"Über die Variationsgleichungen für affine geodätische Linien und nichtholonome, nichtkonservative dynamische Systeme," Prace Mat. Fiz. (1931), 129-147.

On the variational equations for affine geodetic lines and non-holonomic, non-conservative dynamical systems

By

A. Wundheiler

Translated by D. H. Delphenich

In the year 1926, **Levi-Civita** ([1]) generalized the Jacobi formula for the geodetic deviation in two dimensions to arbitrary **Riemann**ian spaces. **Vranceanu** ([2]) and **Synge** ([3]) then adapted it to non-holonomic, but always Riemannian, spaces. **Vranceanu** worked with congruences and obtained rather opaque formulas that cannot be generalized to affine spaces, and they are very hard to understand for the reader that is familiar with **Schouten**'s symbolism. In 1928, **Synge** gave a much simpler tensorial form to the non-holonomic variational equations of geodetics that could not be adapted to affine spaces, either. The dynamical applications are restricted to just the case of a conservative, scleronomic system of given total energy, so the classical theorem of Jacobi that identifies the trajectories under such a constraint with the geodetics of a Riemannian space can be employed.

In the present paper, we will:

1. Derive the equations of deviation for geodetic lines for a general *affine* non-holonomic space from a new viewpoint ([4]). The derivation is very simple, and the desired equations are obtained almost immediately from the commutation formula for the covariant differential. Thanks to the introduction of a curvature tensor that did not occur to any of the cited authors (not even **Schouten**), the equations of deviation will take on precisely the same form in the non-holonomic case that they have in the holonomic case. The method will then be applied to *scleronomic* dynamical systems.

2. A connection between the geodetics of a suitably-chosen multidimensional space and the motions of an arbitrary rheonomic, linear, non-holonomic dynamical system will

---

([1]) **T. Levi-Civita**, "Sur l'écart géodésique," Math. Ann. **97** (1926).
([2]) **G. Vranceanu**, "Studio geometrico dei sistemi anolonomi," Ann. di Mat. (4) **6** (1928-29). Summary of the author's results on non-holonomic spaces.
([3]) **J. L. Synge**, "Geodesics in non-holonomic geometry," Math. Ann. **99** (1928).
([4]) **A. Wundheiler**, "Une simple démonstration de la formule de l'écart géodésique," Rend. dei Lincei **12** (1930), pp. 644.

be presented on an *affine foundation* that is entirely independent of Jacobi's theorem. In that way, the stability problem for arbitrary systems will be reduced to the equations of geodetic variation.

The treatment of the problem will be carried out in connection with **Schouten**'s symbolism ([1]). In **§§ 1** and **2**, we present the concepts that are necessary for the reader to understand the definitions and theorems on non-holonomic spaces, which are unknown to the relevant work by **Schouten**. However, the presentation is rather different in many places ([2]).

§ 1. – Generalities on non-holonomic spaces.

1. Definitions of a constrained A_n^m. – A_n denotes an *n*-dimensional affine space in which a *symmetric* ([3]) parallel displacement is given by means of its components Γ_{il}^k, which are otherwise-arbitrary functions of position. We denote the *unit affinor* in this space by A_i^k, which is then:

$$A_i^k = \begin{cases} 1 & (i = k) \\ 0 & (i \neq k). \end{cases}$$

Now let an *m*-direction be defined arbitrarily at every point of that A_n. That is known to happen when one is given *m* linearly-independent vectors at that point. Any linear combination of those vectors is a vector that falls along that direction. We assign an $m' = (n - m)$-direction to every point of the A_n that has no 1-direction in common with the previously-defined *m*-direction and call it the *pseudo-orthogonal* direction.

Now, any vector can be decomposed into two components, one of which falls along the local *m*-direction, while the other one falls along the pseudo-orthogonal direction. We call the first of those components the *projection of the vectors into the local m-direction* and the other one, *the projection along the pseudo-orthogonal direction.* As an image of those projections, one can now define an affinor as follows:

If u^k is a vector in A_n then:

$$u'^k = B_i^k u^i$$

([1]) **J. A. Schouten**, *Der Ricci-Kalkül*, Berlin, 1924; "Über nichtholonome Überträgungen in einer L_n," Math. Zeit. **30** (1929) (cited as "Schouten").

The last paper of that author: **J. A. Schouten** and **E. R. van Kampen**, "Zur Einbettungs- und Krümmungstheorie nichtholonomer Gebilde," Math. Ann. **103** (1930), can remain unconsidered for our purposes.

([2]) Cf., e.g., Formulas (4), (8.1), (15), the derivation of (13).

([3]) That restriction is entirely inessential. However, since the equation for geodetic lines:

$$\frac{d^2x^k}{ds^2} + \Gamma_{ij}^k \frac{dx^i}{ds}\frac{dx^j}{ds} = 0$$

depends upon only the symmetric part of Γ_{ij}^k, so upon:

$$\frac{1}{2}(\Gamma_{ij}^k + \Gamma_{ji}^k),$$

the consideration of asymmetric displacements would be superfluous for our purposes.

will be its projection onto the local *m*-direction. The components of the affinor B_i^k can be calculated easily as soon as one is given *m* vectors that span the *m*-direction and m' vectors that span the pseudo-orthogonal direction ([1]).

We call the totality of local m-directions an A_n^m. If a pseudo-orthogonal direction is given at each point then we will speak of a *constrained* A_n^m and call B_i^k *its unit affinor*. If that unit affine B_i^k is known then the local *m*-direction, as well as the constrained m'-direction will be given at each point.

If certain integrability conditions ([2]) are fulfilled then the *m*-elements can be assembled into an $(n - m)$-parameter family of A_m that are embedded in A_n. If that is not true then we will be dealing with the general case, and we will call the totality of *m*-elements a *non-holonomic m-dimensional space* A_n^m *that is embedded in* A_n.

