^2)(\alpha^2\xi^1)\}-\frac{1}{2}(\alpha^1\alpha^2)\cdot(\xi^1\xi^2)} \\ &= \frac{(\xi^1 \mid \xi^2)+i\sqrt{D}\cdot(\xi^1\xi^2)}{(\xi^1 \mid \xi^2)-i\sqrt{D}\cdot(\xi^1\xi^2)}. \end{aligned}$$

This invariant contains the roots of the discriminant of the quadratic form. It is therefore not a rational invariant, but the simplest example of an irrational invariant.

The fact that an expression for the double ratio must appear that is capable of taking on two different values was clear from the outset, since one must still decide between the two sequences of zero loci of the quadratic form. The quadratic form $(\xi \mid \xi)$ will be oriented by choosing the sign of $\sqrt{D}$.

65. Distance and angle in elliptic geometry. If we substitute the expression (2) for $(\xi \mid \xi)$ in (10) and the points 1 : 0 and 0 : 1 for ξ^1 and ξ^2 then it will follow that:

(11) $$DV\,(\alpha^1 \alpha^2\, \xi^1 \xi^2) = \frac{(xy)+i\sqrt{(xx)(yy)-(xy)^2}}{(xy)-i\sqrt{(xx)(yy)-(xy)^2}}\,.$$

However, we shall not take this double ratio itself (which indeed already represents an invariant of motion) to be the "distance" between the points x, y in elliptic geometry, but define:

(12) $$\text{dist}\ xy = \frac{1}{2i}\ln DV(\alpha^1\alpha^2\,\xi^1\xi^2).$$

We would like to explain the consequences of this [1]. If ξ^1, ξ^2, ξ^3 are the parameters of three points that lie on the lines α^1, α^2 then:

(13) $$DV(\alpha^1\alpha^2\,\xi^1\xi^2)\cdot DV(\alpha^1\alpha^2\,\xi^2\xi^3)\cdot DV(\alpha^1\alpha^2\,\xi^3\xi^1) = 1,$$

so

(14) $$DV(\alpha^1\alpha^2\,\xi^1\xi^2)\cdot DV(\alpha^1\alpha^2\,\xi^2\xi^3) = DV(\alpha^1\alpha^2\,\xi^1\xi^3)\,.$$

Adjacent segments of a line are then multiplied. However, if we desire that such segments should be added as in elementary geometry then we will have to introduce the logarithm.

In the case of elliptic geometry, the (real) connecting line of two real points x, y will cut the absolute surface at two complex-conjugate points α^1, α^2. The double ratio (11) will then be a complex number of absolute value 1, so its logarithm will be pure imaginary. In order to obtain a real value for the distance between two real points, one must divide by i.

Finally, the factor 1 / 2 is explained as follows: In elliptic geometry, the angle between two planes u, v:

(15) $$\text{ang}\ u\,v = \frac{1}{2i}\ln DV(\alpha^1\alpha^2\,\xi^1\xi^2)$$

is dual to the distance between two points, in which ξ^1, ξ^2 are the parameters of u, v in the pencil of planes that they span, and α^1, α^2 are the parameters of the absolute planes that are contained in the pencil.

We now take (as we do in elementary geometry) two planes to be *orthogonal* when they are conjugate to each other relative to the absolute structure:

(16) $$(u\,u) = 0.$$

However, in that case, one will have $DV(\alpha^1\alpha^2\,\xi^1\xi^2) = -1$, and:

(17) $$\text{ang}\ u\,v = \frac{1}{2i}\cdot\pi i = \frac{\pi}{2}.$$

We have then included the factor 1 / 2, in order to obtain the angle $\pi / 2$ in the case of orthogonal planes, just as in elementary geometry.

From equation (11), and with the help of the formula:

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(1) **F. Klein**, "Über die sogenannte Nichteuklidische Geometrie," Math. Ann. **4** (1871) = Ges. Werke I, (1921), pp. 254.

$$\frac{1}{2i}\ln\frac{1+i\varphi}{1-i\varphi} = \text{arc tan } \varphi, \tag{18}$$

we will derive the equation:

$$\tan \text{dist } xy = \frac{\sqrt{(xx)(yy)-(xy)^2}}{(xy)}, \tag{19}$$

and from that:

$$\left\{\begin{aligned} \cos \text{dist } xy &= \frac{(xy)}{\sqrt{(xx)}\sqrt{(yy)}}, \\ \sin \text{dist } xy &= \frac{\sqrt{(xx)(yy)-(xy)^2}}{\sqrt{(xx)}\sqrt{(yy)}}. \end{aligned}\right. \tag{20}$$

Dual equations are true for the angle between two planes.

66. Distance and angle in hyperbolic geometry. The complete duality between the distance between two points and the angle between two planes no longer exists in hyperbolic geometry. Namely, in the real approach to the absolute surface, the manifold of real points will split into two parts. Equation (12) shows that the distance from a point that does not lie on the absolute surface to an absolute point will be infinite. The absolute points can therefore also be called *infinitely-distant.* A person inside of the absolute surface (which one can imagine to be a sphere) that goes forth with a finite velocity in one and the same direction for a finite length of time will not reach the absolute surface. We therefore distinguish the *reachable domain* (inside of the absolute surface) from the *unreachable domain,* and from now on, we shall restrict ourselves to the treatment of the reachable domain.

The connecting line between two reachable points always cuts the absolute surface in two real points. The double ratio (10) will then be real, and in fact positive, and one will therefore *not* need to divide by i in the definition of distance.

Things are different for the angle between two planes that intersect in a reachable line. The two lines in these planes that lie on the absolute surface will be conjugate imaginary (as in the case of elliptic geometry). The factor 1 : i must then be introduced.

67. The common normals to two lines. In non-Euclidian geometry, as in Euclidian geometry, two lines are called mutually orthogonal (i.e., perpendicular) when they are conjugate relative to the absolute structure, and thus, when one line cuts the absolute polar of the other one. We would now like to find the common normals to two lines $\mathfrak{G}$, $\mathfrak{H}$; i.e., the lines that meet $\mathfrak{G}$ and $\mathfrak{H}$ perpendicularly. These normals will then be the common lines of intersection of $\mathfrak{G}$, $\mathfrak{H}$, and their absolute polars $\mathfrak{G}'$, $\mathfrak{H}'$, resp.

For that, we assume that $\mathfrak{H}$ is not the absolute polar to $\mathfrak{G}$ and that $\mathfrak{G}$ and $\mathfrak{H}$ do not contact the absolute surface, such that $\mathfrak{G}\mathfrak{G}'$, $\mathfrak{H}\mathfrak{H}'$ will be two distinct pairs of skew lines that define a line system of rank $r \geq 3$.

Since the four lines define a figure that is polar to itself relative to the absolute surface, the figure of its lines of intersection will also be polar to itself relative to the absolute surface. In the case of $r = 4$, one will then (no. **27**) be dealing with two mutually-polar lines, which can also coincide with the generators of the absolute surface or a pencil of tangents to the absolute surface. This case will occur when the $\mathfrak{G}$ and $\mathfrak{H}$ intersect at a point of the absolute surface, so $\mathfrak{G}'$ and $\mathfrak{H}'$ will then be two lines of the tangential plane to the point of intersection (or dually).

In the case $r = 3$, one deals with a regulus that corresponds to itself in the absolute polar system. The polarity will induce an involution on that regulus that has two fixed lines. Those fixed lines will be the generators of the absolute surface, and indeed, since they are skew, they will be generators of the same kind. In that case, the lines $\mathfrak{G}$, $\mathfrak{H}$, $\mathfrak{G}'$, $\mathfrak{H}'$ will intersect the regulus of the absolute surface at points of one and the same pair of generators.

If the generators of the pair coincide then the regulus will degenerate into a pair of polar restricted pencils with one generator as their common line.

If we restrict ourselves to real, reachable lines in *hyperbolic geometry* then we will have merely two cases to distinguish. A generator of the absolute surface has, in fact, one and only one real point, namely, its point of intersection with the complex-conjugate generator that lies in the other family of generators. Two real, reachable lines that do not intersect on the absolute plane will then cut the absolute surface in points of different generators, and will then have two distinct, mutually-polar normals. However, if the two lines do intersect on the absolute surface then they will have no common normal in the reachable domain, but the pencil of tangents to the point of intersection in the unreachable domain can qualify as a pencil of common normals.

Now, a real pair of lines can be a pair of complex-conjugate lines or a pair of real lines. One has the following theorem here:

Theorem 1: *A real pair of skew polars in hyperbolic geometry is a pair of real lines, one of which is reachable and the other of which is not.*

Proof: The real pair of polars cuts the absolute surface in a real "elementary quadrilateral" that consists of two generators $\mathfrak{R}_x$, $\mathfrak{R}_y$ of the first kind and two generators $\mathfrak{L}_x$, $\mathfrak{L}_y$ of the second kind. Since the complex-conjugate generators on the absolute surface belong to different families of generators, one will have – say – $\mathfrak{L}_x = \bar{\mathfrak{R}}_x$, $\mathfrak{L}_y = \bar{\mathfrak{R}}_y$, such that the points $\mathfrak{L}_x\mathfrak{R}_x$ and $\mathfrak{L}_y\mathfrak{R}_y$ will be real, and will yield a real, reachable line when they are connected, while the points $\mathfrak{L}_x\mathfrak{R}_y$ and $\mathfrak{L}_y\mathfrak{R}_x$ will be complex-conjugate, and when they are connected they will yield an unreachable line. We summarize this as:

Theorem 2: *Two real, reachable lines that do not intersect on the absolute surface have a real, reachable, common normal in hyperbolic geometry.*

There are no real absolute points in *elliptic geometry*. Two real lines cannot intersect on the absolute surface then. If they intersect in points of one and the same generator $\mathfrak{R}$

then they will also intersect the complex-conjugate generator $\bar{\mathfrak{R}}$ that belongs to the same family. There are thus, in turn, two possible cases:

Theorem 3: *In elliptic geometry, two real lines that are not absolute polars will have either a real pair or a real regulus of common normals.*

In order to clarify the reality behavior, we prove the theorem:

Theorem 4: *A real polar pair in elliptic geometry is a pair of real lines.*

Proof: The real pair, which is always a pair of skew lines, cuts the absolute surface in a real elementary quadrilateral that consists of the generators $\mathfrak{R}, \bar{\mathfrak{R}}$ of the first kind and the generators $\mathfrak{L}, \bar{\mathfrak{L}}$ of the second kind. The points $\mathfrak{RL}$, $\bar{\mathfrak{R}}\bar{\mathfrak{L}}$ and $\mathfrak{R}\bar{\mathfrak{L}}$, $\bar{\mathfrak{R}}\mathfrak{L}$ are pairs of complex-conjugate points, and thus yield a real connecting line. Q. E. D.

It then follows from the theorem that was just proved that the regulus that was mentioned in Theorem 3 (which indeed consists of real polar pairs) will also possess a real character.

68. Clifford parallels. In non-Euclidian geometry, two lines are said to be *parallel* to each other when they intersect at an absolute point. (The angle between two parallel lines will be zero, to the extent that it is defined.) Whereas there are real, reachable, parallel lines in hyperbolic geometry (viz., two parallels to a line through a point that does not lie on it), there are no real parallels in elliptic geometry.

One can, however, still introduce the concept of parallel lines of a different kind, such that real parallels will be possible in elliptic geometry, as well, and therefore certain properties of Euclidian parallels will remain preserved.

Two real lines that cut the absolute surface in the same (complex-conjugate, resp.) left (right, resp.) generator shall be called left (right, resp.) paratactic (i.e., "parallel in the Clifford sense").

There are then two congruences of lines that are paratactic to a real line $\mathfrak{G}$. $\mathfrak{G}$ cuts the absolute surface in two complex-conjugate generators of both kinds. Since one can draw one and only one line of the two congruences through a real point (and therefore it certainly does not belong to any generator), it will follow immediately that:

Theorem 5: *In elliptic geometry, there is a well-defined left-sided line and a well-defined right-sided line that is paratactic to a given line through a given point.*

This already exhibits an analogy to the situation in Euclidian geometry, which the following theorem further emphasizes:

Theorem 6: *Two paratactic lines have* ∞^1 *common normals.*

The fact that two lines that have ∞^1 common normals cut the absolute surface in points of one and the same pair of generators was shown already above. The converse is also true. Namely, we assume that the given **Clifford** parallels $\mathfrak{G}$ and $\mathfrak{H}$ cut the generators $\mathfrak{R}_1$, $\mathfrak{R}_2$ of the same family at the points γ_1, γ_2 and η_1, η_2, resp. Their absolute polars then cut the generators $\mathfrak{R}_1$ and $\mathfrak{R}_2$ in any event. Let the points of intersection be γ_1', γ_2'; η_1', η_2'. If we now perform the absolute polarity then the points γ_1, η_1, γ_1', η_1' will be transformed into four planes, and indeed, the planes that connect $\mathfrak{R}_1$ with $\mathfrak{G}'$, $\mathfrak{H}'$, $\mathfrak{G}$, $\mathfrak{H}$, resp., and thus intersect in the points γ_2', η_2', γ_2, η_2, resp. One therefore has:

$$\gamma_1\eta_1\gamma_1'\eta_1' \;\overline{\wedge}\; \gamma_2'\eta_2'\gamma_2\,\eta_2 \;\overline{\wedge}\; \gamma_2\,\eta_2\gamma_2'\eta_2'.$$

However, four lines that determine the same cast (*Wurf*) from two skew, common lines of intersection will belong to a regulus (cf., no. **27**, rem.). Q. E. D.

A one-parameter group of elliptic motions, and thus automorphic collineations of the absolute surface, will be defined (no. **25**) by the two distinguished generators $\mathfrak{R}_1$, $\mathfrak{R}_2$ of the absolute surface. They will be induced by the binary projectivities that leave the lines $\mathfrak{R}_1$ and $\mathfrak{R}_2$ in the family of generators $\mathfrak{R}$ invariant individually, and each line of the families of generators individually fixed. Each of the ∞^2 lines in the congruence that is determined by $\mathfrak{R}_1$, $\mathfrak{R}_2$ will then be translated into itself. Since the group will fix the lines of a *paratactic congruence* individually, we shall speak of a *group of displacements.* One must distinguish right-sided displacements form left-sided ones.

Theorem 7: *A paratactic congruence admits a one-parameter group of displacements.*

The common normals to two paratactic lines will be permuted transitively by the associated group of displacements; i.e., it will be possible to take each of the ∞^1 normals to each of the other ones by a displacement. Since a displacement (as a motion) leaves the distance between two points invariant, it will ultimately follow that:

Theorem 8: *Two paratactic lines will cut out the same segment on all of their common normals.*

One can call that segment the "distance" between the two lines.

69. Distance between two lines. One refers to the segment that is cut out from a common normal to two lines $\mathfrak{X}$, $\mathfrak{Y}$ that are in general position as the *distance between them*. Since the two lines have two common normals, and the distance along each of those normals is determined only up to sign, one must expect a quadratic equation for the square of the distance. The convention on the sign of the roots of the discriminant of that equation must distinguish the two normals. The discriminant must then agree (up to a numerical factor) with the determinant (4) in § 9 that is defined by the lines $\mathfrak{X}$, $\mathfrak{Y}$, and their absolute polars $\mathfrak{X}'$, $\mathfrak{Y}'$.

We would like to determine the distance between two lines $\mathfrak{X}$, $\mathfrak{Y}$ in elliptic geometry. For that, we denote the line equation of the absolute surface (no. **23**) by $(\mathfrak{X} \mid \mathfrak{X}) = 0$, and the intersection points of the lines $\mathfrak{X}$, $\mathfrak{Y}$ with their common normals by x, y and x', y', resp., such that:

$$\mathfrak{X} = \widehat{x\,x'}, \qquad \mathfrak{Y} = \widehat{y\,y'}, \qquad (xx') = (xy') = (yx') = (yy') = 0. \tag{21}$$

The squares of the cosines d_1, $d_2 = [\cos \operatorname{dist} \mathfrak{X}\,\mathfrak{Y}]^2$ of the distance, when measured along the normals, will then be [from (20)]:

$$d_1 = \frac{(xy)^2}{(xx)(yy)}, \qquad d_2 = \frac{(x'y')^2}{(x'x')(y'y')}. \tag{22}$$

These expressions shall now be expressed as functions of the coordinates of $\mathfrak{X}$ and $\mathfrak{Y}$ alone. For the sake of simplicity, we assume in this that the determinant of the absolute surface possesses the value 1. One will then have, in fact, the following auxiliary formula:

$$(x^1\,x^2\,x^3\,x^4)^2 = (x^1\,x^2\,x^3\,x^4) \cdot (u^1\,u^2\,u^3\,u^4), \tag{23}$$

in which u^i denotes the absolute polar plane to x^i [cf., no. **26**, (20), (21)], and furthermore, from the multiplication theorem for determinants, it will equal:

$$= |\,(x^i\,u^k)\,| = |\,(x^i\,x^k)\,|.$$

Moreover:

$$\begin{aligned}(x^1\,x^2 \mid x^3\,x^4) &= (x^1\,x^2,\,u^3\,u^4)\\ &= (x^1\,u^3)\,(x^2\,u^4) - (x^1\,u^4)\,(x^2\,u^3) \qquad [\S\ 1, (20)]\\ &= (x^1\,x^3)\,(x^2\,x^4) - (x^1\,x^4)\,(x^2\,x^3)\,.\end{aligned} \tag{24}$$

With the use of these auxiliary formulas, one will have:

$$(\mathfrak{X} \mid \mathfrak{X}) = (xx' \mid xx') = (xx \mid x'x'), \qquad (21), (24), \tag{25}$$

$$(\mathfrak{Y} \mid \mathfrak{Y}) = (yy' \mid yy') = (yy \mid y'y'), \tag{26}$$

$$(\mathfrak{X} \mid \mathfrak{Y}) = (xx' \mid yy') = (xy \mid x'y'), \tag{27}$$

and

$$(\mathfrak{X}\mathfrak{Y})^2 = (xx' \mid yy')^2 = \begin{vmatrix} (xx) & (xy) & 0 & 0 \\ (yx) & (yy) & 0 & 0 \\ 0 & 0 & (x'x') & (x'y') \\ 0 & 0 & (y'x') & (y'y') \end{vmatrix} \qquad (23) \tag{28}$$

$$= \{(x\,x)\,(y\,y) - (x\,y)^2\}\{(x'\,x')\,(y'\,y') - (x'\,y')^2\}$$
$$= (\mathfrak{X} \mid \mathfrak{X})\,(\mathfrak{Y} \mid \mathfrak{Y}) + (\mathfrak{X} \mid \mathfrak{Y})^2 \qquad (25), (26), (27)$$
$$- (xx)\,(yy)\,(x'\,y')^2 - (x'\,x')\,(y'\,y')\,(xy)^2.$$

It will then follow that:

$$(29) \qquad d_1 \cdot d_2 = \frac{(xy)^2 (x'y')^2}{(xx)(yy)(x'x')(y'y')} = \frac{(\mathfrak{X} \mid \mathfrak{Y})^2}{(\mathfrak{X} \mid \mathfrak{X})(\mathfrak{Y} \mid \mathfrak{Y})},$$

$$(30) \qquad d_1 + d_2 = \frac{(xy)^2 (x'x')(y'y') + (x'y')^2 (xx)(yy)}{(xx)(yy)(x'x')(y'y')}$$
$$= \frac{(\mathfrak{X} \mid \mathfrak{X})(\mathfrak{Y} \mid \mathfrak{Y}) + (\mathfrak{X} \mid \mathfrak{Y})^2 - (\mathfrak{X}\mathfrak{Y})^2}{(\mathfrak{X} \mid \mathfrak{X})(\mathfrak{Y} \mid \mathfrak{Y})},$$

so:

$$(31) \qquad (d - d_1)\,(d - d_2)$$
$$= d^2 + \frac{(\mathfrak{X}\mathfrak{Y})^2 - (\mathfrak{X} \mid \mathfrak{X})(\mathfrak{Y} \mid \mathfrak{Y}) - (\mathfrak{X} \mid \mathfrak{Y})^2}{(\mathfrak{X} \mid \mathfrak{X})(\mathfrak{Y} \mid \mathfrak{Y})}\, d + \frac{(\mathfrak{X} \mid \mathfrak{Y})^2}{(\mathfrak{X} \mid \mathfrak{X})(\mathfrak{Y} \mid \mathfrak{Y})}.$$

The desired equation will then read:

$$(32) \qquad (\mathfrak{X} \mid \mathfrak{X})\,(\mathfrak{Y} \mid \mathfrak{Y})\,[\cos \operatorname{dist} \mathfrak{X}\,\mathfrak{Y}]^4$$
$$+ \{(\mathfrak{X}\,\mathfrak{Y})^2 - (\mathfrak{X} \mid \mathfrak{X})\,(\mathfrak{Y} \mid \mathfrak{Y}) - (\mathfrak{X} \mid \mathfrak{Y})^2\}\,[\cos \operatorname{dist} \mathfrak{X}\,\mathfrak{Y}]^2 + (\mathfrak{X}\,\mathfrak{Y})^2 = 0.$$

For $[\cos \operatorname{dist} \mathfrak{X}\,\mathfrak{Y}]^2$ itself, one finds from this that:

$$(33) \qquad [\cos \operatorname{dist} \mathfrak{X}\,\mathfrak{Y}]^2 = \frac{1}{2}\frac{(\mathfrak{X} \mid \mathfrak{X})(\mathfrak{Y} \mid \mathfrak{Y}) + (\mathfrak{X} \mid \mathfrak{Y})^2 - (\mathfrak{X}\mathfrak{Y})^2 \pm \sqrt{\Delta}}{(\mathfrak{X} \mid \mathfrak{X})(\mathfrak{Y} \mid \mathfrak{Y})}.$$

In this, Δ means the discriminant of equation (32), which is, as we would expect, the determinant:

$$(34) \qquad \Delta = \begin{vmatrix} * & (\mathfrak{X}\mathfrak{Y}) & (\mathfrak{X} \mid \mathfrak{X}) & (\mathfrak{X} \mid \mathfrak{Y}) \\ (\mathfrak{Y}\mathfrak{X}) & * & (\mathfrak{Y} \mid \mathfrak{X}) & (\mathfrak{Y} \mid \mathfrak{Y}) \\ (\mathfrak{X} \mid \mathfrak{X}) & (\mathfrak{X} \mid \mathfrak{Y}) & * & (\mathfrak{X}\mathfrak{Y}) \\ (\mathfrak{X} \mid \mathfrak{Y}) & (\mathfrak{Y} \mid \mathfrak{Y}) & (\mathfrak{Y}\mathfrak{X}) & * \end{vmatrix}.$$

One derives the formula for $\sin \operatorname{dist} \mathfrak{X}\mathfrak{Y}$ from (33) with the help of the formula $\sin^2 t + \cos^2 t = 1$. One finds that:

$$(35) \qquad (\mathfrak{X} \mid \mathfrak{X})\,(\mathfrak{Y} \mid \mathfrak{Y})\,[\sin \operatorname{dist} \mathfrak{X}\,\mathfrak{Y}]^4$$
$$+ \{(\mathfrak{X} \mid \mathfrak{Y})^2 - (\mathfrak{X} \mid \mathfrak{X})\,(\mathfrak{Y} \mid \mathfrak{Y}) - (\mathfrak{X}\,\mathfrak{Y})^2\}\,[\sin \operatorname{dist} \mathfrak{X}\,\mathfrak{Y}]^2 + (\mathfrak{X}\,\mathfrak{Y})^2 = 0.$$

The *angle between two lines* will be defined dually to the distance between two lines.

