"Rheonome Geometrie. Absolute Mechanik," Prace Mat. Fiz. **40** (1932), 97-142.

Rheonomic geometry. Absolute mechanics (*).

By

A. Wundheiler

Translated by D. H. Delphenich

The main result of this article is a new and entirely-simple set of *equations for non-holonomic and rheonomic systems.* They read (§ **14**):

$$\frac{\delta v^i}{dt} + W^i_{\cdot k}\, v^k = Q^i + S^i,$$

and all terms in that have a mechanical meaning [i.e., they are invariant quantities (§ **23**)]. However, those equations are closely linked with an idea that leads to an *adequate theory of rheonomic, non-holonomic systems* (§ **22**). Up to now, such a thing did not exist, because the usual definition of scleronomic systems is itself useless, since the dependency of the kinetic potential on time might vanish for different choices of parameters. Those and similar arguments necessarily lead to the realization that a correctly-constructed theory of *mechanics is an invariant theory for the group of* ***time-dependent*** *coordinate transformations*. Thanks to that new concept, mechanics will become identical to the *multi-dimensional geometry of deformable spaces*, which we will call *rheonomic geometry* (§ **1**). We shall build the foundations for the two theories in (§§ **2-20**), in which we appeal to the constructions of tensor calculus. However, we shall employ a *stronger conception of tensor calculus* (§ **4**), since we shall consider tensors under a larger group than that of the point-transformations.

The applications are scattered throughout different places. For example, we say *scleronomity* (§ **26**) and *holonomity conditions* (§ **20**) to mean the conditions for the *existence of an "energy integral" for rheonomic systems* (§ **28**) *under the infinitesimal bending of a Riemann space* (§ **10**). Everything is expressed in an invariant language. A *theory of the reaction forces* in a general dynamical system (§§ **30**, **31**) concludes the paper.

We shall assume that the elements of the ordinary tensor calculus are known.

(*) The present article represents a summary of an inaugural dissertation that was submitted to the mathematics-natural science faculty at the University of Warsaw.

Notations

Indices. – Following the example of **Schouten**'s school ([1]), the defining data of a quantity will always have the same "kernel" symbol in different coordinate systems, and the difference in coordinate systems will result exclusively in the type of indices. Different defining data for the same quantities in one and the same system will then be distinguished by symbols ("signatures") that are connected with them; e.g.:

$$v^1, \ldots, v^n; \qquad v_x, v_y, v_z; \qquad x^{\bar{1}}, x^{\bar{2}}, \ldots, x^{\bar{n}}.$$

Different coordinate systems have basically-different systems for the distinguishing symbols (signatures); e.g.:

$$1, 2, \ldots, n; \qquad \mathit{1}, \mathit{2}, \ldots, n; \qquad \mathbf{1}, \mathbf{2}, \ldots, \boldsymbol{n}; \qquad \bar{1}, \bar{2}, \ldots, \bar{n}, \quad \text{etc.}$$

Different indices run through basically-different series of numbers, such that, e.g., x^i and x^a will be different systems of numbers, when not expressly stated otherwise. We shall write a coordinate transformation as:

$$x^i = x^i(x^\lambda).$$

In what follows, we shall maintain the conventions:

$$h, i, j, k = 1, 2, \ldots, n; \qquad \alpha, \beta, \gamma = \underline{1}, \underline{2}, \ldots, \underline{m};$$

$$I, K = \bar{1}, \bar{2}, \ldots, \bar{n}; \qquad \lambda, \mu = \underline{1}', \underline{2}', \ldots, \underline{m}'.$$

Derivatives. – We shall always write:

$$\partial_i, \text{ instead of } \frac{\partial}{\partial x^i}, \qquad \partial_t, \text{ instead of } \frac{\partial}{\partial t}, \qquad \dot{x}^i = \frac{dx^i}{dt},$$

in which t will mean time exclusively for us. δ will always mean the covariant differential.

Summation signs. – As is now becoming generally accepted, doubled indices that appear in a monomial will be automatically summed over.

We shall often employ an abbreviated notation in which tensors can be written without any indices, without any special explanation. That will happen in the cases where the formulas can be calculated with an easy deciphering.

We shall denote spaces and subspaces by upper-case German symbols without exhibiting their dimension numbers. $\mathfrak{A}$ will always mean the initial space. $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{E}$ are

([1]) **J. A. Schouten** and **E. R. van Kampen**, "Zur Einbettungs- und Krümmungstheorie nichtholonomer Gebilde," Math. Ann. **103** (1930).

subspaces. We shall always denote tensors by only Latin symbols (except for dx^i, $\dot{x}^i$), while non-tensors will be denoted by Greek symbols (exception: arc length element $d\sigma$).