If a vector falls in the local *m*-direction then we say that it *belongs to* A_n^m. We call a curve whose tangent vector at each point belongs to A_n^m a *curve in* A_n^m.

The totality of pseudo-orthogonal m'-directions defines an $A_n^{m'}$, in its own right, and we can consider the original local *m*-direction to be its constraint direction; i.e., the $A_n^{m'}$ can be constrained by the A_n^m in their own right. That constraint defines a unit affinor on the $A_n^{m'}$, which we would like to denote by C_l^k. If u^k denotes a vector in A_n^m and v^k denotes a vector in $A_n^{m'}$, then we will obviously have:

$$B_l^k u^l = u^k, \qquad C_l^k u^l = 0,$$
$$B_l^k v^l = 0, \qquad C_l^k v^l = v^k.$$

Hence:

$$A_l^k = B_l^k + C_l^k .$$

2. Projection relative to a given index. – Just like the contravariant vectors, the covariant vectors can also be projected into A_n^m by means of the affinor B_l^k:

$$B_l^k u_k = u'_l .$$

Corresponding formulas are true for the projection into $A_n^{m'}$:

$$C_l^k u^l = u'^k, \qquad C_l^k u_k = u''_l .$$

([1]) **Schouten**, pp. 156, (31).
([2]) **Schouten**, pp. 157, (35).

However, higher affinors can also be projected into the A_n^m, and indeed *relative to arbitrary indices*. For example:

$$T'^{k}_{i} = B^{k}_{k'} T^{k'}_{i}$$

is the projection of the affinor T_i^k into A_n^m relative to the index k. Likewise, one has, e.g.:

$$T''^{k}_{i} = B^{i'}_{i} T^{k}_{i'}$$

for its projection into A_n^m relative to the index i. We can also project the affinor into A_n^m relative to the indices h, i and into $A_n^{m'}$ relative to the index j:

$$T''^{hk}_{ij} = B^{h}_{h'} B^{i'}_{i} C^{j'}_{j} T^{h'k}_{i'j'} .$$

If such a projection is equal to the affinor itself, so, e.g.:

$$B^k_h T^h_i = T^k_i ,$$

then we will say that this affinor *lies in* A_n^m *with respect to that index*, or more simply, that *the index lies* A_n^m. The projection of that affinor into the constraint space relative to that index is then obviously zero, so:

$$C^k_h T^h_i = 0.$$

In particular, the unit affinor with two indices lies in the space to which it belongs, so:

$$B^k_h B^h_l = B^k_l , \quad C^k_h C^h_l = C^k_l , \quad C^k_h B^h_l = C^k_h B^h_l = 0.$$

We [like **Schouten** ([1])] shall employ the symbols $a, \ldots, g$ exclusively for indices that lie in A_n^m and the symbols $p, \ldots, w$ for the ones that lie in $A_n^{m'}$. The indices $h, \ldots, l$ can be employed without any restriction.

3. The parallel translation that is induced in A_n^m. – The covariant differential of an arbitrary vector in A_n^m:

(1) $$\delta u^k = du^k + \Gamma^k_{ij} u^i dx^j$$

can be decomposed into its two projections into each of our spaces:

$$\delta u^k = \delta' u^k + \delta'' u^k,$$

([1]) **Schouten-Kampen**, pp. 760.

where

(2) $$\delta' u^k = B_i^k \delta u^i ,$$

and

(3) $$\delta'' u^k = C_i^k \delta u^i .$$

For higher-rank affinors, we set, e.g.:

$$\delta' T_i^{\ k} = B_{k'}^k B_l^{l'} \delta T_{i'}^{\ k'} ,$$

etc. We now calculate the components of the induced translation. Let u^k be an arbitrary vector in A_n^m. We then have:

$$\begin{aligned}\delta' u^k &= B_i^k \delta u^i = B_h^k (du^h + \Gamma_{ij}^h u^i\, dx^j) \\ &= d(B_h^k u^h) - u^i dB_i^k + B_h^k \Gamma_{ij}^h u^i\, dx^j \\ &= du^k + (B_h^k \Gamma_{ij}^h - \partial_j B_i^k) u^i\, dx^j \qquad \left(\partial_j = \frac{\partial}{\partial x^j}\right).\end{aligned}$$

If we now set ([1]):

(4) $$\Gamma'^{\,k}_{ij} = B_h^k \Gamma_{ij}^h - \partial_j B_i^k$$

then that will give:

(5) $$\delta' u^k = du^k + \Gamma'^{\,k}_{ij} u^i\, dx^j .$$

If we set:

(6) $$\bar{\Gamma}'^{\,k}_{ij} = B_i^h \Gamma_{hj}^k + \partial_j B_i^k$$

then we will find, analogously:

(7) $$\delta' u_l = du_l - \bar{\Gamma}'^{\,k}_{ij} u_k\, dx^j .$$

([1]) The treatment of the Γ_{ij}^k for the induced translation in **Schouten** is different from ours, since he employed non-holonomic coordinates in the A_n. We have avoided the rewriting of our formula for non-holonomic coordinates in order to get around inessential complications in the derivation of the commutation formula (27). The reader that is familiar with Schouten's work will easily find the alteration that must be made to the affinor (15) when one has non-holonomic coordinates in A_n.

§ 2. – Curvature affinors and the commutation formula for A_n^m.

1. Some fundamental formulas. – One first has:

$$\delta A_i^k = \delta B_i^k + \delta C_i^k = 0,$$

so:

$$\delta B_i^k = - \delta C_i^k .$$

One further has:

$$\delta' B_i^k = - \delta' C_i^k = 0. \tag{8}$$

In fact:

$$\delta' B_i^k = B_{k'}^k B_i^{l'} \delta B_{l'}^{k'} = - B_{k'}^k B_i^{l'} \delta C_{l'}^{k'} = B_{k'}^k C_{l'}^{k'} \delta B_i^{l'} = 0. \tag{8.1}$$

In a sense, δB_i^k gives the change in direction in A_n^m when it moves to a neighboring point. From (8), the "intrinsic" change in direction of A_n^m is then zero, so A_n^m is autoparallel with respect to itself.