§ 21. Passing to the limit of Euclidian geometry.

70. Degeneracy of the absolute surface. In order to carry out the passage to the limit of Euclidian geometry in the formulas of the preceding paragraphs, we put the absolute surface into the form:

(1) $$(x\,x) \equiv x_0^2 + k^2(x_1^2 + x_2^2 + x_3^2) = 0,$$

(2) $$(u\,u) \equiv k^2u_0^2 + u_1^2 + u_2^2 + u_3^2 = 0,$$

in which k is real or imaginary, and k^2 can correspondingly be positive or negative according to whether one is dealing with an elliptic or hyperbolic metric, respectively. Under the passage to the limit $k \to 0$, the absolute surface will go to the doubly-counted imaginary plane $x_0^2 = 0$ as a locus of points and the absolute conic section:

(3) $$[u\,u] = u_1^2 + u_2^2 + u_3^2 = 0$$

as a locus of planes.

The line equation for the absolute surface ([1]), namely:

(4) $$(\mathfrak{X} \mid \mathfrak{X}) = \frac{1}{k}(\mathfrak{X}_{01}^2 + \mathfrak{X}_{02}^2 + \mathfrak{X}_{03}^2) + k(\mathfrak{X}_{23}^2 + \mathfrak{X}_{31}^2 + \mathfrak{X}_{12}^2) = 0,$$

will become:

(5) $$[\mathfrak{X} \mid \mathfrak{X}] \equiv \mathfrak{X}_{01}^2 + \mathfrak{X}_{02}^2 + \mathfrak{X}_{03}^2 = 0$$

under the passage to the limit, which is the complex of lines of intersection with the absolute conic section.

71. Distance between two points. Angle between two planes. We would now like to pass to the limit in the formulas of elliptic geometry. To that end, we alter our definition up to now and write $k \cdot \text{dist}\, xy$ for what we previously denoted by $d = \text{dist}\, xy$. From (20) in § 20, one will then have:

(6) $$\sin k \,\text{dist}\, xy = k\,d - \frac{1}{3!}k^3\,d^{\,3} \pm \ldots = \frac{\sqrt{(xx)\cdot(yy)-(xy)^2}}{\sqrt{(xx)}\sqrt{(yy)}}.$$

Thus:

(7) $$\lim_{k\to 0}\frac{1}{k}\sin k\,\text{dist}\, xy = d$$

$$= \lim_{k\to 0}\frac{1}{k}\frac{\sqrt{(x_0^2+k^2x_1^2+\cdots)(y_0^2+k^2y_1^2+\cdots)-(x_0y_0+k^2x_1y_1+\cdots)^2}}{\sqrt{x_0^2+k^2x_1^2+\cdots}\sqrt{y_0^2+k^2y_1^2+\cdots}}$$

([1]) In the derivation of the formulas for the distance and angle between two lines in no. **69**, we assumed that the absolute surface had the determinant 1. For that reason, the factor k in $(\mathfrak{X} \mid \mathfrak{X})$ must be arranged as it is (4).

$$= \frac{1}{x_0 y_0}\sqrt{(x_0y_1 - x_1y_0)^2 + (x_0y_2 - x_2y_0)^2 + (x_0y_3 - x_3y_0)^2}\,.$$

This is the Euclidian distance between the two points x, y (when written in homogeneous coordinates).

One can apply the passage to the limit to the angle between two planes u, v with no further preparations and get:

$$\text{(8)}\qquad \cos\angle\, uv = \frac{[uv]}{\sqrt{[uu]}\sqrt{[vv]}}, \qquad \sin\angle\, uv = \frac{\sqrt{[uu][vv]-[uv]^2}}{\sqrt{[uu]}\sqrt{[vv]}}\,.$$

72. Distance and angle between two lines. If one introduces (4), and sets kd in place of dist $\mathfrak{X}\mathfrak{Y}$, as in no. **71**, then equation (35) in § 20 will become:

$$\text{(9)}\qquad \left\{\begin{aligned} &+\left(\frac{1}{k^2}[\mathfrak{X}\,|\,\mathfrak{X}][\mathfrak{Y}\,|\,\mathfrak{Y}]+\cdots\right)(k^4d^4+\cdots)\\ &+\left(\frac{1}{k^2}[\mathfrak{X}\,|\,\mathfrak{Y}]^2-\frac{1}{k^2}[\mathfrak{X}\,|\,\mathfrak{X}][\mathfrak{Y}\,|\,\mathfrak{Y}]+\cdots\right)(k^2d^2+\cdots)\\ &+(\mathfrak{X}\mathfrak{Y})^2=0,\end{aligned}\right.$$

and when one passes to the limit $k \to 0$:

$$\text{(10)}\qquad -d^2\{[\mathfrak{X}|\mathfrak{X}][\mathfrak{Y}|\mathfrak{Y}] - [\mathfrak{X}|\mathfrak{Y}]^2\} + (\mathfrak{X}\mathfrak{Y})^2 = 0,$$

so

$$\text{(11)}\qquad d^2 = \frac{(\mathfrak{X}\mathfrak{Y})^2}{[\mathfrak{X}\,|\,\mathfrak{X}][\mathfrak{Y}\,|\,\mathfrak{Y}]-[\mathfrak{X}\,|\,\mathfrak{Y}]^2}\,.$$

In order to obtain the *Euclidian angle* between the lines $\mathfrak{X}$, $\mathfrak{Y}$, we start with the formula that is dual to (35) in § 20:

$$\text{(12)}\qquad \begin{aligned}&(\mathfrak{X}|\mathfrak{X})(\mathfrak{Y}|\mathfrak{Y})\,[\sin \text{ang } \mathfrak{X}\mathfrak{Y}]^4\\ &\quad+\{(\mathfrak{X}|\mathfrak{Y})^2 - (\mathfrak{X}|\mathfrak{X})(\mathfrak{Y}|\mathfrak{Y}) - (\mathfrak{X}\mathfrak{Y})^2\}[\sin \text{ang } \mathfrak{X}\mathfrak{Y}]^2 + (\mathfrak{X}\mathfrak{Y})^2 = 0,\end{aligned}$$

multiply it by k^2, and let $k \to 0$. It will then follow that:

$$\text{(13)}\qquad [\mathfrak{X}|\mathfrak{X}][\mathfrak{Y}|\mathfrak{Y}]\,[\sin\angle\,\mathfrak{X}\mathfrak{Y}]^4 + \{[\mathfrak{X}|\mathfrak{Y}]^2 - [\mathfrak{X}|\mathfrak{X}][\mathfrak{Y}|\mathfrak{Y}]\}[\sin\angle\,\mathfrak{X}\mathfrak{Y}]^2 = 0,$$

and thus, when one splits off $[\sin\angle\,\mathfrak{X}\mathfrak{Y}]^2$:

(14) $$\sin \angle \mathfrak{X}\mathfrak{Y} = \frac{\sqrt{[\mathfrak{X} \mid \mathfrak{X}][\mathfrak{Y} \mid \mathfrak{Y}] - [\mathfrak{X} \mid \mathfrak{Y}]^2}}{\sqrt{[\mathfrak{X} \mid \mathfrak{X}]}\sqrt{[\mathfrak{Y} \mid \mathfrak{Y}]}}.$$

It will follow from this that:

(15) $$\cos \angle \mathfrak{X}\mathfrak{Y} = \frac{[\mathfrak{X} \mid \mathfrak{Y}]}{\sqrt{[\mathfrak{X} \mid \mathfrak{X}]}\sqrt{[\mathfrak{Y} \mid \mathfrak{Y}]}}$$

and

(16) $$\tan \angle \mathfrak{X}\mathfrak{Y} = \frac{\sqrt{[\mathfrak{X} \mid \mathfrak{X}][\mathfrak{Y} \mid \mathfrak{Y}] - [\mathfrak{X} \mid \mathfrak{Y}]^2}}{[\mathfrak{X} \mid \mathfrak{Y}]}.$$

Finally, the formulas that we derived yield the following relations, which will be important in the sequel:

(17) $$\text{dist}\ \mathfrak{X}\mathfrak{Y} \cdot \tan \angle \mathfrak{X}\mathfrak{Y} = \frac{(\mathfrak{X}\mathfrak{Y})}{[\mathfrak{X} \mid \mathfrak{Y}]}$$

and

(18) $$\text{dist}\ \mathfrak{X}\mathfrak{Y} \cdot \sin \angle \mathfrak{X}\mathfrak{Y} = \frac{(\mathfrak{X}\mathfrak{Y})}{\sqrt{[\mathfrak{X} \mid \mathfrak{X}]}\sqrt{[\mathfrak{Y} \mid \mathfrak{Y}]}}.$$

With these formulas, formulas (14)-(16), which depend upon only the angle, will take on their geometric interpretations in regard to the transformations of an enveloping group, namely, the group of similarity transformations.

73. The group of similarity transformations. The group of non-Euclidian motions and transfers is characterized completely as the group of (real and imaginary) automorphic collineations of a regular, second-order surface. Corresponding to the ∞^{15} collineations and the ∞^9 second-order surfaces in R_3, we will have $\infty^{15} : \infty^9 = \infty^6$ non-Euclidian motions and transfers. The groups g_6, h_6 of Euclidian motions and transfers will then arise from these groups by passing to the limit. However, that group will no longer be characterized by saying that their transformations fix the absolute conic section. Indeed, there are ∞^8 singular surfaces of class two, and therefore ∞^7 automorphic collineations of a conic section. They define the group g_7 of *similarity transformations:*

(1) $$\begin{cases} x_0' = x_0, \\ x_1' = a_1 x_0 + \lambda(a_{11}x_1 + a_{12}x_2 + a_{13}x_3), \\ x_2' = a_2 x_0 + \lambda(a_{21}x_1 + a_{22}x_2 + a_{23}x_3), \\ x_3' = a_3 x_0 + \lambda(a_{31}x_1 + a_{32}x_2 + a_{33}x_3), \end{cases}$$

in which, the a_{ik} define an orthogonal matrix. (Cf., no. **103**)

The group of motions and transfers is contained in (1). It comes about when one sets $\lambda = 1$.

Now, let g be a subgroup of an arbitrary group G. We subject an arbitrary transformation t of g to a transformation T of G; i.e., we form $t' = T^{-1}tT$. When the resulting transformation t' is once more a transformation of g, independently of how t and T are chosen, one will call g an *invariant subgroup* of G.

One confirms directly that g_6 is contained invariantly in g_7 .

One will now obtain all similarity transformations when one composes the motions (and transfers) with the one-parameter group of *stretchings from the origin:*

$$(2) \qquad \begin{cases} x_0' = x_0, \\ x_1' = \lambda x_1, \; x_2' = \lambda x_2, \; x_3' = \lambda x_3. \end{cases}$$

These stretchings fix the origin and every point of the imaginary plane individually.

The manifold of stretchings (viz., *perspective similarity transformations*) of all points in space – together with the displacements – defines a group g_4:

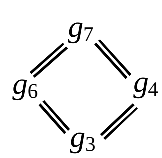
$$(3) \qquad \begin{cases} x_0' = x_0, \\ x_1' = a_1 x_0 + \lambda x_1, \\ x_2' = a_2 x_0 + \lambda x_2, \\ x_3' = a_3 x_0 + \lambda x_3, \end{cases}$$

which is likewise contained invariantly in g_7. g_6 and g_4 intersect in the group g_3 of *displacements:*

$$(4) \qquad \begin{cases} x_0' = x_0, \\ x_1' = a_1 x_0 + x_1, \\ x_2' = a_2 x_0 + x_2, \\ x_3' = a_3 x_0 + x_3, \end{cases}$$

whose transformations commute pair-wise. g_3 is contained invariantly in g_6 and g_4 .

We clarify the relationship between the four groups with a diagram, in which a double line shall represent the relationship of being contained invariantly:



§ 22. The axes of a linear complex.

74. Definition of an axis-pair in non-Euclidian geometry. A linear complex $\mathfrak{C}$ is linked invariantly with the polar complex $\mathfrak{C}'$, relative to the absolute surface F^2, namely, the locus of all polars to the lines that are contained in $\mathfrak{C}$. Under the map to R_5, the reguli of F^2 will be mapped to two conic sections in planes E and E', resp., and one will obtain the image points of the polar complexes of $\mathfrak{C}$ when one subjects the image points of $\mathfrak{C}$ to the involutory collineations that are determined by E and E' as incidence domains.

Now, the complexes $\mathfrak{C}$ and $\mathfrak{C}'$ span a pencil of linear complexes that generally belong to two distinct singular complexes. Its guiding lines are called *axes of the linear complex.*

Since the pencil of complexes that is spanned by $\mathfrak{C}$, $\mathfrak{C}'$ goes to itself under the absolute polarity, the pair of axes will also be fixed by that polarity. It will then consist of a pair of polar lines. **E. Study** has called the figure that consists of two mutually-polar lines a *line-cross.* We shall then speak of the *axis-cross of a linear complex.*

One can also arrive at the figure of the axis-cross in the following way: The complex $\mathfrak{C}$ cuts each of the two reguli of F^2 in two distinct lines. They determine an *elementary quadrilateral* that goes to itself under polarity in F^2. The generators of the elementary quadrilateral will also belong to the polar complex $\mathfrak{C}'$ then, and thus, to the intersection congruence of the pencil of complexes that is spanned by $\mathfrak{C}$ and $\mathfrak{C}'$. The guiding lines of the congruence, which are the missing counterparts to the aforementioned elementary quadrilateral, are the mutually-polar axes of the complexes.

75. Classification of the figures that consist of a linear complex and a regular, second-order complex. In the previous number, we assumed a general case that is characterized by the fact that the complex $\mathfrak{C}$ cuts the two generating reguli of the absolute surface in two distinct generators. In R_5, the connecting line $\mathfrak{C}\mathfrak{C}'$ will cut M_4^2 at two distinct points in this case. *Should these points coincide* (i.e., should the axis-pair become a doubly-counted axis, which must then be polar to itself and therefore a generator of the absolute surface), then the connecting line $\mathfrak{C}\mathfrak{C}'$ would have to contact M_4^2 at a point of E (or E'). The polar R_4 would then have to contact M_4^2 at the same point. It would then contact the conic section in M_4^2 that lies in the plane E (or E'), but not the one that lies in the other plane. The elementary quadrilateral would then degenerate on F^2 accordingly: The two generators of the one family would then coincide with the axis of the complex, while the other two would remain distinct.

Should the line $\mathfrak{C}\mathfrak{C}'$ belong to M_4^2 completely, then $\mathfrak{C}$ would have to be a tangent to F^2 that is different from a generator. $\mathfrak{C}'$ would then be a tangent to a pencil that is determined by $\mathfrak{C}$, and the entire pencil of tangents would consist of axes of $\mathfrak{C}$.

The line $\mathfrak{C}\mathfrak{C}'$ will be undetermined when $\mathfrak{C}$ coincides with $\mathfrak{C}'$. That will happen if and only if $\mathfrak{C}$ belongs to one of the planes E or E'. The polar R_4 will then contain E' (E, resp.), while the complex $\mathfrak{C}$ will contain one of the two reguli of F^2. We shall then call it a "generating complex." Such a complex can be regular or singular. In the latter case, its guiding line will be a generator of F^2.

In summary, we must then distinguish between the following cases:

I. *Complexes in general position.* (Regular or singular complexes) Each of the reguli of F^2 intersect in two distinct generators, which will be two skew polar axes.

II. *Contact complexes:*

a) *Regular complex.* One of the two reguli intersects in two coincident generators, while the other one intersects in two distinct ones. The doubly-counted generators will be doubly-counted axes.

b) *Singular complexes.* The guiding line is a tangent (that cuts both reguli at two coincident generators). One will have ∞^1 axes that fill up the pencil of tangents that is determined by the given tangent.

III. *Generating complexes.* They contain one of the two reguli. The axes will be undetermined.

a) *Regular complexes.* The second regulus intersects in two distinct generators.

b) *Singular complex.* The second regulus intersects in one doubly-counted generator, namely, the guiding line of the given complex.

76. Axes in elliptic and hyperbolic geometry. Since the absolute surface does not possess real generators in elliptic and hyperbolic geometry, but a real complex cuts the absolute surface in a real generator figure, possibilities IIa and IIIb will no longer apply to either geometry. Since there are no real absolute tangents in elliptic geometry, IIb will also drop out for elliptic geometry. Finally, the case IIIa is rejected in hyperbolic geometry, since two complex-conjugate generators of the sphere will belong to different families of generators. We summarize the results, to the extent that they are concerned with hyperbolic geometry:

Theorem 9: *A real, regular, linear complex always has a well-defined reachable axis in hyperbolic geometry. A real, singular complex with a reachable guiding line will have that guiding line for a reachable axis. All of the lines of the pencil of tangents that are determined by the given tangents will belong to an absolute tangent as axes.*

77. Axes in Euclidian geometry. If the absolute surface degenerates into the absolute conic then the axis-pair of the complex (in the sense of non-Euclidian geometry) will become the axis-pair in the sense of Euclidian geometry. However, that axis-pair is linked invariantly with the complex with respect to similarity transformations, as well.

As in non-Euclidian geometry, a polar complex $\mathfrak{C}'$ is coupled to a complex $\mathfrak{C}$ in the geometry of similarity transformations. If $[\mathfrak{X} \mid \mathfrak{X}] = 0$ is the line equation of the absolute conic then the polar complex will have the equation:

$$[\mathfrak{X} \mid \mathfrak{X}] = 0. \tag{1}$$

It will then have the coordinates:

$$0 : 0 : 0 : \mathfrak{C}_{01} : \mathfrak{C}_{02} : \mathfrak{C}_{03}\,, \tag{2}$$

and it will itself be singular. Its guiding line will be called the *auxiliary axis* of the complex $\mathfrak{C}$. It can also be obtained as the polar to the null point:

$$0 : \mathfrak{C}_{01} : \mathfrak{C}_{02} : \mathfrak{C}_{03} \tag{3}$$

to the imaginary plane relative to the absolute conic. *The auxiliary axis will be undetermined if and only if $\mathfrak{C}$ is an imaginary line.*

Under the assumption that $\mathfrak{C}$ is not an imaginary line, $\mathfrak{C}$ will be different from its auxiliary axis. Both of them collectively span the pencil:

$$(\mathfrak{D}\mathfrak{X}) \equiv \xi_1\,(\mathfrak{C}\mathfrak{X}) + \xi_2\,[\mathfrak{C} \mid \mathfrak{X}] = 0\,. \tag{4}$$

Since one of the singular complexes of the pencil is known already, the other one can be determined linearly. The equation:

$$(\mathfrak{D}\mathfrak{D}) \equiv \xi_1\{\xi_1(\mathfrak{C}\mathfrak{C}) + \xi_2\,[\mathfrak{C} \mid \mathfrak{C}]\} = 0 \tag{5}$$

yields:

$$\xi_1 : \xi_2 = -\,2\,[\mathfrak{C} \mid \mathfrak{C}] : (\mathfrak{C}\mathfrak{C})\,. \tag{6}$$

The equation of the second axis, – viz., the *principal axis* of the complex $\mathfrak{C}$ – will then read:

$$(\mathfrak{P}\mathfrak{X}) \equiv -\,[\mathfrak{C} \mid \mathfrak{C}]\,(\mathfrak{C}\mathfrak{X}) + \tfrac{1}{2}(\mathfrak{C}\mathfrak{C})\,[\mathfrak{C} \mid \mathfrak{X}] = 0. \tag{7}$$

From no. **11**, the principal axis $\mathfrak{P}$ is the null polar to the auxiliary axis. It will coincide with the auxiliary axis when the expression $[\mathfrak{C} \mid \mathfrak{C}]$ vanishes for the regular complex $\mathfrak{C}$, so when $\mathfrak{C}$ is a regular, *isotropic complex.*

If $\mathfrak{C}$ is a singular, isotropic complex then the minimal line $\mathfrak{C}$ will cut the auxiliary axis, and both of them will span a pencil of axes of the complex.

We summarize the results, to the extent that they are concerned with real figures:

Theorem 10: *A real, regular, linear complex always possesses a well-defined, real, proper line as its principal axis: viz., the null polar of its auxiliary axis.*

A real, singular, linear complex with a proper guiding line has that guiding line for its principal axis.

A singular linear complex with an imaginary line has no well-defined axis.

78. Metric properties of linear complexes. Let $(\mathfrak{C}\ \mathfrak{X}) = 0$ be a regular complex in a real domain, and let (7) be its axis. Now, if $\mathfrak{Y}$ is a line of the complex $\mathfrak{C}$, so:

(8) $$(\mathfrak{C}\ \mathfrak{Y}) = 0, \qquad \text{but} \qquad [\mathfrak{C} \mid \mathfrak{Y}] \neq 0,$$

then one will have:

(9) $$(\mathfrak{P}\ \mathfrak{Y}) = \tfrac{1}{2}(\mathfrak{C}\ \mathfrak{X}) \cdot [\mathfrak{C} \mid \mathfrak{Y}], \qquad [\mathfrak{P} \mid \mathfrak{Y}] = -[\mathfrak{C} \mid \mathfrak{C}] \cdot [\mathfrak{C} \mid \mathfrak{Y}],$$

and it will follow from this that:

(10) $$-\frac{(\mathfrak{P}\mathfrak{Y})}{[\mathfrak{P} \mid \mathfrak{Y}]} = \frac{\frac{1}{2}(\mathfrak{C}\mathfrak{C})}{[\mathfrak{C} \mid \mathfrak{C}]} = k.$$

This equation is also true for the lines with $[\mathfrak{C} \mid \mathfrak{Y}] = 0$ that were excluded above, on the grounds of continuity.

By assumption, the constant k – namely, the *parameter of the linear complex* – is non-zero. It is an absolute invariant of the motion. Since:

(11) $$-\frac{(\mathfrak{P}\mathfrak{Y})}{[\mathfrak{P} \mid \mathfrak{Y}]} = -\operatorname{dist} \mathfrak{P}\mathfrak{Y} \cdot \tan \angle\, \mathfrak{P}\mathfrak{Y} = k \qquad [\S\ 21, (17)],$$

one will now have:

Theorem 11: *If $\mathfrak{P}$ is the principal axis of a real, regular, linear complex $\mathfrak{C}$, and $\mathfrak{X}$ is a proper, real line of the complex then the product* (*distance between $\mathfrak{P}$ and $\mathfrak{X}$*) times (*tangent of the angle between $\mathfrak{P}$ and $\mathfrak{X}$*) *will be constant.*

This theorem allows us to illustrate the distribution of real lines of a real, linear complex from the standpoint of real, metric geometry. If we think of dist $\mathfrak{P}\mathfrak{Y}$ as being constant then it will follow that $\tan \angle\, \mathfrak{P}\mathfrak{Y}$ is also constant; i.e.:

Theorem 12: *Lines of the linear complex with the same distance from the principal axis define the same angle with that principal axis.*

Now, the larger that dist $\mathfrak{P}\mathfrak{Y}$ becomes, the smaller that tan $\angle\,\mathfrak{P}\mathfrak{Y}$ will be, and the smaller that the angle that the lines of the linear complex make with the principal axis will be, as well. For dist $\mathfrak{P}\mathfrak{Y} = 0$, that angle will become $\pi / 2$: i.e., the *transverse lines* will be perpendicular to the main axis.