I

Strong invariants of the inhomogeneous quadratic form. Rheonomic geometry.

1. Concept of rheonomic geometry. – We will understand *rheonomic geometry* to mean *the geometry of a deformable space*. We construct its using the model of a surface that moves in three-dimensional space, which will also often connect us with the case of the deformable surface. We suggest that the reader should refer all of our concepts to that case as an illustration. In any event, the case of an arbitrary moving three-dimensional medium can be drawn upon in order to make things more intuitive.

We will be compelled naturally into multi-dimensional geometry. If we would like to arrive at natural quantities then we must operate with invariants. However, what group shall we use as a basis? In ordinary Riemannian geometry, we choose the group of point-transformations:

$$x^i = x^i(x^I) \qquad i = 1, \dots, n; \quad I = \bar{1}, \bar{2}, \dots, \bar{n}, \tag{1}$$

and seek the invariants of the quadratic differential form:

$$ds^2 = a_{ik}\, dx^i\, dx^k.$$

That will occur because in **Riemann** space all of the coordinate systems that are linked by (1) are equivalent (in the general non-Euclidian case). What form do things take in a deformed space?

According to our program, if we consider the deformable surface in three-dimensional space then we will see that (when the surface is not too rigid) one cannot speak of the "same point at different moments." That is because if we do not think of the surface as being material in its manifestation then how are we to recognize the point at different moments? If one chooses a representation:

$$x^\lambda = x^\lambda(x^i, t) \qquad\qquad (\lambda = \underline{1}, \underline{2}, \underline{3};\ i = 1, 2)$$

then "the same point" will always correspond to the same x^i. However, one can just as well choose a different representation:

$$x^\lambda = x^\lambda(x^I, t) \qquad\qquad I = \bar{1}, \bar{2},$$

in which:

$$x^i = x^i(x^I, t), \tag{2}$$

and neither of the two representations is distinguished from the others (in the general case). Naturally, the transformation (2) will change the "identity" of the point on the surface. According to what was done above, the quantities of "rheonomic geometry" must be invariant under the transformations (2), since the "identities" of their points will not represent any property of the moving surface, except perhaps when it is rigid. We then clarify that *rheonomic geometry is the invariant theory of the* ***kinematical group*** :

(3) $$x^i = x^i\,(x^I, t) \qquad\qquad i = 1, \ldots, n\,;\, I = \bar{1}, \ldots, \bar{n}\,.$$

Naturally, we can also introduce time-dependent coordinates into an ordinary **Riemann** space and thus create the appearance of rheonomity. In such a case, we shall speak of a *strongly-scleronomic space*. It will often be useful to see what the general rheonomic quantities will be in the strongly-scleronomic case. That will often allow us to understand their meaning.

2. The elementary displacement. – In ordinary differential geometry, we denote an elementary displacement by an infinitesimal vector with the components dx^i. That notion cannot be expressed in rheonomic geometry: We must characterize the elementary displacement by means of the system of quantities dx^i, dt. There are two arguments in favor of that viewpoint:

1. If we consider all coordinate systems that are coupled by way of (2):

$$x^i = x^i\,(x^I, t)$$

to be equivalent then the components of the displacement dx^i in the system $\{i\}$ will still not be determined by the dx^I, since they also depend upon dt. Two different displacements will have different components in different coordinate systems for the same dx^I and different dt. A displacement that is determined exactly – i.e., uniquely in all rheonomic systems – must also possess a well-define dt.

2. We consider the case of a moving surface. A displacement – i.e., two infinitely-close points A and B – is considered to be determined when it likewise corresponds to a well-defined displacement in the surrounding space. Now, when the surface moves, the point-pair that covers A and B will depend upon the moment at which A and B were fixed in the surrounding space, so it will also depend upon the duration of the displacement, and thus, upon dt.

We then propose that:

We refer to the system of differentials dx^i, dt as an elementary displacement in a rheonomic space.