2. The induced curvature. – As always, we set:

$$\delta B_i^k = \nabla_j B_i^k \delta x^j .$$

Now, if one projects both indices *k* and *i* of that affinor into A_n^m then, from (8), one will get zero. By contrast, if one projects only one of the indices then one will get the two affinors:

$$H'^{\cdot\cdot i}_{jk} = B_j^{j'} B_k^{k'} \nabla_{j'} B_{k'}^l , \qquad L'^{\cdot i}_{j\cdot k} = B_j^{j'} B_{l'}^l \nabla_{j'} B_k^{l'} , \tag{9}$$

which one can employ as a measure of the relative change in A_n^m. Note that the first two indices of these two affinors lie in A_n^m, but, from (8), the last one lies in $A_n^{m'}$. One can then write: $H'^{\cdot\cdot r}_{ba}$, $L'^{\cdot a}_{b\cdot r}$.

In parallel with that, we introduce the following quantities that are referred to $A_n^{m'}$:

$$H''^{\cdot\cdot k}_{ji} = C_j^{j'} C_i^{l'} \nabla_{j'} C_{l'}^k , \qquad L'^{\cdot i}_{j\cdot k} = C_j^{j'} C_{k'}^k \nabla_{j'} C_i^{k'} . \tag{10}$$

Following **Schouten**, we call those affinors *the induced curvature affinors*.

3. Holonomity condition. – The following equation is true for vectors in A_n^m, which justifies the name of, e.g., curvature affinors for the $H'^{\cdot\cdot k}_{ji}$:

(11) $$\delta u^k = \delta' u^k + H'^{\cdot\cdot k}_{ji} u^i \, dx^j .$$

In fact:

$$\delta u^k = \delta(B_i^k u^i) = B_i^k \delta u^i + \delta B_i^k \cdot u^i = \delta' u^k + \nabla_{j'} B_{k'}^k \cdot B_i^{k'} u^i \cdot B_j^{j'} du^j .$$

Similarly, one has:

(12) $$\delta u_l = \delta' u_l + L'^{\cdot\cdot k}_{j\cdot i} u_k \, dx^j .$$

Equation (11) allows us to say when the A_n^m is holonomic. By definition, that will be the case when the *m*-direction elements can be combined into an *m*-dimensional space, which will then give an m'-parameter family. In order for that to be true, it is necessary and sufficient that any curve that lies in A_n^m (so one whose direction is always included in the local *m*-direction) lies completely in an A_m. If we then displace a vector parallel to A_n^m along a path that lies in A_n^m then it must always belong to the same A_m. In particular, in the holonomic case, an infinitesimal parallelogram that is constructed in an A_n^m must lie completely in one A_m, so *it must be closed.*

We now set $MM' = \delta x^k$, $MN' = \bar{\delta} x^k$, $M'N'' \parallel M N'$, $N'N''' \parallel MM'$ relative to A_n^m. Now, N'' and N''' must coincide in the holonomic case, so we must have $\delta \bar{\delta} x^k - \bar{\delta}\, \delta x^k = 0$. On the one hand, we have:

$$\bar{\delta}' \delta x^k = 0, \qquad \delta' \bar{\delta} x^k = 0,$$

and on the other hand, from (11):

$$\delta \bar{\delta} x^k = \delta' \bar{\delta} x^k + H'^{\cdot\cdot k}_{ji} \bar{\delta} x^i \delta x^j , \qquad \bar{\delta}\, \delta x^k = \bar{\delta}' \delta x^k + H'^{\cdot\cdot k}_{ji} \delta x^i \bar{\delta} x^j .$$

It will then follow that:

$$(\delta \bar{\delta} - \bar{\delta}\, \delta)\, x^k = (H'^{\cdot\cdot k}_{ji} - H'^{\cdot\cdot k}_{ij})\, \bar{\delta} x^i \delta x^j .$$

Should that be true for δx^k and $\bar{\delta} x^k$ then one must have:

(13) $$H'^{\cdot\cdot k}_{ji} = H'^{\cdot\cdot k}_{ij} .$$

That is the desired holonomity condition. It is also sufficient ([1]).

([1]) **Schouten**, pp. 161, (61).

4. Commutation formula for A_n^m. – We shall adapt the well-known commutation formula:

$$(\bar{\delta}\,\delta - \delta\,\bar{\delta})\,u^k = R_{jli}^{\cdot\cdot\cdot k}\,dx^j\,\bar{d}x^l u^j$$

to A_n^m. In regard to the chosen notations, we establish that the first index of the curvature affinor refers to the first differentiation, the second index, to the second differentiation, and third one is linked with the vector index.

As always, δ and $\bar{\delta}$ will denote two displacements that commute in A_n (but not A_n^m !). We recall that this assumption is equivalent to $d\,\bar{d}x^k = \bar{d}\,dx^k$. We calculate, in the usual way:

$$\begin{aligned}\bar{\delta}'\,\delta' u^c &= \bar{d}\,(\delta' u^c) + \Gamma'^{c}_{bl}\,\delta' u^b\,\bar{d}x^l \\ &= \bar{d}\,(du^c + \Gamma'^{c}_{aj}\,u^a\,dx^l) + \Gamma'^{c}_{bl}\,(du^b + \Gamma'^{b}_{aj}\,u^a dx^j\,)\,\bar{d}x^l \\ &= \underline{\underline{\bar{d}\,du^c}} + \partial_i\,\Gamma'^{c}_{aj}\,u^a\,dx^j\,\bar{d}x^i + \Gamma'^{c}_{bl}\,\partial_i u^b + \underline{\Gamma'^{b}_{aj}\,u^a dx^j\,\bar{d}x^l} \\ &\quad + \underline{\underline{\Gamma'^{b}_{aj}\,u^a\bar{d}\,dx^j}} + \underline{\Gamma'^{c}_{bl}\,\partial_j u^b dx^j\,\bar{d}x^l} + \Gamma'^{c}_{bl}\,\Gamma'^{b}_{aj}\,u^a\,dx^j\,\bar{d}x^i .\end{aligned}$$

