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Metric field and gravitational field

Note

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1. – In order to make the concepts of "metric field" and "gravitational field" more precise, and in that way, to hopefully meet up with the viewpoint of an intuitively-minded physicist, we define a *metric field* here to be a space-time domain with a well-defined metric and a *gravitational field* to be a metric field with a well-defined reference observer. In that sense, a disc that is rotating relative to a **Galilean** system will determine a gravitational field, and indeed independently of the calibration of the coordinate chronoscopes. Dropping an **Einsteinian** elevator corresponds to a transition from one gravitational field to another.

It is inevitable that no definition of the concept of a "gravitational field" can satisfy all of the nuances that the various authors will give to that concept. The one that is proposed here has the advantage that it at least leads to a well-defined physical and mathematical statement of the problem.

Speaking four-dimensionally, a gravitational field, with the definition that was given above, is equivalent to a space-time domain with a well-defined metric and a well-defined congruence of everywhere-timelike world-lines. Let it be stressed expressly here that we are dealing with a distinguished line congruence – hence, a direction field – and by no means with a distinguished *vector field*.

2. – In a given gravitational field, we can perform an arbitrary coordinate transformation of the form:

$$\xi^{\alpha'} = \xi^{\alpha'}(\xi^1, \xi^2, \xi^3), \tag{1 a}$$
$$\xi^{4'} = \xi^{4'}(\xi^1, \xi^2, \xi^3, \xi^4). \tag{1 b}$$

Here and in what follows, the Greek indices shall run from 1 to 3 and Latin ones from 1 to 4. Since we would like to consider nothing but proper space-time coordinate systems with the fourth axis as the time axis, the four functions in (1) must satisfy certain conditions, in addition to the usual regularity conditions that are expressed as inequalities

([1]). We shall not write out those inequalities here, since we will make no explicit use of them.

Equations (1) differ from the most general proper space-time coordinate transformations by the lack of ξ^4 in (1 a).

3. – In order to find the characteristic tensors of the gravitational field, we shall pursue a physically intuitive path here, and first pose the problem of ascertaining the state of motion in the given reference observer of the ξ^i-coordinate system S in a sufficiently-small space-time domain relative to a local inertial system. To that end, we consider an arbitrary space-time point P_0 and assume, for the sake of simplicity, that its ξ^i-coordinates vanish.

The known equations ([2]):

$$\xi^i = x^i - \frac{1}{2}\begin{Bmatrix} i \\ k\,l \end{Bmatrix} x^k x^l \tag{2}$$

(in which, the curly brackets mean **Christoffel** three-index symbols *at the point* P_0) mediate the transition between S and a locally-geodetic x^i-coordinate system L at P_0, whose axis directions at P_0 coincide with those of S.

In what follows, we shall consider the x^i (as well as the ξ^i) to be first-order infinitesimals and then neglect the terms or order three and higher. In that approximation, the curvatures of the space-time continuum in the neighborhood of P_0 will play no role, and it is one of the basic assumptions of general theory of relativity that (in this approximation) one can consider the x^i-coordinates to be "linear coordinates"; i.e., affine coordinates in a **Galilean** system ([3]). We can go further from these coordinates to ordinary **Lorentzian** coordinates by a suitably-chosen linear substitution, but the arbitrariness of the clarity that is present in the following arguments would probably suffer as a result.

The equations are easily solved for the x^i with the desired approximation, since one must only simple replace x^i with ξ^i, and conversely. If we make partial use of that freedom and single out the spatial indices (α, β, γ = 1, 2, 3), and especially the temporal index 4, in the double summation then we will get:

$$x^\alpha = \xi^\alpha + \frac{1}{2}\begin{Bmatrix} \alpha \\ \beta\,\gamma \end{Bmatrix}\xi^\alpha \xi^\beta + \begin{Bmatrix} \alpha \\ 4\,\beta \end{Bmatrix}\xi^\beta x^4 + \frac{1}{2}\begin{Bmatrix} \alpha \\ 4\,4 \end{Bmatrix} x^4 x^4 . \tag{3}$$

4. – Those equations can be regarded as the equations of motion of the reference observer in **Lagrangian** form (with the "**Lagrangian** coordinates" ξ^α, which characterize the individual particles of the reference observer), in the event that x^4 can be

([1]) Cf., e.g., Jan von **Weyssenhoff**, "Anschauliches zur Relativitätstheorie, I" Zeit. Phys. **95** (1935), 391. Cited as A I in what follows.

