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Ruled surfaces of fixed twist with constantly-twisted bands of striction

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1. Introduction. – A concept that was first suggested by G. Sannia [**12**] and later developed by E. Kruppa in vector notation [**6**] was that of a “natural geometry” of ruled surfaces in three-dimensional Euclidian space, which employed an orthonormal *moving triad* that is attached to the central point Z and generates the surface and consists of the generating vector $\boldsymbol{e}_1$, the surface normal $\boldsymbol{e}_2$, and the central tangent $\boldsymbol{e}_3 = \boldsymbol{e}_1 \times \boldsymbol{e}_2$. If $\boldsymbol{z} = \boldsymbol{z}(s)$ is the vectorial representation of the line of striction or throat curve that goes through the central point Z, when referred to the arc length s, then the ruled surface Φ can be described by means of the independent parameters s and t :

$$\boldsymbol{x} = \boldsymbol{z}(s) + t \cdot \boldsymbol{e}_1(s) . \tag{1.1}$$

Assuming that they are suitably differentiable, the derivatives of the triad vectors with respect to s will satisfy the fundamental *differential equations:*

$$\boldsymbol{e}_1' = \kappa \boldsymbol{e}_2 , \qquad \boldsymbol{e}_2' = - \kappa \boldsymbol{e}_2 + \tau \boldsymbol{e}_3 , \qquad \boldsymbol{e}_3' = - \tau \boldsymbol{e}_2 . \tag{1.2}$$

The invariants of motion $\kappa =$ and $\tau = \boldsymbol{e}_2' \boldsymbol{e}_3 = - \boldsymbol{e}_3' \boldsymbol{e}_1$ that appear in them are called the natural *curvature* (*torsion*, resp.) of the ruled surface Φ. As a third invariant, one can add the *angle of striction* $\sigma = \sphericalangle\, \boldsymbol{e}_1 \boldsymbol{z}'$ that the generator makes with the line of striction k ; it is defined by:

$$\boldsymbol{z}' = \boldsymbol{e}_1 \cos \sigma + \boldsymbol{e}_2 \sin \sigma , \tag{1.3}$$

and will usually be restricted to the interval $-\pi / 2 < s \leq \pi / 2$.

The surface Φ is determined completely, up to motions (and thus, in form), by the “natural equations” $\kappa = \kappa(s)$, $\tau = \tau(s)$, and $\sigma = \sigma(s)$ [**6**]. One has $\sigma \equiv 0$ for the torse (†) Φ that is defined by the tangents to the space curve k, and the differential equations (1.2) are

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(†) Translator: Apparently, the German word *Torse* refers to a developable surface.

the well-known formulas of F. Frenet, in which κ means the curvature of k, and τ means the torsion.

The ruled surface Ψ that is defined by the central tangents will be referred to as the *band of striction* of Φ, as long as not all central tangents coincide, which is what happens for the erect conoid. With the exception of those surfaces, as well as the conoidal surfaces, which are characterized by $\tau \equiv 0$ (and their generators are parallel to a fixed plane of directions, such that their central tangents fill up a cylinder), Φ and the band of striction Ψ that is represented by:

$$\text{(1.4)} \qquad \boldsymbol{y} = \boldsymbol{z}\,(s) + t \cdot \boldsymbol{e}_3\,(s)$$

have the throat curve k in common, and have a reciprocal relationship; i.e., Φ is the band of striction of Ψ [**2**, **3**, **7**]. If $\sigma \not\equiv 0$, moreover (and therefore Φ is not developable), and $\sigma \not\equiv \pi / 2$ (Ψ is not developable) then the two surfaces will contact each other along their common line of striction k. The possibility of *osculation* along k, which has not been observed up to now, was examined in [**14**]; the identity:

$$\text{(1.5)} \qquad \kappa \cos \sigma + \tau \sin \sigma = 0$$

is definitive for that phenomenon.