3. The inhomogeneous quadratic differential form. – Riemann geometry is the invariant theory of a quadratic differential form:

$$ds^2 = a_{ik}\, dx^i\, dx^k.$$

Our model of the deformable surface teaches us that rheonomic geometry must be based upon the inhomogeneous form:

$$ds^2 = a_{ik}\, dx^i\, dx^k + 2\alpha_i\, dx^i\, dt + \mathsf{A}\, dt^2.$$

If we calculate that form for the moving surface, which is given by:

$$x^\lambda = x^\lambda (x^i, t) \qquad (\lambda = \underline{1}, \underline{2}, \underline{3}; i = 1, 2),$$

then that will give:

(4) $$\alpha_i = a_{\lambda\mu} \frac{\partial x^\lambda}{\partial t}\frac{\partial x^\mu}{\partial x^i}, \qquad \mathsf{A} = a_{\lambda\mu} \frac{\partial x^\lambda}{\partial t}\frac{\partial x^\mu}{\partial t}.$$

(The differentiation $\partial / \partial t$ is done with constant x^i.) α_i is then the projection of the *guiding velocity* $\partial x^\lambda / \partial t$ onto the surface, while the "guiding longitude" A is the *guiding vis viva*. Naturally, neither quantity is *intrinsic*, since they are referred to a well-defined coordinate system. However, we shall derive invariant quantities from them later.

4. Strong tensors. – We call a system v^i of n numbers that transform according to the formulas:

$$v^I = \frac{\partial x^I}{\partial x^i} v^i$$

under:

(5) $$x^i = x^i (x^I, t)$$

a *strong* (contravariant) *vector*, so it transforms like an ordinary vector under the geometric transformation:

(6) $$x^i = x^i (x^I) .$$

Covariant vectors and various tensors are defined similarly according to the well-known models.

To clarify, we remark that unlike the usual situation, the dx^I do not define a vector, since one has:

$$dx^I = \frac{\partial x^I}{\partial x^i} dx^i + \frac{\partial x^I}{\partial t} dt .$$

That fact defines the fundamental difference between the ordinary and the "rheonomic" theory of invariants. By contrast:

$$\frac{\partial f}{\partial x^I}$$

will be a strongly-covariant vector only when f is a strong scalar. In fact, from (6), as well as (5):

$$\frac{\partial f}{\partial x^I} = \frac{\partial f}{\partial x^I}\frac{\partial x^I}{\partial x^i}.$$

The following simple and important theorem will allow one to construct strong tensors:

(7) *If T is a strong tensor that depends upon $\dot{x}^i$, x^i, and t then so is* $\dfrac{\partial T}{\partial \dot{x}^i}$.

In fact, one has:

$$\dot{x}^I = \frac{\partial x^I}{\partial x^i}\dot{x}^i + \frac{\partial x^I}{\partial t},$$

so

$$\frac{\partial \dot{x}^I}{\partial \dot{x}^i} = \frac{\partial x^I}{\partial x^i}.$$

However, the $\dfrac{\partial x^I}{\partial x^i}$ are independent of $\dot{x}^i$. If one then differentiates, e.g.:

$$T^K = \frac{\partial x^K}{\partial x^i}T^i, \qquad K = \bar{1}, \ldots, \bar{n}$$

with respect to $\dot{x}^I$ then one will get the theorem directly.

5. The fundamental strong tensors. – We shall now start with the form:

$$2T = a_{ik}\,\dot{x}^i\,\dot{x}^k + 2\alpha_i\,\dot{x}^i + \mathrm{A}, \qquad i, k = 1, \ldots, n,$$

which must be invariant under:

$$x^I = x^I(x^i, t),$$

by assumption.

An application of the aforementioned theorem (7) will imply the strongly-covariant vector:

$$v_i = \frac{\partial T}{\partial \dot{x}^i} = a_{ik}\,\dot{x}^k + \alpha_i,$$

which we shall call the *longitudinal guideline*. Another application of the same theorem (7) will yield the second-rank strongly-covariant "fundamental tensor":

$$\frac{\partial v_i}{\partial \dot{x}^k} = a_{ik} .$$

As usual, we introduce the reciprocal tensor:

$$a^{ij}\, a_{jk} = a^i_k = \begin{cases} 1 & i = k, \\ 0 & i \neq k. \end{cases}$$

If we now set:

$$\alpha^i = a^{ik}\, \alpha_k$$

then we will come to the strongly-contravariant vector:

$$v^i = a^{ik}\, v_k = \dot{x}^i + \alpha^i .$$

Naturally, dt is a strong scalar. Hence:

$$\delta x^i = v^i\, dt = dx^i + \alpha^i\, dt$$

will be a strong (infinitesimal) vector that we shall call the *absolute elementary displacement*. It enters in place of dx^i in our calculus.

In the strongly-scleronomic case, one will have $\alpha^i = 0$ for a suitably-chosen coordinate system. We see that the absolute coordinates of the element coincide with the ordinary (distinguished) ones in this case. The absolute components of the element then cancel the apparent rheonomity that was introduced by false coordinates, to some extent.