([2]) Cf., e.g., **L. P. Eisenhart**, *Riemannian Geometry*, Princeton, 1921, pp. 91 (18.13).

([3]) A I, § 7; **H. Weyl**, *Raum, Zeit, Materie*, 4th ed., pp. 160.

regarded as "time." However, the x^i-coordinates (as well as the ξ^i-coordinates) are not "orthochronous" ([1]), in general. It is only when we have changed the mutual calibrations of the coordinate chronoscopes such that this orthochronocity ([2]) is attained that we can regard the givens of the chronoscopes as being proportional to "time." The L-observer will then remain unchanged.

One can represent that transition computationally as follows: In L, one will have ([3])([4]):

$$s^2 = g_{ik}\, x^i\, x^k = \sigma^2 + g_{44}\left(x^4 + \frac{g_{4\alpha}}{g_{44}}\,x^\alpha\right)^2, \tag{4}$$

in which:

$$\sigma^2 = \gamma_{\alpha\beta}\, x^\alpha\, x^\beta \tag{5}$$

and

$$\gamma_{\alpha\beta} = g_{\alpha\beta} - \frac{g_{4\alpha}\, g_{4\beta}}{g_{44}}. \tag{6}$$

If we now set:

$$ct = \sqrt{-g_{44}}\left(x^4 + \frac{g_{4\alpha}}{g_{44}}\,x^\alpha\right) \tag{7}$$

then we will have

$$s^2 = \sigma^2 - c^2\, t^2, \tag{8}$$

and the coordinates x^1, x^2, x^3, x^4 will obviously represent the desired orthochronous coordinate system L^* with the same reference observer as L; in that, t gives the time directly in seconds.

We can now convert equations (3) as follows:

$$x^\alpha = \xi^\alpha + \frac{1}{2}\begin{Bmatrix}\alpha\\ \beta\,\gamma\end{Bmatrix}\xi^\beta\xi^\gamma + \begin{Bmatrix}\alpha\\ 4\,\beta\end{Bmatrix}\xi^\beta\left(\frac{ct}{\sqrt{-g_{44}}} - \frac{g_{4\gamma}}{g_{44}}\,\xi^\gamma\right) + \frac{1}{2}\begin{Bmatrix}\alpha\\ 4\,4\end{Bmatrix}\left(\frac{ct}{\sqrt{-g_{44}}} - \frac{g_{4\gamma}}{g_{44}}\,\xi^\gamma\right)^2. \tag{9}$$

These are three equations of the form:

$$x^\alpha = x^\alpha(\xi^1, \xi^2, \xi^3, t), \tag{10}$$

which can obviously be regarded as the equations of motion in **Lagrangian** form for the S-reference observer relative to L^* in a sufficiently-small neighborhood of the point $\xi^\alpha = 0$ and during a sufficiently-small time-interval.

([1]) A I, § 2.
([2]) The four-dimensional geometrical interpretation of this process raises no special difficulties.
([3]) Cf., e.g., A I, eqs. (25), (26), and (27).
([4]) Here, as in what follows, the g_{ik} (as well as the three-index symbols) mean the values of those quantities at P_0. They must then be held constant for the further differentiations that one must perform.

5. – We will get:

$$\gamma^{\alpha} = -\frac{c^2}{g_{44}}\begin{Bmatrix}\alpha\\ 4\,4\end{Bmatrix} \tag{11}$$

from this by a double partial differential with respect to t, which is the "acceleration of the reference observer" (of the point $\xi^{\alpha} = 0$ of the reference observer) relative to the local inertial system L^*.

We obtain the velocity components as linear functions of the "Lagrangian coordinates" ξ^{α} from (9) by a single differentiation with respect to t and from them, we get the "deformation tensor of the reference observer" $\varepsilon^{\alpha}_{\beta}$ by using the general formula:

$$\varepsilon^{\alpha}_{\beta} = \frac{\partial}{\partial x^{\beta}}\frac{\partial x^{\alpha}}{\partial t} = \frac{\partial^2 x^{\alpha}}{\partial \xi^{\gamma}\,\partial t}\cdot\frac{\partial \xi^{\gamma}}{\partial x^{\beta}}. \tag{12}$$