The doubly-underlined terms are symmetric in d and $\bar{d}$. The same thing will be true for the sum of the singly-underlined terms. If we now form the difference $(\bar{\delta}'\,\delta' - \delta'\,\bar{\delta}')\,u^c$ then the aforementioned terms will cancel, and that will give:

$$(\bar{\delta}'\,\delta' - \delta'\,\bar{\delta}')\,u^c = R'^{\cdot\cdot\cdot c}_{jla}\,dx^j\,\bar{d}x^l u^a\,, \tag{14}$$

when we set:

$$R'^{\cdot\cdot\cdot c}_{jla} = \partial_i\,\Gamma'^{\,c}_{aj} - \partial_j\,\Gamma'^{\,c}_{ai} + \Gamma'^{\,b}_{aj}\,\Gamma'^{\,c}_{bi} - \Gamma'^{\,b}_{ai}\,\Gamma'^{\,c}_{bj}\,.$$

As the notations for the indices would suggest, the first two indices of that affinor lie in A_n, while the last two lie in A_n^m. It has, so to speak, an intermediate position between A_n and A_n^m, and we have that to thank for the simplicity of the commutation formula that we obtain. We remark that this does not occur in either the cited works of **Vranceaunu** or **Schouten**. The latter author skipped over a commutation formula for A_n^m completely ([1]). We suggest that it will offer a great advantage for the problem of geodesic deviation.

§ 3. – Deviation equations in a A_n^m.

1. Two arbitrary curves in A_n^m. – Let C and C' be two curves in A_n^m; i.e., their directions fall into the local m-direction at each point. We relate them to each other by a one-to-one correspondence of their points and now assume that the curves are

([1]) **Schouten**, pp. 162, line 13 from the bottom.

neighboring; i.e., that corresponding points are infinitely close. If M is an arbitrary point of the curve C then we denote the corresponding point of C' by M' and set:

$$MM' = \bar{\delta} x^k .$$

We generally let $\bar{\delta}$ denote the covariant differential that corresponds to the displacement $\bar{\delta} x^k$; we shall call it the *covariant variation*.

If the A_n^m is holonomic then C, as well as C', will fall in an A_m. However, $\bar{\delta} x^k$ will no longer lie in the A_n^m, in general ([1]), since the A_m of C and C' would be different then. However, if $\bar{\delta} x^k$ lies in A_n^m for a particular M then it will already lie in A_n^m along all of C, and C and C' would have to fall in the same A_m. *That is not true in the non-holonomic case.* In fact, it follows directly from the condition that it must lie in A_n^m:

$$C_i^k \delta x^i = 0$$

(δ means the covariant differential that corresponds to a displacement along C that commutes with $\bar{\delta} x^k$) that:

$$C_i^k \bar{\delta}\, \delta x^i + \bar{\delta} C_i^k \delta x^i = 0. \tag{16}$$

We set $\bar{\delta} x^k = \xi^k$ and decompose ξ^k into two components: ξ'^k, which is in A_n^m, and ξ''^k, which is in $A_n^{m'}$:

$$\xi'^k = B_i^k \xi^i , \qquad \xi''^k = C_i^k \xi^i .$$

If we observe that δ and $\bar{\delta}$ commute and we employ (16) then we will get:

$$\begin{aligned} \delta \xi''^k = \delta(C_i^k \xi^i) &= C_i^k \delta \bar{\delta} x^i + \delta C_i^k \xi^i = - \bar{\delta} C_i^k \delta x^i + \delta C_i^k \xi^i \\ &= -\nabla_j C_i^k \xi^j \delta x^i + \nabla_j C_i^k \xi^i \delta x^j . \end{aligned} \tag{17}$$

However, we represent $\nabla_j C_i^k$ in terms of the curvature affinor:

$$\begin{aligned} \nabla_j C_i^k &= (B_{k'}^k + C_{k'}^k)(B_j^{j'} + C_j^{j'}) \nabla_{j'} C_i^{k'} \\ &= -\,(H'^{\cdot\cdot k}_{ji} + L'^{\cdot k}_{j \cdot i})(H''^{\cdot\cdot k}_{ji} + L''^{\cdot k}_{j \cdot i}) . \end{aligned} \tag{18}$$

If we then set $u^k = \delta x^k / ds$, in which u^k means the tangent vector to C, then we will get:

([1]) The discovery of that almost-banal fact has its own history. Cf., **Vranceanu**, "Sur l'écart géodésique dans les espaces non-holonomes," Ann. Scient. Univ. Jassy **15** (1928), pp. 7 and pp. 309, as well as **E. Cartan**, "Sur l'écart géodésique et quelques notions connexes," Rend. dei Lincei **5** (1927), pp. 609.

(19) $$\frac{\delta\xi''^k}{ds} = [(H'^{..k}_{ij} - H'^{..k}_{ji})\xi'^i - (L'^{\cdot k}_{j\cdot i} + L''^{\cdot k}_{j\cdot i})\xi''^i]u^j .$$

We have substituted (18) in (17), then set $\xi^k = \xi'^k + \xi''^k$, and finally multiplied out the parentheses, while observing the position of the indices in the curvature affinor. Products like:

(20) $$H'^{..k}_{ij}\xi''^i, \qquad H''^{..k}_{ij}\delta x^i, \qquad L'^{.k}_{i\cdot j}\xi'^i, \qquad L''^{.k}_{i\cdot j}\delta x^j, \qquad \text{etc.,}$$

in which an index from A_n^m is concatenated with one from $A_n^{m'}$, will vanish.