Finally, another important invariant for ruled surfaces is the so-called *twist*, which represents a measure for the winding of the surface strips along a generator and can be described as the limiting value for the quotient of the distance and angle between two generators as they move closer together. According to [**2**, **6**], two corresponding generators of the surfaces Φ (1.1) and Ψ (1.4) have the following values for their twists:

$$\text{(1.6)} \qquad p = \frac{\sin \sigma}{\kappa}, \qquad q = \frac{\cos \sigma}{\tau}.$$

One will have the relation $p + q = 0$ for osculating pairs of surfaces as a result of (1.5). As was shown in [**14**], the possibility of hyperosculation is characterized by fixed values of the twists p and $q = - p$. That fact immediately raises the question of whether there are pairs of surfaces Φ, Ψ with mutually-independent *constant values of their twists p* and q, in general. That problem is the topic of the present investigation, which is naturally based in the conditions (1.6) with p = const. and q = const. The results will possibly make a welcome contribution to the sphere of questions that relate to constantly-twisted ruled surfaces, which were expressed many times in connection with J. Krames [**5**, **7**], and above all, H. Brauner and his students [**1**].

2. Pairs of surfaces with prescribed direction cones at the central torse. – A prescribed *central torse* Γ that contacts a skew ruled surface Φ (and its band of striction Ψ) along the throat curve k is especially significant for all metric-related questions that pertain to that surface. As the envelope of the family of central planes, it is determined by:

$$(2.1) \qquad (\boldsymbol{x} - \boldsymbol{z}(s)) \cdot \boldsymbol{e}_2(s) = 0 .$$

As differentiation with respect to the parameter s of the family will show, its generator will have the direction vector:

$$(2.2) \qquad \boldsymbol{d} = \boldsymbol{e}_2 \times \boldsymbol{e}_2' = \tau \boldsymbol{e}_1 + \kappa \boldsymbol{e}_3 .$$

That is the so-called *Darboux rotation vector*, with whose help, the differentiation formulas (1.2) can be combined into $\boldsymbol{e}_i' = \boldsymbol{d} \times \boldsymbol{e}_i$ (i = 1, 2, 3). It has a magnitude:

$$(2.3) \qquad |\,\boldsymbol{d}\,| = |\,\boldsymbol{e}_2'\,| = \sqrt{\kappa^2 + \tau^2} = \lambda ,$$

which is referred to as *Lancretian* or "*total*" *curvature* [**2**]. $\boldsymbol{d}$ plays the role of instantaneous axes for motion of the triad ($\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$) that is attached to the origin O that results when the central point Z moves along the throat curve k.

The aforementioned collective motion from O can be regarded in terms of the elements of spherical kinematics as the rolling of the plane that is spanned by $\boldsymbol{e}_1$ and $\boldsymbol{e}_3$ on the *direction cone of the central torse* Γ that is filled up by the instantaneous axes $\boldsymbol{d}$. Its trace curve g on the unit cone at O will be described by the vector $\boldsymbol{g} = \boldsymbol{d} / \lambda$, which is normalized to have unit length. $\boldsymbol{g}$ is "polar" to the spherical image of the normals $c_2 : \boldsymbol{e}_2 = \boldsymbol{e}_2(s)$, in the sense of spherical geometry, while the spherical image of the generators c_3: $\boldsymbol{e}_3 = \boldsymbol{e}_3(s)$, of Φ appears as the spherical evolvent of g.

As was mentioned before in [**14**], if one is then given the spherical image of the central torse $\boldsymbol{g}$, referred to its arc length u, by way of:

$$(2.4) \qquad \boldsymbol{g} = \boldsymbol{g}(u) \qquad \text{with} \qquad \boldsymbol{g}^2 = \dot{\boldsymbol{g}}^2 = 1$$

then one will have the following representation of the triad vectors:

$$(2.5) \qquad \boldsymbol{e}_1 = \boldsymbol{g}\cos u - \dot{\boldsymbol{g}}\sin u, \qquad \boldsymbol{e}_2 = \dot{\boldsymbol{g}} \times \boldsymbol{g}, \qquad \boldsymbol{e}_3 = \boldsymbol{g}\sin u + \dot{\boldsymbol{g}}\cos u .$$