We shall now rewrite the quadratic fundamental form by means of the absolute displacement in order to arrive at new invariants:

$$ds^2 = a_{ik}\,(dx^i + \alpha^i\, dt)\,(dx^k + \alpha^k\, dt) + (\mathsf{A} - a_{ik}\, \alpha^i\, \alpha^k)\, dt^2,$$

$$ds^2 = a_{ik}\, \delta x^i\, \delta x^k + (\mathsf{A} - \alpha_i\, \alpha^i)\, dt^2.$$

The left-hand side and the first summand on the right are strongly invariant. The same thing will also be true for:

$$\mathcal{A} = \mathsf{A} - \alpha_i\, \alpha^i$$

then. We call $\mathcal{A}$ the *transverse viv viva*. The reader might verify that $\mathcal{A}$ yields the square of the transverse component of the guiding velocity in the case of a rigid moving surface.

Naturally, one has $\mathcal{A} = 0$ in the strongly-scleronomic case, because $\mathsf{A} = \alpha_i = 0$ in the distinguished coordinate system, since the form is homogeneous.

6. Strongly-covariant differential. – We must now go on to the definition of a "strong" differential. Naturally, in order to duplicate the known theories, we must demand that [2]:

(8)
1. The differential of a scalar is equal to the ordinary differential:
$$\delta p = dp.$$
2. The differential of a strong tensor is a tensor of the same rank and type.
3. The differential is additive and "partial":
$$\delta(U+V) = \delta U + \delta V; \quad \delta(UV) = U\,\delta V + V\,\delta U.$$

Along with these absolutely-necessary conditions, we add that:

4. The differential is also partially-applicable to a scalar product:
$$\delta(U_i V^i) = U_i\,\delta V^i + V^i\,\delta U_i.$$
5. The differential of the fundamental tensor is zero:
$$\delta a_{ik} = 0.$$

Postulate 1. can be considered to be the definition of the covariant differential of a scalar. If we set:

$$\delta v^i = dv^i + \omega^i_k\, v^k,$$
$$\delta v_i = dv_i + \tilde{\omega}^i_k\, v_i,$$

(ω and ϖ are differential forms in dx^i and dt), and analogously in the known way for tensors, then (8.3) will be fulfilled. If we then require that:

$$\omega^i_k + \varpi^i_k = 0$$

then we will also achieve (8.4). However, the most complicated postulates (8.2) and (8.5) still remain.

We will achieve our goal by means of a method that is very remarkable, although it is applied relatively little in tensor calculus. We will employ it often, and we shall request that the reader should direct his attention to it. It consists of introducing new tensors as the coefficients of scalar forms, which are, on the other hand, represented by the sums of ordinary (i.e., scalar) differentials of scalar forms. That will permit one to exhibit the tensor property very easily, and in a sense by going back to its origin, namely, the differential of a scalar. In that way, one can often be spared many calculations that are ordinarily connected with coordinate transformations.

We consider the expression:

$$\varphi = \underset{1}{\delta}(a_{ik}\,\underset{2}{\delta}x^i\,\underset{3}{\delta}x^k) + \underset{2}{\delta}(a_{ik}\,\underset{1}{\delta}x^i\,\underset{3}{\delta}x^k) - \underset{3}{\delta}(a_{ik}\,\underset{1}{\delta}x^i\,\underset{2}{\delta}x^k), \tag{9}$$

---

[2] **J. A. Schouten**, *Der Ricci-Kalkul*, Berlin, Springer, 1924, pp. 63. Our treatment is different from the one that was given there.

and propose to rewrite it as a differential form in $\underset{3}{\delta} x^k$. The coefficients of that form will then give us a tensor that proves to be the strongly-covariant differential $\underset{1}{\delta}\underset{2}{\delta} x^k$. However, in order for that to be possible, we must choose the elementary displacement suitably: That is the essential gist of the method ([3]).

We set (invariantly):

$$\underset{2}{d} t = \underset{3}{d} t = 0, \tag{10}$$

so

$$\underset{2}{\delta} x^i = \underset{2}{d} x^i , \qquad \underset{3}{\delta} x^i = \underset{3}{d} x^i , \tag{11}$$

and also choose the displacements $\underset{1}{d}$, $\underset{2}{d}$, $\underset{3}{d}$ to be commutable:

$$\underset{a}{d}\,\underset{b}{d}\, x^i = \underset{b}{d}\,\underset{a}{d}\, x^i \qquad (a, b = 1, 2, 3). \tag{12}$$