We now ask what the condition would be for the vanishing of ξ''^k for every s to follow from the vanishing of ξ''^k_0 for a certain $s = s_0$. In order for that to be true, it is obviously necessary and sufficient that ξ'^k must not enter into equations (19); i.e.:

$$H'^{..k}_{ji} - H'^{..k}_{ij} = 0,$$

which implies the holonomity condition (13). If the A_n^m are not holonomic then $\xi''^k_0 = 0$ can follow from $\xi''^k = 0$ only in exceptional cases.

(19) represents only *n – m independent equations for the deviation* ξ^k that are true for *arbitrary* pairs of C and C'. We shall give some more equations of deviation that can be useful in many investigations.

We can rewrite (16) in the form:

$$\frac{\delta\xi''^k}{ds} = -\bar{\delta}C_i^k \frac{\delta x^i}{ds} = -\nabla_j C_i^k u^i \bar{\delta} x^j .$$

Hence, from (18) ([1]):

(21) $$\boxed{\frac{\delta''\xi^k}{ds} = (H'^{..k}_{ji} - L''^{.k}_{j\cdot i})\xi^j u^i .}$$

We further have:

$$\delta''\delta'\xi^k = C_h^k\delta(B_i^k\delta\xi^i) = C_h^k\nabla_j B_i^k\delta\xi^i\delta x^j ,$$
$$\delta'\delta''\xi^k = B_h^k\delta(C_i^k\delta\xi^i) = B_h^k\nabla_j C_i^k\delta\xi^i\delta x^j .$$

Hence:

(22) $$\boxed{\delta''\delta'\xi^k = H'^{..k}_{ji}\delta'\xi^j\delta u^i}$$

and

(23) $$\boxed{\delta'\delta''\xi^k = -L'^{.k}_{j\cdot i}\,\delta''\xi^j\delta u^i .}$$

([1]) Those are essentially eqs. (6.42) in **Synge** (Geodesics, etc.) and (43′) of **Vranceanu** (Studio geometrico, etc.)

2. Geodetic lines in A_n^m. – We call a curve in A_n^m whose direction with respect to the A_n^m does not change (so it is autoparallel relative to the A_n^m) a *geodetic line in* A_n^m. If t denotes a parameter on a curve C then $u^k = \delta x^k / dt$ will be its tangent vector. If its direction does not change then we must have:

$$\frac{\delta' u^k}{dt} = \alpha u^k.$$

For a certain choice of parameter $s = f(t)$, that equation will go to:

$$\frac{\delta' u^k}{ds} = 0. \tag{24}$$

How does one describe a geodetic in A_n^m relative to the translation in A_n? We immediately get from (11) that:

$$\frac{\delta u^k}{ds} = \frac{\delta' u^k}{ds} + H_{ji}^{\cdot\cdot k} u^i u^j,$$

so:

$$\frac{\delta u^k}{ds} = H_{ji}^{\cdot\cdot k} u^i u^j.$$

A geodetic in A_n^m is then a curve whose curvature lies in $A_n^{m'}$ for a suitable choice of parameter s. We call that parameter the *affine* parameter.

3. Geodetic deviation. – Now let C and C' be two infinitely-close geodetic lines in A_n^m. In that way, their directions shall also differ infinitely little when we relate them to each other in a suitable one-to-one way. If the point M of C corresponds to the point M' of C' then we call the vector MM', which we denote by $\bar{\delta} x^k = \xi^k$, as in § **3**, no. 1, the *geodesic deviation*. As we said before, we shall call the covariant differential that corresponds to the displacement $\bar{\delta} x^k$ the *covariant variation*.

Along with that covariant variation, we shall introduce the differential along C (C', resp.), which corresponds to the displacement δx^k. The displacements δ and $\bar{\delta}$ shall commute; i.e., if δ moves from M to N along C then δ will move from M' to N' along C'. If M corresponds to the parameter value s on C and M', to the parameter value s' on C' then we will have:

$$ds' = ds + \bar{\delta}\, ds = (1 + \mu)\, ds,$$

in which have set:

$$\mu = \frac{\bar{\delta}\, ds}{ds}. \tag{25}$$

μ is infinitely small.

We first have:

$$\bar{\delta} u^k = \bar{\delta}\frac{\delta x^k}{ds} = \frac{ds\,\bar{\delta}\,\delta x^k - \delta x^k\,\bar{\delta}\,ds}{ds^2} = \frac{\delta\xi^k}{ds} - \mu\,u^k,$$

since one has $\bar{\delta}\,\delta x^k = \delta\,\bar{\delta}\,x^k = \delta\xi^k$. We project into A_n^m and get:

(26) $$\bar{\delta}' u^k = \frac{\delta'\xi^k}{ds} - \mu\,u^k,$$

since $B_i^k u^i = u^k$.

4. The Levi-Civita formula. – We set u^c equal to the tangent vector $\delta x^c / ds$ of our geodetic in the commutation formula (14):

$$(\delta'\,\bar{\delta}' - \bar{\delta}'\,\delta')\,u^c = R'^{\dots c}_{jia}\,u^a\,\bar{\delta}x^j\,\delta x^i,$$

in which we recall (24). Due to (26), we will get:

$$\delta'\left(\frac{\delta'\xi^c}{ds} - \mu\,u^c\right) = R'^{\dots c}_{jia}\,u^a\,\xi^j\,\delta x^i$$

or [1]:

(27) $$\boxed{\frac{\delta'^2\xi^c}{ds^2} - \frac{d\mu}{ds}u^c = R'^{\dots c}_{jia}\,\xi^j u^i u^a.}$$

That is the **Levi-Civita** formula for non-holonomic spaces.