One has:

$$(2.6) \qquad \dot{\boldsymbol{e}}_1 = -(\boldsymbol{g} + \ddot{\boldsymbol{g}})\sin u, \quad \dot{\boldsymbol{e}}_2 = \ddot{\boldsymbol{g}} \times \boldsymbol{g}, \qquad \dot{\boldsymbol{e}}_3 = -(\boldsymbol{g} + \dot{\boldsymbol{g}})\cos u$$

for the derivatives with respect to u . One easily convinces oneself that $\boldsymbol{g} + \ddot{\boldsymbol{g}}$ has the same direction as $\boldsymbol{e}_2$ with the help of the relations $\boldsymbol{g}\,\dot{\boldsymbol{g}} = 0$, $\dot{\boldsymbol{g}}\,\ddot{\boldsymbol{g}} = 0$, which follow from the conditions (2.1), and $\boldsymbol{g}\,\ddot{\boldsymbol{g}} = -1$. If one has:

$$(2.7) \qquad \boldsymbol{g} + \ddot{\boldsymbol{g}} = -\mu\,\boldsymbol{e}_2 ,$$

accordingly, then one will see from:

$$(2.8) \qquad \dot{\boldsymbol{e}}_2 = -\mu\, \boldsymbol{e}_2 \times \boldsymbol{g} = \mu\, \boldsymbol{g} \times (\dot{\boldsymbol{g}} \times \boldsymbol{g}) = \mu\, \dot{\boldsymbol{g}}$$

that:

$$(2.10) \qquad \mu = (\boldsymbol{g},\ \dot{\boldsymbol{g}},\ \ddot{\boldsymbol{g}})$$

means the *conical curvature of the central torse* Γ. With consideration given to $\dot{\boldsymbol{e}}_1 = \boldsymbol{e}_1' \cdot \dot{\boldsymbol{s}}$, a comparison of the differential formulas (1.2) and (2.6) will then yield the relations:

$$(2.10) \qquad \mu \dot{\boldsymbol{s}} = \mu \sin u, \qquad \tau \dot{\boldsymbol{s}} = \mu \cos u, \qquad \lambda \dot{\boldsymbol{s}} = \mu .$$

With the use of the prescribed (constant) values of the twists p and q, (1.6) then yield the following relations for the *angle of striction* σ:

$$(2.11) \quad \sin \sigma = p\ \kappa = p\ \lambda \sin u, \qquad \cos \sigma = q\ \tau = p\ \lambda \cos u, \qquad \tan \sigma = (p\ /\ q) \tan u .$$

That implies the following expressions for the *curvature* and *torsion:*

$$(2.12) \qquad \kappa = \lambda \sin u, \qquad \tau = \lambda \cos u, \qquad \text{with} \qquad \lambda = 1/\sqrt{p^2 \sin^2 u + q^2 \cos^2 u} .$$

The form of the desired ruled surface Φ will be determined once the basic invariants κ; τ, and σ are known as functions of u and the connection between the auxiliary parameter u and the arc length s of the throat curve is given by $\dot{s} = \mu\ /\ \lambda$, with μ as in (2.9). The spatial position is also fixed, up to translations, on the basis of the prescribed direction cone (2.4) of the central torse. In order to ascertain Φ, it is, above all, necessary for one to know the tangent vector to the throat curve, and according to (1.3) and (2.5), it will be represented by:

$$(2.13) \qquad \boldsymbol{z}' = \boldsymbol{g} \cos (\sigma - u) + \dot{\boldsymbol{g}}\ \sin (\sigma - u) .$$

Eliminating σ by means of (2.11) then yields:

$$(2.14) \qquad \dot{z} = \boldsymbol{z}' \dot{s} = [(p \sin^2 u + q \cos^2 u)\, \boldsymbol{g} + (p - q) \sin u \cos u \cdot \dot{\boldsymbol{g}}\,]\, \lambda \dot{s} ,$$

and finally when one integrates this, while keeping (2.10) in mind, one will get the *line of striction k* :

$$(2.15) \qquad \boldsymbol{z} = \tfrac{1}{2} \int \{[(p+q) - (p-q) \cos 2u]\, \boldsymbol{g} + (p-q) \sin 2u \cdot \dot{\boldsymbol{g}}\}\ \mu\, du ;$$

μ is as it is in (2.9) in this. – A parametric representation of the desired *ruled surface* Φ with (1.1) as a model will then read:

(2.16) $$\boldsymbol{x} = \boldsymbol{z}(u) + t \cdot \boldsymbol{e}_1(u), \quad \text{with} \quad \boldsymbol{e}_1 = \boldsymbol{g}\cos u - \dot{\boldsymbol{g}}\sin u .$$

One can simply replace $\boldsymbol{e}_1$ with $\boldsymbol{e}_1 = \boldsymbol{g}\sin u + \dot{\boldsymbol{g}}\cos u$ for the *band of striction* Ψ.