We now develop (9), employ (10), (11), (12) once more, and obtain a form in $\underset{3}{\delta} x^i$. The calculation takes the following form:

$$\begin{aligned}
\underset{1}{\delta}(a_{ik}\,\underset{2}{\delta} x^i\,\underset{3}{\delta} x^i) &= a_{ik}\,\underset{1}{d}\,\underset{2}{d}\, x^i\,\underset{3}{\delta} x^k + a_{ik}\,\underset{1}{d}\,\underset{3}{d}\, x^k\,\underset{2}{\delta} x^i + \underset{1}{d}\, a_{ik}\,\underset{2}{d}\, x^i\,\underset{3}{\delta} x^k \\
&= a_{ik}\,\underset{1}{d}\,\underset{2}{d}\, x^i\,\underset{3}{\delta} x^k + \underline{a_{ik}\,\underset{1}{d}\,\underset{3}{d}\, x^k\,\underset{2}{\delta} x^i} + \underline{\underline{\partial_j\, a_{ik}\,\underset{2}{\delta} x^i\,\underset{1}{d}\, x^j\,\underset{3}{\delta} x^k}} + \partial_t\, a_{ik}\,\underset{2}{\delta} x^i\,\underset{2}{d}\, t\,\underset{3}{\delta} x^k ,
\end{aligned}$$

$$\begin{aligned}
\underset{2}{\delta}(a_{ik}\,\underset{1}{\delta} x^i\,\underset{3}{\delta} x^i) &= \underset{2}{d}\,(a_{ik}\,\underset{1}{d}\, x^i\,\underset{3}{\delta} x^k) + \underset{2}{d}(\alpha_k\,\underset{3}{\delta} x^i\,\underset{1}{d}\, t) \\
&= a_{ik}\,\underset{2}{d}\,\underset{1}{d}\, x^i\,\underset{3}{\delta} x^k + \underline{a_{ik}\,\underset{1}{d}\, x^i\,\underset{2}{d}\,\underset{3}{d}\, x^k} + \underline{\underline{\partial_j\, a_{ik}\,\underset{2}{\delta} x^j\,\underset{1}{d}\, x^i\,\underset{3}{\delta} x^k}} + \partial_j\, \alpha_k\,\underset{2}{\delta} x^i\,\underset{1}{d}\, t\,\underset{3}{\delta} x^k + \underline{\alpha_k\,\underset{2}{d}\,\underset{3}{d}\, x^k\,\underset{1}{d}\, t} ,
\end{aligned}$$

$$\begin{aligned}
\underset{3}{\delta}(a_{ik}\,\underset{1}{\delta} x^i\,\underset{2}{\delta} x^i) &= \underset{3}{d}\,(a_{ik}\,\underset{1}{d}\, x^i\,\underset{2}{\delta} x^k) + \underset{3}{d}(\alpha_k\,\underset{2}{\delta} x^i\,\underset{1}{d}\, t) \\
&= \underline{a_{ij}\,\underset{3}{d}\,\underset{1}{d}\, x^i\,\underset{2}{d}\, x^j} + \underline{a_{ik}\,\underset{1}{d}\, x^i\,\underset{3}{d}\,\underset{2}{d}\, x^k} + \underline{\underline{\partial_k\, a_{ij}\,\underset{2}{\delta} x^j\,\underset{1}{d}\, x^i\,\underset{3}{\delta} x^k}} + \partial_k\, \alpha_j\,\underset{2}{\delta} x^j\,\underset{1}{d}\, t\,\underset{3}{\delta} x^k + \underline{\alpha_j\,\underset{3}{d}\,\underset{2}{d}\, x^j\,\underset{1}{d}\, t} .
\end{aligned}$$

The singly-underlined terms remain. The rest possess $\underset{3}{\delta} x^k$ as a "factor" and must then define a strong tensor. The doubly-underlined terms define the cyclic objects whose coefficients are:

([3]) One encounters an expression of the form (9) in **Th. de Donder**, *Théorie des invariants intégraux*, Paris, 1927, pp. 114. Naturally, the covariant differential can be derived in only ordinary **Riemann** space for him. A suitable selection of displacements does not come into play there.

(13) $$2\,\Gamma_{k,\,ij} = \partial_i\, a_{jk} + \partial_j\, a_{ki} - \partial_k\, a_{ij}\,; \qquad \Gamma^k_{ij} = a^{kh}\,\Gamma_{h,\,ij}$$

in the way that follows from **Riemann**ian geometry.