For the special case of Riemannian space, so when the A_n^m form a V_n, we can get the formula for μ that **Levi-Civita** gave directly with the help of covariant variation [2]. It is then known that the affine parameter s is the arc-length, so:

$$\bar{\delta}\,ds^2 = \bar{\delta}(g_{ik}\,dx^i dx^k),$$

or

$$ds\,\bar{\delta}\,ds = g_{ik}\,\delta x^i\,\bar{\delta}\,\delta x^k,$$

or after dividing by ds^2:

(28) $$\mu = \frac{\bar{\delta}\,ds}{ds} = g_{ik}\,u^i\,\frac{\delta\xi^k}{ds}.$$

---

[1] Compare the simplicity of this to formulas (6.32) and (6.52) of **Synge** and (44) of **Vranceanu**. In the latter, one must consider the defining equations for the symbols that enter into them.

[2] **Levi-Civita**, pp. 314, (35′).

5. Dynamical equations of deviation for scleronomic systems. – As is known ([1]), the Lagrange equations of motion for a scleronomic system with a *vis viva* of $2T = g_{ik}\,\dot{x}^i\,\dot{x}^k$ that is acted upon by the generalized force Q_i, which are the equations:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}^i} - \frac{\partial T}{\partial x^i} = Q_i\,,$$

can be written in the form:

$$\frac{\delta v^k}{dt} = \frac{\delta}{dt}\frac{\delta x^k}{dt} = Q^k = g^{ki}\,Q_i\,,$$

in which δ means the covariant differential in the Riemannian space with the fundamental form $ds^2 = g_{ik}\,dx^i\,dx^k$. The x^k are the independent parameters of the system then.

If scleronomic and non-holonomic constraints are added to the system then we can replace them with a constraint force. The constraints define an *m*-direction field in which the velocity vector must fall. We denote the unit affinor of the V_n^m that is defined in that way and constrained orthogonally by B_i^k. The aforementioned constraint force, which must be perpendicular to the virtual displacement, since it does no work, must fall along the orthogonal constraint direction. If we denote it by R^k then we will also have $B_i^k R^i = 0$. We now project the equations of motion:

$$\frac{\delta v^k}{dt} = Q^k + R^k$$

into V_n^m and obtain ([2]):

$$\frac{\delta v^k}{dt} = Q'^k. \tag{29}$$

We now relate two neighboring paths of the system to each other in a one-to-one way, while preserving the notation of footnote ([1]), except that we write *t*, instead of *s*. If we once more apply the commutation formula, but this time to the vector $v^k = \delta x^k / dt$, then we will get:

$$\delta'\left(\frac{\delta'\xi^c}{dt} - \mu\, v^c\right) - \bar{\delta}'(Q'^c\,dt) \;=\; R'^{\;\;\ldots c}_{jia}\,\xi^j\,\delta x^i v^a\,,$$

since $\delta' v^k = Q'^k\,dt$, from the equations of motion (29). If we perform the differentiations, once more replace $\delta' v^k$ with $Q'^k\,dt$, and divide by dt then that will give:

$$\boxed{\frac{\delta'^2\xi^c}{dt^2} - \frac{d\mu}{dt}v^c = R'^{\;\;\ldots c}_{jia}\,\xi^j v^i v^a + 2\mu\, Q'^c + \bar{\delta}' Q'^c.} \tag{30}$$

([1]) See, e.g., **E. Cartan**, *Leçons sur la géométrie des espaces de Riemann*, Paris, 1928, pp. 42.

([2]) **Schouten**, pp. 171, (115). **Vranceanu**, "Sopra le equazioni del moto di un sistema anolonomi," Rend. dei Lincei **4** (1926), pp. 508.

If the curves C and C' are "isochronously" related to each other, so $\bar{\delta} dt = 0$, $\mu = 0$, then (30) will take the simple form:

$$\boxed{\frac{\delta'^2 \xi^c}{dt^2} = R'^{\;\dots c}_{jia}\, \xi^j v^i v^a + \bar{\delta}' Q'^c.} \tag{30.1}$$

Remark. – (27), as well as (30) represent *only m* independent equations, since the two sides of those equations will fall in the A_n^m. The remaining $n - m$ equations are then obtained by the equations (21), which are true for *any* curve in A_n^m, among which, there are similarly only $n - m$ of them.

However, how do we calculate the $\frac{\delta^2 \xi^k}{dt^2}$ themselves? We first have:

$$\delta^2 \xi^k = \delta'^2 \xi^k + \delta'' \delta' \xi^k + \delta' \delta'' \xi^k + \delta''^2 \xi^k.$$

We obtain the first three summands on the right from (27), (22), and (23), resp. As far as the fourth one is concerned, δ''-differentiating (21) will give:

$$\frac{\delta''^2 \xi^k}{ds^2} = C_h^k \nabla_l (H'^{\;\cdot\cdot h}_{ji} - L''^{\;\cdot\cdot h}_{ji}) \cdot \xi^j u^i u^l, \tag{31}$$

since the other summands that arise from differentiation will vanish due to (20). Adding (27), (22), (23), and (31) will yield the formula:

$$\frac{\delta^2 \xi^k}{ds^2} - \mu u^k = (H'^{\;\cdot\cdot h}_{ji} - L'^{\;\cdot\cdot h}_{ji}) \frac{\delta \xi^i}{ds} u^j + [R'^{\;\dots h}_{jil} + C_h^k \nabla_l (H'^{\;\cdot\cdot h}_{ji} - L''^{\;\cdot\cdot h}_{ji})] \xi^j u^i u^l.$$

§ 4. – Motions as geodetic lines.

Deviation equations for arbitrary dynamical systems

1. – We understand a *motion* of a dynamical system with n degrees of freedom whose position is determined by the parameter x^k to mean a curve in the $(n + 1)$-dimensional space (x^k, t) that is defined by the equations $x^k = x^k(t)$ that satisfy the differential equations of motion. That curve then determines not only the path, but also the way that it is traversed.