3. Special cases. – If the twist quotient p / q assumes one of the distinguished values 0, , + 1, or – 1 then that will define ratios of a special kind.

a) $p = 0$, $q \neq 0$. As a result of (2.11), the ruled surface Φ is a *torse* ($\sigma = 0$) with a *ridge line* (†) *k of constant torsion* $\tau = 1/q$ and the band of striction Ψ is its binormal surface. The curve k is the *geodetic line of the central torse* Γ and will be represented according to (2.15) by:

(3.1) $$\boldsymbol{z} = q\int \mu(\boldsymbol{g}\cos u - \dot{\boldsymbol{g}}\sin u)\cos u\, du .$$

b) $q = 0$, $p \neq 0$. Conversely, Φ is the *binormal surface of a curve of constant torsion* $1 / p$ in this, and Ψ is its tangent surface.

c) $p = q > 0$. As a result of (2.12), Φ and Ψ are *ruled surfaces of constant Lancretian curvature* $\lambda = 1 / p$ with the throat curve $k : \boldsymbol{z} = p \int \mu\, \boldsymbol{g}\, du$. Due to (2.11), one has $\sigma = u$ for the angle of striction, such that from (2.13), one will have $\boldsymbol{z}' = \boldsymbol{g}$. That means that the generator of the central torse always coincides with the tangent to the line of striction k, so the *throat curve is the line of osculation* of the surfaces Φ and Ψ, which agrees with the criterion $\kappa\cos\sigma = \tau\sin\sigma$ that E. Kruppa [**6**] gave for an asymptotic line of striction. – If the generators of the surface Φ are carried along with the flattening (*Verebnung*) of the central torse Γ then from a known theorem of G. Darboux [**2**, **4**, **6**], it would go to the rays of a pencil of parallels. That property – viz., that the generators of the surface are geodetically parallel along the line of striction – is characteristic of the line of striction. If all generators of Φ were rotated through the same angle α in the central plane around its central point then that property would still be true, and a new ruled surface Φ_α would arise that would have the throat curve k and the central torse Γ in common with $\Phi = \Phi_0$. Since the surface normal $\boldsymbol{e}_2$ has not changed, the Lancretian curvature λ (2.3) will also be preserved. All that will change is that σ will change to $\bar{\sigma} = \sigma + \alpha$, and corresponding to the rotation of $\boldsymbol{e}_1$ to $\bar{\boldsymbol{e}}_1 = \boldsymbol{e}_2\cos\alpha - \boldsymbol{e}_3\sin\alpha$, the curvature κ will go to $\bar{\kappa} = \kappa\cos\alpha + \tau\sin\alpha$. In the present case, in agreement with a remark of J. Krames [**7**, pp. 147] and a formula for the twist of Φ_α that H. Sachs [**10**] derived, that will imply that *all derived surfaces* Φ_α *possess the same constant twist p* (including the band of striction $\Psi = \Phi_{\pi/2}$).

d) $p = -q < 0$. In this case, the ruled surfaces Φ and Ψ also have *constant Lancretian curvature* $\lambda = 1 / q$, as a result of (2.12). Furthermore, due to (2.11), one will have $\sigma = -u$ for the angle of striction, such that the *osculation condition* (1.5) is fulfilled.

(†) Translator: If a torse is a developable surface then presumably its ridge line is its edge of regression.

As was proved in [**14**], one will even have *hyper-osculation* between the surfaces Φ and Ψ as a result of the constant twist. – From (2.15), the throat curve k will be represented by:

$$\mathbf{z} = q\int \mu(\mathbf{g}\cos 2\mu - \dot{\mathbf{g}}\sin 2u)\,du\,. \tag{3.2}$$

4. Pairs of surfaces with a central torse of constant slope (*Böschung*)**.** – In order to illustrate the developments in Section **2**, we will further invoke the assumption that the central torse Γ is a *slope torse.* The direction cone is then a right cone with a well-defined vertex angle 2ω, in which we will assume that $0 < \omega < 2\pi$, in order to exclude Φ that are cylinders or planes. The spherical image of the torse g will then be a minor circle of the unit sphere that might be set to:

$$\mathbf{g} = \left(n\cos\frac{u}{n}, n\sin\frac{u}{n}, m\right), \qquad \text{with } n = \sin\omega,\ m = \cos\omega. \tag{4.1}$$

The conical curvature of the central torse has the value $\mu = m / n = \cot\omega$.