In addition, we set:

(14) $$2\,\Gamma_{kj} = \partial_t\, a_{jk} + \partial_j\, a_k - \partial_k\, a_j\,; \qquad \Gamma^k_{\ j} = a^{kh}\,\Gamma_{hj}$$

analogously and get:

$$\varphi \;=\; \underset{3}{\delta}\, x^k [2a_{ik}\, \underset{1}{d}\, \underset{2}{\delta}\, x^i + 2\,\Gamma_{k,hj}\, \underset{2}{\delta}\, x^h\, \underset{1}{d}\, x^j + 2\,\Gamma_{kh}\, \underset{2}{\delta}\, x^h\, \underset{1}{d}\, t]$$

$$=\; 2a_{ik} [\underset{1}{d}\, \underset{2}{\delta}\, x^i + \Gamma^i_{hj}\, \underset{2}{\delta}\, x^h\, \underset{1}{d}\, x^j + \Gamma^i_{\ h}\, \underset{2}{\delta}\, x^h\, \underset{1}{d}\, t]\, \underset{3}{\delta}\, x^k\,.$$

If we set:

(15) $$\delta v^i = dv^i + \Gamma^i_{hj}\, v^h dx^j + \Gamma^i_{\ h}\, v^h dt$$

or

(16) $$\delta v_h = dv_h + \Gamma^i_{hj}\, v_i\, dx^j + \Gamma_h^{\ i}\, v_i\, dt\,,$$

resp., for a strong vector v^i, by definition, then it will be easy to see that this differential fulfills the conditions (8.2) and (8.5). At first, we have:

(14a) $$\varphi = 2a_{ik}\, \underset{1}{\delta}\, \underset{2}{\delta}\, x^i\, \underset{3}{\delta}\, x^k\,,$$

and since φ is a strong scalar and $\underset{3}{\delta}\, x^k$ is an arbitrary vector, it will follow from this that $a_{ik}\, \underset{1}{\delta}\, \underset{2}{\delta}\, x^k$ is a *strong vector*, and therefore $\underset{1}{\delta}\, \underset{2}{\delta}\, x^i$, as well.

We further have:

$$\underset{1}{\delta}(v_i\, \underset{2}{\delta}\, x^i) = v_i\, \underset{1}{\delta}\, \underset{2}{\delta}\, x^i + \underset{2}{\delta}\, x^i\, \underset{1}{\delta}\, v_i\,.$$

Here, the left-hand side is a strong scalar, while $\underset{1}{\delta}\, \underset{2}{\delta}\, x^i$ is a strong vector, as was just proved. Hence, $\underset{2}{\delta}\, x^i\, \underset{1}{\delta}\, v_i$ is also a scalar, and therefore $\underset{1}{\delta}\, v_i$ is a strong vector. If we start from the expression $v_i\, w^i$ then we will prove analogously that δw^i is a strong vector; things are similar for the higher tensors.

Finally, in order to prove (8.5) – i.e., $\delta\, a_{ik} = 0$ – it suffices to set $\underset{2}{\delta}\, x^k = \underset{3}{\delta}\, x^k = \delta x^k$ in (9). We then get:

$$\varphi = \underset{1}{\delta}(a_{ik}\, \delta x^i\, \delta x^k) \;=\; 2\, a_{ik}\, \underset{1}{\delta}\, \delta x^i\, \delta x^k\,.$$

On the other hand, when we evaluate that expression in a partially-covariant way:

$$\varphi = \underset{1}{\delta} a_{ik}\, \delta x^i\, \delta x^k + 2\, a_{ik}\, \underset{1}{\delta}\, \delta x^i\, \delta x^k .$$

Due to (14a), it follows from this that:

$$\underset{1}{\delta} a_{ik}\, \delta x^i\, \delta x^k = 0$$

for arbitrary δx^l, so:

(17) $$\delta a_{ik} = 0.$$

Q. E. D.

By covariant differentiation:

$$a^{ij}\, a_{jk} = a_k^i = \left\{ \begin{array}{c} 1 \\ 0 \end{array} \right. ,$$

we derive from that:

(18) $$\delta a^{ik} = 0.$$

Equations (17) and (18) insure that strong differentiation commutes with the raising and lowering of indices. However, that shows the formal equivalence of the strong and ordinary covariant differentiation.

If there is a coordinate transformation that makes the Γ_h^i vanish everywhere and always (e.g., in the strongly-scleronomic case) then we will have a special case that relates to the general case in the same way that Euclidian geometry relates to Riemannian, in a certain sense.