In our case, we have $\partial\xi^{\gamma} / \partial x^{\beta} = \delta^{\gamma}_{\beta}$, and an easy calculation will yield:

$$\varepsilon^{\alpha}_{\beta} = \frac{c}{\sqrt{-g_{44}}}\left(\begin{Bmatrix}\alpha\\ 4\,\beta\end{Bmatrix} - \frac{g_{4\beta}}{g_{44}}\begin{Bmatrix}\alpha\\ 4\,4\end{Bmatrix}\right). \tag{13}$$

By contraction, that will give the "dilatation velocity of the reference observer":

$$\varepsilon^{\alpha}_{\alpha} = \frac{c}{\sqrt{-g_{44}}}\frac{\partial}{\partial x^4}\ln\sqrt{\gamma}, \tag{14}$$

in which γ means the three-dimensional determinant of the $\gamma_{\alpha\beta}$. In the derivation of this formula we have made use of the known formula:

$$\begin{Bmatrix}i\\ 4\,i\end{Bmatrix} = \begin{Bmatrix}\alpha\\ 4\,\alpha\end{Bmatrix} + \begin{Bmatrix}4\\ 4\,4\end{Bmatrix} = \frac{\partial}{\partial x^4}\ln\sqrt{-g},$$

where $g = |\,g_{ik}\,|$, as well as the easily-verified relation $g = g_{44}\,\gamma$.

In order to decompose the deformation tensor (13) into its symmetric and antisymmetric parts ([1]), we must first lower the upper index, which must come about as a result of the three-dimensional fundamental spatial tensor (6). After some conversions, we will then get:

$$\varepsilon_{\alpha\beta} = \gamma_{\alpha\beta}\,\varepsilon^{\delta}_{\beta}, \tag{15}$$

$$\tfrac{1}{2}(\varepsilon_{\alpha\beta} + \varepsilon_{\beta\alpha}) = \frac{1}{2}\frac{c}{\sqrt{-g_{44}}}\frac{\partial\gamma_{\alpha\beta}}{\partial x^4}, \tag{16}$$

([1]) Cf., **Weyl**, *loc. cit.*, 5th ed., pp. 42.

$$\omega_{\alpha\beta} = \tfrac{1}{2}(\varepsilon_{\alpha\beta} - \varepsilon_{\beta\alpha}) = \frac{1}{2}\frac{c}{\sqrt{-g_{44}}}\left[\frac{\partial a_\beta}{\partial x^\alpha} - \frac{\partial a_\alpha}{\partial x^\beta} - \left(a_\alpha \frac{\partial a_\beta}{\partial x^4} - a_\beta \frac{\partial a_\alpha}{\partial x^4}\right)\right], \tag{17}$$

in which we have set:

$$a_\alpha = \frac{g_{4\alpha}}{g_{44}}. \tag{18}$$

The latter bivector is especially interesting, since it gives the angular velocity of the reference observer relative to the local inertial system L^*.

6. – We have thus arrived at various expressions that are defined by the g_{ik} and their first derivatives and that behave like the components of three-dimensional tensors under the transformation (1) – i.e., with the introduction of new coordinates without changing the reference observer. We would now like to establish that state of affairs computationally. As is known, the general transformation equations of the contravariant and covariant four-vectors read:

$$a^{l'} = A_l^{l'} a^l, \qquad a_{l'} = A_{l'}^l a_l, \tag{19}$$

in which:

$$A_{l'}^l = \frac{\partial x^l}{\partial x^{l'}}, \qquad A_l^{l'} = \frac{\partial x^{l'}}{\partial x^l}. \tag{20}$$

In our special case:

$$A_{4'}^\alpha = A_4^{\alpha'} = 0, \tag{21}$$

and for that reason, we will also have:

$$A_{4'}^4 \cdot A_4^{4'} = 1. \tag{22}$$

The transformation formulas (19) will then assume the following forms:

$$a^{\alpha'} = A_\alpha^{\alpha'} a^\alpha, \qquad a_{\alpha'} = A_{\alpha'}^\alpha a_\alpha + A_{\alpha'}^4 a_4, \tag{23}$$

$$a^{4'} = A_\alpha^{4'} a^\alpha + A_4^{4'} a^4, \qquad a_{4'} = A_{4'}^4 a_4. \tag{23}$$

All other four-dimensional tensors transform like the correspondingly-chosen products of such vectors.