It also follows directly from Theorem 11 that:

Theorem 13: *A linear complex $\mathfrak{C}$ admits an arbitrary rotation around its principal axis and an arbitrary displacement along that principal axis; hence (when combined), an arbitrary screw around that principal axis.*

§ 23. The axis surface of a pencil of linear complexes.

79. The axis surface in non-Euclidian geometry. We consider a regular, second-order surface F^2 in space of a regular pencil of linear complexes of types I and IIa. (Cf., no. **75**) The locus of axis-crosses of the linear complex is called the *axis surface of the pencil of complexes.* The properties of the axis surface are obtained most simply by mapping it to R_5 .

The pencil of complexes will be mapped to a line G that meets neither of the two polar image planes E, E' to the generating complex of F^2 relative to M_4^2. In order to obtain the image point of the axis, one must subject the points of the line G to the involutory collineations that are determined by E and E' and intersect the connecting line of corresponding points with M_4^2. However, upon transforming the point sequence on G one will obtain a projective point sequence on a second line G'. By assumption, the lines G and G' are skew, and they span an R_3 that intersects **Plücker**'s M_4^2 in an M_2^2. In the sequel we shall assume that it is regular. The two projective points sequences of the lines G and G' generate a regulus whose lines cut M_2^2 in the image points of the desired axis surface. Those image points will then fill up the curve of intersection of two second-order surfaces, which is a fourth-order curve C^4 of the first kind.

80. Fourth-order space curves. The fact that the curve of intersection of two second-order surfaces is a fourth-order curve, and will thus intersect any plane at four points, is explained thusly: Any plane will intersect each of the two second-order surfaces in a second-order curve, and these two second-order curves will intersect at four points, namely, the points of intersection of the curves with the plane. With the first two second-order surfaces, the curve will also contain all surfaces of the pencil that they span, and in particular, the four cones of the pencil. As the basic curve of a pencil of second-order surfaces, the fourth-order curve of the first kind is to be contrasted with the fourth-order curve of the second kind, which contains only *one* second-order surface. The latter curve is rational – i.e., capable of a rational parametric representation – while the former is elliptic. A more precise treatment of these curves would require the theory of elliptic functions, which would fall beyond the scope of this book.

81. Fourth-order ruled surfaces of the first kind. An R_4 cuts the connecting R_3 of G and G' in a plane, and thus cuts C^4 in four points. Therefore, the axis surface in R_3 will be cut by a line (that hereafter meets four generators) at four points. The axis surface will then be a fourth-order ruled surface. As the image of a fourth-order curve of the first kind, it is called the *fourth-order ruled surface of the first kind.*

Since the image manifold of the generators of our axis surface is contained in an R_3, *the generators will belong to a linear complex*, and it is easy to give its guiding lines. To that end, we think of the image R_3 as being spanned by the four points at which the lines G and G' cut the M_2^2. These four points are the image points of the two singular complexes of the pencil of complexes that we started with and the pencil of complexes that is polar to it. *The desired guiding lines are then the two, mutually-polar, common lines of these four singular complexes.* The guiding lines of the congruence are also called the guiding lines of the ruled surface.

A (2, 2)-correspondence between the generators of the M_2^2 that is the image manifold of our congruence is defined by the C^4: A generator of the first kind cuts the regulus, and therefore the C^4, at two points, and two generators of the second kind will emanate from those two points, and conversely. We will then have a relationship that associates one generator of the one kind with two generators of the other kind. Since the generators of the M_2^2 correspond to the points of the guiding lines of our congruence, we will find that the ruled surface is generated by a (2, 2)-correspondence between the points of its guiding lines.

Theorem 14: *A regular pencil of linear complexes of type I and IIa (cf., no.* **75**), *together with the pencil that is polar to it relative to a given, regular* F^2, *will span a regular pencil of complexes. The axis surface of the pencil of complexes is then a fourth-order ruled surface of the first kind with two skew guiding lines, between which, the generators define a* (2, 2)-*correspondence.*

A more precise examination would teach us that the (2, 2)-correspondence is not the most general one. We shall pass over the examination of the special cases and an analytical presentation of our results that would lead to elegant formulas by the use of **Weitzenböck**'s complex symbolism.

82. The cylindroid. If the absolute conic appears in place of the absolute surface in the construction of the axis surface then the axis surface will decompose into a plane – namely, the imaginary plane – and a third-order ruled surface, namely, the (**Plücker**) *cylindroid.*

The axes of a linear complex relative to the absolute conic were explained in no. **77**. The null points of the imaginary plane relative to the null system of the pencil of complexes define a point sequence in the imaginary plane, namely, the polar complexes (or auxiliary axes) of the complexes of the pencil that is the polar pencil of lines relative to the absolute conic.

We once more apply the map to R_5 ! The pencil of complexes will be mapped to a line G, and the pencil of the polar complexes, to a generator G' of M_4^2. We once more assume that the two lines span an R_3 that cuts a regular M_2^2 out of M_4^2. The regulus that is generated by the lines G and G', which are related projectively to each other, now cuts M_2^2 in a C^4, from which, the generator G' will split off. The remaining intersection is a cubic space curve C^3.

We would like to call the line G' a generator of the first kind of the M_2^2. Any generator of the first kind will then cut C^3 at two points, and any generator of the second kind, at a single point. (The second point of intersection of the generator with the decomposable C^4 will lie on the line G'.) *The C^3 mediates a* (1, 2)-*correspondence between the generators of the first and second kind of the M_2^2.*

Just as the C^4 decomposes into a line and a third-order space curve in image space, the ruled surface will decompose into a pencil of lines and a third-order ruled surface. Since the C^3 belongs to an R_3, the ruled surface will belong to a linear congruence. One of the guiding lines of the congruence will be imaginary. It will be the guiding line whose pencil of lines will be mapped to the generators of the second kind of the M_2^2; any generator of the second kind will then, in fact, cut the line G'. Every pencil of lines that corresponds to a generator of the second kind will then contain an imaginary line.

Just as the C^3 defines a (1, 2)-correspondence between the lines of the two reguli of M_2^2, the generators of the ruled surface will mediate a (1, 2)-correspondence between the points of the two guiding lines: Every point of the real guiding line will correspond to two points of the imaginary one.

Theorem 15: *A regular pencil of linear complexes contains no imaginary lines and no minimal lines, and it spans a regular bush of complexes, together with the pencil that is polar to it relative to the absolute conic. The axis surface of the pencil will then decompose into a pencil of imaginary lines and a "cylindroid": viz., a ruled surface of order and class 3 with two skew guiding lines that define a line-cross. The generators mediate a* (1, 2)-*correspondence between the real and imaginary guiding lines.*

A simple construction of the cylindroid then gives:

Theorem 16: *One constructs the common normals to a fixed and a variable generator in a regulus. The result will be a cylindroid.*

The theorem is proved easily with the help of the map to R_5.

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Chapter Six

Ray geometry

§ 24. Study's conversion principle ([1]).

83. The mapping equations. Let $\mathfrak{X}_{ik}$ be the coordinates of a real complex, which are themselves real. The equations:

$$(1) \qquad \begin{cases} X_1 = \mathfrak{X}_{01} + i\mathfrak{X}_{23}, \\ X_2 = \mathfrak{X}_{02} + i\mathfrak{X}_{31}, \\ X_3 = \mathfrak{X}_{03} + i\mathfrak{X}_{12} \end{cases}$$

will then give a map of a point X in a complex plane to a real, linear complex in R_3. If we multiply the coordinates (1) by a complex factor $\rho = \sigma + i\tau$:

$$(2) \qquad \begin{cases} \rho X_1 = (\sigma\mathfrak{X}_{01} - \tau\mathfrak{X}_{23}) + i(\sigma\mathfrak{X}_{23} + \tau\mathfrak{X}_{01}), \\ \rho X_2 = (\sigma\mathfrak{X}_{02} - \tau\mathfrak{X}_{31}) + i(\sigma\mathfrak{X}_{31} + \tau\mathfrak{X}_{02}), \\ \rho X_3 = (\sigma\mathfrak{X}_{03} - \tau\mathfrak{X}_{12}) + i(\sigma\mathfrak{X}_{12} + \tau\mathfrak{X}_{03}) \end{cases}$$

then the mapped point X will remain the same, due to homogeneity. However, its image will change; it will consist of a pencil of real, linear complexes. We would like to replace this somewhat non-intuitive figure with a simpler one.

The aforementioned pencil will be spanned by the two complexes:

$$(3) \qquad \begin{cases} K_1: \quad \mathfrak{X}_{01} : \mathfrak{X}_{02} : \mathfrak{X}_{03} : \mathfrak{X}_{23} : \mathfrak{X}_{31} : \mathfrak{X}_{12} \\ K_2: \quad -\mathfrak{X}_{23} : -\mathfrak{X}_{31} : -\mathfrak{X}_{12} : \mathfrak{X}_{01} : \mathfrak{X}_{02} : \mathfrak{X}_{03} \end{cases}$$

We see that K_1 and K_2 are polar relative to the second-order surface:

$$(4) \qquad (\mathfrak{X} \mid \mathfrak{X}) = \mathfrak{X}_{01}^2 + \mathfrak{X}_{02}^2 + \mathfrak{X}_{03}^2 - \mathfrak{X}_{23}^2 - \mathfrak{X}_{31}^2 - \mathfrak{X}_{12}^2 = 0;$$

if we polarize:

$$(5) \qquad (\mathfrak{X} \mid \mathfrak{Y}) = \mathfrak{X}_{01}\mathfrak{Y}_{01} + \mathfrak{X}_{02}\mathfrak{Y}_{02} + \mathfrak{X}_{03}\mathfrak{Y}_{03} - \mathfrak{X}_{23}\mathfrak{Y}_{23} - \mathfrak{X}_{31}\mathfrak{Y}_{31} - \mathfrak{X}_{12}\mathfrak{Y}_{12} = 0$$

and fix $\mathfrak{X}$ then that will yield the equation for K_2.

(4) is the line equation of the *unit sphere:*

([1]) **E. Study**, *Geometrie der Dynamen*, Leipzig, 1903, § 23. "Über Nichteuklidische und Liniengeometrie," Jahresb. d. Deutschen Mathematiker-Vereinigung **11** (1902), pp. 342, *et seq.*

(6) $$-x_0^2 + x_1^2 + x_2^2 + x_3^2 = 0,$$

which will play a distinguished role in what follows. *We are then dealing with hyperbolic geometry in the image space.*

Since K_1 and K_2 are polar to each other relative to the absolute surface, all of the complexes of the pencil that they span will have the same axis. We then speak of a *pencil of coaxial complexes.*

From Chap. V, Theorem 9, we now have two cases to distinguish. In the first case, one is dealing with a pencil with two real, two-dimensional, mutually-polar, skew axes, one of which is reachable, while the other one is unreachable, and in the second case, one is dealing with a real pencil of tangents.

If we now restrict ourselves to the reachable domain as before then we (with **E. Study**) will call the reachable piece of a reachable line a *proper ray* and a point of the absolute sphere an *improper ray.* A pencil of the first kind will determine a proper ray in a one-to-one and invertible way, while a pencil of the second kind will determine an improper ray will one-to-one and invertible manner:

Theorem 1: *The points of the complex plane are mapped to the (proper and improper) rays in hyperbolic space by equations* (1).

84. Ray coordinates. A ray will then be characterized by six coordinates $\mathfrak{X}_{ik}$. The ray coordinates of a line are distinguished from its **Plücker** coordinates by the fact that the former do not need to satisfy the **Plücker** identity. One can thus assume that there are ∞^5 rays, while R_3 contains only ∞^4 lines. However, we must imagine that the ray coordinates possess an extended homogeneity: X will not change when one multiplies then by a complex number, which will contain *two* real quantities. *One can always obtain the Plücker coordinates of a line from its ray coordinates by a suitable choice of that factor.*

85. The fundamental conic section. Which points of the complex plane now correspond to the improper rays in hyperbolic space? In order to answer that question, we consider the expression:

(7) $$(X\ Y) = X_1\ Y_1 + X_2\ Y_2 + X_3\ Y_3\,,$$

and substitute it into (1). It will then follow that:

(8) $$(X\ Y) = (\mathfrak{X} \mid \mathfrak{Y}) + i\,(\mathfrak{X}\mathfrak{Y}).$$

The two equations:

(9) $$(\mathfrak{X} \mid \mathfrak{Y}) = 0, \quad (\mathfrak{X}\mathfrak{Y}) = 0$$

between real quantities follow from the one equation $(X\ Y) = 0$ between complex lines; i.e.:

Theorem 2: *Two points X, Y that are conjugate with respect to the "fundamental conic section"* $(X\ X) = 0$ *will be mapped to two rays that intersect each other at right angles.*

The figure that consists of two proper rays that intersect at right angles does not need to be clarified further. In order to find a geometric interpretation in the remaining cases, we propose to let $\mathfrak{X}$ and $\mathfrak{Y}$ be the **Plücker** coordinates of straight lines by multiplying by suitable factors. If, say, $\mathfrak{X}$ were then an absolute tangent, and should equations (9) be verified then $\mathfrak{Y}$, as well as $\mathfrak{X}$, would also have to be the polar of $\mathfrak{X}$, and they would then cut a line that belongs to the same pencil of tangents as $\mathfrak{X}$, and would then have to run through the vertex of the pencil. Finally, should the tangents $\mathfrak{X}$ and $\mathfrak{Y}$ be simultaneously conjugate and incident, then they would need to have the same contact point.

Theorem 3: *A proper ray will intersect an improper ray at right angles when it contains the point of the improper ray. Two improper rays will intersect at right angles when they coincide.*

The improper ray is characterized by the equations $(\mathfrak{X} \mid \mathfrak{X}) = 0$, $(\mathfrak{X}\mathfrak{X}) = 0$. The desired result will then follow from that:

Theorem 4: *The points of the fundamental conic will be mapped to improper rays.*

We make the following remark in regard to that map: The fundamental conic is a binary domain as the locus of its points, which, just like a binary domain of points on a line, can be mapped to the **Gaussian** plane or the **Riemann** sphere in a one-to-one and invertible way. The map that one obtains in this way will be identical to the one in Theorem 4, which we assert without proof. Hence, one can also obtain the map of the points in the complex plane to the rays of hyperbolic space as follows ([1]):

Point of the complex plane $\rightarrow$ point-pair on the fundamental conic (**Hesse**'s conversion principle) $\rightarrow$ point-pair on the **Riemann** sphere $\rightarrow$ ray.

86. Rays of the second sheet. Now, let one of two conjugate points *X*, *Y* – e.g., *X* – be fixed, and let *Y* run through the polars to *X*. The image ray of *Y* will then run through the *normal congruence* of the ray *X*. We now consider only the axis of that congruence. We can then say: *A line of the complex will be mapped to the axis of a normal congruence.* However, that axis is, at the same time, also the image ray of the pole *X* of the line that we are considering. If one would wish to distinguish image rays from points and lines then *that would imply the necessity of doubly-covering the hyperbolic ray space:* From now on, we shall distinguish between rays *X* that are the image rays of points in the complex plane and rays *U* that are the image rays of lines in it.

([1]) **F. Klein**, "Eine Übertragung des Pascalschen Satzes auf Raumgeometrie," Math. Ann. **22** = Ges. Werke I, pp. 406.

87. The group of dual collineations. The conversion principle allows us to convert theorems and figures in the complex plane to ones in ray space. Under it, properties of points that are invariant under the group of complex collineations in the plane:

$$(10)\qquad \left\{\begin{aligned} X_1' &= A_{11}X_1 + A_{12}X_2 + A_{13}X_3, \\ X_2' &= A_{21}X_1 + A_{22}X_2 + A_{23}X_3, \\ X_3' &= A_{31}X_1 + A_{22}X_2 + A_{33}X_3, \end{aligned}\right. \qquad |A_{ik}| \neq 0$$

will correspond to properties of rays of the first sheet that are invariant under the transformations of the group G_{16}^* of *dual collineations in hyperbolic space* that will follow from (10) by means of the conversion principle.

The conversion formulas yield:

$$(11)\qquad \mathfrak{X}_{01}' + i\mathfrak{X}_{23}' = (a_{11} + i\ a_{11}')(\mathfrak{X}_{01} + i\ \mathfrak{X}_{23}) + (a_{12} + i\ a_{12}')(\mathfrak{X}_{02} + i\ \mathfrak{X}_{31}) + (a_{13} + i\ a_{13}')(\mathfrak{X}_{03} + i\ \mathfrak{X}_{12})$$

and two analogous equations, and after splitting them into their real and imaginary parts:

$$(12)\qquad \left\{\begin{aligned} \mathfrak{X}_{01}' &= a_{11}\mathfrak{X}_{01} + a_{12}\mathfrak{X}_{02} + a_{13}\mathfrak{X}_{03} - a_{11}'\mathfrak{X}_{23} - a_{12}'\mathfrak{X}_{31} - a_{13}'\mathfrak{X}_{12}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \mathfrak{X}_{23}' &= a_{11}'\mathfrak{X}_{01} + a_{12}'\mathfrak{X}_{02} + a_{13}'\mathfrak{X}_{03} + a_{11}\mathfrak{X}_{23} + a_{12}\mathfrak{X}_{31} + a_{13}\mathfrak{X}_{12}, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned}\right.$$

Just as the collineation (10) induces a contragredient transformation of the line U, (12) will also induce a *contragredient transformation of the rays of the second sheet* in such a way that two rays U and X that intersect at right angles – viz., $(U\ X) = 0$ – will go to two rays U', X' of the same kind.

By contrast, the property of two rays from the same sheet intersecting at right angles is obviously not invariant under dual collineations. In particular, the property of a ray being proper or improper will not be invariant under those transformations, since a collineation (10) does not also, in fact, fix the fundamental conic, in general: The three-parameter group of automorphic collineations of the fundamental conic corresponds to the *group* G_6^* *of hyperbolic motions.*

One obtains the family of *transfers* from the group of motions when one composes a transformation of the group with a special transfer. One special transfer is the reflection through the center of the sphere:

$$(13)\qquad x_0' = -x_0, \qquad x_1' = x_1, \qquad x_2' = x_2, \qquad x_3' = x_3,$$

which will give:

$$(14) \qquad \begin{cases} \mathfrak{X}'_{01} = +\mathfrak{X}_{01}, & \mathfrak{X}'_{02} = +\mathfrak{X}_{02}, & \mathfrak{X}'_{03} = +\mathfrak{X}_{03}, \\ \mathfrak{X}'_{23} = -\mathfrak{X}_{23}, & \mathfrak{X}'_{31} = -\mathfrak{X}_{31}, & \mathfrak{X}'_{12} = -\mathfrak{X}_{12} \end{cases}$$

in line coordinates, so in point coordinates, one will have *conjugation:*

$$(15) \qquad X'_1 = \bar{X}_1, \qquad X'_2 = \bar{X}_2, \qquad X'_3 = \bar{X}_3,$$

which associates every point of the complex plane with its complex-conjugate point.

If one composes the group (10) with conjugation then one will obtain the family:

$$(16) \qquad \begin{cases} X'_1 = A_{11}\bar{X}_1 + A_{12}\bar{X}_2 + A_{13}\bar{X}_3, \\ X'_2 = A_{21}\bar{X}_1 + A_{22}\bar{X}_2 + A_{23}\bar{X}_3, \\ X'_3 = A_{31}\bar{X}_1 + A_{22}\bar{X}_2 + A_{33}\bar{X}_3 \end{cases}$$

of *anti-collineations.* An anti-collineation (16) induces a contragredient anti-collineation of the line U in such a way that one will have $(U' \, X') = (\bar{U} \, \bar{X})$. Therefore, a pair of mutually-perpendicular intersecting rays of different sheets will also go to a pair of the same kind under a pair of contragredient, dual, anti-collineations. The anti-collineations that fix the fundamental conic will be mapped to the hyperbolic transfers.

We summarize:

Theorem 5: *Under the conversion principle, the groups of collineations and anti-collineations of the complex plane will correspond to the groups* G_{16}^*, H_{16}^* *of dual collineations and anti-collineations of hyperbolic space, resp. The subgroups of automorphic collineations and anti-collineations will then correspond to the subgroups* G_6^*, H_6^* *of hyperbolic motions and transfers.*

The dual collineations and anti-collineations are associated pair-wise as contragredient transformations. A pair of contragredient transformations will take a pair of perpendicular rays of different sheets to a pair of the same kind.

§ 25. The configuration of Petersen and Morley.

88. The common normal to two rays. If we wanted to find the common normal ray to two given rays by the methods of ordinary analytical geometry then we would have to perform some rather cumbersome calculations. With the help of ray coordinates, we can write down the result directly: Just as two different points X and Y of the complex plane produce a connecting line $U = XY$ without exception, one will have:

Theorem 6: *Two different rays X and Y will always have one and only one common normal ray U with the coordinates:*

$$(6) \qquad U_1 = X_2 \, Y_3 - X_3 \, Y_2, \qquad U_2 = X_3 \, Y_1 - X_1 \, Y_3, \qquad U_3 = X_1 \, Y_2 - X_2 \, Y_1 \, .$$

89. Plücker's theorem. We now consider a figure that arises by repeated construction of the common normals to three given lines. We start with a theorem from elementary geometry.

The altitude theorem says that the altitudes of a triangle intersect in a point. That theorem is capable of being generalized to non-Euclidian geometry and can be expressed as follows:

Theorem 7: *Two triangles that are polar with respect to a conic section are perspective.*

The connecting lines of corresponding points of the two triangles are the altitudes of each of the two triangles, in the sense of the metric that refers to the conic section. Two lines are then said to be orthogonal to each other in the non-Euclidian sense when they are conjugate relative to the absolute conic.

When two triangles are perspective, from **Desargues**'s theorem, the points of intersection of corresponding sides of the two triangles will lie along a line, namely, the axis of the perspectivity. The figure that consists of two triangles and their sides, the center of perspectivity and the axis of perspectivity, the intersection points of corresponding sides, and the connecting lines of corresponding vertices defines a $(10_3, 10_3)$ configuration – i.e., each of the ten points of the figure contains three of the lines, and each of the ten lines of the figure contains three of the points – which is the *Desargues configuration.* One can easily show that the configuration that is provided by **Plücker**'s theorem (viz., Theorem 7) is polar with respect to the absolute conic: The two starting triangles by which the configuration is determined completely are switched by the polarity. Any line of the configuration will then be the polar to a point of the configuration.

90. The Petersen-Morley configuration ([1]). The *map that is mediated by the conversion principle* will take the **Desargues** configuration that was just described (which is polar to itself with respect to the fundamental conic) to a figure that consists of ten rays (each ray of which is, at the same time, a ray of the first and second sheet) such that each ray will intersect three other ones at right angles.

Theorem 8: *In hyperbolic geometry, there exists a figure of ten rays such that each of the rays cuts three other ones at right angles (viz., the* ***Petersen-Morley*** *figure).*

Just as the existence of the **Desargues** figure is based upon a *closing theorem*, the existence of the **Petersen-Morley** configuration is, as well. The theorem reads: Construct the common normals to each two of three given rays 1, 2, 3, and obtain the rays 1′, 2′, 3′. Then, construct the common normals to the rays 11′, 22′, 33′. Those three rays will admit a common normal. The aforementioned ten rays define a **Petersen-Morley** configuration. Naturally, the starting rays must be assumed to be in "general position" in

([1]) **T. Kubota**, "An application of binary quadratic forms to geometry," Science Rep. Tôhuku Imp. Un. **6**. **H. Beck**, "Über einen Satz von Herrn Kubota," Tôhuku Math. Journ. **24** (1925).

such a way that the given construction would make sense. However, we shall pass over the task of presenting the assumptions as inequalities.

