"Anschauliches zur Relativitätstheorie. II. Raumzeitmessungen in Gravitationalsfeldern," Zeit. Phys. **107** (1937), 64-72.

# Intuitive aspects of the theory of relativity

## II. Space-time measurements in gravitational fields.

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The basic assumptions pertaining to space-time measurements in gravitational fields will be discussed by means of yardsticks and chronometers, and it will be shown that – contrary to a very widespread opinion – the "rigidity" of the yardsticks does not need to be assumed, and that an arbitrarily-moving clock does not generally show its "proper time." The elastic properties of the materials that the yardstick and clocks are composed of first play a role for the "measurements of the second kind" by means of a measuring device that is at rest in the gravitational field.

Occasionally, the possibility of defining the gravitational field, as distinct from the metric field, will be mentioned, and the condition for the "irrotationality" of a gravitational field will be given.

**1.** – In the first part ([1]), I discussed the role of linear (hence, non-Galilean, in general) coordinates and the physical meanings of the individual $g_{ik}$ coefficients in special relativity from the standpoint of an intuitively-minded physicist. That mainly happened in order to ease the intuitive comprehension of the connection between special and general relativity, since the simplest transition from general to local geodetic space-time coordinates leads directly to (approximate) linear coordinates and not to (approximate) **Galilean** coordinates.

In the present article, we shall mainly discuss the ways by which space-time measurements in gravitational fields will differ from those in **Galilean** domains. In it, we shall generally understand space-time measurements to mean the various kinds of length and time measurements and the measurements of the $g_{ik}$ coefficients that are connected with them, and regard the intuitive physical meanings of those measurements in special relativity, as in A I, as being sufficiently clarified.

It should be remarked expressly here that, just as in A I, we are not proposing to either criticize relativity theory or attempt to axiomatize it, but merely to raise some points that are often represented in a manner that is either not sufficiently clear, imprecise, or completely false.

Even those – like **Carathéodory, Reichenbach, Robb, Weyl**, *et al*. – who cared to embark upon that axiomatization, based upon the foundation of "light geometry" could not help but introduce assumptions about the behavior of bodies in gravitational fields in

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([1]) "Anschauliches zur Relativitätstheorie. I. Lineare Koordinate und $g_{ik}$-Koeffizienten in der speziellen Relativitätstheorie," Zeit. Phys. **95** (1935), 391-408. Cited as A I in what follows.

order to encompass all of relativity theory. They were then also obliged to address their place among the questions that will be discussed below.

**2. Measuring with soft rubber yardsticks. –** For the moment, we base ourselves upon the foundations of special relativity and, in particular, direct our attention to the fact that a reference system that is fixed on the Earth can no longer be considered to be a Galilean system as in pre-relativistic mechanics – with a gravitational field that acts in it – and that relativistic measurements in a true Galilean system (an "Einsteinian elevator" that falls freely without drag, a Moon rocket that moves freely without rotation, the center of the Earth) can take on a *much simpler* form than they do in our laboratories on Earth.

Since no perturbing influence of a gravitational field makes itself known, one does not need either "rigid" or "almost rigid" yardsticks in order to carry out length measurements in such a system. Any unforced, rotationless, freely-suspended body can be employed as a yardstick. Indeed, since any unforced fluid mass will assume a spherical form when it is at rest in a Galilean system, one can also use such spheres for length measurements, in principle!

One can eliminate a flaw in the current manner of representation with that obvious remark that consists of the fact that one uses "rigid" – if not also "infinitely small" yardsticks in order to base geometry, and then proves that a "rigid" body is impossible according to the laws of relativity. The degree of "rigidity" of the "yardstick" first plays a role for certain relativistic measurements in a gravitational field, as we will explain below.

It should be pointed out briefly here that one might probably conclude from the "equivalence of all Galilean systems" that equal bodies that move the same way relative to two Galilean systems must behave the same in all respects, although it by no means follows from this that two yardsticks that are made from different materials and have the same length at rest next to each other in a Galilean system will also remain that way for arbitrary common motions. In contrast, we have grounds for believing that this is not so. For the foundation of the theory, it is, in principle, unimportant that this difference is negligibly small in many cases. If one therefore often introduces the explicit assumption that "two line segments (that are defined by any two points that are marked out on a solid body) that can be made to overlap at one point in time can be made to overlap always and everywhere" then one must probably understand that to mean that both line segments can be compared only in a "free Galilean system at rest."

**3. Concept of the gravitational field. –** It is known that different gravitational fields can arise in one and the same relativistic domain according to the coordinate system that is used as a basis. However, in order to be able to assume the viewpoint of an intuitively-minded physicist, it is advisable to not refer to any relativistic coordinate transformation as a change in the gravitational field, but merely the ones that can be regarded as a transition to a new reference observer ([1]).