7. The strongly-covariant derivative. – Along with the strongly-covariant differential, we consider the strongly-covariant derivative. The differential is a linear form in the elementary displacement; e.g.:

$$\delta v^i = \frac{\partial v^i}{\partial x^k} dx^k + \frac{\partial v^i}{\partial t} dt + \Gamma_{hk}^i\, v^h\, dx^k + \Gamma_h^i\, v^h\, dt .$$

The coefficients of this form give rise to the definition of counterparts to the partial derivatives. In fact, it follows from theorem (7) that the coefficient of dx^k, namely:

$$\frac{\partial}{\partial \dot{x}^k} \frac{\delta v^i}{dt} = \frac{\partial v^i}{\partial x^k} + \Gamma_h^i\, v^h$$

is a strong tensor. We denote it by $\nabla_k\, v^i$ and the corresponding operation by ∇_k, in general. It is identical to the ordinary covariant differentiation that we employ in

Riemannian geometry. It proves to be not only an ordinary tensor, but also a strong one $\left(\text{cf.}, \dfrac{\partial f}{\partial x^k}\right)$.

By contrast, the coefficient of *dt* is *not* a strong tensor, and we must then look deeper to find the counterpart of the partial derivative with respect to time. To that end, we transform the covariant differential into the form of an *absolute* elementary displacement. The coefficient of δx remains the same as it was before with dx^i ; by contrast, for *dt*, we get, e.g.:

$$\frac{\partial v^i}{\partial t} + \Gamma^i_h v^h - \alpha^j \nabla_j v^i .$$

We call that construction the *strongly-covariant partial derivative with respect to time* and denote it by $\nabla_t v^i$.

For a strong scalar, we get, e.g.:

$$\nabla_t f = \frac{\partial f}{\partial t} - \alpha^j \frac{\partial f}{\partial x^j} ,$$

instead of $\partial f / \partial t$. In general, in order to arrive at the strong partial derivative with respect to *t*, we must pick off the quantity $\alpha^j \nabla_j$ from the coefficient of *dt*. We will often write the strong differential in the totally-strong form:

$$dT = \nabla_k T\, \delta x^k + \nabla_t T\, dt. \tag{19}$$

8. The rate of rate of strain tensor. – We will now discover a tensor that defines a peculiarity of rheonomic geometry, in a sense, and possesses no analogue in **Riemann**ian geometry. It proves to be definitive of the elongation of a space and vanishes for a moving rigid space. Since a simply-infinite family of surfaces can always be regarded as a moving surface, it will also be important in the problem of infinitesimal isometries. In that case (and in the case of a hypersurface, more generally), it is closely related to the second fundamental form [pp. 15, (24)].

That tensor is intrinsic to rheonomic space in the sense that it is expressed solely in terms of the inhomogeneous fundamental form. We will pursue it by means of the method that was set down on pp. 8, § **6**.

We consider the scalar form:

$$\psi = \bar{\delta}\,(a_{ik}\, \delta x^i\, \delta x^k) = \bar{d}\, a_{ik}\;\; \delta x^i\, \delta x^k,$$

and choose the *commuting* displacements in the following (invariant) way:

$$dt = 0, \qquad \bar{\delta} x^i = 0, \qquad \text{so} \qquad dx^i = -\alpha^i\, dt .$$

The displacement $\bar{\delta}$ is then purely-temporal, so to speak, and corresponds to a partial differentiation with respect to time, to some extent. By contrast, the displacement δ is an interval between two "simultaneous" points. ψ will give the dilatation of a purely-"spatial" interval during the time *dt*. In the strongly-scleronomic case, all of that is true verbatim, and ψ is naturally zero. The dilatation continues to exist for a deformed surface. Let us now calculate ψ.

$$\begin{aligned}\psi = \bar{\delta}\,(a_{ik}\,\delta x^i\,\delta x^k) &= \bar{d}a_{ik}\,\delta x^i\,\delta x^k + 2a_{ij}\,\bar{d}\,\delta x^j\,\delta x^i \\ &= (\partial_t\,a_{ik} - \alpha^j\partial_j\,a_{ik})\,\delta x^i\,\delta x^k\,\bar{d}t - 2a_{ij}\,\partial_k\alpha^j\,\delta x^j\,\delta x^i\,\bar{d}t \\ &= (\partial_t\,a_{ik} - \alpha^j\partial_j\,a_{ik} - a_{ij}\partial_k\,\alpha^j - a_{kj}\partial_i\,\alpha^j)\,\delta x^i\,\delta x^k\,\bar{d}t\,.\end{aligned}$$