With the use of the abbreviation:

$$v = u / n, \tag{4.2}$$

one will find the following components for the tangent vector $\dot{\mathbf{z}} = (\dot{z}_1, \dot{z}_2, \dot{z}_3)$ to the line of striction:

$$\begin{aligned}
\dot{z}_1 &= \frac{m}{2}(p+q)\cos v + \frac{m}{4n}(p-q)\,[(1-n)\cos(v+2u) - (1+n)\cos(v-2u)],\\
\dot{z}_2 &= \frac{m}{2}(p+q)\sin v + \frac{m}{4n}(p-q)\,[(1-n)\sin(v+2u) - (1+n)\sin(v-2u)],\\
\dot{z}_3 &= \frac{m^2}{2n}(p+q) - \frac{m^2}{2n}(p-q)\cos 2u\,.
\end{aligned} \tag{4.3}$$

If one interprets the parameter u as time then (4.3) will represent the *velocity diagram* of the central point Z as it moves along the throat curve k. According (2.14), the end point of the velocity vector moves on a circle in the tangent plane to the right cone of revolution that is spanned by the orthogonal vectors $\mathbf{g}$ and $\dot{\mathbf{g}}$:

$$\dot{\mathbf{z}} = \frac{m}{2n}[(p+q) - (p-q)\cos 2u]\,\mathbf{g} + \frac{m}{2n}(p-q)\sin 2u\cdot\dot{\mathbf{g}} \tag{4.4}$$

with constant angular velocity 2, while as a result of (4.1) that plane will rotate around the axis of the cone with angular velocity $1 / n$. One then sees that the curve of the diagram is a *spherical cycle.* It moves on the cone:

$$\dot{z}_1^2 + \dot{z}_2^2 + \dot{z}_3^2 - \frac{p+q}{n}\dot{z}_3 + \frac{m^2}{n^2}pq = 0, \tag{4.5}$$

and it will arise when one rolls a cone of revolution with vertex angle $\pi - 2\omega$ on a cone of revolution with vertex angle 4ω.

If one integrates (4.3), while recalling (4.2), then one will arrive at the *throat curve k:*

$$z_1 = \frac{mn}{2}(p+q)\sin v + \frac{m}{4}(p-q)\left[\frac{1-n}{1+2n}\sin(v+2u) - \frac{1+n}{1-2n}\sin(v-2u)\right],$$

$$z_2 = -\frac{mn}{2}(p+q)\cos v - \frac{m}{4}(p-q)\left[\frac{1-n}{1+2n}\cos(v+2u) - \frac{1+n}{1-2n}\cos(v-2u)\right], \tag{4.6}$$

$$z_3 = \frac{m^2}{2}(p+q)\,v - \frac{m^2}{4n}(p-q)\sin 2u, \qquad \text{in which} \qquad u = n\,v.$$

It will generally be assumed that $n \neq 1/2$ in this. The assumption that $n = 1/2$ requires special treatment and shall not be pursued further here. – In the case of $n \neq 1/2$, one can read off from the complex combination:

$$z_1 + i\,z_2 = -\frac{im}{4}e^{iv}\left[2n(p+q) + (p-q)\left(\frac{1-n}{1+2n}e^{2iu} - \frac{1+n}{1-2n}e^{-2iu}\right)\right] \tag{4.7}$$

that the line of striction *k* generally traverses a *helicoid*, which will be described in terms of the independent parameters *u* and *v* by the representation (4.6). It will arise, perhaps, by screwing the profile curve $v = 0$ around the x_3-axis, and thus an ellipse that moves in the plane:

$$m(1-4n^2)\,z_1 + 2n(1+2n^2)\,z_3 = 0, \tag{4.8}$$

which is independent of *p* and *q*. The pitch of the screw has the value $c = m^2(p+q)/2$. Under a uniform screwing motion, the central point that describes the throat curve will also move on the profile ellipse with constant surface velocity.