The figure in Theorem 8 is invariant under hyperbolic motions and transfers, but not under all dual collineations and anti-collineations. In general, if rays that overlap each other on different sheets are separated by the application of a general collineation or anti-collineation (just as a pair of distinct, mutually-polar **Desargues** figures will arise from the **Desargues** configuration that has a special relationship to the fundamental conic by an application of a general collineation or anti-collineation in the plane) then a *figure that consists of two systems of ten rays such that every ray of the one system cuts three rays of the other system at right angles* will come about in space.

§ 26. Chains.

91. Chains in the complex plane. The complex plane is four-dimensional as a locus of its points. If one counts the real parameters then it will contain ∞^4 points that will be mapped to the ∞^4 rays of hyperbolic space by the conversion principle. A curve that lies in the complex plane is two-dimensional as a locus of points. Its image will be an (entirely special) ray congruence.

One-dimensional point manifolds in the complex plane were called *strings* by the founder of complex geometry **C. Segre**. The simplest of them is the *chain.*

In the complex projective geometry of lines of **Ch. v. Staudt**, one understands a chain to mean the locus of all points that define a real double ratio with three given fixed points. Since one can always take the three points to three real points by a suitable collineation, that definition will be equivalent to the following one: *A chain is a locus of points that is projectively equivalent to a sequence of real lines.* The real sequence of lines is itself a simple example of a chain.

The latter definition of a chain leads to a simpler *representation of the chain in the complex plane, as well.* Let p and q be two distinct, real points in the plane, and let $\xi_1 : \xi_2$ be their parameters. The expression:

$$\xi_1\, p + \xi_2\, q \qquad [p = \bar{p}\,,\ q = \bar{q}\,] \tag{1}$$

will then represent the real sequence of connecting lines pq. If one performs an arbitrary, complex collineation then (1) will give:

$$\xi_1\, p^* + \xi_2\, q^*, \tag{2}$$

in which ξ_1, ξ_2 will be real parameters, as before, but p^*, q^* will be two arbitrary complex points. *Any chain in the plane can be represented in the form* (2). (Here, a chain seems to be determined by two points. However, in order for the parametric representation to make sense, one must demand that p^* and q^* should be homogeneous in the same way, and thus obtain the third point $p^* + q^*$ for the parameter pair $\xi_1, \xi_2 = 1 : 1$.)([1])

([1]) A chain will be mapped to a circle under the map of the complex line to the **Gaussian** plane of the **Riemann** sphere.

A *chain of lines* is defined dually.

92. Chains of rays. Under the conversion, a point-chain will become a one-dimensional locus of rays that is called a *chain of rays.* The image of (2) will then become a real sequence in a real pencil of linear complexes. One seeks the axis-pair (axis-pencil, resp.) in each of those complexes, and only the reachable axis in each pair, and only the reachable part of it will be considered. It will then follow that:

Theorem 9: *A chain of rays is the part of the axis surface of a real pencil of complexes that lies in the absolute sphere.*

We immediately derive some simple theorems about chains of rays by means of the conversion principle:

Theorem 10: *There are* ∞^7 *chains of rays.*

Proof: There are ∞^4 straight lines in the complex plane. There are ∞^3 chains on each of those lines. Thus, there are ∞^7 chains in the plane, and just as many chains of rays in space.

Theorem 11: *Any chain of rays belongs to a normal congruence.*

In fact: A normal congruence is the image of the lines along which the corresponding point-chains lie.

Since a point-chain is determined by three distinct points of a line, it will ultimately follow that:

Theorem 12: *A chain of rays will be determined by three distinct rays of a normal congruence.*

Just as the "binary" chain is defined to be a figure that is complex-projectively equivalent to a sequence of real lines, the "ternary" line is defined to be the locus of points that is complex-projectively equivalent to a sequence of real planes. Such a chain will be mapped to a certain ray congruence. We cannot concern ourselves with these figures in more detail here, but refer to **E. Study**'s ground-breaking book *Geometrie der Dynamen*, Leipzig, 1903 for the Euclidian geometry of loci of rays. For the hyperbolic theory, we refer to **H. Beck**, "The Strahlenketten im hyperbolischen Raume," Diss. Bonn, 1905. For elliptic geometry, we refer to **J. L. Coolidge**, "Die dual-projektive Geometrie im elliptischen und sphärischen Raum," Diss. Greifswald, 1904.

§ 27. Passing to the limit of Euclidian geometry.

93. Dual numbers. The complex number $a + ib$ is composed of the two units 1, i with real coefficients a, b, in which one has the convention that $i^2 = -1$. One can make a more general convention and set $i^2 = -k^2$, in which k represents a non-zero real number. The system of complex numbers that one obtains in that way will not differ essentially from the system of ordinary complex numbers. However, if one goes to the limit $k \to 0$ then one will obtain a number system with new properties, namely, the *system of dual numbers:*

(1) $$a + \varepsilon b \qquad [a, b \text{ real}, \varepsilon^2 = 0].$$

Dual numbers are added and multiplied like ordinary complex numbers (while observing the convention that $\varepsilon^2 = 0$). Division by a dual number involves calculating the reciprocal to the dual number. The equation:

(2) $$(a + \varepsilon b) \cdot (x + \varepsilon y) = 1$$

will then lead to the system of equations:

(3) $$ax = 1; \qquad ay + bx = 0,$$

which possess a solution only when $a \neq 0$. It will then follow that:

(4) $$x + \varepsilon y = (a + \varepsilon b)^{-1} = \frac{1}{a^2}(a - \varepsilon b).$$

94. The conversion formulas of Euclidian geometry. We now pass to the limit in the conversion formulas of (1) in § 24, as well. Once again, the equations:

(5) $$\begin{cases} X_1 = \mathfrak{X}_{01} + \varepsilon \mathfrak{X}_{23}, \\ X_2 = \mathfrak{X}_{02} + \varepsilon \mathfrak{X}_{31}, \\ X_3 = \mathfrak{X}_{03} + \varepsilon \mathfrak{X}_{12} \end{cases}$$

will give a map of the point X in the dual plane to a real linear complex in R_3. However, whereas the map of complexes to points is single-valued, a point will be associated with ∞^1 complexes that one obtains when one multiplies the coordinates X_i by a dual proportionality factor $\rho = \sigma + \varepsilon \tau$:

(6) $$\begin{cases} \rho X_1 = \sigma \mathfrak{X}_{01} + \varepsilon(\sigma \mathfrak{X}_{23} + \tau \mathfrak{X}_{01}), \\ \rho X_2 = \sigma \mathfrak{X}_{02} + \varepsilon(\sigma \mathfrak{X}_{31} + \tau \mathfrak{X}_{02}), \\ \rho X_3 = \sigma \mathfrak{X}_{03} + \varepsilon(\sigma \mathfrak{X}_{12} + \tau \mathfrak{X}_{03}). \end{cases}$$

In general, a pencil of linear complexes that are spanned by the complexes:

(7) $$\begin{cases} K_1: & \mathfrak{X}_{01} : \mathfrak{X}_{02} : \mathfrak{X}_{03} : \mathfrak{X}_{23} : \mathfrak{X}_{31} : \mathfrak{X}_{12} \\ K_2: & 0 : \quad 0 : \quad 0 : \mathfrak{X}_{01} : \mathfrak{X}_{02} : \mathfrak{X}_{03} \end{cases}$$

will come about. K_2 is the *auxiliary axis* of the complex K_1 in the sense of the *Euclidian metric* that refers to the absolute conic:

$$[\mathfrak{X} \mid \mathfrak{X}] \equiv \mathfrak{X}_{01}^2 + \mathfrak{X}_{02}^2 + \mathfrak{X}_{03}^2 = 0. \tag{8}$$

As we know (Chap. V, Theorem 10), the complex K_2 is undetermined if and only if K_1 is an improper line. If we exclude that case then we will obtain a pencil of coaxial linear complexes, in the sense of Euclidian geometry, which determines a principal axis uniquely. We call the figure of such a principal axis a *proper ray* (of Euclidian geometry). The coordinates $\mathfrak{X}_{ik}$ are called *ray coordinates*. They differ, in turn, from the **Plücker** line coordinates by the fact that they do not need to satisfy the **Plücker** identity. Upon multiplying the X_i by a suitable dual proportionality factor, however, they will go to the **Plücker** coordinates of the corresponding ray.

We shall first speak of improper rays later on, since their introduction will create greater difficulties than it does in hyperbolic geometry. For the time being, all rays that occur will be assumed to be proper. Nevertheless, we summarize the results so far:

Theorem 13: *Equations* (5) *mediate a map of the points in the dual plane to the rays of Euclidian space.*

95. Applications ([1]). We now pass to the limit with the figures and theorems that we derived as applications of the conversion principle in hyperbolic geometry, as well. We first point out the equation:

$$(X\,Y) = [\mathfrak{X} \mid \mathfrak{Y}] + \varepsilon\,(\mathfrak{X}\mathfrak{Y}): \tag{9}$$

Theorem 14: *Two points of the dual plane that are conjugate relative to the fundamental conic* $(X\,X) = 0$ *will be mapped to two lines that intersect at right angles, in the sense of Euclidian geometry.*

The ∞^2 points of a line U that has the point X for its pole will then be mapped to the normal net of the line X. As in hyperbolic geometry, it will then prove to be convenient to cover the ray manifold with two sheets. For two rays X and Y on the first sheet, one then obtains the common normal ray U of the second sheet as the image ray of the connecting line XY.

A Euclidian analogue to the *Petersen-Morley configuration* will arise by passing to the limit ([2]).

Finally, a chain of the dual plane will be mapped to a ray-chain in Euclidian space, and just as a chain in the dual plane will generally contain no point of the fundamental conic, the chain of lines will also generally contain only proper rays:

Theorem 15: *The image of a point-chain in the dual plane that does not meet the fundamental conic is ray-chain that consists of only proper rays. That figure is identical*

([1]) **W. Blaschke** discussed some applications to differential geometry in his *Vorlesungen über Differentialgeometrie I* (1921), Chap. 7.

([2]) This is the figure that **Petersen** and **Morley** gave originally. Cf. **E. Study**, *Geometrie der Dynamen*, Leipzig, 1903, pp. 107.

with the figure of ∞^1 *real generators of a cylindroid or the figure of a pencil of lines with a proper vertex.*

The second case occurs when the pencil of complexes is a pencil of lines with a proper vertex.

96. The group of radial collineations. By conversion, the groups of collineations and anti-collineations of the dual plane will go to the groups G_{16}, H_{16} of dual collineations and anti-collineations of Euclidian ray space, resp. Similar to the ones in (12), § 24, the equations for the collineations will read:

$$(10) \qquad \begin{cases} \mathfrak{X}'_{01} = a_{11}\mathfrak{X}_{01} + a_{12}\mathfrak{X}_{02} + a_{13}\mathfrak{X}_{03}, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \mathfrak{X}'_{23} = a'_{11}\mathfrak{X}_{01} + a'_{12}\mathfrak{X}_{02} + a'_{13}\mathfrak{X}_{03} + a_{11}\mathfrak{X}_{23} + a_{12}\mathfrak{X}_{31} + a_{13}\mathfrak{X}_{12}, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{cases}$$

Like the collineations and anti-collineations of the plane, they are paired off as contragredient transformations, and such pairs will have the property that they take rays from different sheets that intersect at right angles to rays with the same property.

As opposed to hyperbolic geometry, however, the groups G_{16}, H_{16} are not the largest groups of transformations that have that property. A one-parameter group of transformations that are not contained in G_{16}, H_{16}, but still have the stated property, is defined by the stretchings about the origin, which are given by the equations:

$$(11) \qquad x'_0 = x_0, \qquad x'_1 = \lambda \cdot x_1, \qquad x'_2 = \lambda \cdot x_2, \qquad x'_3 = \lambda \cdot x_3,$$

in point coordinates, and thus by the equations:

$$(12) \qquad \begin{cases} \mathfrak{X}'_{01} = \mathfrak{X}_{01}, \quad \mathfrak{X}'_{02} = \mathfrak{X}_{02}, \quad \mathfrak{X}'_{03} = \mathfrak{X}_{03} \\ \mathfrak{X}'_{23} = \lambda\mathfrak{X}_{23}, \; \mathfrak{X}'_{31} = \lambda\mathfrak{X}_{31}, \; \mathfrak{X}'_{12} = \lambda\mathfrak{X}_{12} \end{cases}$$

in line coordinates. If one combines these with (10) then one will obtain the *group* G_{17} *of radial collineations:*

$$(13) \qquad \begin{cases} \mathfrak{X}'_{01} = a_{11}\mathfrak{X}_{01} + a_{12}\mathfrak{X}_{02} + a_{13}\mathfrak{X}_{03}, \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ \mathfrak{X}'_{23} = a'_{11}\mathfrak{X}_{01} + a'_{12}\mathfrak{X}_{02} + a'_{13}\mathfrak{X}_{03} + \lambda\{a_{11}\mathfrak{X}_{23} + a_{12}\mathfrak{X}_{31} + a_{13}\mathfrak{X}_{12}\}, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \end{cases}$$

The transformations of this group are also paired off as contragredient, and contragredient transformation have the property that they take pairs of perpendicular, incident rays of different sheets to pairs of the same kinds.

G_{16} (as well as G_{16}, H_{16}) is contained in this group invariantly. However, G_{17} contains yet another invariant subgroup, namely, the *group G_9 of radial collineations that fix every point of the imaginary plane* (any bundle of parallels, as a whole). The first three equations of such a transformation must have the form:

$$\mathfrak{X}'_{01} = \mathfrak{X}_{01}, \qquad \mathfrak{X}'_{02} = \mathfrak{X}_{02}, \qquad \mathfrak{X}'_{03} = \mathfrak{X}_{03}. \tag{14}$$

If one sets $\mathfrak{X}_{01} = \mathfrak{X}_{02} = \mathfrak{X}_{03} = 0$ (as one must if one is to be dealing with the transformation of an improper line) then the last three of equations (13) will reduce to:

$$\mathfrak{X}'_{23} = \lambda\, \mathfrak{X}_{23}, \quad \mathfrak{X}'_{31} = \lambda\, \mathfrak{X}_{31}, \quad \mathfrak{X}'_{12} = \lambda\, \mathfrak{X}_{12} \qquad [\lambda \neq 0]. \tag{15}$$

The equations for the group G_9 will then read:

$$\left\{\begin{aligned} &\mathfrak{X}'_{01} = \mathfrak{X}_{01},\ \mathfrak{X}'_{02} = \mathfrak{X}_{02},\ \mathfrak{X}'_{03} = \mathfrak{X}_{03}, \\ &\mathfrak{X}'_{23} = \lambda\mathfrak{X}_{23} + a'_{11}\mathfrak{X}_{01} + a'_{12}\mathfrak{X}_{02} + a'_{13}\mathfrak{X}_{03}, \\ &\mathfrak{X}'_{31} = \lambda\mathfrak{X}_{31} + a'_{21}\mathfrak{X}_{01} + a'_{22}\mathfrak{X}_{02} + a'_{23}\mathfrak{X}_{03}, \\ &\mathfrak{X}'_{12} = \lambda\mathfrak{X}_{12} + a'_{31}\mathfrak{X}_{01} + a'_{32}\mathfrak{X}_{02} + a'_{33}\mathfrak{X}_{03}. \end{aligned}\right. \tag{16}$$

One might believe that the general transformation of the group depends upon ten constants. However, only nine of the constants are essential, since the ray coordinates $\mathfrak{X}_{ik}$ can be replaced with coordinates:

$$\mathfrak{X}_{01} : \mathfrak{X}_{02} : \mathfrak{X}_{03} : \mathfrak{X}_{23} + \tau\mathfrak{X}_{01} : \mathfrak{X}_{31} + \tau\mathfrak{X}_{02} : \mathfrak{X}_{12} + \tau\mathfrak{X}_{03} \tag{17}$$

without changing the ray. However, the quantities:

$$a'_{11} + \lambda\tau, \qquad a'_{22} + \lambda\tau, \qquad a'_{33} + \lambda\tau \tag{18}$$

will appear in place of a'_{11}, a'_{22}, a'_{33}, and one can choose the value of τ in such a way that the sum of these quantities will vanish. Therefore, one can already assume that:

$$a'_{11} + a'_{22} + a'_{33} = 0 \tag{19}$$

in (16) from the outset.

The groups G_{16} and G_9 intersect in the group G_8, whose representation one will obtain when one sets $\lambda = 1$ in (16). That group will be contained invariantly in G_{16} and G_9 and its transformations will commute pair-wise.

We can once more illustrate the connections between the groups by a diagram:

$$\begin{tikzcd} & G_{17} \arrow[dl, dash, double] \arrow[dr, dash, double] & \\ G_{16} \arrow[dr, dash, double] & & G_9 \arrow[dl, dash, double] \\ & G_8 & \end{tikzcd}$$

This diagram not only represents an analogy with the diagram in no. **73**, but it is also a generalization of it, in the sense that the groups G contain the corresponding groups g, and will even reduce to them as long as one demands that the transformations should not separate overlapping rays on different sheets.

97. Improper rays. Up to now, we have spoken of only proper rays; i.e., it was assumed that at least one of the ray coordinates $\mathfrak{X}_{01}$, $\mathfrak{X}_{02}$, $\mathfrak{X}_{03}$ was non-zero. In the case when $\mathfrak{X}_{01} = \mathfrak{X}_{02} = \mathfrak{X}_{03} = 0$, it seems natural to regard the line:

$$0 : 0 : 0 : \mathfrak{X}_{23} : \mathfrak{X}_{31} : \mathfrak{X}_{12} \tag{20}$$

as the representative of the corresponding *improper ray*. By that convention, the open continuum of proper rays (in **E. Study**'s terminology) will be closed into the *irregular continuum* of proper and improper rays. The radial collineations are defined everywhere and continuous in that irregular continuum.

The continuum is called *irregular* due to the fact that it exhibits remarkable behavior when one wishes to go from a proper ray to an improper one continuously. We shall exhibit that behavior in the example of a pencil of parallels.

Let a pencil of parallels be spanned by the improper line $0 : 0 : 0 : \mathfrak{X}_{23} : \mathfrak{X}_{31} : \mathfrak{X}_{12}$ and a line $\mathfrak{Y}$ that is incident with it:

$$(0 : 0 : 0 : \mathfrak{X}_{23} : \mathfrak{X}_{31} : \mathfrak{X}_{12}) + t\,(\mathfrak{Y}_{01}, \mathfrak{Y}_{02}, \mathfrak{Y}_{03}, \mathfrak{Y}_{23}, \mathfrak{Y}_{31}, \mathfrak{Y}_{12}). \tag{21}$$

If one lets a line of the pencil wander about that pencil then in the limiting case $t = 0$, it will fall upon the improper line $\mathfrak{X}$. In *Plücker line geometry*, that behavior will describe a *line* by running through a pencil of parallels. However, in ray geometry, things are different.

We multiply the ray coordinates:

$$X_1 = t\,\mathfrak{Y}_{01} + \varepsilon\{\mathfrak{X}_{23} + t\,\mathfrak{Y}_{23}\}, *, * \tag{22}$$

by the proportionality factor $\rho = \sigma(t) + \varepsilon\,\tau(t)$:

$$\rho\,X_1 = \sigma(t)\,t\,\mathfrak{Y}_{01} + \varepsilon\{\sigma(t)\,\mathfrak{X}_{23} + \sigma(t)\,t\,\mathfrak{Y}_{23} + \tau(t)\,t\,\mathfrak{Y}_{01}\}, *, *. \tag{23}$$

If one now goes to the limit and chooses the arbitrary functions $\sigma(t)$, $\tau(t)$ in such a way that:

(24) $$\lim_{t\to 0} \sigma(t) = \sigma_0 , \qquad \lim_{t\to 0} \tau(t) \cdot t = \tau_0$$

then equations (23) will go to:

(25) $$X_1^* = \varepsilon\{\sigma_0\, \mathfrak{X}_{23} + \tau_0\, \mathfrak{Y}_{01}\}, *, *$$

as $t \to 0$; i.e., one will obtain an arbitrary ray of the pencil that is spanned by the improper rays:

(26) $$\begin{cases} 0 : 0 : 0 : \mathfrak{X}_{23} : \mathfrak{X}_{31} : \mathfrak{X}_{12} \\ 0 : 0 : 0 : \mathfrak{Y}_{01} : \mathfrak{Y}_{02} : \mathfrak{Y}_{03} \end{cases}$$

The second of these two rays will be the absolute polar (i.e., auxiliary axis) of the ray $\mathfrak{Y}$.

Theorem 16: *If one lets a ray in the irregular ray continuum run through a pencil of parallels with the vertex p then one will obtain a pencil of improper rays upon going to the improper domain. The vertex of the pencil will be the point of intersection p′ of the absolute polar of p with the improper line of the plane of the given pencil of parallels.*

In order to avoid the difficulty that comes from the fact that a pencil of parallels will first become a closed continuum when one adds a pencil of improper rays to it, one can agree to combine the ∞^1 rays of the pencil into a new concept of *point-ray*. One can then imagine that a ray that runs through a pencil of parallels (just as in hyperbolic geometry) will become a point (viz., the point p') when one passes to the limit. The introduction of point-rays led to **Study**'s first, regular ray-continuum ([1]).

§ The invariant ($X\ Y\ Z$).

98. Conversion of the invariant. One of the fundamental invariants ($U\ X$) of projective geometry in the dual plane can be translated into the language of Euclidian ray geometry using equation (9) of the previous paragraph. We would now like to consider the second fundamental invariant ($X\ Y\ Z$), when we, in turn, restrict ourselves to proper rays. Its vanishing says that three points X, Y, Z in a plane belong to a line U. Hence:

Theorem 17: *The vanishing of the invariant* ($X\ Y\ Z$) *represents the necessary and sufficient condition for the three rays X, Y, Z to admit a common normal.*

The conversion of the invariant yields:

([1]) For an analytical representation of this continuum and the second, regular ray continuum, we refer to **E. Study**, *Geometrie der Dynamen*, Leipzig, 1903, § 27.

$$(1) \qquad (X\,Y\,Z) = (\mathfrak{X}\,\mathfrak{Y}\,\mathfrak{Z}) + \varepsilon\,\{\mathfrak{X}\,\mathfrak{Y}\,\mathfrak{Z}\},$$

in which we have set:

$$(2) \qquad (\mathfrak{X}\,\mathfrak{Y}\,\mathfrak{Z}) = \begin{vmatrix} \mathfrak{X}_{01} & \mathfrak{X}_{02} & \mathfrak{X}_{03} \\ \mathfrak{Y}_{01} & \mathfrak{Y}_{02} & \mathfrak{Y}_{03} \\ \mathfrak{Z}_{01} & \mathfrak{Z}_{02} & \mathfrak{Z}_{03} \end{vmatrix},$$

$$(3) \qquad \{\mathfrak{X}\,\mathfrak{Y}\,\mathfrak{Z}\} = \begin{vmatrix} \mathfrak{X}_{23} & \mathfrak{X}_{02} & \mathfrak{X}_{03} \\ \mathfrak{Y}_{23} & \mathfrak{Y}_{02} & \mathfrak{Y}_{03} \\ \mathfrak{Z}_{23} & \mathfrak{Z}_{02} & \mathfrak{Z}_{03} \end{vmatrix} + \begin{vmatrix} \mathfrak{X}_{01} & \mathfrak{X}_{31} & \mathfrak{X}_{03} \\ \mathfrak{Y}_{01} & \mathfrak{Y}_{02} & \mathfrak{Y}_{03} \\ \mathfrak{Z}_{01} & \mathfrak{Z}_{02} & \mathfrak{Z}_{03} \end{vmatrix} + \begin{vmatrix} \mathfrak{X}_{01} & \mathfrak{X}_{02} & \mathfrak{X}_{12} \\ \mathfrak{Y}_{01} & \mathfrak{Y}_{02} & \mathfrak{Y}_{12} \\ \mathfrak{Z}_{01} & \mathfrak{Z}_{02} & \mathfrak{Z}_{12} \end{vmatrix}.$$

We would now like to interpret the vanishing of the two invariants (2) and (3) individually. For that, we shall assume (as would be permissible) that the ray coordinates that occur become **Plücker** line coordinates upon multiplication by suitable factors.