The problem of reducing the examination of a dynamical system to the examination of the geodetics of a suitably-chosen multidimensional space has been taken up several times before. We first recall the classical theorem of **Jacobi**, according to which, *the trajectory of a scleronomic, conservative system of well-defined total energy* is given by the geodetics of Riemannian space with:

$$ds^2 = 2\,(h - V)\,T\,dt^2$$

in which h means the total energy and V means the potential. Obviously, that space is still not determined by the system itself ([1]). Should the corresponding space be associated with the system in a one-to-one way, then it would have to have $n + 1$ dimensions. Namely, the totality of the motions is (in general) $2n$-dimensional, whereas the totality of the geodetics in an n-dimensional space is only $(2n - 2)$-dimensional. **Eisenhart** ([2]) gave such a Riemannian space, but only for *conservative, scleronomic, and holonomic systems.* Unfortunately, its $(n + 1)$ coordinate is not time, but *a parameter that is not holonomic for the system* (which is therefore not determined by the position of the system), namely, a linear combination of the Jacobian action and time. For rheonomic (but always holonomic and conservative systems), there is an $(n + 2)$-dimensional Riemannian space that achieves the desired objective ([3]).

In these paragraphs, we shall indeed dispense with the construction of a *Riemannian* space that solves the problem, and satisfy ourselves with an *affine* one ([4]). However, in return, the problem will be solved for *arbitrary,* but only *linear,* non-holonomic systems that can otherwise be subject to rheonomic and arbitrary forces.

2. Holonomic systems. – We consider an arbitrary holonomic system with n degrees of freedom that is referred to the parameters x^k and is given by its kinetic energy:

$$T = \tfrac{1}{2}\, g_{\alpha\beta}\, \dot{x}^\alpha\, \dot{x}^\beta$$

and the generalized forces that act upon it. Moreover, we establish in these paragraphs that Greek indices range through the values 0, 1, 2, …, n, and Latin indices range through only the values 1, 2, …, n.

We understand x^0 to mean the time t ; $\dot{x}^0$ then stands for 1, the form for T is generally inhomogeneous. We first specify the Lagrange equations under the assumption that the $g_{\alpha\beta}$ also depend upon $x^0 = t$. We have:

$$\frac{\partial T}{\partial \dot{x}^i} = g_{i\alpha}\,\dot{x}^\alpha, \qquad \frac{d}{dt}\frac{\partial T}{\partial \dot{x}^i} = g_{ij}\,\ddot{x}^j + \partial_\beta\, g_{i\alpha}\cdot \dot{x}^\alpha \dot{x}^\beta, \qquad \frac{\partial T}{\partial x^i} = \tfrac{1}{2}\partial_i g_{i\alpha}\cdot \dot{x}^\alpha \dot{x}^\beta,$$

so:

$$Q_i = \frac{d}{dt}\frac{\partial T}{\partial \dot{x}^i} - \frac{\partial T}{\partial x^i} = g_{ij}\,\ddot{x}^j + [i, \alpha\beta]\,\dot{x}^\alpha \dot{x}^\beta, \qquad [\gamma, \alpha\beta] = \tfrac{1}{2}(\partial_\alpha\, g_{\beta\gamma} + \partial_\beta\, g_{\gamma\alpha} - \partial_\gamma\, g_{\alpha\beta})\,.$$

([1]) Cf., e.g., **P. Appell**, *Traité de mécanique rationelle*, t. II, Paris, 1923, pp. 453.

([2]) **L. P. Eisenhart**, "Dynamical Trajectories and Geodesics," Ann. Math. **30** (1929), pp. 603.

([3]) **L. P. Eisenhart**, *loc. cit.*, pp. 593.

([4]) Cf., also **J. L. Synge**, "On the Geometry of Dynamics," Phil. Trans. Roy. Soc. London, vol. **226**, pp. 35, line 17 from the top.

The fact that such a Riemannian space cannot exist for arbitrary systems is self-evident. However, we hope to prove on another occasion that it does not exist for a conservative, scleronomic system either, in general.

If we let g^{ki} denote the n-dimensional matrix that is reciprocal to g_{ij} (but not to $g_{\alpha\beta}$!!) then we can rewrite that equation in the form:

$$\ddot{x}^k + g^{ki}[i, \alpha\beta]\,\dot{x}^\alpha \dot{x}^\beta = Q^k \qquad (Q^k = g^{ki} Q_i). \tag{32}$$

We now define an affine $(n + 1)$-dimensional translation as follows:

$$\Gamma^k_{i\beta} = g^{kj}\,[j, i\,\beta], \qquad \Gamma^k_{00} = g^{kj}\,[j, 0\,0], \qquad \Gamma^0_{\alpha\beta} = 0. \tag{33}$$

It is easy to verify that the geodetics that belong to this translation are the motions of our system. In fact, it follows from the equations:

$$\frac{d^2x^\lambda}{ds^2} + \Gamma^\lambda_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = 0$$

that for $\lambda = 0$, one has:

$$\frac{d^2x^0}{ds^2} + \Gamma^0_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} = \frac{d^2t}{ds^2} = 0\,; \tag{34}$$

hence, $s = k\,t + s_0$. We now choose $s = t$, precisely. That will allow us to write the remaining equations in the form:

$$\frac{d^2x^k}{dt^2} + \Gamma^k_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} = 0.$$

We further specify this by introducing the values (33) for $\Gamma^k_{\alpha\beta}$ and then get equations (32).

The geodetic lines of the translation (33) can coincide with the equations of an arbitrary holonomic system.