With the use of the normal vector $\boldsymbol{e}_2 = \dot{\boldsymbol{g}} \times \boldsymbol{g} = (m\cos v,\ m\sin v,\ -n)$ that is established by (2.5), in which *v* once more stands for u / n, one will get the equation:

$$x_1\cos v + x_1\sin v - \frac{n}{m}x_3 = \frac{3m(p-q)}{4(1-4n^2)}\sin 2u - \frac{m}{2}(p+q)\,u \tag{4.9}$$

for the central plane (2.1). If one again sets $u = n\, v$ on the right-hand side then one will see (in general, and always under the assumption that $n \neq 1/2$) that the family (4.9) will run through a plane that has been subjected to a "progressive harmonic reversal." That is a motion that is composed of a uniform screwing motion around the x_3-axis (with the aforementioned pitch c) and a harmonic oscillation along that axis (with frequency $2n$) [**13**]. With the name that O. Obůrka [**8**] introduced, the *central torse* Γ that is enveloped by the planes will then be a *vibratory torse.*

As was shown in [**13**], the ridge line of such a vibratory torse appears in outline as the parallel curve of an epicycloid or hypocycloid and will likewise go to another such curve when one develops the torse. In the present case (although we shall not go further into the basis for this), under the assumption that $n \neq 1/2$, as the outline of a parallel curve that moves at a distance $mn\,(p + q) / 2$, it will be a cycloid of the family $(1 - 2n) : (1 + 2n)$ with an azimuthal radius of $3mn\,(p - q) / 2\,(1 - 4n^2)$, while as a flattening, it will always be a *parastroid*, namely, a parallel curve that runs at a distance $m\,(p + q) / 2n$ from an astroid (viz., a hypocycloid with four vertices) with an vertex circle of radius of $m\,(p - q) / 2n$. Under the flattening of the central torse, the throat curve k (4.6) will go to an *ellipse* with a semi-axes mp / n and mq / n that contacts the parastroid at the four vertices.

Finally, the desired ruled surface Φ of constant twist, as well as its band of striction Ψ, can be written down with no difficulty on the basis of the representation (2.16), although we shall go into that. From Darboux (cf., Section 3c), under flattening of the central torse, the generators of the two surfaces will go to mutually orthogonal families of parallels whose directions will be those of the axes of the aforementioned ellipse.

a) $p = 0$, $q \neq 0$. Here, the line of striction k (4.6) is a *curve of constant torsion with a fixed inclination of the principal normal*, which occasionally appeared in E. Salkowski [**11**]; it is a geodetic line of the vibratory torse Γ (4.9). The associated ruled surface Φ is developable and consists of the tangents to k ; the band of striction Ψ is the binormal surface of k.

b) $q = 0$, $p \neq 0$. Here, the roles of the surfaces Φ and Ψ in *a*) are switched.

c) $p = q \neq 0$. In this case, the throat curve k (4.6) takes the form of a *helix:*

$$z_1 = mnp \sin v, \qquad z_2 = -\, mnp \cos v, \qquad z_3 = m^2\, p\, v, \tag{4.10}$$

which is simultaneously the ridge of the central torse Γ (4.9). The ruled surface Φ then belongs to surfaces that were investigated by J. Krames [**4**], which have a helical torse for their central torse and one of the helices that belong to it as the throat curve. One is then dealing with the most general ruled surfaces that are congruent to all of the ones that are derived from them by rotating the generators. The fact that each special case of them whose line of striction coincides with the ridge of the helical torse is distinguished by constant twist was pointed out in [**4**]. The value of the twist $p = c / m^2$ is equal to the radius of torsion of the ridge helix, which agrees with more general facts (cf., [**7**, pp. 147], [**1**], [**10**]).

d) $p = -q < 0$. In this case, the pitch c vanishes, such that (for $n \neq 1/2$) a *reversal torse* (with frequency $2n$), which W. Kautny [**3**] considered, will appear in place of the vibratory torse (4.9) as the *central torse*:

$$x_1 \cos v + x_2 \sin v - \frac{n}{m} x_3 = \frac{3mq}{2(1-4n^2)} \sin 2nv . \tag{4.11}$$