The equation $(\mathfrak{X}\,\mathfrak{Y}\,\mathfrak{Z}) = 0$ says that the improper points of the three lines are collinear.

99. A theorem from the metric geometry of second-order surfaces. In order to be able to interpret the equation $\{\mathfrak{X}\,\mathfrak{Y}\,\mathfrak{Z}\} = 0$, we first make the following remark: If the bilinear invariant $\sum_{i,k=1}^{3} a_{ik}\,p_{ik}$ vanishes for a second-order curve $\sum_{i,k=1}^{3} a_{ik}x_i x_k = 0$ and a second-class curve $\sum_{i,k=1}^{3} p_{ik}u_i u_k = 0$ then the order curve will be said to be *apolar* to the class curve. The ∞^1 polar triangles of the class curve can then be inscribed in the order curve (and dually). If it is possible to inscribe *one* polar triangle of a class curve in an order curve then the bilinear invariant of the two curves will vanish, and there will exist ∞^1 triangles of the stated kind.

One can apply this theorem to the metric theory of second-order surfaces. A regular, second-order surface $\sum_{i,k=0}^{3} a_{ik}x_i x_k = 0$ cuts the imaginary plane in the second-order curve $\sum_{i,k=1}^{3} a_{ik}x_i x_k = 0$. We then consider that curve, together with the absolute conic:

$$\sum_{i,k=1}^{3} p_{ik}u_i u_k = u_1^2 + u_2^2 + u_3^2 = 0.$$

It then follows immediately that:

Theorem 18: *If one can find three generators in a family of generators of a regular second-order surface that are pair-wise perpendicular then the regulus will contain ∞^1 triples of that kind, and the second regulus will contain ∞^1 triples of the same kind.*

We would like to call a surface with that property *orthogonal*. The condition for the surface $\sum_{i,k=0}^{3} a_{ik} x_i x_k = 0$ to be orthogonal is that its imaginary curve of order 2 must be apolar to the absolute conic, and thus reads:

(4) $$a_{11} + a_{22} + a_{33} = 0.$$

100. Interpretation of the equation $\{\mathfrak{X}\ \mathfrak{Y}\ \mathfrak{Z}\} = 0$. If we assume that the lines $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$ are pair-wise skew then we can consider the second-order surface that is determined by those lines. We can write down its equation directly with the help of **Weitzenböck**'s complex symbolism and calculate the coordinates a_{11}, a_{22}, a_{33}. It shows that these coordinates are proportional to the three, three-rowed determinants that enter into $\{\mathfrak{X}\ \mathfrak{Y}\ \mathfrak{Z}\}$. We can then set:

(5) $$\{\mathfrak{X}\ \mathfrak{Y}\ \mathfrak{Z}\} = a_{11} + a_{22} + a_{33}.$$

It will then follow that:

Theorem 19: *Let* $\mathfrak{X}\ \mathfrak{Y}\ \mathfrak{Z}$ *be three pair-wise-skew proper lines.* $\{\mathfrak{X}\ \mathfrak{Y}\ \mathfrak{Z}\} = 0$ *is then the necessary and sufficient condition for the second-order surface that is determined by the three lines to be orthogonal.*

Finally, one can ask how it would follow from the simultaneous validity of the equations $(\mathfrak{X}\mathfrak{Y}\mathfrak{Z}) = 0$ and $\{\mathfrak{X}\ \mathfrak{Y}\ \mathfrak{Z}\} = 0$ that the lines $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$ would admit a common normal. It follows from $(\mathfrak{X}\mathfrak{Y}\mathfrak{Z}) = 0$ that the surface that $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$ determines is a paraboloid that cuts the imaginary plane in a pair of lines. Since $\{\mathfrak{X}\ \mathfrak{Y}\ \mathfrak{Z}\} = 0$, that will be a pair of lines u, v that are polar with respect to the absolute conic. If the lines $\mathfrak{X}$, $\mathfrak{Y}$, $\mathfrak{Z}$ cut the imaginary plane at points of the line u then the generator of the second kind through the absolute polar of u (which lies on v) will be the desired common normal ([1]).

§ 20. The dual angle.

101. Distance and angle between rays. In the geometry of automorphic collineations of a regular conic section $(X\ X) = 0$, two points X, Y are associated with an absolute invariant, namely, the distance between the two points in the non-Euclidian geometry that relates to the conic (cf., no. **65**). One can also introduce such a definition of distance in the non-Euclidian geometry of the dual plane that refers to the fundamental conic and set:

(1) $$\cos \operatorname{dist} X\,Y = \frac{(XY)}{\sqrt{(XX)}\sqrt{(YY)}}.$$

([1]) The contents of this number go back to **G. Reuschenbach.**

We would now like to carry this absolute invariant over to ray space. We will then obtain an invariant with respect to such collineations that does not affect the overlapping of two rays on two different sheets, so (from no. **96**, conclusion), it will be a *Euclidian* invariant of motion.

We set:

(2) $$\text{dist}\, X\, Y = \Theta + \varepsilon H.$$

One will then have:

(3) $$\begin{aligned}&\cos(\Theta + \varepsilon H)\\ &= \cos\Theta\left(1 - \frac{\varepsilon^2 \cdot H^2}{2!} + \cdots\right) - \sin\Theta\left(\varepsilon H - \frac{\varepsilon^3 \cdot H^3}{3!} + \cdots\right)\\ &= \cos\Theta - \varepsilon H \sin\Theta.\end{aligned}$$

The right-hand side of (1) converts to:

(4) $$\frac{(XY)}{\sqrt{(XX)}\sqrt{(YY)}} = \frac{[\mathfrak{X}\,|\,\mathfrak{Y}]}{\sqrt{[\mathfrak{X}\,|\,\mathfrak{X}]}\sqrt{[\mathfrak{Y}\,|\,\mathfrak{Y}]}} + \varepsilon\frac{(\mathfrak{X}\mathfrak{Y})}{\sqrt{[\mathfrak{X}\,|\,\mathfrak{X}]}\sqrt{[\mathfrak{Y}\,|\,\mathfrak{Y}]}}.$$

A comparison with the formulas (15), (18) of § 21 the yields:

Theorem 20: *If one sets* $\cos(\Theta + \varepsilon H) = \dfrac{(XY)}{\sqrt{(XX)}\sqrt{(YY)}}$ *then* Θ *will be the angle, and H will be the distance between the rays X, Y.*

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Chapter Seven

Kinematics ([1])

§ 30. Ternary orthogonal transformations.

102. Connection with ray geometry. Euclidian kinematics is the geometry whose spatial elements are the motions and transfers of Euclidian space. Therefore, the introduction of coordinates for those motions and transfers – i.e., a parametric representation for those transformations – is fundamental for that geometry. In no. **87**, we obtained hyperbolic motions by converting the complex, automorphic collineations of the fundamental conic. A parametric representation of hyperbolic motions will then be obtained from a parametric representation of those collineations by the conversion principle. One would obtain a parametric representation of Euclidian motions by passing to the limit of Euclidian geometry.

(We will learn about further connections between ray geometry and kinematics that rest upon deeper-lying analogies in §§ 34, 35.)

103. Orthogonal matrices. In order to achieve that goal, we must next look for a parametric representation of the automorphic collineations of the conic section:

$$(1) \qquad (x\,x) \equiv x_1^2 + x_2^2 + x_3^2 = 0.$$

We would like to appeal to another, somewhat more intuitive, geometric interpretation of that problem by regarding x_1, x_2, x_3 as inhomogeneous coordinates for Euclidian space, not as homogeneous coordinates for the plane. If we demand that a linear transformation:

$$(2) \qquad \begin{cases} x_1' = c_{11}x_1 + c_{12}x_2 + c_{13}x_3, \\ x_2' = c_{21}x_1 + c_{22}x_2 + c_{23}x_3, \\ x_3' = c_{31}x_1 + c_{32}x_2 + c_{33}x_3 \end{cases}$$

should imply the identity:

$$(3) \qquad x_1'^2 + x_2'^2 + x_3'^2 = x_1^2 + x_2^2 + x_3^2$$

then we would be demanding that it should fix the square of the distance of the point x from the origin. We would then characterize it as a motion or transfer with the origin as its fixed point.

If we substitute the value (2) into (3) then it will follow that:

$$(4) \qquad x_1'^2 + \ldots = (c_{11}\,x_1 + c_{12}\,x_2 + c_{13}\,x_3)^2 + \ldots = x_1^2 + \ldots,$$

([1]) **E. Study**'s investigations into kinematics are found in the Appendix to *Geometrie der Dynamen*, Leipzig 1903, and in the Berlin lecture "Grundlagen und Ziele der analytischen Kinematik," Sitz. d. Berl. Math. Ges. **12** (1913). The following numbers **102** to **118** go back to an elaboration upon **Study**'s lecture "Ausgewählte Kapitel aus der höheren Geometrie" that the author completed in the year 1926.

and from that, by comparing coefficients:

$$(5)\qquad \begin{cases} c_{11}^2 + c_{21}^2 + c_{31}^2 = 1, & c_{12}c_{13} + c_{22}c_{23} + c_{32}c_{33} = 0, \\ c_{12}^2 + c_{22}^2 + c_{32}^2 = 1, & c_{13}c_{11} + c_{23}c_{21} + c_{33}c_{31} = 0, \\ c_{13}^2 + c_{23}^2 + c_{33}^2 = 1 & c_{11}c_{12} + c_{21}c_{22} + c_{31}c_{32} = 0. \end{cases}$$

In words: The inner product of each column with itself must have the value 1, while the inner product of any two different columns must have the value 0. A matrix with that property will be called *orthogonal*.

From (5), the square of the determinant of an orthogonal matrix will have the value 1, so the determinant itself will have the value + 1 or – 1. One thus distinguishes *proper* and *improper orthogonal matrices.* This distinction arises from the difference between motions and transfers. In what follows, we shall assume proper-orthogonal matrices.

If we denote the algebraic complement to c_{ik} by C_{ik} then we will get the three equations:

$$(6)\qquad \begin{cases} c_{11}C_{11} + c_{21}C_{21} + c_{31}C_{31} = 1, \\ c_{12}C_{11} + c_{22}C_{21} + c_{32}C_{31} = 0, \\ c_{13}C_{11} + c_{23}C_{21} + c_{33}C_{31} = 0. \end{cases}$$

If one regards the C_{ik} in them as unknowns then, since the determinant $|\, c_{ik}\,|$ is non-zero, one will have one and only one system of solutions, and from (5), it will be:

$$(7)\qquad C_{11} = c_{11}, \qquad C_{21} = c_{21}, \qquad C_{31} = c_{31};$$

in general, one will have:

$$C_{ik} = c_{ik}\,.$$

Theorem 1: *Any element of a proper orthogonal matrix is equal to its algebraic complement.*

One has the relations:

$$(8)\qquad \begin{cases} c_{11}C_{11} + c_{12}C_{12} + c_{13}C_{13} = 1, \\ c_{21}C_{11} + c_{22}C_{12} + c_{23}C_{13} = 0, \\ c_{31}C_{11} + c_{32}C_{12} + c_{33}C_{13} = 0, \end{cases}$$

corresponding to (6). If one replaces the C_{ik} in them with c_{ik}, using Theorem 1, then it will follow that the inner product of any row with itself must have the value 1, while the inner product of two different rows must possess the value 0.

104. Parametric representation of proper orthogonal transformations ([1]). Equations (5) impose six independent conditions on the nine quantities c_{ik}. Only three of the coefficients c_{ik} are essential then. One then attempts to represent the c_{ik} as functions of three of them. The coefficients c_{11}, c_{22}, c_{33} will serve as parameters, in their own right. In order to represent the remaining coefficients as functions of these parameters, we start from the equations [cf. (5)]:

$$\text{(9)}\qquad \begin{cases} -1 = -c_{11}^2 - c_{12}^2 - c_{13}^2, \\ +1 = +c_{12}^2 + c_{22}^2 + c_{32}^2, \\ +1 = +c_{13}^2 + c_{23}^2 + c_{33}^2, \end{cases}$$

from which, it will follow by addition that:

$$\text{(10)}\qquad c_{23}^2 + c_{32}^2 = 1 + c_{11}^2 - c_{22}^2 - c_{33}^2.$$

We then infer the equation:

$$\text{(11)}\qquad 2\, c_{23}\, c_{32} = -\, 2c_{11} + 2c_{22}\, c_{33}$$

from Theorem 1, and can now calculate the squares of the sums and differences of the quantities c_{23}, c_{32}:

$$\text{(12)}\qquad \begin{cases} (c_{23} + c_{32})^2 = (1 - c_{11})^2 - (c_{22} - c_{33})^2, \\ (c_{23} - c_{32})^2 = (1 + c_{11})^2 - (c_{22} + c_{33})^2. \end{cases}$$

This representation for c_{23}, c_{32} is irrational. In order to obtain a representation in rational form, we set:

$$\text{(13)}\qquad \begin{cases} 4m_0 = 1 + c_{11} + c_{22} + c_{33}, \\ 4m_1 = 1 + c_{11} + c_{22} - c_{33}, \\ 4m_2 = 1 - c_{11} + c_{22} - c_{33}, \\ 4m_3 = 1 - c_{11} - c_{22} + c_{33}. \end{cases}$$

We will then have:

$$\text{(14)}\qquad \begin{cases} 1 = m_0 + m_1 + m_2 + m_3, \\ c_{11} = m_0 + m_1 - m_2 - m_3, \\ c_{22} = m_0 - m_1 + m_2 - m_3, \\ c_{33} = m_0 - m_1 - m_2 + m_3, \end{cases}$$

and

$$\text{(15)}\qquad \begin{cases} c_{23} = 2\left\{\sqrt{m_2}\sqrt{m_3} + \sqrt{m_0}\sqrt{m_1}\right\}, \\ c_{32} = 2\left\{\sqrt{m_2}\sqrt{m_3} - \sqrt{m_0}\sqrt{m_1}\right\}. \end{cases}$$

([1]) Cf., **E. Study**, "Die Hauptsätze der Quaternionentheorie," Mitteilungen des nat. Vereins für Neuvorpommern und Rügen. 31 Jahrg. 1899.

We will obtain all remaining coefficients from (15) by cyclic permutation. Since the roots have a value that is fixed once and for all, we can set:

(16) $$\sqrt{m_0} = \alpha_0 , \quad \sqrt{m_1} = \alpha_1 , \quad \sqrt{m_2} = \alpha_2 , \quad \sqrt{m_3} = \alpha_3 ,$$

and then represent the orthogonal transformations in a rational form (viz., **Euler**'s formulas).

Thus, from (14), the relation:

(17) $$1 = \alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2$$

exists between the α_i . The α_i are therefore not independent of each other. However, we can satisfy equation (17) identically when we divide the α_i by the square root of the sum of their squares. To that end, we set:

(18) $$c_{ik} = a_{ik} : a_{00}$$

and

(19) $$a_{00} = \alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 .$$

The equations:

(20) $$\begin{cases} a_{00} = \alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 , \\ a_{11} = \alpha_0^2 + \alpha_1^2 - \alpha_2^2 - \alpha_3^2 , \\ a_{22} = \alpha_0^2 - \alpha_1^2 + \alpha_2^2 - \alpha_3^2 , \\ a_{33} = \alpha_0^2 - \alpha_1^2 - \alpha_2^2 + \alpha_3^2 , \end{cases}$$

$$\begin{array}{ll} a_{23} = 2 \{\alpha_2\, \alpha_3 + \alpha_0\, \alpha_1\}, & a_{31} = 2 \{\alpha_3\, \alpha_1 + \alpha_0\, \alpha_2\}, \\ a_{32} = 2 \{\alpha_2\, \alpha_3 - \alpha_0\, \alpha_1\}, & a_{13} = 2 \{\alpha_3\, \alpha_1 - \alpha_0\, \alpha_2\}, \end{array}$$
$$a_{12} = 2 \{\alpha_1\, \alpha_2 + \alpha_0\, \alpha_3\},$$
$$a_{21} = 2 \{\alpha_1\, \alpha_2 - \alpha_0\, \alpha_3\}$$

will then give the desired *parametric representation of the coefficients of a proper orthogonal transformation.*

Every quadruple of homogeneous quantities $\alpha_0 : \alpha_1 : \alpha_2 : \alpha_3$ then corresponds to the coefficient system of a well-defined proper orthogonal transformation. However, does every proper orthogonal transformation correspond conversely to a quadruple of numbers? That is in fact the case. One can calculate the α_i from the coefficients a_{ik} and find that:

(21)
$$\begin{array}{llll} \alpha_0 & : \alpha_1 & : \alpha_2 & : \alpha_3 \\ = a_{00} + a_{11} + a_{22} + a_{33} & : a_{23} - a_{32} & : a_{31} - a_{13} & : a_{12} - a_{21}, \\ = a_{23} - a_{32} & : a_{00} + a_{11} - a_{22} - a_{33} & : a_{12} + a_{21} & : a_{31} + a_{13}, \\ = a_{31} - a_{13} & : a_{12} + a_{21} & : a_{00} - a_{11} + a_{22} - a_{33} & : a_{23} + a_{32}, \\ = a_{12} - a_{21} & : a_{31} + a_{13} & : a_{23} + a_{32} & : a_{00} - a_{11} - a_{22} + a_{33} . \end{array}$$

It is possible that one of the proportions breaks down, since all terms vanish simultaneously. However, as one easily shows, not all four proportions can be simultaneously useless.

105. Composition of two proper orthogonal transformations. We compose two proper orthogonal transformations S_a, S_b into a third one S_c . How do the parameters γ_i of the product transformations depend upon the parameters α_i, β_i? If:

(22)
$$\begin{cases} a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3 = a_{00}x_i', & S_a \quad (i = 1, 2, 3) \\ b_{j1}x_1 + b_{j2}x_2 + b_{j3}x_3 = b_{00}x_j'', & S_b \quad (j = 1, 2, 3) \\ c_{k1}x_1 + c_{k2}x_2 + c_{k3}x_3 = c_{00}x_k''', & S_c \quad (k = 1, 2, 3) \end{cases}$$

are the three transformations, and the three determinants $|a_{ik}|$, $|b_{ik}|$, $|c_{ik}|$ are denoted by Δ_a, Δ_b, Δ_c, respectively, then one will have:

(23)
$$\Delta_a \, \Delta_b = \Delta_c \, .$$

Now, one has:

(24)
$$D_a = \frac{1}{a_{00}^3} \, |a_{ik}| = \frac{\Delta_a}{a_{00}^3} = 1, *, *$$

for the determinant of the proper orthogonal transformation that corresponds to S_a . Thus:

(25)
$$\Delta_a = a_{00}^3 , \qquad \Delta_b = b_{00}^3 , \qquad \Delta_c = c_{00}^3 .$$

For that reason, from (23), one will have:

(26)
$$a_{00}^3 \cdot b_{00}^3 = c_{00}^3 .$$

We can, and would like to, establish that:

(27)
$$a_{00} \cdot b_{00} = c_{00} \, .$$

The expressions for the remaining c_{ik} are also bilinear in the a_{ik} and b_{ik}:

(28)
$$a_{1k} \, b_{i1} + a_{2k} \, b_{i2} + a_{3k} \, b_{i3} = c_{ik} \, .$$

If one replaces the a_{ik} in (21) with these values then one will obtain bilinear functions of the a_{ik}, b_{ik} for the corresponding parameters γ_i :

$$(29)\qquad \left\{\begin{aligned}
&\rho_0\gamma_0 = \frac{c_{00}+c_{11}+c_{22}+c_3}{2}; && \rho_1\gamma_0 = \frac{c_{23}-c_{32}}{2};\\
&\rho_0\gamma_1 = \frac{c_{23}-c_{32}}{2}; && \rho_1\gamma_1 = \frac{c_{00}+c_{11}-c_{22}-c_3}{2};\\
&\rho_0\gamma_2 = \frac{c_{31}-c_{13}}{2}; && \rho_1\gamma_2 = \frac{c_{12}+c_{21}}{2};\\
&\rho_0\gamma_3 = \frac{c_{12}-c_{21}}{2}; && \rho_1\gamma_3 = \frac{c_{31}+c_{13}}{2};\\
&\rho_2\gamma_3 = \frac{c_{31}-c_{13}}{2}; && \rho_3\gamma_0 = \frac{c_{12}-c_{21}}{2};\\
&\rho_2\gamma_3 = \frac{c_{12}+c_{21}}{2}; && \rho_3\gamma_1 = \frac{c_{31}+c_{31}}{2};\\
&\rho_2\gamma_1 = \frac{c_{00}-c_{11}+c_{22}-c_3}{2}; && \rho_3\gamma_2 = \frac{c_{23}+c_{32}}{2};\\
&\rho_2\gamma_3 = \frac{c_{23}+c_{32}}{2}; && \rho_3\gamma_3 = \frac{c_{00}-c_{11}-c_{22}+c_3}{2};
\end{aligned}\right.$$

and thus, quadratic functions of the α_i and β_i, by means of (20). If one now introduces the abbreviations:

$$(30)\qquad \left\{\begin{aligned}
\alpha_0\beta_0-\alpha_1\beta_1-\alpha_2\beta_2-\alpha_3\beta_3 &= \Phi_0(\alpha,\beta),\\
\alpha_0\beta_1+\alpha_1\beta_0+\alpha_2\beta_3-\alpha_3\beta_2 &= \Phi_1(\alpha,\beta),\\
\alpha_0\beta_2+\alpha_2\beta_0+\alpha_2\beta_0-\alpha_1\beta_3 &= \Phi_2(\alpha,\beta),\\
\alpha_0\beta_3+\alpha_3\beta_0+\alpha_2\beta_0-\alpha_2\beta_1 &= \Phi_3(\alpha,\beta)
\end{aligned}\right.$$

then equations (29) can be written in the form:

$$(31)\qquad \rho_0\gamma_i = \Phi_0\,\Phi_i\,,\qquad \rho_1\gamma_i = \Phi_1\,\Phi_i\,,\qquad \rho_2\gamma_i = \Phi_2\,\Phi_i\,,\qquad \rho_3\gamma_i = \Phi_3\,\Phi_i\,.$$

Now, not all Φ_i can vanish here. If all $\Phi_i = 0$ then one would have that all γ_i and all c_{ik} would vanish, and the product transformation would not be a proper orthogonal transformation. Therefore, at least one Φ_i is non-zero. We can identify it with the corresponding ρ_i and thus obtain:

$$(32)\qquad \left\{\begin{aligned}
\alpha_0\beta_0-\alpha_1\beta_1-\alpha_2\beta_2-\alpha_3\beta_3 &= \gamma_0,\\
\alpha_0\beta_1+\alpha_1\beta_0+\alpha_2\beta_3-\alpha_3\beta_2 &= \gamma_1,\\
\alpha_0\beta_2+\alpha_2\beta_0+\alpha_2\beta_0-\alpha_1\beta_3 &= \gamma_2,\\
\alpha_0\beta_3+\alpha_3\beta_0+\alpha_2\beta_0-\alpha_2\beta_1 &= \gamma_3.
\end{aligned}\right.$$

Finally, we can state that the choice of proportionality factor that made corresponds to the convention (27).