([1]) We assume that such a thing is present, since we would like to restrict ourselves to the consideration of merely proper relativistic coordinates in the present treatise.

Since a relativistic domain with a well-defined metric can also be referred to as a *metric field*, I propose to clarify that a *gravitational field* is a metric field with a well-defined reference observer. In it, one can perform arbitrary coordinate transformations of the form:

$$x^{\alpha'} = x^{\alpha'}(x^1, x^2, x^3), \qquad (\alpha' = 1, 2, 3)$$
$$x^{4'} = x^{4'}(x^1, x^2, x^3), \tag{1}$$

[with the well-known restriction that under coordinate transformations, a true relativistic coordinate system should go to another such system ([1])]. Speaking intuitively, one can cover the points of the reference observer with arbitrary "curvilinear spatial coordinates" and also calibrate the "coordinate chronoscopes" that are embedded in it in a largely arbitrary way. However, their "mutual calibration" must satisfy certain inequalities, in addition to the obvious conditions of continuity and "smoothness," if one is to ensure that the **Einsteinian** coordinate system that now arises will remain a true relativistic coordinate system ([2]).

In the four-dimensional **Minkowski** diagram, a gravitational field in the sense above is nothing but a relativistic domain with a given metric and a given congruence of everywhere time-like lines. I have treated the physical and invariant-theoretic problem that this implies on another occasion ([3]).

We will refer to a point with constant $x^\alpha$ coordinates ($\alpha$ = 1, 2, 3) as being *at rest in a gravitational field*; by contrast, we will speak of points that are *instantaneously at rest* when their velocities vanish relative to the reference observer (without the higher derivatives of the spatial coordinates with respect to $x^4$ needing to vanish). In the former case, the world-lines of the point considered will coincide with the lines of the given congruence, while in the latter, they will only contact them.

**4. Transforming away the gravitational field.** – Now, as is known, the transition from special to general relativity is accomplished in such a way that one distinguishes a special class of coordinate systems in the neighborhood of any event-point $P_0$ . In **Riemannian** geometry, one calls those coordinate systems "geodetic at $P_0$". In physics (and in a certain approximation that we would not like to make more precise here), they take on the role of the inertial systems of special relativity, and will be called "local inertial systems."

Intuitively, the transition to a local inertial system will be referred to as transforming away the gravitational field, although one often exhibits the realization of such a transition simply when one says that a freely-falling, sufficiently-small box will realize such a system, as long as that box does not rotate, in general. We once more reiterate that if one wishes to obtain a local inertial system then it will not suffice to release a box that is at rest in a gravitational field suddenly, but without any jerk, but one must generally apply a correctly-chosen rotational impulse when one releases it, in general. If the box

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([1]) Cf., e.g., A I, § 4 and 25.

([2]) Cf., e.g., A I, § 18 to 23 and 25.

([3]) "Metrisches Feld und Gravitationsfeld," Bull. Acad. Polon. Sci. et Lett. (1937), 252-159. Cited as MG in what follows.

already falls freely then one can establish by correspondingly-chosen experiments whether it does or does not rotate relative to the class of all local inertial systems that correspond to the sufficiently-small relativistic domain considered, but there is no general process of releasing the box without rotation (or throwing it). The old problem of absolute rotation once more appears here in new clothing!

The aforementioned angular velocity by which a sufficiently-small box that is initially in a gravitational field at rest and then released without jerk will begin to rotate ([1]) is nothing but the angular velocity of a "particle" in the reference observer relative to a local inertial system. It amounts to ([2]):

$$\omega_{\alpha\beta} = \frac{1}{2}\frac{c}{\sqrt{-g_{44}}}\left\{\frac{\partial a_\beta}{\partial x^\alpha} - \frac{\partial a_\alpha}{\partial x^\beta} - \left(a_\alpha \frac{\partial a_\beta}{\partial x^4} - a_\beta \frac{\partial a_\alpha}{\partial x^4}\right)\right\}, \tag{2}$$

in which:

$$a_\alpha = \frac{g_{4\alpha}}{g_{44}} \qquad (\alpha, \beta = 1, 2, 3).$$

As an example, we appeal to a disc that rotates uniformly relative to a Galilean system, and which determines a unique gravitational field in the sense that was defined above when regarded as a reference observer (and indeed independently of the calibration of the coordinate chronoscopes on the disc). It is clear that an arbitrarily-small body that is initially fixed on the disc and then set free suddenly will fly away along a circular tangent and rotate about its center of mass with the angular velocity of the disc. The gravitational field on the disc is just a "gravitational vortex field," in which $\omega_{\alpha\beta} \neq 0$.