The ridge line of such a torse is the slope line of a quadric of rotation and appears in outline as an epicycloid or a hypocycloid and goes to an astroid under flattening of the torse. The *throat curve* k, which is described by:

$$\begin{aligned} z_1 &= -\frac{mq}{2}\left[\frac{1-n}{1+2n}\sin(v+2u) - \frac{1+n}{1-2n}\sin(v-2u)\right], \\ z_2 &= \frac{mq}{2}\left[\frac{1-n}{1+2n}\cos(v+2u) - \frac{1+n}{1-2n}\cos(v-2u)\right], \\ z_3 &= \frac{m^2 q}{2n}\sin 2u, \qquad \text{(with } u = nv\text{)}, \end{aligned} \tag{4.12}$$

traverses a quadric of rotation and maps to a cycloid (but with no vertex) under horizontal projection. Due to the fact that $\sigma = -u$, one finds the constant value $\gamma = 1/\dot{s} = \lambda / \mu = n/mq$ for its geodetic curvature from (2.10). The line of striction k then goes to a *circle* of radius mq / n under the flattening of the central torse, and indeed to the vertex circle of the aforementioned astroid. – The associated *ruled surfaces* Φ and Ψ belong to the ruled surfaces that were recently studied by G. Pillwein [**9**], which have fixed Lancretian curvature, a constantly-inclined central torse, and throat curve of constant geodetic curvature. The ones that appear here are distinguished by *hyper-osculating bands of striction*, and are noteworthy for that reason, since algebraic surfaces belong to them, namely, for rational $n \neq 1/2$. The algebraic character was remarked in [**14**]; the simplest example occurs as a surface of degree seven when $n = 1/4$.

5. Pairs of surfaces with conical central torse. – If the vertex of the *central cone* G that is now assumed is employed as the coordinate origin then the throat curve k can be set to:

$$\boldsymbol{z} = r \cdot \boldsymbol{g}, \qquad \text{with} \quad r = r(u), \qquad \boldsymbol{g} = \boldsymbol{g}(u), \qquad \boldsymbol{g}^2 = \dot{\boldsymbol{g}}^2 = 1, \tag{5.1}$$

based upon (2.4). A comparison of the tangent vector:

$$\dot{\boldsymbol{z}} = \dot{r}\,\boldsymbol{g} + r\,\dot{\boldsymbol{g}} \tag{5.2}$$

with the decomposition (2.13) will lead to:

(5.3) $$\frac{\dot r}{r} = \cot(\sigma - u) = \frac{\cos\sigma\cos u + \sin\sigma\sin u}{\sin\sigma\cos u - \cos\sigma\sin u}.$$

With hindsight of (2.11), one will then have for $p \neq q$:

(5.4) $$\frac{\dot r}{r} = \frac{q\cos^2 u + p\sin^2 u}{(p-q)\sin u\cos u} = m\tan u + n\cot u, \quad \text{with} \quad m = \frac{p}{p-q}, \quad n = \frac{q}{p-q}.$$

Now, with:

(5.5) $$r = c\sin^n u / \cos^m u \quad (\text{whereby } m - n = 1),$$

integration will yield the polar equation for the *flattened line of striction k*. After going to Cartesian coordinates $x = r\cos u$, $y = r\sin u$, it can be represented by:

(5.6) $$x^m y^{1-m} = c \qquad \text{or} \qquad y = c' \cdot x^{p/q}.$$

As a result of (5.3), the generators of the surface-pair Φ, Ψ will take on the directions of the x-axis (y-axis, resp.) under the flattening of the central cone.

One finds that the *conical curvature* $\mu = \lambda\dot s$ of the still-to-be-determined central cone Γ, with $\dot s^2 = \dot z^2 = r^2 + \dot r^2 = r^2 / \sin^2(\sigma - u)$, is:

(5.7) $$\mu = \frac{\pm\lambda r}{\sin(\sigma - u)} = \frac{\pm r}{(p-q)\sin u\cos u} = \frac{\pm c}{p-q}\cdot\frac{\sin^{m-2} u}{\cos^{m+1} u}.$$

If one imagines that u means the arc length of the spherical image of the central cone g : viz., $\boldsymbol{g} = \boldsymbol{g}(u)$ and μ means its geodetic curvature then the form of the *natural equation* $\mu = \mu(u)$ of that curve g that is known from (5.7) will be determined. An explicit coordinate representation is not generally possible, even in special cases. In order to arrive at a definite picture for the course of the spherical curve g, one might first ascertain the *plane curve* $\bar g$ that is established by the dependency of its curvature μ on the arc length s. It can be described in Cartesian coordinates in a known way by:

(5.8) $$\bar x = \int \cos\varphi \cdot du, \qquad \bar y = \int \sin\varphi \cdot du \qquad \text{with} \qquad \varphi = \int \mu\, du,$$

which can each be established by means of graphical or numerical integration. If one then imagines a narrow strip of paper with a midline $\bar g$ being “ironed onto” the unit sphere then it will assume the form of g, which can be quite complicated.