Theorem 2: *The proper orthogonal transformations can be represented exhaustively as bilinear combinations with the help of a quadruple of homogeneous parameters* α_i.

§ 31. Quaternions.

106. Fundamental definitions. Hamilton's quaternions are quadruples of real numbers (α_0, α_1, α_2, α_3) with which one can calculate by using certain rules. Two quaternions are said to be *equal* when they have the same coordinates:

(1) $$(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (\beta_0, \beta_1, \beta_2, \beta_3), \qquad \text{when } \alpha_i = \beta_i .$$

The *addition of two quaternions* is defined by the formula:

(2) $$(\alpha_0, \alpha_1, \alpha_2, \alpha_3) + (\beta_0, \beta_1, \beta_2, \beta_3) = (\alpha_0 + \beta_0, \alpha_1 + \beta_1, \alpha_2 + \beta_2, \alpha_3 + \beta_3)$$

and obeys the same rules as the addition of real numbers. The quaternion (0, 0, 0, 0) will be called the *zero quaternion*.

We define the *multiplication of the quaternion* (α_0, α_1, α_2, α_3) by the *scalar* factor c by the equation:

(3) $$c \cdot (\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (c \cdot \alpha_0, c \cdot \alpha_1, c \cdot \alpha_2, c \cdot \alpha_3),$$

and finally, a third quaternion can be derived from two quaternions by the formula:

(4) $$(\alpha_0, \alpha_1, \alpha_2, \alpha_3) \cdot (\beta_0, \beta_1, \beta_2, \beta_3) = (\gamma_0, \gamma_1, \gamma_2, \gamma_3),$$

in which the γ_i are defined by equations (32) of the previous paragraph. We also write:

(5) $$\alpha \cdot \beta = \gamma$$

in place of (4), more briefly. *The multiplication that is thus defined is not commutative.* In general, one has:

(6) $$\alpha\beta \neq \beta\alpha .$$

We shall treat commuting quaternions later on.

By contrast, one does have the *distributivity laws:*

(7) $$(\alpha + \alpha') \beta = \alpha\beta + \alpha'\beta \qquad \text{and} \qquad \alpha (\beta + \beta') = \alpha\beta + \alpha\beta',$$

and the associativity law:

(8) $$(\alpha\beta) \gamma = \alpha (\beta\gamma) = \alpha\beta\gamma.$$

107. Units. In order to be able to summarize the multiplication formulas for two quaternions more briefly, we shall compose a quaternion from four *units:*

(9) $(1, 0, 0, 0) = e_0, \quad (0, 1, 0, 0) = e_1, \quad (0, 0, 1, 0) = e_2, \quad (0, 0, 0, 1) = e_3,$

such that:

(10) $$(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = \alpha_0\, e_0 + \alpha_1\, e_1 + \alpha_2\, e_3 + \alpha_3\, e_3\,.$$

The quaternions $\alpha_i\, e_i$ are called *components* of the quaternion α (as opposed to its coordinates). The multiplication formulas give the following values for the multiplication of the units:

(11) $$\begin{array}{c|cccc} & e_0 & e_1 & e_2 & e_3 \\ \hline e_0 & e_0 & e_1 & e_2 & e_3 \\ e_1 & e_1 & -e_0 & e_3 & -e_2 \\ e_2 & e_2 & -e_3 & -e_0 & e_1 \\ e_3 & e_3 & e_2 & -e_1 & -e_0 \end{array}$$

One infers, e.g., the relation $e_2 \cdot e_3 = e_1$ from this *multiplication table.* Moreover, it is clear from this table that the principal unit e_0 behaves like 1 under multiplication. No contradiction will then arise if we identify e_0 with 1 in what follows. The numbers $\alpha_0\, e_0$ will then be set equal to ordinary real numbers. Every quaternion can then be decomposed into the sum of its *scalar* and *vectorial components:*

(17) $$\alpha = S\alpha + V\alpha = \alpha_0 + (\alpha_1\, e_1 + \alpha_2\, e_2 + \alpha_3\, e_3).$$

108. The inverse quaternion. Similarly to what we do in the theory of ordinary complex numbers, we now define:

(13) $$\tilde{\alpha} = S\alpha - V\alpha = \alpha_0 - (\alpha_1\, e_1 + \alpha_2\, e_2 + \alpha_3\, e_3) \qquad \text{(“}\alpha\text{ circumflex”)}$$

to be the *conjugate quaternion* and refer to *norm* of the quaternion α when we are dealing with its product with its conjugate:

(14) $$Na = \alpha\tilde{\alpha} = \alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2\,.$$

In order be able to calculate the *norm of a product*, we must first look for the quaternion that is conjugate to the product $\alpha\beta$. The multiplication formulas yield:

(15) $$\widetilde{\alpha\beta} = \tilde{\beta}\,\tilde{\alpha}\,,$$

and thus [1]:

$$N(\alpha\beta) = \alpha\beta\cdot\widetilde{\alpha\beta} = \alpha N\beta\,\tilde{\alpha} = \alpha\tilde{\alpha}\cdot N\beta = N\alpha\cdot N\beta. \tag{16}$$

From now on, we shall assume that the norms of all quaternions that occur shall be non-zero. For such quaternions, one can always solve the equations:

$$\xi\alpha = 1, \qquad \alpha\eta = 1, \tag{17}$$

and one will find that:

$$\xi = \frac{\tilde{\alpha}}{N\alpha} = \eta, \tag{18}$$

as one can easily confirm.

The quaternion that thus belongs to a quaternion of non-vanishing norm will be referred to as the *quaternion that is inverse* or *reciprocal* to α, and denoted by α^{-1}.

109. Commuting quaternions. We now return to the question of commuting quaternions that we suggested above. We pose the somewhat more general question: Under what assumptions are the two products $\alpha\beta$ and $\beta\alpha$ linearly-dependent upon each other?

$$\alpha\beta = \rho\,\beta\alpha. \tag{19}$$

We take the norm of both sides:

$$N(\alpha\beta) = \rho^2 N(\beta\alpha); \quad \rho^2 = 1. \tag{20}$$

We must then distinguish between the cases $\rho = +1$ and $\rho = -1$, and thus speak of *proper* and *improper commuting quaternions*, accordingly.

1. $\rho = +1$. Should one have $\alpha\beta = \beta\alpha$ then one would need to have $V(\alpha\beta) = V(\beta\alpha)$ or:

$$\alpha_1 : \alpha_2 : \alpha_3 = \beta_1 : \beta_2 : \beta_3\,. \tag{21}$$

2. $\rho = -1$. Should one have $\alpha\beta = -\beta\alpha$ then the scalar part of the product would have to vanish:

$$\alpha_0\,\beta_0 - \alpha_1\beta_1 - \alpha_2\,\beta_2 - \alpha_3\,\beta_3 = 0. \tag{22}$$

In addition, it would follow that:

$$\alpha_0\,\beta_1 + \alpha_1\,\beta_0 = \alpha_0\,\beta_2 + \alpha_2\,\beta_0 = \alpha_0\,\beta_3 + \alpha_3\,\beta_0 = 0. \tag{23}$$

Since the rank of this system of equations (case 1 having been dealt with) is 3, it will follow that $\alpha_0 = \beta_0 = 0$, and the relationship (22) will assume the simplified form:

[1] The symbols *N*, *S*, *V* always refer to only the quaternion that follows *N*, *S*, *V* immediately. If the norm, the scalar part, or the vectorial part of a product is intended then the product will be placed in brackets.

(24) $$\alpha_1\beta_1 + \alpha_2\,\beta_2 + \alpha_3\,\beta_3 = 0\,.$$

We call two quaternions that satisfy equation (24) *orthogonal*.

Theorem 3: *Properly commuting quaternions are quaternions with proportional vectorial parts. Improperly commuting quaternions have orthogonal vectorial parts.*

§ 32. Rotations.

110. The axis of a rotation. Let x_1, x_2, x_3 be inhomogeneous coordinates in space, and let:

(1) $$x_i' = \sum_{k=1}^{3} c_{ik}\,x_k \qquad (i = 1, 2, 3)$$

be a proper orthogonal transformation. We assert that we are dealing with a *rotation* around an axis that runs through the origin. In fact, in order to find a fixed point, we set $x_i' = x_i$ in (1) and obtain the system of equations:

(2) $$\left\{\begin{array}{rrrl} (c_{11}-1)x_1 + & c_{12}x_2 + & c_{13}x_3 = & 0, \\ c_{21}x_1 + & (c_{22}-1)x_2 + & c_{23}x_3 = & 0, \\ c_{31}x_1 + & c_{32}x_2 + & (c_{33}-1)x_3 = & 0. \end{array}\right.$$

The determinant of this system of equations vanishes. If one develops it in powers of – 1 then it will follow, with consideration given to Theorem 1, that:

(3) $$\begin{aligned} \mathfrak{D} &= D \cdot (-1)^0 + (C_{11} + C_{22} + C_{33}) \cdot (-1)^1 + (c_{11} + c_{22} + c_{33}) \cdot (-1)^2 + (-1)^3 \\ &= 1 - (c_{11} + c_{22} + c_{33}) + (c_{11} + c_{22} + c_{33}) - 1 = 0. \end{aligned}$$

The system will then have, in fact, ∞^1 solutions, in general, which corresponds to a line that is fixed point-wise.

Theorem 4: *A proper orthogonal transformation that is different from the identity is a rotation around a well-defined axis.*

If denote the imaginary point of the fixed line (viz., the ratio of the direction cosines) by $r_1 : r_2 : r_3$ then we will have [depending upon which row of (2) one uses to calculate with]:

(4) $$\left\{\begin{array}{rlll} r_1 : r_2 : r_3 = & 1 + c_{11} - c_{22} - c_{33} : & c_{12} + c_{21} : & c_{31} + c_{13} : \\ = & c_{12} + c_{21} : & 1 + c_{11} + c_{22} - c_{33} : & c_{23} + c_{32} : \\ = & c_{31} + c_{13} : & c_{23} + c_{32} : & 1 - c_{11} - c_{22} + c_{33} : \end{array}\right.$$

If one thinks of the parameter α_i as being introduced in place of the c_{ik}, as in (20), § 30, then it will follow that:

(3) $$r_1 : r_2 : r_3 = \alpha_1 : \alpha_2 : \alpha_3 .$$

111. Commuting rotations. The composition of two rotations corresponds to the composition of the corresponding quaternions. Commuting rotations then correspond to commuting quaternions. The relation (5), together with Theorem 3, allows us to give all types of commuting rotations directly.

Properly-commuting rotations are then rotations around the same axis. *Improperly-commuting rotations* have mutually-orthogonal axes. However, it follows further from $\alpha_0 = \beta_0 = 0$ that $a_{ik} = a_{ki}$ and $b_{ik} = b_{ki}$. These rotations will then be (cf., Theorem 1) identical with the corresponding inverse rotations, so they will be involutory. We call involutory rotations (i.e., rotations through the angle π) *reversals.* With that, we have the result:

Theorem 5: *There are two types of commuting rotations: Properly-commuting rotations are rotations around one and the same axis. Improperly-commuting rotations are reversals around orthogonal axes.*

We remark that the product of two commuting reversals possesses the parameters:

(6) $$\begin{cases} \gamma_0 = 0, \\ \gamma_1 = \alpha_2\beta_3 - \alpha_3\beta_2,\ \gamma_2 = \alpha_3\beta_1 - \alpha_1\beta_3,\ \gamma_3 = \alpha_1\beta_2 - \alpha_2\beta_1. \end{cases}$$

From that, one sees that one is again dealing with a reversal, and indeed one around an axis that is orthogonal to the first two. The product of three reversals is then the identity. (That is, one is dealing with the four-group)

112. Representation of rotations by quaternion formulas. In order to write down, not only the composition of parameters, but the rotation formulas themselves, with the help of quaternions, we start with the point x as a vectorial quaternion:

$$x = x_1\, e_1 + x_2\, e_2 + x_3\, e_3 .$$

The expression:

(7) $$\alpha^{-1}\, x\, \alpha = x' \qquad (N\alpha \neq 0)$$

is a proper orthogonal transformation. In fact, it follows that:

(8) $$Nx' = N\alpha^{-1} Nx\, N\alpha = Nx.$$

The transformation thus-represented is then orthogonal. The facts that it is proper orthogonal and that the group of proper orthogonal transformations is exhausted in that way follow from the fact that (7) merely summarizes the explicit formulas that we used previously.

The new representation yields an especially simple process for the composition of two rotations α and β. It follows from:

(9) $$\alpha^{-1} x \alpha = x', \qquad \beta^{-1} x' \beta = x''$$

that

(10) $$\beta^{-1} \alpha^{-1} \, x \, \alpha \, \beta = x''.$$

If we write $\alpha\beta = \gamma$ then, from no. **108**, we can write:

(11) $$\gamma^{-1} x \gamma = x'',$$

in place of (10).

113. The rotation angle. In particular, every rotation can be written as the product of two reversals in ∞^1 ways. In fact, the presence of the rotational axis will imply the existence of a fixed plane through the origin that is perpendicular to the rotational axis. The plane will be rotated into itself, and indeed the motion in space will be determined completely by that rotation. Now, the rotation in space can be decomposed into the product of two reflections in ∞^1 ways. Spatially, such a reflection corresponds to a reversal around the axis of reflection.

Theorem 6: *A rotation can be decomposed into the product of two reversals whose axes are perpendicular to the rotational axis in* ∞^1 *ways.*

Any of these axes can be chosen arbitrarily. The other one will then be well-defined: If ϑ is the angle between the two reversal axes then the angle of rotation will be 2ϑ.

Now, if α and β are two reversals – so $\alpha_0 = \beta_0 = 0$ – then $\alpha\beta = \gamma$ will be the associated rotation, and the quaternion γ will take on the coordinates:

(12) $$\begin{cases} \gamma_0 = -\alpha_1\beta_1 - \alpha_2\beta_2 - \alpha_3\beta_3, \\ \gamma_1 = \quad \alpha_2\beta_3 - \alpha_3\beta_2, \\ \gamma_2 = \quad \alpha_3\beta_1 - \alpha_3\beta_1, \\ \gamma_3 = \quad \alpha_1\beta_2 - \alpha_2\beta_1. \end{cases}$$

This argument allows us to give an expression for the rotation angle 2ϑ of the rotation ϑ. The angle ϑ between the reversal axes will be determined by [(15), no. **72**]:

(13) $$\cos \vartheta = \frac{\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3}{\sqrt{N\alpha}\sqrt{N\beta}}.$$

If we establish the dependency:

(14) $$\sqrt{N\alpha}\sqrt{N\beta} = \sqrt{N\gamma}$$

between the roots of the norms then we can also write:

(15) $$\cos\vartheta = \frac{-\gamma_0}{\sqrt{N\gamma}}$$

instead of (13). We then further obtain:

(16) $$\sin\vartheta = \sqrt{1-\cos^2\vartheta} = \frac{\sqrt{N\gamma-\gamma_0^2}}{\sqrt{N\gamma}}$$

for ϑ. Finally, dividing (15) by (16) will give:

(17) $$\cot\vartheta = \frac{-\gamma_0}{\sqrt{N\gamma-\gamma_0^2}}$$

for the *half-angle* ϑ of rotation. With that, – in conjunction with (5) – the ratios $\gamma_0 : \gamma_1 : \gamma_2 : \gamma_3$ can be interpreted geometrically as the parameters of a rotation.

§ 33. Motions and transfers.

114. Parametric representation of motions in point coordinates. We combine two quaternions α and β into the dual combination of a *biquaternion:*

(1) $$A = \alpha + \varepsilon\beta.$$

The rules for the addition and multiplication of biquaternions are obtained from this. We define:

(2) $$\tilde{A} = \tilde{\alpha} + \varepsilon\tilde{\beta}$$

to be the biquaternion that is conjugate to A. We define the norm of the biquaternion A to be:

(3) $$NA = A\tilde{A} = (\alpha + \varepsilon\beta)(\tilde{\alpha} + \varepsilon\tilde{\beta}) = \alpha\tilde{\alpha} + \varepsilon(\alpha\tilde{\beta} + \beta\tilde{\alpha}).$$

Under the assumption that $N\alpha \neq 0$ (cf., no. **108**), we find that the reciprocal of the biquaternion $\alpha + \varepsilon\beta$ is

(4) $$A^{-1} = (\alpha + \varepsilon\beta)^{-1} = \frac{1}{NA}(\tilde{\alpha} + \varepsilon\tilde{\beta}).$$

If we now apply the conversion principle in no. **94** to the rotation formula $\alpha^{-1}x\alpha = x'$ then we will obtain the equation:

$$A^{-1}\,X\,A = X'. \tag{5}$$

From the remarks that were made at the beginning of this chapter, we can surmise that this equation represents the **Euclidian** motions in ray coordinates. Starting with that parametric representation, it must then be possible to arrive at a parametric representation of the **Euclidian** motions in point coordinates with the help of the eight homogeneous parameters α_i, β_i. Since the calculations that are necessary for that prove to be laborious, it is preferable to start with the parametric representation in point coordinates and then once more arrive at equation (9).

We deviate from the convention in the previous paragraph and set:

$$x = x_0 + x_1\,e_1 + x_2\,e_2 + x_3\,e_3\,. \tag{6}$$

In this, x_0 shall be the *mass* of the point x; we exclude points of mass 0. Let $x_0 : x_1 : x_2 : x_3$ be the homogeneous coordinates of the point x. In what follows, they themselves shall have a geometric interpretation, and not just their ratios, in order to completely exhaust the contents of the formulas that will be derived.

It now follows from:

$$\alpha^{-1}x\alpha = x' \tag{7}$$

that:

$$Sx = Sx'; \qquad \alpha^{-1}Vx\alpha = Vx' \qquad [N\alpha \neq 0]. \tag{8}$$

Hence, the mass of a point will be preserved by the transformation. The point itself will experience a rotation around the origin.

We will now obtain a general motion when we compose this rotation with a translation:

$$Sx = Sx'; \qquad \alpha^{-1}Vx\alpha + x_0\,\xi = Vx'. \tag{9}$$

In this, ξ shall denote a vectorial quaternion. If ξ and α are given then we can always determine β in such a way that we will have:

$$\xi = -\,2\alpha^{-1}\beta \qquad [\beta = -\tfrac{1}{2}\,\alpha\,\xi]. \tag{10}$$

Since ξ is vectorial, the relationship:

$$(\alpha\beta) = \alpha_0\,\beta_0 + \alpha_1\,\beta_1 + \alpha_2\,\beta_2 + \alpha_3\,\beta_3 = 0 \tag{11}$$

must exist in this. We shall call a biquaternion whose coordinates satisfy equation (11) a *bound biquaternion*. The norm of a bound biquaternion $\alpha + \varepsilon\beta$ will be $N\alpha$. By introducing β and considering (11), formulas (9) will assume the following form:

$$Sx = Sx', \qquad \alpha^{-1}Vx\alpha - 2x_0\,\frac{\tilde{\alpha}\beta}{N\alpha} = Vx' \qquad [N\alpha \neq 0]. \tag{12}$$

In order to combine these two formulas into a single equation, we remark that the first one can also be written as:

(13) $$\alpha^{-1} Sx\alpha = Sx'.$$

However, one further has:

(14) $$\tilde{\alpha}\beta + \tilde{\beta}\alpha = \tilde{\alpha}\beta + \widetilde{\tilde{\alpha}\beta} = 2S(\tilde{\alpha}\beta) = 2(\alpha\beta) = 0,$$

and for that reason:

(15) $$2\tilde{\alpha}\beta = \tilde{\alpha}\beta - \tilde{\beta}\alpha .$$

Due to (15), the second equation (12) can also be written in the form:

(16) $$\alpha^{-1} Vx\alpha - \{\tilde{\alpha}\beta - \tilde{\beta}\alpha\}\frac{x_0}{N\alpha} = Vx'.$$

Finally, by composing (13) and (16), we will obtain the new equation:

(17) $$(\alpha + \varepsilon\beta)^{-1}(Sx + \varepsilon Vx)(\alpha - \varepsilon\beta) = Sx' + \varepsilon Vx' \qquad [N\alpha \neq 0].$$

In fact, expanding the left-hand side will give:

(18) $$\begin{aligned}&\frac{1}{N\alpha}(\alpha + \varepsilon\beta)^{-1}(Sx + \varepsilon Vx)(\alpha - \varepsilon\beta)\\ &= \frac{1}{N\alpha}\{\tilde{\alpha}\,Sx\,\alpha\} + \frac{\varepsilon}{N\alpha}\{\tilde{\alpha}Vx\,\alpha - \tilde{\alpha}\,Sx\,\beta + \tilde{\beta}\,Sx\,\alpha\}\\ &= \alpha^{-1}Sx\alpha + \varepsilon\{\alpha^{-1}Vx\alpha - \frac{x_0}{N\alpha}(\tilde{\alpha}\beta - \tilde{\beta}\alpha)\}\end{aligned}$$ Q. E. D.

Theorem 7: *Equation* (17) *gives an exhaustive representation of all Euclidian motions in terms of eight homogeneous parameters* α_i , β_i *that are coupled by the equations* $(\alpha\beta) = 0$ *and the inequality* $N\alpha \neq 0$.

For:

(19) $$1 : 0 : 0 : 0 : 0 : 0 : 0 : 0,$$

one will get the *identity motion.*

115. Composition of two motions. Let α, β be the parameters of the first motion, and let α', β' be the parameters of the second one. The equation of the product of the two motions will then read:

(20) $$(\alpha' + \varepsilon\beta')^{-1}(\alpha + \varepsilon\beta)^{-1}(Sx + \varepsilon Vx)(\alpha - \varepsilon\beta)(\alpha' - \varepsilon\beta') = Sx' + \varepsilon Vx'.$$

Now, should one have:

(21) $$(\alpha - \varepsilon\beta)\,(\alpha' - \varepsilon\beta') = \alpha'' - \varepsilon\beta''$$
then one would have to set:
(22) $$\alpha\alpha' = \alpha''; \qquad \alpha\beta' + \beta\alpha' = \beta''.$$
One would also have:
(23) $$\begin{aligned}(\alpha' + \varepsilon\,\beta')^{-1}(\alpha + \varepsilon\,\beta)^{-1} &= [(\alpha + \varepsilon\,\beta)(\alpha' + \varepsilon\,\beta')]^{-1} \\ &= [\alpha\alpha' + \varepsilon\,(\alpha\beta' + \beta\alpha')]^{-1} = (\alpha'' + \varepsilon\,\beta'')^{-1}\end{aligned}$$

then, and one would finally have ([1]):

(24) $$\begin{aligned}(\alpha''\beta'') &= S\,\{\alpha''\tilde{\beta}''\} = S\,\left\{\alpha\alpha'\,[\widetilde{\alpha\beta' + \beta\alpha'}]\right\} \qquad (22) \\ &= S\,\left\{\alpha\alpha'\tilde{\beta}'\tilde{\alpha} + \alpha\alpha'\tilde{\alpha}'\tilde{\beta}\right\} \\ &= S\,\{\alpha'\tilde{\beta}'\}\cdot N\alpha + S\,\{\alpha\tilde{\beta}\}\cdot N\alpha' \\ &= N\alpha\cdot(\alpha'\beta') + N\alpha'\cdot(\alpha\beta) = 0.\end{aligned}$$

With that, we have shown that the parameters of two motions are composed according to the formulas (21).