If we vary the forces that act upon the system then the translation must also vary if the equations are to remain geodetic lines, as before. Namely, if forces R^k are added to the Q^k then, from (33), the Γ^k_{00} must be diminished by R^k. The equations of motion will then read:

$$\frac{d^2x^\lambda}{ds^2} + \Gamma^\lambda_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} - R^\lambda\left(\frac{dx^0}{ds}\right)^2 = 0$$

when we set $R^0 = 0$. In the space with the translation (33), the equations of motion of a holonomic system will then read:

$$\frac{\delta^2 x^\mu}{ds^2} = R^\mu\left(\frac{dt}{ds}\right)^2, \tag{35}$$

if R^k means the forces that act upon the system, besides Q^k, which are not considered when one forms the Γ^k_{00}.

3. Non-holonomic systems. – We understand A_{n+1} to mean the affine space with the translation (33) that was just defined and assume that our system is subject to certain non-holonomic conditions:

(36) $$\underset{p}{e}_\alpha \, \delta x^\alpha = 0 \qquad (p, q = 1, 2, \ldots, n - m),$$

in which the $\underset{p}{e}_\alpha$ depend upon time. Those equations obviously determine an A^{m+1}_{n+1}. How should we constrain it, and how do we find the unit affinor?

We choose the constraint direction to be the one that falls along the constraining force that replaces the couplings (36). If we denote it by R_i then it must follow from:

$$\underset{p}{e}_i \, \delta x^i = 0$$

(for which the virtual displacement is $\delta x^0 = \delta t = 0$!) that $R_i \, \delta x^i = 0$. We will then have:

$$R_i = \underset{p}{R} \, \underset{p}{e}_i .$$

It follows from this that:

$$R^k = g^{ki} R_i = \underset{p}{R} \, g^{ki} \underset{p}{e}_i .$$

If we set $R_0 = 0$ then we can write:

(37) $$R^\mu = \underset{p}{R} \, \underset{p}{e}^\mu ,$$

where the vectors $\underset{p}{e}^\mu$ are explained by the equations:

(38) $$\underset{p}{e}^\mu = g^{ki} \underset{p}{e}_i , \qquad \underset{p}{e}^0 = 0.$$

From equation (37), the constraining force falls in the space that is spanned by the vectors $\underset{p}{e}^\mu$, and we choose it to be the pseudo-orthogonal constraint direction. We leave it to the reader to prove that the unit affinor of the A^{m+1}_{n+1} that is constrained in that way is given by the formulas:

(39) $$B^\mu_\lambda = A^\mu_\lambda - C^\mu_\lambda , \qquad C^\mu_\lambda = h^{pq} \underset{p}{e}_\lambda \underset{q}{e}^\mu , \qquad h^{pq} h_{qr} = \delta^p_r , \quad h_{qr} = \underset{q}{e}^\mu \underset{r}{e}_\mu$$

$$(p, q, r = 1, \ldots, n - m).$$

Now, from (35), the equations of the motion of the constrained system read:

$$\frac{\delta^2 x^\mu}{ds^2} = R^\mu \left(\frac{dt}{ds}\right)^2 .$$

If we now project into A_{n+1}^{m+1} then, since R^μ falls in the constraint direction, we will get:

(40) $$\frac{\delta'^2 x^\mu}{ds^2} = 0.$$

The system will then describe a geodetic line in A_{n+1}^{m+1}. Q. E. D.

Those are only $m + 1$ independent equations that determine the parameter s. In addition, we have $n - m$ first-order condition equations:

(41) $$\underset{p}{e}_\alpha \frac{\delta x^\alpha}{ds} = 0.$$

Equations (40) and (41) define the most general equations of motion for linear non-holonomic systems. *We then write down the equations of deviation as in* § **3**, nos. 1 and 3.

4. Summary. – We have arrived at the following results:

If an arbitrary, linear, non-holonomic dynamical system is given by its kinetic energy:

$$T = \tfrac{1}{2} g_{\alpha\beta}\, \dot{x}^\alpha\, \dot{x}^\beta , \qquad x^0 = t \qquad (\alpha, \beta = 0, 1, \ldots, m),$$

along with the generalized forces Q_i that act upon it and the non-holonomic conditions:

$$\underset{p}{e}_\alpha\, \delta x^\alpha = 0 \qquad (p = 1, \ldots, n - m),$$

then we will get its equations of deviation in the form (when we choose the variation to be isochronous):

(42) $$\boxed{\begin{aligned} \frac{\delta'^2 \xi^\mu}{dt^2} &= R'^{\cdots\mu}_{\omega\lambda\nu} \xi^\omega v^\lambda v^\nu , \\ \frac{\delta'' \xi^\mu}{dt} &= (H'^{\cdot\cdot\mu}_{\lambda\nu} - L''^{\cdot\mu}_{\lambda\cdot\nu})\, \xi^\lambda v^\nu , \end{aligned}} \qquad \begin{aligned} &(v^\lambda = \dot{x}^\lambda) \\ &[\text{cf.}, (27), (21)]. \end{aligned}$$

[We have taken $s = t$, which is obviously permissible, from (34).] Here, we have:

$$R'^{\cdots\mu}_{\omega\lambda\nu} = \partial_\lambda \Gamma'^{\mu}_{\nu\omega} - \partial_\omega \Gamma'^{\mu}_{\nu\lambda} + \Gamma'^{\alpha}_{\nu\omega} \Gamma'^{\mu}_{\alpha\lambda} - \Gamma'^{\alpha}_{\nu\lambda} \Gamma'^{\mu}_{\alpha\omega} \qquad \text{[see (15)]}$$

$$\Gamma'^{\mu}_{\nu\omega} = B^{\mu}_{\alpha} \Gamma^{\alpha}_{\nu\omega} - \partial_\omega B^{\mu}_{\nu} \qquad \text{[see (4)]}$$

$\Gamma^{\alpha}_{\nu\omega}$ is defined by (33), while B^{μ}_{ν} is defined by (39). The former depends upon only $g_{\alpha\beta}$ and Q_i , while the latter depends upon the g_{ik} and the non-holonomic conditions.