Let us remark here, by the way, that a "particle" in the reference observer that relates to the local inertial system, in addition to a collective acceleration ([3]) (which can vanish for certain – viz., neutral – fields, such as in the **Einstein** or **Lemaître** universe) and a collective rotation, will suffer a deformation in general ([4]), which will, however, vanish in all stationary gravitational fields ([5]). The process of releasing the box will be further complicated by their presence. Nonetheless, those complications are only concomitant to the special Gedanken experiments that are used to "generate" the local inertial systems. However, in the main definition of the transition to a local inertial system, such an experiment is not necessary with no further conditions. The transforming away of the gravitational field indeed rests, in the final analysis, upon a change of reference observer, relative to which we describe phenomena. The new reference observer will be determined by a rotationless, freely-suspended, unforced body without that body needing to be initially at rest in the given gravitational field.

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([1]) As far as what will happen later is concerned, that depends not only upon the gravitational field in all of the domain that it describes, but also on the distribution of mass in its interior. It is easy to construct an example in which a box that falls freely in a static field (in which $\omega_{\alpha\beta}$ always vanishes) will take on an ever larger angular velocity.

([2]) MG. Equation (17).

([3]) MG. Equation (11).

([4]) MG. Equation (16).

([5]) As is known, one calls a gravitational field *stationary* when one can define a relativistic coordinate system with $g_{ik}$ coefficients that are independent of $x^4$. A stationary field is *static* when one has $g_{4\alpha} = 0$ in that coordinate system, as well.

One can also think of the transforming away of the gravitational field as being performed in such a way that the new reference observer is defined by freely-moving points. In the event that these points are thought of as initially at rest in the given gravitational field, one must distribute them with (infinitely-small) velocities that are chosen according to the manner of release in order to compensate for the aforementioned rotational motion, as well as any deformation velocity that might be present. The aforementioned common acceleration will then be cancelled automatically. The distribution of a common translational velocity (relative to the local inertial system) will be unaffected.

**5. Space-time measurements of the first and second kind. –** Already in § 2, we have confirmed the difference in principle between making relativistic measurements in a Galilean system and a gravitational field. In order to exhibit the difficulties in making precise measurements in a gravitational field intuitively, we can imagine either measuring lengths in a laboratory on the Earth in which we have, however, only yardsticks and apparatuses made of soft rubber to work with or making length measurements with ordinary metal yardsticks in gravitational fields that are much more intensive than that of the Earth (e.g., like the ones that many stars produce), or ones that we can produce artificially, say, by means of a centrifuge.

In particular, direct your attention to the fact that saying that a body is "at rest in a gravitational field" will mean nothing precise as long as one knows nothing more specific about the way that its state was established. So, for example, a suspended yardstick and an identical one that is erected close to it will exhibit a different length when one is precise. (One should also confer the statements at the end of § 2.)

Previously, it was believed that all of those difficulties would go away when one assumed the existence of an "absolutely rigid body." However, such an assumption finds no support in experiments, since all bodies are more or less compressible and bendable. Moreover, an "absolutely-rigid body" cannot be approximated with arbitrary precision by passing to the limit of ever larger elastic constants, since such a passage will, in principle, be impracticable in relativity theory, due the existence of the upper limit on the speed of signals.

One also tries to compel the "rigidity" of the yardstick by reducing its dimensions, since obviously all of the effects that were mentioned here and at the end of § 2 will become all the smaller when one chooses smaller yardsticks (and clocks). Now it is plausible that such a passage to the limit is something fundamentally different from the ones for which the definition of the local inertial system plays a role and which are, in fact, inevitable for the foundation of general relativity. Whether a relativistic domain can been chosen to be sufficiently small that one can cover it with a coordinate system that is practically equivalent to a Galilean system will depend upon its curvature field. One can obviously move any solid body, no matter how small, in it in such a violent way that it cannot be considered to be "rigid" for that motion.

Although the latter passage to the limit (or one that is equivalent to it) must certainly play a role for any precise founding of the relativistic theory of elasticity, we do not need to consider it further here, since, in principle, neither an "absolutely rigid body" of finite dimensions nor an "infinitely small" one is necessary for the founding of relativistic measurements in a gravitational field. In order to clarify that as briefly and succinctly as possible, we will need to distinguish between two types of relativistic measurements in gravitational fields.