Theorem 8: *The composition of two motions corresponds to the composition of the corresponding (bound) biquaternions.*

116. Motions in rod and spar coordinates. Since we have concerned ourselves with mass points up to now, in the representation of motions in line space, we will employ, not ordinary **Plücker** coordinates, but *rod coordinates*. If one sets $\mathfrak{X} = \widehat{xy}$ and displaces the points x and y along the line $\mathfrak{X}$ then the **Plücker** coordinates will be multiplied by a factor. However, if one keeps the distance between the two points fixed during the displacement, for which one can introduce a kind of weight, then the ratios of the line coordinates will no longer have the only geometric interpretation, but also the line coordinates themselves. In that context, we shall call them *rod coordinates*, and denote them by p_1, p_2, p_3; q_1, q_2, q_3, such that:

(25) $$p_1\,q_1 + p_2\,q_2 + p_3\,q_3 = 0.$$

Finally, we combine each of the three rod coordinates by the equations:

(26) $$\begin{cases} p = p_1e_1 + p_2e_2 + p_3e_3 = Sx\cdot Vy - Sy\cdot Vx, \\ q = q_1e_1 + q_2e_2 + q_3e_3 = V(Vx\cdot Vy). \end{cases}$$

Now, in order to find the transformation equations for the p and q, we form the product of the equations that follow from (17):

([1]) $$\begin{aligned}2S\{\alpha\gamma\tilde{\alpha}\} &= \alpha\gamma\tilde{\alpha} + \alpha\tilde{\gamma}\tilde{\alpha} = \alpha(\gamma + \tilde{\gamma})\tilde{\alpha} \\ &= \alpha\cdot 2S\gamma\cdot\,\tilde{\alpha} = 2S\gamma N\alpha.\end{aligned}$$

$$(27)\qquad \begin{cases} (\alpha+\varepsilon\beta)^{-1}(Sx+\varepsilon Vx)(\alpha-\varepsilon\beta)=Sx'+\varepsilon Vx', \\ (\tilde{\alpha}-\varepsilon\tilde{\beta})^{-1}(Sx-\varepsilon Vx)(\tilde{\alpha}+\varepsilon\tilde{\beta})=Sy'-\varepsilon Vy'. \end{cases}$$

That yields:

$$(28)\qquad \begin{aligned} &(Sx' + \varepsilon\, Vx')(Sy' - \varepsilon\, Sy') = Sx'\, Sy' + \varepsilon\{Sy'\, Vx' - Sx'\, Vy'\} \\ &= (\alpha+\varepsilon\beta)^{-1}(Sx + \varepsilon\, Vx)\, (\alpha-\varepsilon\beta)\, (\tilde{\alpha}-\varepsilon\tilde{\beta})\, (Sy - \varepsilon\, Vy)\, (\tilde{\alpha}+\varepsilon\tilde{\beta})^{-1} \\ &= (\alpha+\varepsilon\beta)^{-1}(Sx + \varepsilon\, Vx)\, (Sy - \varepsilon\, Vy)\, (\tilde{\alpha}+\varepsilon\tilde{\beta})^{-1}\cdot N\,(\alpha-\varepsilon\beta) \\ &= (\alpha+\varepsilon\beta)^{-1}(Sx\, Sy + \varepsilon\,\{Sy\, Vx - Sx\, Vy\})(\alpha+\varepsilon\beta), \end{aligned}$$

and from this:

$$(29)\qquad p' = (\alpha+\varepsilon\beta)^{-1}\cdot p\cdot(\alpha+\varepsilon\beta).$$

In order to find the transformation equations for q, we start from the equations (12) and (14). From that:

$$(30)\qquad \begin{cases} Vx' = \alpha^{-1}Vx\alpha + 2\dfrac{\tilde{\beta}\alpha}{N\alpha}\cdot Sx, \\ Vy' = \alpha^{-1}Vy\alpha - 2\dfrac{\tilde{\alpha}\beta}{N\alpha}\cdot Sy, \end{cases}$$

and therefore:

$$(31)\qquad \begin{aligned} &Vx'\cdot Vy' \\ &= \alpha^{-1}Vx\, Vy\;\alpha + \frac{2}{N\alpha}\tilde{\beta}\cdot Vy\cdot\alpha\cdot Sx - \frac{2}{N\alpha}\tilde{\alpha}\cdot Vx\cdot\beta\cdot Sy - 4\frac{N\beta}{N\alpha}Sx\cdot Sy, \end{aligned}$$

$$(32)\qquad \begin{aligned} &V(Vx'\cdot Vy') \\ &= V(\alpha^{-1}Vx\, Vy\;\alpha) + \frac{1}{N\alpha}\left\{\tilde{\beta}\cdot Vy\cdot\alpha - \tilde{\alpha}\widetilde{Vy}\beta\right\}Sx - \frac{1}{N\alpha}\left\{\tilde{\alpha}\cdot Vy\cdot\beta - \tilde{\beta}\widetilde{Vx}\alpha\right\}Sy, \\ &= \alpha^{-1}V(Vx\, Vy)\alpha + \frac{1}{N\alpha}\left\{\tilde{\beta}\cdot Vy\cdot\alpha + \tilde{\alpha}Vy\beta\right\}Sx - \frac{1}{N\alpha}\left\{\tilde{\alpha}\cdot Vy\cdot\beta + \tilde{\beta}Vx\alpha\right\}Sy, \end{aligned}$$

$$(33)\qquad q' = \alpha^{-1}q\alpha + \frac{1}{N\alpha}\{\tilde{\alpha}p\beta + \tilde{\beta}p\alpha\}.$$

Equations (29) and (33) can be combined into the equation:

$$(34)\qquad (\alpha+\varepsilon\beta)^{-1}(p + \varepsilon\, q)(\alpha+\varepsilon\beta) = p' + \varepsilon\, q',$$

which we started with at the beginning of the chapter. [Cf., (5), no. **114**].

Finally, we ask about a *parametric representation of motions in plane coordinates.* However, here, as well, we shall not stay in the plane coordinates u_i, but assign a meaning to the coordinates themselves. That will come about when we endow the plane u with a weight $\sqrt{u_1^2+u_2^2+u_3^2}$ that can be interpreted as the area of an oriented parallelogram that lies in the plane. We then speak of a *spar*. In spar coordinates, the equations of motion will read:

$$(35)\qquad (\alpha + \varepsilon\beta)^{-1}(Vu + \varepsilon\, Su)(\alpha - \varepsilon\beta) = Vu' + \varepsilon\, Su'.$$

In fact: We will verify that (35) represents the transformation that is contragredient to (17) when we confirm the existence of the identity $(u'\,x') = (u\,x)$. One will have:

$$(36)\qquad \begin{aligned}(Sx' + \varepsilon\, Vx')(Vu' - \varepsilon\, Su) &= Sx'\, Vu' - \varepsilon\,\{Sx'\, Su' - Vx'\, Vu'\}\\ &= (\alpha + \varepsilon\beta)^{-1}(Sx + \varepsilon\, Vx)(Vu - \varepsilon\, Su)(\alpha + \varepsilon\beta)\\ &= (\alpha + \varepsilon\beta)^{-1}(Sx\, Vu - \varepsilon\,\{Sx\, Su - VxVu\})\,(\alpha + \varepsilon\beta)\,,\end{aligned}$$

and therefore (cf., rem., pp. ?):

$$(37)\qquad \begin{aligned}(u'\,x') &= S\,\{Sx'\, Su' - Vx'\, Vu'\}\\ &= S\,\{Sx\, Su - Vx\, Vu\} = (u\,x). \qquad\qquad \text{Q. E. D.}\end{aligned}$$

We summarize the three parametric representations that we just obtained as:

$$(38)\qquad \begin{cases}(\alpha+\varepsilon\beta)^{-1}(Sx+\varepsilon Vx)(\alpha-\varepsilon\beta) = Sx'+\varepsilon Vx',\\ (\alpha+\varepsilon\beta)^{-1}(p+\varepsilon q)\quad(\alpha+\varepsilon\beta) = p'+\varepsilon q',\\ (\alpha+\varepsilon\beta)^{-1}(Vu+\varepsilon Su)(\alpha-\varepsilon\beta) = Vu'+\varepsilon Su'.\end{cases}$$

117. Parametric representation of transfers. Parametric representations for the transfers will follow from these formulas when we compose the transformations (38) with an arbitrary fixed transfer – e.g., the *reflection in the origin:*

$$(39)\qquad Vx' = -\,Vx; \qquad p' = -\,p; \qquad Su' = -\,Su.$$

If we then replace Vx', p', Su' with $-\,Vx'$, $-\,p'$, $-\,Su'$ on the right-hand sides of (38) then we will obtain a parametric representation of the transfers. We reorganize them by replacing ε with $-\,\varepsilon$ everywhere and denoting the parameters of a transfer by γ, δ:

$$(40)\qquad \begin{cases}(\gamma-\varepsilon\delta)^{-1}(Sx-\varepsilon Vx)(\gamma+\varepsilon\delta) = Sx'+\varepsilon Vx',\\ -(\gamma-\varepsilon\delta)^{-1}(p-\varepsilon q)\quad(\gamma+\varepsilon\delta) = p'+\varepsilon q',\\ (\gamma-\varepsilon\delta)^{-1}(Vu-\varepsilon Su)(\gamma+\varepsilon\delta) = Vu'+\varepsilon Su'.\end{cases}$$

By what rules do we compose the motions (38) *with the transfers* (40) *now?* If we perform the transfer γ', δ' on the motion α, β then we will get:

$$(41)\qquad \begin{aligned}&(\gamma' - \varepsilon\delta')^{-1}(\alpha - \varepsilon\beta)^{-1}\,(Sx - \varepsilon\, Vx)(\alpha + \varepsilon\beta)(\gamma' + \varepsilon\delta')\\ &\qquad = Sx'' + \varepsilon\, Vx'' = (\gamma'' - \varepsilon\delta'')^{-1}(Sx - \varepsilon\, Vx)(\gamma'' + \varepsilon\delta''),\end{aligned}$$

in which we have set:

(42) $$\gamma'' + \varepsilon\delta'' = (\alpha + \varepsilon\beta)(\gamma' + \varepsilon\delta') .$$

One obtains the remaining composition formulas in the same way:

(43) $$\begin{cases} (\alpha+\varepsilon\beta)(\alpha'+\varepsilon\beta') = \alpha''+\varepsilon\beta'', \\ (\alpha+\varepsilon\beta)(\gamma'+\varepsilon\delta') = \gamma''+\varepsilon\delta'', \\ (\gamma+\varepsilon\delta)(\alpha'-\varepsilon\beta') = \gamma''+\varepsilon\delta'', \\ (\gamma+\varepsilon\delta)(\gamma'-\varepsilon\delta') = \alpha''+\varepsilon\beta''. \end{cases}$$

With that, we have also shown analytically that the motions and transfers define a (laminated) group.

If we are dealing with the problem of *finding the inverse to just a motion* α, β *or a transfer* γ, δ then, as formulas (43) show, we must replace α, β with $\tilde{\alpha}$, $\tilde{\beta}$ and γ, δ with $\tilde{\gamma}$, $\tilde{\delta}$; we will then have [(14), (19)]:

(44) $$\begin{cases} (\alpha+\varepsilon\beta)(\tilde{\alpha}+\varepsilon\tilde{\beta}) = \alpha\tilde{\alpha}+\varepsilon\{\alpha\tilde{\beta}+\beta\tilde{\alpha}\} = N\alpha, \\ (\gamma+\varepsilon\delta)(\tilde{\gamma}+\varepsilon\tilde{\delta}) = \gamma\tilde{\gamma}+\varepsilon\{\gamma\tilde{\delta}+\delta\tilde{\gamma}\} = N\gamma. \end{cases}$$

118. Involutory motions and transfers. We can now easily determine the involutory motions and transfers. From what we just said, we get the proportions:

(45) $$\begin{array}{c} \alpha_0 : \alpha_1 : \alpha_2 : \alpha_3 : \beta_0 : \beta_1 : \beta_2 : \beta_3 \\ \alpha_0 : -\alpha_1 : -\alpha_2 : -\alpha_3 : \beta_0 : -\beta_1 : -\beta_2 : -\beta_3 \end{array}$$

for *involutory motions*. In this, there are two possibilities to distinguish:

$$\underline{1.\ \alpha_1 = \alpha_2 = \alpha_3 = \beta_1 = \beta_2 = \beta_3 = 0.}$$

It also follows from $(\alpha\beta) = 0$ that $\alpha_0\,\beta_0 = 0$, and since $N\alpha \neq 0$, that will also imply that $\beta_0 = 0$. We then have the *identity*, which we do not count among the involutory transformations.

$$\underline{2.\ \alpha_0 = \beta_0 = 0, \text{ and since } (\alpha\beta) = 0, \text{ also } \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 = 0.}$$

Since we already know one type of involutory motion – namely, the reversal (no. **111**) – and our derivation shows that only one type exists, we have the conditions for the motion α, β to be a *reversal*. In order to find the reversal axis, we remark that the given condition equation has the form of the **Plücker** identity. For that reason, we surmise that the reversal axis possesses the **Plücker** coordinates:

(46) $$\alpha_0 : \alpha_1 : \alpha_2 : \beta_1 : \beta_2 : \beta_3 .$$

In order to prove that, we seek the fixed points of the motion. The first of equations (38) yields:

(47) $$\begin{cases} (V\alpha + \varepsilon V\beta)^{-1}(Sx + \varepsilon Vx)(V\alpha - \varepsilon V\beta) = Sx + \varepsilon Vx, \\ \qquad (Sx + \varepsilon Vx)(V\alpha - \varepsilon V\beta) = (V\alpha + \varepsilon V\beta)(Sx + \varepsilon Vx), \\ \qquad VxV\alpha - V\alpha Vx - 2SxV\beta = 0, \\ (x_2\alpha_3 - x_3\alpha_2)\,e_1 + (x_3\alpha_1 - x_1\alpha_3)\,e_2 + (x_1\alpha_2 - x_2\alpha_1)\,e_3 - x_0(\beta_1 e_1 + \beta_2 e_2 + \beta_3 e_3) = 0. \end{cases}$$

However, from (14), § 1, that is the condition for the point x to lie on the line (46). The points of that line are therefore fixed points. Q. E. D.

Theorem 9: *There is only one kind of involutory motion, namely, the reversal. The motion α, β is a reversal when:*

$$\alpha_0 = \beta_0 = 0 \qquad \textit{and} \qquad \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 = 0.$$

Its axis then has the Plückerian coordinates:

$$\alpha_1 : \alpha_2 : \alpha_3 : \beta_1 : \beta_2 : \beta_3 .$$

For *involutory transfers*, one will have the proportion:

(48) $$\begin{aligned} &\gamma_0 : \gamma_1 : \gamma_2 : \gamma_3 : \delta_0 : \delta_1 : \delta_2 : \delta_3 \\ &= \gamma_0 : -\gamma_1 : -\gamma_2 : -\gamma_3 : -\delta_0 : \delta_1 : \delta_2 : \delta_3 . \end{aligned}$$

There are two cases to distinguish here, as well:

$$\underline{1.\ \gamma_1 = \gamma_2 = \gamma_3 = \delta_0 = 0.}$$

We are dealing with a *reflection through the point:*

(49) $$\gamma_0 : \delta_1 : \delta_2 : \delta_3 .$$

Namely, in reference to the first of equations (40), if we ask what the fixed point of the transfer is then it will follow that:

$$(50)\quad \begin{cases} (\gamma_0 - \varepsilon V\delta)^{-1}(Sx - \varepsilon Vx)(\gamma_0 + \varepsilon V\delta) = Sx + \varepsilon Vx, \\ (Sx + \varepsilon Vx)(\gamma_0 + \varepsilon V\delta) = (\gamma_0 - \varepsilon V\delta)(Sx + \varepsilon Vx), \\ Sx\, V\delta + V\delta\, Sx - Vx\,\gamma_0 - \gamma_0\, Vx = 0, \\ Sx\, V\delta = \gamma_0\, Vx, \\ \dfrac{Vx}{Sx} = \dfrac{V\delta}{\gamma_0}. \qquad \text{Q.E.D.} \end{cases}$$

$\underline{2.\ \gamma_0 = \delta_1 = \delta_2 = \delta_3 = 0.}$

We are dealing with a *reflection in the plane:*

$$(51)\qquad \delta_0 : \gamma_1 : \gamma_2 : \gamma_3 ;$$

the first equation in (40) will then yield:

$$(52)\quad \begin{cases} (Sx - \varepsilon Vx)(V\gamma + \varepsilon\delta_0) = (V\gamma - \varepsilon\delta_0)(Sx - \varepsilon Vx), \\ -Vx\,V\gamma + \delta_0\, Sx - V\gamma Vx + \delta_0\, Sx = 0, \\ +2(x_1\gamma_1 + x_2\gamma_2 + x_3\gamma_3) + 2\delta_0 x_0 = 0, \\ \delta_0 x_0 + x_1\gamma_1 + x_2\gamma_2 + x_3\gamma_3 = 0. \end{cases}$$

for the fixed point now. Every point of this plane will then be fixed individually.Q. E. D.

Theorem 10: *There are two types of involutory transfers: Reflections through points and reflection through planes. The reflection through the point* $\gamma_0 : \delta_1 : \delta_2 : \delta_3$ *will have the coordinates:*

$$\gamma_0 : 0 : 0 : 0 : 0 : \delta_1 : \delta_2 : \delta_3 ,$$

while the reflection in the plane $\delta_0 : \gamma_1 : \gamma_2 : \gamma_3$ *will have the coordinates:*

$$0 : \gamma_1 : \gamma_2 : \gamma_3 : \delta_0 :\ 0 : 0 : 0.$$

119. Geometric interpretation of the parameters of motion. From Theorem 9, the parameters of motion can be interpreted in one special case in such a way that one can characterize the motion completely with the help of that interpretation. That shall now be done in general.

We first ask what the *proper fixed line of the motion* $\alpha + \varepsilon\beta$ would be. Obviously, every motion will possess *one* proper fixed line, which will be either the line with the ray coordinates:

$$(53)\qquad (\alpha + \varepsilon\beta)^{-1}\,(V\alpha + \varepsilon V\beta)\,(\alpha + \varepsilon\beta) = (V\alpha + \varepsilon V\beta) \qquad \text{[cf., } (38)_2\text{]},$$

or (when that line is undetermined) the proper line of the bundle of parallels with the vertex $0 : \beta_1 : \beta_2 : \beta_3$. As we know, the latter case will occur when $V\alpha$ vanishes, and thus when we are dealing with a translation.

If there is no translation present then there will be two cases to distinguish: Either there is a fixed point on the fixed line, and one is dealing with a rotation around that point, or there is no fixed point on the fixed axis. The motion will then be the product of a rotation around the axis and a translation along that axis, namely, a *screw*. We would also like to count the rotations among the screwing motions.

Theorem 11: *Every motion that is different from the identity that is not a translation is a screwing motion around a well-defined proper axis.*

Now, such a screwing motion is characterized by two things in addition to its axis: The *magnitude of the translation* 2η and the angle of rotation 2ϑ. We would like to show how one calculates these quantities from the coordinates of motion. To that end, we start with equation (12):

(54) $$\alpha^{-1}\, Vx\, \alpha - 2x_0 \frac{\tilde{\alpha}\beta}{N\alpha} = Vx' \qquad [Na \neq 0].$$

In this, the rotations around the origin are given by the first part of the equation. They are then characterized by the fact that the β_i vanish. Their coordinates are then:

(55) $$\alpha_0 : \alpha_1 : \alpha_2 : \alpha_3 : 0 : 0 : 0 : 0.$$

From (17), § 32, the angle of rotation is given by:

(56) $$\cot\, \vartheta = \frac{-\alpha_0}{\sqrt{N\alpha - \alpha_0^2}}.$$

A translation should be regarded as the product of the identity rotation $V\alpha = 0$ with a transformation that is characterized by the quaternion β. If one substitutes $V\alpha = 0$ in (54), and then, since $(\alpha\beta) = 0$, $\beta_0 = 0$, one will obtain a *translation:*

(57) $$\alpha_0 : 0 : 0 : 0 : 0 : \beta_1 : \beta_2 : \beta_3$$

whose magnitude is:

(58) $$2\sqrt{\frac{\tilde{\alpha}\beta\tilde{\beta}\alpha}{(N\alpha)^2}} = 2 \cdot \frac{\sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2}}{\alpha_0}.$$

One sets:

(59) $$\eta = \frac{\sqrt{\beta_1^2 + \beta_2^2 + \beta_3^2}}{-\alpha_0}$$

and calls 2η the *magnitude of translation* of the translation.

Theorem 12: *A motion $\alpha + \varepsilon\beta$ that is different from the identity and has $V\alpha = 0$, $\beta_0 = 0$ is a translation in the direction $\beta_1 : \beta_2 : \beta_3$ whose magnitude is 2η.*

We now go on to the *screwing motions* and first consider one whose axis runs through the origin. It is composed of a rotation α and a translation $\sigma_0 + \varepsilon V\alpha$ along the axis of rotation. The coordinates of that screw $\alpha(\sigma_0 + \varepsilon V\alpha)$ are then:

(60)
$$\alpha'_0 : \alpha'_1 : \alpha'_2 : \alpha'_3 : \beta'_0 : \beta'_1 : \beta'_2 : \beta'_3$$
$$= \sigma_0\,\alpha_0 : \sigma_1\,\alpha_1 : \sigma_2\,\alpha_2 : \sigma_3\,\alpha_3 : -(\alpha_1^2 + \alpha_2^2 + \alpha_3^2) : \alpha_0\,\alpha_1 : \alpha_0\,\alpha_2 : \alpha_0\,\alpha_3 .$$

Its *translation magnitude* and *angle of rotation* are determined by:

(61)
$$\begin{cases} \eta = \dfrac{\sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}}{-\sigma_0} = \dfrac{\beta'_0}{\sqrt{\alpha_1'^2 + \alpha_2'^2 + \alpha_3'^2}}, \\ \cot\vartheta = \dfrac{-\alpha_0}{\sqrt{\alpha_1'^2 + \alpha_2'^2 + \alpha_3'^2}} = \dfrac{-\alpha'_0}{\sqrt{\alpha_1'^2 + \alpha_2'^2 + \alpha_3'^2}}. \end{cases}$$

If one now subjects a given screw to a suitable translation then one can always arrange that the axis of the screw runs through the origin. Conversely: If one subjects the screw $\alpha + \varepsilon\beta$ to the translation $\Sigma : \tau_0 + \varepsilon V\tau$ then one will obtain the most general screw $\Sigma^{-1} S\,\Sigma = \alpha'' + \varepsilon\beta''$. The quaternion product:

(62)
$$\alpha'' + \varepsilon\beta'' = (\tau_0 + \varepsilon V\tau)^{-1}\,(\alpha + \varepsilon\beta)\,(\tau_0 + \varepsilon V\tau)$$
$$= \frac{1}{\tau_0}\{\tau_0\,\alpha' + \varepsilon[2\,(0, \alpha'_2\tau_3 - \alpha'_3\tau_2\,,\ \alpha'_3\tau_1 - \alpha'_1\tau_3\,,\ \alpha'_1\tau_2 - \alpha'_2\tau_1) + \tau_0\,\beta']\}$$

will now have the coordinates:

(63)
$$\begin{cases} \alpha''_0 = \tau_0\alpha'_0, & \beta''_0 = \tau_0\beta'_0, \\ \alpha''_1 = \tau_0\alpha'_1, & \beta''_1 = \tau_0\beta'_1 + 2(\alpha'_2\tau_3 - \alpha'_3\tau_2), \\ \alpha''_2 = \tau_0\alpha'_2, & \beta''_2 = \tau_0\beta'_2 + 2(\alpha'_3\tau_1 - \alpha'_1\tau_3), \\ \alpha''_3 = \tau_0\alpha'_3, & \beta''_3 = \tau_0\beta'_3 + 2(\alpha'_1\tau_2 - \alpha'_1\tau_1). \end{cases}$$

The quantities of translation and rotation remain unchanged under the transformation. We can then infer them from formulas (61) and find that:

(64)
$$\begin{cases} \eta = \dfrac{\beta'_0}{\sqrt{\alpha_1'^2 + \alpha_2'^2 + \alpha_3'^2}} = \dfrac{\beta''_0}{\sqrt{\alpha_1''^2 + \alpha_2''^2 + \alpha_3''^2}}, \\ \cot\vartheta = \dfrac{-\alpha'_0}{\sqrt{\alpha_1'^2 + \alpha_2'^2 + \alpha_3'^2}} = \dfrac{-\alpha''_0}{\sqrt{\alpha_1''^2 + \alpha_2''^2 + \alpha_3''^2}}. \end{cases}$$

It will then follow that:

Theorem 13: *The screw $\alpha + \varepsilon\beta$ has an axis with ray coordinates $V\alpha + \varepsilon V\beta$. Its magnitude of translation and its angle of rotation are determined by:*

$$\eta = \beta_0 : \sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2},$$
$$\cot \vartheta = -\alpha_0 : \sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}.$$

An immediate consequence of this is that:

Theorem 14:

For $\beta_0 = 0$, one is dealing with a rotation.