It next transpires that the nine $g^{\alpha\beta}$ components of the four-dimensional g^{ik} tensor only transform amongst each other, and in fact, like the components of a spatial tensor, according to the formula:

$$g^{\alpha'\beta'} = A_\alpha^{\alpha'} A_\beta^{\beta'} g^{\alpha\beta}. \tag{25}$$

We have not yet encountered this "spatial tensor" along the physically-intuitive path, but it represents nothing but the doubly-contravariant form of the "fundamental spatial tensor" $\gamma_{\alpha\beta}$, as a simple calculation will yield:

$$g^{\alpha\beta}\,\gamma_{\beta\gamma} = \delta^{\alpha}_{\gamma}\,. \tag{26}$$

The tensor properties of the $\gamma_{\alpha\beta}$ are obvious from (23), (24) directly.

In order to verify the tensor properties of (11) and (13), we must direct our attention to the transformation formulas for the three-index symbols:

$$\begin{Bmatrix} i' \\ j'\,k' \end{Bmatrix} = A^{l'}_{l} A^{j}_{j'} A^{k}_{k'} \begin{Bmatrix} i \\ j\,k \end{Bmatrix} + A^{l'}_{l} \frac{\partial^2 x^l}{\partial x^{j'}\,\partial x^{k'}}\,. \tag{27}$$

As is known, the last term has the effect of saying that this symbol does not define a four-dimensional tensor. However, in our case, that term will vanish especially for the **Christoffel** symbols that appear in (11) and (13).

7. – As is known, in a metric field, one can always choose the space-time coordinates in such a way that $g_{4\alpha}$ vanishes for the entire field. However, the reference observer must change, in general. It is easy to give a *necessary and sufficient condition for the possibility of introducing everywhere-orthochronous coordinates into a given gravitational field.*

To that end, we consider the following transformation formulas, in which the conditions (21) have already been considered:

$$g_{4\alpha} = A^{4'}_{4} A^{\alpha'}_{\alpha}\, g_{4'\alpha'} + A^{4'}_{4} A^{4'}_{\alpha}\, g_{4'4'}\,, \tag{28}$$

$$g_{44} = A^{4'}_{4} A^{4'}_{4}\, g_{4'4'}\,. \tag{29}$$

If we now demand that $g_{4'\alpha'} = 0$ then when we divide the first equation by the second one and consider (18) and (20), we will obtain:

$$\frac{\partial x^{4'}}{\partial x^{\alpha}} - a_{\alpha}\frac{\partial x^{4'}}{\partial x^{4}} = 0. \tag{30}$$

That is a system of three partial differential equations in four variables. In order for it to possess a non-constant solution, it must be *complete* ([1]). One must then have:

$$\left(\frac{\partial}{\partial x^{\alpha}} - a_{\alpha}\frac{\partial}{\partial x^{4}}\right)\left(\frac{\partial}{\partial x^{\beta}} - a_{\beta}\frac{\partial}{\partial x^{4}}\right) = 0. \tag{31}$$

The calculation of this operator implies the desired condition, which one can give in the form:

$$\omega_{\alpha\beta} = 0. \tag{32}$$

([1]) See, e.g., **L. P. Eisenhart**, *Continuous Groups of Transformations*, 1933. Theorem (2.1).

We then see, e.g., that it is impossible to introduce everywhere-orthochronous space-time coordinates on a rotating disc (relative to a **Galilean** system) – viz., the disc is considered to be the reference observer. In other words: It is impossible to calibrate the clocks on the disc in such a way that all light signals will run symmetrically; i.e., with the same velocities in opposite directions.

8. – The characteristic tensors of the gravitational field allow one to sort those fields into various types. Of particular interest are, e.g., gravitational fields with vanishing γ^{α}. One can call them, say, *neutral* when they "point in no direction." Free mass-points at rest will remain at rest in them, or – speaking four-dimensionally – the world-lines of points at rest are geodetic. In recent times, the neutral gravitational fields have played the role of "cosmological backgrounds." In **Newton**'s theory of gravitation, a "neutral" field was identical with a "vanishing" field – i.e., with a **Galilean** system – which is obviously not the case here in the general theory of relativity.

When $\omega_{\alpha\beta}$ vanishes, we can speak of an *irrotational* field (the fields of gravitational vortices have still not been investigated very much), and in the same way, we can speak of a *dilatation-free* field when $\varepsilon_{\alpha}^{\alpha} = 0$, etc.

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