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Conditions on a flexible, inextensible surface

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We consider the problem: How to recognize whether a one-parameter family of *m*-dimensional spaces that is embedded in an *n*-dimensional Riemannian space represents a sequence of positions of a flexible and inextensible space (a sequence of applicable spaces). The method employed is a generalization of the tensor calculus and will permit us to write out the desired conditions in a form that is independent of the mode of representation of the family.

The equations of our family are:

$$y_\alpha = f_\alpha (x_k , t) \qquad (\alpha = 1, \dots, n ; k = 1, \dots, m) . \tag{1}$$

Moreover, that representation establishes a correspondence between the various V_m (t) of the family (for equal values of x_k), which generally has no relationship with the isometry that possibly exists between all the (1). Here, one is dealing with a property that is independent of the chosen representation and persists for the kinematical transformations:

$$\bar{x}_i = x_i (x_k , t) \qquad (i = 1, \dots, m) . \tag{2}$$

We say that a vector is *strongly-contravariant* when it consists of a system of *m* functions u^i that transform under (2) according to the well-known formulas:

$$\bar{u}^i = \frac{\partial \bar{x}_i}{\partial x_k} u^k , \tag{3}$$

which do not depend upon *t*, like (2). One defines strongly-covariant and strongly-contravariant tensors analogously.

The sequence V_m (t) generates an (m + 1)-dimensional Riemannian space V_{m+1} whose metric form can be written:

$$ds^2 = a_{ik}\, dx_i\, dx_k + 2\, h_i\, dx_i\, dt + k\, dt^2 . \tag{4}$$

(x_i, t) define a coordinate system in V_{m+1} , so (2) is a change of coordinates that does not alter the spaces t = const. The metric in V_{m+1} for a given value of t will be given by $ds^2 = a_{ik}\, dx_i\, dx_k$. We let ∇ denote the covariant derivative that corresponds to the form (always for a given t).

Fundamental theorem:

The system:

$$W_{ik} = \frac{\partial a_{ik}}{\partial t} - \nabla_k h_i - \nabla_i h_k \tag{5}$$

is a strong tensor.

One can prove that theorem by direct calculation.

Theorem:

The necessary and sufficient condition for the spaces in the sequence (1) *to all be isometric is the existence of a system e_i that satisfies the equation:*

$$\nabla_k e_i + \nabla_i e_k = W_{ik} . \tag{6}$$

Proof: If there exists an isometry between the V_m (t) then one can take the t-lines to be the curves that pass through the corresponding points under that isometry. That amounts to a "kinematical" transformation of the coordinates (2). In such a coordinate system, one will have $\partial a_{ik} / \partial t = 0$, and therefore:

$$-\nabla_k h_i - \nabla_i h_k = W_{ik} . \tag{7}$$

Now let e_i denote the strong vector whose components in the system that was just described are $-h_i$. It will verify the invariant relation (6). Conversely, if (6) have a solution e_i then one can choose the coordinate system in V_{m+1} in such a way that one will get $-h_i = e_i$, which is obviously possible. One will then have $\partial a_{ik} / \partial t = 0$.

As an application of that, one sees that one will always have an isometry when $W_{ik} = 0$. The solution of (6) is $e_i = 0$, and one establishes an isometry for $h_i = 0$, which means that the t-lines are orthogonal to the spaces V_m (t). We call that isometry "orthogonal," and we can state the following theorem:

The necessary and sufficient condition for there to exist an orthogonal isometry between the spaces V_m (t) is that $W_{ik} = 0$.

Observe that this is a completely-explicit condition that demands only differentiations.

Infinitesimal isometries. – In order for the space V_m (0) to be isometric to the infinitely-close spaces in the sequence (1), it is necessary and sufficient that (6) must be true for at least $t = 0$. Suppose that this condition is fulfilled. Since the spaces in the sequence (1) are well-defined, the only arbitrary step in establishing the isometry is choosing the angle that the t-lines in V_{m+1} form with V_m (0), which amounts to a convenient choice of h_i . One will then get other isometries from the possible values of h_i . Let h_i' and h_i'' be two of those values. Obviously, one can assume that the change of coordinates that takes h_i' to h_i'' does not alter the coordinates on V_m (0), i.e., it reduces to an identity for $t = 0$. Consequently, e_i will have the same coordinates in the two systems in which (6) will become:

$$-\nabla_k h_i' - \nabla_i h_k' = -\nabla_k h_i'' - \nabla_i h_k'' = \nabla_k e_i + \nabla_i e_k ,$$

respectively, so:

$$\nabla_k (h_i' - h_i'') + \nabla_i (h_k' - h_k'') = 0 ,$$

which is the well-known Killing equation, which is the condition for $h_i' - h_i''$ to represent a rigid motion in V_m (0). We have obtained the theorem:

One gets all of the infinitesimal isometries between one space and a sequence of infinitesimally-close spaces by superimposing just one of them with all the rigid motions in that space.

One sees the close link between the infinitesimal transformation and the rigid motions of a space into itself. Furthermore, (6) is a type of generalization of the Killing equation.

Observe, moreover, that if there exists an orthogonal infinitesimal isometry then one will determine it by means of a total differential equation with just one unknown if $n = m + 1$.

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