For $\beta_0 = 0$, one is dealing with an "unscrewing" (cot $\vartheta = 0$, $\vartheta = \pi / 2$, $2\vartheta = \pi$.)

When $\alpha_0 = \beta_0 = 0$, one is dealing with a reversal ([1]).

§ 34. Map of motions to the points of an M_6^2 in R_7.

120. Right-handed and left-handed somas. When a motion is considered to be a spatial element, we (with **E. Study**) would like to call it a *soma*. In order to be able to link the concept of a soma to something more intuitive, we proceed as follows: We refer to a dreibein of three mutually-perpendicular unit vectors that is fixed once and for all (i.e., a coordinate system) as a *protosoma*. The protosoma will go to another dreibein under a motion $\alpha + \varepsilon\beta$, and that dreibein will be associated with the motion $\alpha + \varepsilon\beta$ in a single-valued and invertible way. We would like to call that dreibein a *soma* and call the quantities $\alpha_0 : \alpha_1 : \alpha_2 : \alpha_3 : \beta_0 : \beta_1 : \beta_2 : \beta_3$ the *coordinates* of the soma. Along with the *right-handed somas* that are defined in that way, we shall also consider *left-handed* ones:

$$\gamma_0 : \gamma_1 : \gamma_2 : \gamma_3 : \delta_0 : \delta_1 : \delta_2 : \delta_3$$

that emerge from the protosoma by transfers.

121. Pseudo-somas. We now interpret the coordinates α_i, β_i of the soma as the homogeneous coordinates of a point in R_7. A regular M_6^2 in it will then be distinguished by the equation:

(1) $$(\alpha\beta) = \alpha_0\beta_0 + \alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 = 0$$

([1]) Cf., **E. Study**, "Von den Bewegungen und Umlegungen," Math. Ann. **39** (1891).

as the locus of the image points of all (right-handed) somas. However, while every soma corresponds to an image point in M_6^2, not every point of M_6^2 will correspond to a soma. Points for which $N\alpha = 0$ yield no motion.

In order to define the map with no gaps, we then introduce the new concept of *pseudo-soma*. A pseudo-soma has coordinates of the form:

$$0 : 0 : 0 : 0 : \beta_0 : \beta_1 : \beta_2 : \beta_3 . \tag{2}$$

We would like to attempt to associate pseudo-somas with an intuitive figure in R_3. For that, we can be guided by the analogy to line geometry.

A improper line $\mathfrak{X}$ in R_3 will be illustrated by the proper lines of its complex of lines of intersection. Those lines will be represented in the image R_5 of line space by points in the tangential R_4 to M_4^2 at the point $\mathfrak{X}$. We also correspondingly construct the tangential R_6 to the M_6^2 at the point (2) in the image R_7 of soma space. In the running coordinates α'', β'', it will have the equation:

$$S(\tilde{\beta}\alpha'') = \alpha_0''\beta_0 + \alpha_1''\beta_1 + \alpha_2''\beta_2 + \alpha_3''\beta_3 = 0. \tag{3}$$

However, this equation is itself satisfied by the ∞^5 somas that one obtains when one composes a soma $\beta + \varepsilon\beta^*$ (with an arbitrary β^*) with all unscrewings $V\alpha' + \varepsilon\beta'$ (cf., Theorem 14):

$$\alpha'' + \varepsilon\,\beta'' = (\beta + \varepsilon\beta^*)(V\alpha' + \varepsilon\beta'). \tag{4}$$

In fact, it follows that:

$$\alpha'' = \beta \cdot V\alpha', \tag{5}$$

and thus:

$$S(\tilde{\beta}\alpha'') = S(\tilde{\beta}\beta V\alpha'') = N\beta \cdot S(V\alpha') = 0. \qquad \text{Q. E. D.} \tag{6}$$

Theorem 15: *If one subjects a soma to all unscrewings then one will obtain a system of ∞^5 somas that represent a pseudo-soma.*

Theorem 16: *The manifold of all somas and pseudo-somas can be mapped to the points of a regular M_6^2 in R_7 in a single-valued and invertible way.*

In what follows, we shall restrict ourselves to somas, for the sake of simplicity.

122. Linear spaces in the M_6^2 in R_7. Just as the manifold of improper lines (as a field of planes) can be mapped to a plane in M_4^2, the manifold:

$$\alpha_0 = 0, \qquad \alpha_1 = 0, \qquad \alpha_2 = 0, \qquad \alpha_3 = 0 \tag{7}$$

of all pseudo-somas can be mapped to an R_3 in M_6^2. We shall now direct our attention to the linear spaces that lie in M_6^2.

One first sees that linear spaces of higher dimension than R_3 cannot lie in M_6^2. Namely, a point that lies on M_6^2 must lie on its polar R_6, and thus, an R_p that lies on M_6^2 must lie on its polar R_{6-p}. For that reason, one must have:

$$p \leq 6 - p, \qquad p \leq 3. \tag{8}$$

Now, in order to find all R_3 that lie in M_6^2, we project M_6^2 stereographically [which is a process that **C. Segre** ([1]) employed] from one of its points p to an R_6 (cf., § 17). The projection is, in general, single-valued and invertible. The single-valuedness breaks down only for the points of the cone M_5^2 at which the tangential R_6^* to p cuts M_6^2. The ∞^1 points that lie on a line of that cone that goes through p will be projected to a point of the regular M_4^2 at which the cone M_5^2 cuts R_6. That M_4^2 will lie in the R_5 at which the tangential R_6^* to p cuts through R_6. We would like to call that R_5 the *imaginary* R_5 in R_6.

Under projection, the R_3 in M_6^2 will be projected to an R_3' in R_6 that cuts the imaginary R_5 in a plane that lies on M_4^2. Conversely, an R_3' of that kind will correspond to an R_3 that lies on M_6^2. Now, there are obviously two types of spaces R_3' to distinguish: Ones that cut the M_4^2 in planes of the first kind and ones that cut M_4^2 in planes of the second kind. There are, correspondingly, *two kinds of spaces* R_3 *on* M_6^2 to distinguish, and indeed, we have ∞^3 of each kind, since there are ∞^3 planes of each kind on M_4^2, and $\infty^6 : \infty^3 = \infty^3$ R_3 run through a plane in R_6.

We would now like to look for the *ways that the two kinds of* R_3 *can intersect. Two* R_3' *of the same kind* cut M_4^2 in two planes of the same kind, which will have a point in common, according to Theorem 5 of Chap. II. The two R_3' (which, as R_3''s in R_6, will generally intersect at one and only one point) do not therefore need to have a real point of intersection. Since an imaginary point of intersection (as a point of M_4^2) would not be mapped to a well-defined point of M_6^2, the corresponding R_3 will be skew to M_6^2. However, in the special case where they have a common point that projects to a real point of R_6, the corresponding R_3' will intersect in a line (viz., the connecting line of the real point with the imaginary point of intersection), and the R_3 in M_6^2 will then have a line in common.

Conversely, *two* R_3 *of different kinds* will project onto two R_3' that cut M_4^2 in two planes of different kinds. From Chap. II, Theorem 5, they will generally be skew. The R_3' must then intersect in a real point that corresponds to the point of intersection of the

([1]) **C. Segre**, "Studio sulle quadriche in uno spazio lineare ad un numero qualunque di dimensioni. Sulla geometria della retta e delle sue serie quadratiche," Mem. Acc. Torino (2) **36** (1884).

corresponding R_3 on M_6^2. However, if these R_3 had even one common line G in the special case (which would then project to a real line G' in R_6) then the two planes on M_4^2 would have a common point (viz., the imaginary point of G'), and thus a common line, from Chap. II, Theorem 5. The two R_3' would then intersect in a real plane, namely, the connecting plane of the imaginary line that was just found with the line G'. The corresponding R_3 in M_6^2 then have a plane in common.

Theorem 17: *The highest-dimensional linear spaces in a regular M_6^2 in R_7 are R_3's. An M_6^2 contains ∞^6 R_3's. They are divided into two families: Two R_3's of the same family will be skew, in general. However, if they have a point in common then they will also intersect along a line. Two R_3's from different families will generally intersect at a point. However, if they have a line in common then they will also intersect along a plane.*

The method of proof can be easily generalized. It will then be possible to arrive at a statement about regular, quadratic manifolds M_{n-1}^2 in R_n. We will see that the highest-dimensional linear spaces alternately belong to one system (conic section, M_3^2 in R_4) or two different systems (point-pairs, second-order surface M_4^2 in R_5, M_6^2 in R_7), and that on these latter manifolds M_{2q}^2, two spaces of different kinds will generally be skew for even q and incident for odd q.

123. Linear manifolds of somas ([1]). The linear point manifolds on M_6^2 correspond to linear manifolds of somas in soma space. The points of an R_3 that belongs to M_6^2 are pair-wise conjugate relative to M_6^2. If we would like to learn more about the manifold of somas (which corresponds to the R_3 in M_6^2) then we would first need to examine what properties of two somas correspond to the conjugate position of their image points. We assert:

Theorem 18: *Conjugate points of M_6^2 correspond to somas that can go to each other by a rotation.*

In fact: We will obtain the product:

$$\alpha'' + \varepsilon\beta'' = (\alpha + \varepsilon\beta)^{-1} \cdot (\alpha' + \varepsilon\beta') . \tag{9}$$

It will then follow from this that:

$$\beta'' = \frac{1}{N\alpha}\{\tilde{\alpha}\beta' + \tilde{\beta}\alpha'\} \tag{10}$$

and

([1]) Cf., **H. Beck**, Math. Ann. **81** (1920), **87** (1922).

(11) $$S\beta'' = \frac{1}{N\alpha}[S(\tilde{\alpha}\beta') + S(\tilde{\beta}\alpha')] = \frac{1}{N\alpha}[(\alpha\,\beta') + (\beta\,\alpha')].$$

Therefore, $S\beta'' = 0$ will mean the same thing as:

(12) $$(\alpha\,\beta') + (\beta\,\alpha') = 0,$$

and therefore, from Theorem 14, that will lead to the proof.

An immediate consequence of this is:

Theorem 19: *The* ∞^9 *lines in* M_6^2 *correspond to* ∞^9 *"chains of rotations"; i.e., one-dimensional manifolds of somas that arise when one subjects a soma to the* ∞^1 *rotations around an axis (Limiting case:* ∞^1 *translations in a direction).* (∞^4 axes · ∞^6 somas = ∞^{10} chains of rotations, each of which is counted ∞^1 times, however.)

From the theorem that was just proved, soma-R_3's are three-dimensional manifolds of somas with the property that two of the somas can be linked by a chain of rotations.

We will find a *soma-*R_3 *of the first kind* when we subject a given soma to all rotations around a point. There are ∞^6 soma-R_3's of that kind. (∞^3 points · ∞^6 somas = ∞^9 soma-R_3's, each of which is counted ∞^3 times, however.)

We will find a *soma-*R_3 *of the second kind* when we subject a given left-handed soma to all reflections in all planes in space. In fact, one can link every right-handed soma that is obtained in that way to every other one by a chain of rotations: The product of two reflections in a plane is (as one can see immediately or with the help of the algebra of quaternions) a rotation around the line of intersection of the two planes. There are then ∞^6 left-handed somas that correspond to the ∞^6 soma-R_3's of the second kind. It is interesting to interpret the incidence conditions between the two kinds of R_3 in M_6^2 that was given in Theorem 17 in terms of soma geometry.

§ 35. Analogies with ray geometry.

124. Parallel, hemi-symmetral, and symmetral somas ([1]). In order to make the analogy with ray geometry more evident, we set:

(1) $$\begin{aligned} &\mathfrak{A}_0 = \alpha_0, \quad &&\mathfrak{A}_{01} = \alpha_1, \quad &&\mathfrak{A}_{02} = \alpha_2, \quad &&\mathfrak{A}_{03} = \alpha_3; \\ &\mathfrak{A}_{123} = \beta_0, \quad &&\mathfrak{A}_{23} = \beta_1, \quad &&\mathfrak{A}_{31} = \beta_2, \quad &&\mathfrak{A}_{12} = \beta_3 \end{aligned}$$

and

([1]) **E. Study**, *Geometrie der Dynamen*, Leipzig, 1903, pp. 557.

(2)
$$\begin{cases} A_0 = \mathfrak{A}_0 + \varepsilon \mathfrak{A}_{123}, \\ A_1 = \mathfrak{A}_{01} + \varepsilon \mathfrak{A}_{23}, \\ A_2 = \mathfrak{A}_{02} + \varepsilon \mathfrak{A}_{31}, \\ A_3 = \mathfrak{A}_{03} + \varepsilon \mathfrak{A}_{12}. \end{cases}$$

If we let X_i denote ray coordinates, for the moment, then, from (34), § 33:

(3)
$$A^{-1} X A = X'$$

will represent the motion of A in ray coordinates. We can now do without the requirement that A must be a bound biquaternion; the biquaternion A represents this motion, as does the biquaternion $(\sigma + \varepsilon \tau) \cdot A$, since the equations:

(4)
$$(\sigma + \varepsilon \tau) XA = AX'(\sigma + \varepsilon \tau) \qquad \text{and} \qquad XA = AX'$$

are equivalent, due to the dual homogeneity of the ray coordinates.

The quantities A_i can then be regarded as the dual-homogeneous point-coordinates. That will then yield a *connection with the projective geometry of dual* R_3 .

However, equations (2) further say that *kinematics is a generalization of ray geometry* and contains it as a special case.

In order for some analogies to be able to emerge, we pose the problem of finding the motion A that takes the soma X to the soma Y. We find that:

(5)
$$A = X^{-1} Y,$$

and from it:

(6)
$$\begin{cases} A_0 = X_0 Y_0 + X_1 Y_1 + X_2 Y_2 + X_3 Y_3, \\ A_1 = X_0 Y_1 - X_1 Y_0 - X_2 Y_3 + X_3 Y_2, \\ A_2 = X_0 Y_2 - X_2 Y_0 - X_3 Y_1 + X_1 Y_3, \\ A_3 = X_0 Y_3 - X_3 Y_0 - X_1 Y_2 + X_2 Y_1. \end{cases}$$

It will then follow from this that:

(7)
$$\mathfrak{A}_0 = [\mathfrak{X} \mid \mathfrak{Y}] = \mathfrak{X}_0 \mathfrak{Y}_0 + \mathfrak{X}_{01} \mathfrak{Y}_{01} + \mathfrak{X}_{02} \mathfrak{Y}_{02} + \mathfrak{X}_{03} \mathfrak{Y}_{03} ,$$

(8)
$$\begin{aligned} \mathfrak{A}_{123} = (\mathfrak{X}\, \mathfrak{Y}) = {} & \mathfrak{X}_0 \mathfrak{Y}_{123} + \mathfrak{X}_{01} \mathfrak{Y}_{23} + \mathfrak{X}_{02} \mathfrak{Y}_{31} + \mathfrak{X}_{03} \mathfrak{Y}_{12} \\ & + \mathfrak{X}_{123} \mathfrak{Y}_0 + \mathfrak{X}_{23} \mathfrak{Y}_{01} + \mathfrak{X}_{31} \mathfrak{Y}_{02} + \mathfrak{X}_{12} \mathfrak{Y}_{03} . \end{aligned}$$

We now call two somas *parallel* when one of them emerges from the other one by a translation, *hemi-symmetral* when the one goes to the other by an unscrewing, and *symmetral* when the second emerges from the first by a reversal.

Should the somas X, Y be parallel, then from Theorem 12, one would need to have $\mathfrak{A}_0 = \mathfrak{A}_{02} = \mathfrak{A}_{03} = 0$. The would yield the three equations:

(2)
$$\begin{cases} \mathfrak{X}_0\,\mathfrak{Y}_{01} - \mathfrak{X}_{01}\,\mathfrak{Y}_0 - \mathfrak{X}_{02}\,\mathfrak{Y}_{03} + \mathfrak{X}_{03}\,\mathfrak{Y}_{02} = 0, \\ \mathfrak{X}_0\,\mathfrak{Y}_{02} - \mathfrak{X}_{02}\,\mathfrak{Y}_0 - \mathfrak{X}_{03}\,\mathfrak{Y}_{01} + \mathfrak{X}_{01}\,\mathfrak{Y}_{03} = 0, \\ \mathfrak{X}_0\,\mathfrak{Y}_{03} - \mathfrak{X}_{03}\,\mathfrak{Y}_0 - \mathfrak{X}_{01}\,\mathfrak{Y}_{02} + \mathfrak{X}_{02}\,\mathfrak{Y}_{01} = 0, \end{cases}$$

from which:

(10)
$$\mathfrak{X}_0 : \mathfrak{X}_{01} : \mathfrak{X}_{02} : \mathfrak{X}_{03} = \mathfrak{Y}_0 : \mathfrak{Y}_{01} : \mathfrak{Y}_{02} : \mathfrak{Y}_{03}.$$

Theorem 20: *Two somas are parallel when the scalar parts of their coordinates are proportional.*

Should the somas X, Y be hemi-symmetral or symmetral, then, from Theorem 14, one would need to have $\mathfrak{A}_0 = 0$ or $\mathfrak{A}_0 = \mathfrak{A}_{123} = 0$. Hence:

Theorem 21: *Two somas X, Y are hemi-symmetral when $[\mathfrak{X} \mid \mathfrak{Y}]$ vanishes. They are symmetral when:*

(11)
$$\begin{aligned}(X\,Y) &= X_0\,Y_0 + X_1\,Y_1 + X_2\,Y_2 + X_3\,Y_3 \\ &= [\mathfrak{X} \mid \mathfrak{Y}] + \varepsilon\,(\mathfrak{X}\mathfrak{Y})\end{aligned}$$

vanishes.

With that, the analogy to formula (9), § 27 is found.

We already spoke of somas that can go to each other by a rotation – viz., *conjugate* somas – above.

125. The dual angle between two somas. In order to apply the ideas that were introduced, we define the dual angle between two somas X, Y [in analogy with (1), § 29] by the equation:

(12)
$$\cos(\Theta + \varepsilon\,\mathrm{H}) = \frac{(XY)}{\sqrt{(XX)}\sqrt{(YY)}}$$

$$= \cos\Theta - \varepsilon\,\mathrm{H}\cdot\sin\Theta = \frac{[\mathfrak{X}\mid\mathfrak{Y}]}{\sqrt{[\mathfrak{X}\mid\mathfrak{X}]}\sqrt{[\mathfrak{Y}\mid\mathfrak{Y}]}} + \varepsilon\,\frac{(\mathfrak{X}\mathfrak{Y})}{\sqrt{[\mathfrak{X}\mid\mathfrak{X}]}\sqrt{[\mathfrak{Y}\mid\mathfrak{Y}]}}.$$

In this, it is assumed that the coordinates X_i and Y_i are multiplied by suitable dual factors, such that X, Y become bound biquaternions. One will then have:

$$(\mathfrak{X}\mathfrak{X}) = (\mathfrak{Y}\mathfrak{Y}) = 0.$$

From (12), one will now have:

$$(13)\quad \begin{cases} \cos\Theta = \dfrac{[\mathfrak{X} \mid \mathfrak{Y}]}{\sqrt{[\mathfrak{X} \mid \mathfrak{X}]}\sqrt{[\mathfrak{Y} \mid \mathfrak{Y}]}}, \\ \text{and therefore } \sin\Theta = \dfrac{-\sqrt{[\mathfrak{X} \mid \mathfrak{X}][\mathfrak{Y} \mid \mathfrak{Y}] - [\mathfrak{X} \mid \mathfrak{Y}]^2}}{\sqrt{[\mathfrak{X} \mid \mathfrak{X}]}\sqrt{[\mathfrak{Y} \mid \mathfrak{Y}]}}, \end{cases}$$

so

$$(14)\quad \begin{cases} \cot\Theta = \dfrac{-[\mathfrak{X} \mid \mathfrak{Y}]}{\sqrt{[\mathfrak{X} \mid \mathfrak{X}][\mathfrak{Y} \mid \mathfrak{Y}] - [\mathfrak{X} \mid \mathfrak{Y}]^2}}, \\ \mathrm{H} = \dfrac{(\mathfrak{X}\mathfrak{Y})}{\sqrt{[\mathfrak{X} \mid \mathfrak{X}][\mathfrak{Y} \mid \mathfrak{Y}] - [\mathfrak{X} \mid \mathfrak{Y}]^2}}. \end{cases}$$

If one now considers that from (7), (8), one will have:

$$(15)\quad [\mathfrak{X} \mid \mathfrak{Y}] = \mathfrak{A}_0, \qquad (\mathfrak{X}\mathfrak{Y}) = \mathfrak{A}_{123}$$

and

$$(16)\quad [\mathfrak{X} \mid \mathfrak{X}]\,[\mathfrak{Y} \mid \mathfrak{Y}] - [\mathfrak{X} \mid \mathfrak{Y}]^2 = \mathfrak{A}_{01}^2 + \mathfrak{A}_{02}^2 + \mathfrak{A}_{03}^2$$

then one will get, on the basis of Theorem 13:

Theorem 22: *If one sets:*

$$\cos(\Theta + \varepsilon \mathrm{H}) = \frac{(XY)}{\sqrt{(XX)}\sqrt{(YY)}}$$

then 2Θ *will be the angle, and* $2\mathrm{H}$ *will be the magnitude of translation for the motion that takes the soma X to the soma Y.*