Primarily, only *measurements of the first kind* play a role in the fundamental basis for relativistic measurements, and they are the ones that are performed by means of "absolutely unaccelerated" (i.e., rotationless in a local inertial system), freely-suspended yardsticks and clocks. Hence, e.g., a length measurement of the first kind will be performed by an observer that lives in a (stationary) gravitational field when he throws a normal meter stick upwards without rotation in such a way that it comes to rest along the line segment to be measured (which is at rest in the gravitational field) and must then fall back again. The body that serves as yardstick is then by no means established in that way. It remains in a given gravitational field only momentarily, but during a finite time interval in a local inertial system ([1]). Time measurements of the first kind will be discussed in the next paragraph.

Now, one might justifiably object that, in reality, measurements of the first kind can never be performed and are possible only as Gedanken experiments. For that reason, one must also consider *measurements of the second kind* by means of yardsticks that are at rest in gravitational fields. The elastic properties of the yardstick and the construction of the clocks first play a role for them, so their precise theory can first be constructed upon the basis of the relativistic theory of elasticity. In practice, it is the business of the experimental physicist to consider the corresponding corrections for the influence of the gravitational field on his measuring devices. Those corrections can be made negligibly small, in general, due to the existence of "practically rigid bodies." However, it is, in principle, definitive that measurements of the first kind, for which the elastic properties of the measuring bodies play no role, will suffice for the definition of the concept of space-time in metric fields (see § 7, as well).

**6. Proper time measurements.** – It is often assumed (and even frequently raised to an axiom!) that an arbitrarily-moving (rotationless?) clock will yield its proper time; i.e.: the integral:

$$\int_A^B d\tau \qquad \left(d\tau^2 = -\frac{1}{c^2}\,ds^2\right), \tag{3}$$

when it is taken over the world-line of the clock from the event-point $A$ to the event-point $B$. However, there is no reference point in experiments for making that statement for arbitrary accelerations and clocks that are rotating arbitrarily fast (relative to the local inertial system). In contrast, for a sufficiently large acceleration, any clock will certainly stand still or even go to pieces! One might imagine throwing a Nardin chronometer out of a window!

Nonetheless, one cannot deny that the integral (3) possesses an invariant meaning. However, it can generally be evaluated, not by *one* clock, but by *an infinitude* of them, in

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([1]) It is interesting to point out that the possibility of performing such Gedanken experiments is implied by the rapid damping of elastic oscillations in measuring bodies, and thus, in the final analysis, by the validity of the second law of thermodynamics.

such a way that each of those clocks is a rest in a local inertial system, while the world-line contacts the world-line *AB*. Something similar is true in thermodynamics, where the integral that determines the increase in entropy must be calculated along a "quasi-static" path, in which perhaps one takes a large number of heat reservoirs whose temperatures, in turn, differ by only an infinitely-small amount and by which the bodies in question are brought into thermal contact with each other!

Although the process of evaluating the integral (3) might seem rather artificial, at first, it is still realized with high precision in nature, and to some extent automatically. Namely, if we observe a luminous gas that is at rest in a gravitational field then any luminous gas molecule will play the role of a freely-falling clock. Those clocks are, in fact, not "momentarily at rest" in a gravitational field, but the influence of their irregular motions, which expresses itself in the line breadths, is easily estimated, and will possibly need to be considered.

**7. Space-time measurements in a gravitational field.** – In summary, we can describe the measuring-out of a gravitational field intuitively as follows: One measures the three-dimensional fundamental spatial tensor:

$$\gamma_{\alpha\beta} = g_{\alpha\beta} - \frac{g_{4\alpha}\, g_{4\beta}}{g_{44}} \tag{4}$$

(cf., A I, § 26) in the usual geometric way by means of rotationless, freely-falling bodies that are instantaneously at rest and determine the "rest lengths" and "rest angles" in the gravitational field of a body at rest. The coefficient $g_{44}$ will be measured by a rotationless, free-falling chronometer that is instantaneously at rest in such a way that the time duration of an advance of the neighboring coordinate chronoscope by $\varepsilon$ that one reads from that chronometer will be $\varepsilon\sqrt{-g_{44}}$; hence, $\varepsilon$ must be chosen to be sufficiently small.

One will get the values of the coefficients $g_{4\alpha}$, which are proportional to the difference between the speeds of light in two opposite directions of the $x^\alpha$ parameter line, just as one does in the linear coordinate systems, from measuring the speed of light [A I, equations (29) and (31)] or by means of "dynamical experiments" (A I, § 20 and 26). Ultimately, one can calculate the $g_{\alpha\beta}$ coefficients from (4).

Obviously, this model simplifies noticeably in "irrotational gravitational fields" ([1]) and orthochronous relativistic coordinates.

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([1]) I call a gravitational field irrotational when (2) vanishes for it. As was shown in MG, the vanishing of (2) is a necessary and sufficient condition for the possibility of introducing everywhere-orthochronous relativistic coordinates in a gravitational field.