J. Krames [5] carried out the determination of constantly-twisted ruled surfaces with central cones by starting from the polar equation $r = r(u)$ of the flattened throat curve, and he addressed numerous examples. – In regard to the *special cases* that were mentioned in Section **3**, it should be remarked that:

a) $p = 0$, $q \neq 0$ ($m = 0$). Consistent with the rectilinear flattening (5.6) of their throat curve *k*, the surface Φ will be the *tangent surface to a geodetic cone of constant torsion* $\tau = 1 / q$, and as such, it will *circumscribe a cone.* The band of striction Ψ will be the binormal surface of *k*.

b) $q = 0$, $p \neq 0$. The roles of the surfaces Φ and Ψ in *a*) are switched here.

d) $p = -q$ ($m = 1/2$). The surface-pair Φ, Ψ that is distinguished by hyper-osculation was mentioned already in [**14**], where the flattening of the line of striction was known to be an equilateral hyperbola, which is also in agreement with (5.6).

6. Surface-pairs with cylindrical central torses. – Under the assumption of a *central cylinder* Γ, the image of the central torse will contract to a point, and the spherical image of the generators c_1 of the ruled surface Φ will be a circle. Its moving triad can be exhibited by:

$$\mathbf{e}_1 = (\ a \sin\varphi,\ -a\cos\varphi,\ b),\qquad \mathbf{e}_2 = (\ \cos\varphi,\ \sin\varphi,\ 0),\qquad \mathbf{e}_3 = (-b\sin\varphi,\ b\cos\varphi,\ a),\qquad \text{in which}\quad a^2 + b^2 = 1, \tag{6.1}$$

with non-vanishing constants *a* and *b*. A comparison of the derivatives (1.2) with respect to the arc length *s* of the throat curve with the ones with respect to φ, namely:

$$\dot{\mathbf{e}}_1 = a\,\mathbf{e}_2\,,\qquad \dot{\mathbf{e}}_2 = -a\,\mathbf{e}_1 + b\,\mathbf{e}_3\,,\qquad \dot{\mathbf{e}}_3 = -b\,\mathbf{e}_2\,, \tag{6.2}$$

will yield the relations:

$$\kappa\dot{s} = a,\qquad \tau\dot{s} = b\,. \tag{6.3}$$

With the conditions (1.6), the demand of constant twist for the surface-pair Φ, Ψ will lead to:

$$\tan\sigma = p\kappa / q\tau = ap / bq = \text{const.}, \tag{6.4}$$

and thus, to the *constancy of the angle of striction* σ. However, and again with hindsight of (1.6), one will also have:

$$\kappa = \sin\sigma / p = \text{const.},\quad \tau = \cos\sigma / q = \text{const.} \tag{6.5}$$

However, the constancy of all three basic invariants implies that the surface Φ, and therefore, its band of striction Ψ, as well, will generally be *ruled helicoids.* With the use of (13), and if one recalls that $\dot{s}\sin\sigma = ap$ and $\dot{s}\cos\sigma = bq$, their *throat helices k* will be represented by:

$$z_1 = ab\,(p - q)\cos\varphi\,,\qquad z_2 = ab\,(p - q)\sin\varphi\,,\qquad z_3 = (a^2 p + b^2 q)\,\varphi\,. \tag{6.6}$$

In regard to the *special cases* that were cited in Section **3**, it should be pointed out that:

a) $p = 0$, $q \neq 0$. Φ is a *screw torse*, Ψ is the binormal surface of the ridge helix.
b) $q = 0$, $p \neq 0$. Φ and Ψ switch roles here.

c) $p = q \neq 0$. The line of striction k (6.6) coincides with the screw axis, so Φ and Ψ are then (skew) *closed ruled helicoids.*

d) $p = -q \neq 0$. Here, one is dealing with the *hyper-osculation* of the pair of ruled surfaces Φ, Ψ that was mentioned in [**14**].

The *exceptional case* of vanishing pitch ($a^2 p + b^2 q = 0$) that one can abstract from (6.6) should be emphasized. In that case, the throat curve k will be a circle, and the surface-pair Φ, Ψ will consist of two *hyperboloids of rotation* of one sheet, which will even merge together in the special case *d*) ($p = -q$, $a = b = 1/\sqrt{2}$, $\sigma = 1/4$).

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