The last step was necessary in order to obtain coefficients that are symmetric in *i* and *k*, because only the symmetric part will be determined by the values of a quadratic form. If one then sets:

$$\begin{aligned}W_{ik} &= \tfrac{1}{2}(\partial_t\,a_{ik} - \alpha^j\partial_j\,a_{ik} - a_{ij}\partial_k\,\alpha^j - a_{kj}\partial_i\,\alpha^j) \\ &= \tfrac{1}{2}(\partial_t\,a_{ik} - \nabla_k\alpha_i - \nabla_i\alpha_k)\,,\end{aligned} \tag{21}$$

as one convinces oneself by calculation, then one will have:

$$\bar{\delta}\,(a_{ik}\,\delta x^i\,\delta x^k) = 2W_{ik}\,\delta x^i\,\delta x^k\,\bar{d}t\,, \tag{22}$$

and since δx^i is arbitrary, we conclude that W_{ik} has the character of a tensor. We call it the *rate of strain tensor.*

In order to ultimately justify this name, we consider an arbitrarily-moving surface and choose the "identities" of its points "normally"; i.e., in such a way that the paths of its points will be orthogonal trajectories to the family of all positions of the surface. Naturally, we will then have $\alpha^i = 0$, and the rate of strain tensor will reduce to $\tfrac{1}{2}\partial_t\,a_{ik}$. That follows from the fact that it measures a purely-longitudinal stretching.

It follows immediately from the above that:

The necessary and sufficient condition for a "transversally" moving surface to be rigid is the vanishing of its rate of strain tensor.

The same thing can also be expressed as:

A simply-infinite family of surfaces is orthogonally isometric if and only if its rate of strain tensor vanishes,

in which the parameter that distinguishes the surface must be interpreted as time. We therefore emphasize that it is important for that condition to be strongly-invariant, so it will be entirely independent of the chosen way of representing the family of surfaces.

We can also now give the *necessary and sufficient condition for the strong scleronomity of space.* It reads:

$$W = 0, \qquad \mathcal{A} = 0, \tag{23}$$

in which $\mathcal{A} = \mathsf{A} - \alpha_i\, \alpha^i$ (cf., pp. 5). It is in fact necessary, since the form is homogeneous and independent of time in the distinguished coordinate system, so equations (23) will be true. However, conversely, if (23) are also fulfilled then we shall choose the coordinates according to the condition $\alpha_i = 0$ (which is obviously always possible). Since (23) is invariant, it must also be true in that coordinate system, so one will then have:

$$\partial_t\, a_{ik} = 0, \qquad \mathsf{A} = 0.$$

Q. E. D.

9. Connection with the second fundamental form. – If an m-dimensional space $\mathfrak{B}$ moves in an n-dimensional space $\mathfrak{A}$ then it will sweep out a "tube" that is an m+1-dimensional space $\mathfrak{C}$. $\mathfrak{B}$ is a hypersurface in that space at each moment, and therefore have a well-defined second fundamental form (the induced curvature). We will show that it is connected closely with the rate of strain tensor.

We choose the coordinate system on $\mathfrak{B}$ in such a way that the trajectories of constant x^i prove to be orthogonal to the $\mathfrak{B}$. If we set:

$$x^\lambda = x^i, \qquad \lambda = 1, \ldots, m\,; \;\; x^0 = t$$

then we will have a coordinate system $\{x^\lambda\}$ on the $(m + 1)$-dimensional tube $\mathfrak{C}$. If $\bar{\delta}$ means a displacement with $\bar{\delta} x^i = 0$, as in § **7**, then since $a^i = 0$ and $\delta x^i = dx^i$ here, the displacement will take place along the t-line, so it will be normal to $\mathfrak{B}$ in $\mathfrak{C}$. If we set:

$$\bar{d}x^\lambda = B^\lambda\, \bar{d}t$$

then B^λ will be the lateral velocity of $\mathfrak{B}$. It is a strong vector, by its nature. If we set:

$$B^\lambda = B\, n^\lambda$$

then n^λ will be the unit normal to $\mathfrak{B}$ in $\mathfrak{C}$.

Now let δ be a "purely-spatial" displacement that commutes with $\bar{\delta} t$ when $\bar{\delta} t = 0$. We will then have ($\delta \bar{d}t = 0$!):

$$\bar{\delta}\, \delta x^\lambda = \delta\, \bar{\delta}\, x^\lambda = \delta(B^\lambda \bar{d}t) = \delta B^\lambda\, dt,$$

or:

$$\bar{\delta}\,\delta x_\lambda = \delta B_\lambda\,\bar{d}t\,.$$

We now write:

$$\bar{d}\,(a_{ik}\,\delta x^i \delta x^k) = \bar{d}\,(\delta x_\lambda\,\delta x^\lambda) = 2\bar{\delta}\,\delta x_\lambda\,\delta x^\lambda = 2\delta B_\lambda\,\delta x^\lambda\,\bar{d}t = 2B\,\delta_\lambda\,\delta x^\lambda\,\bar{d}t\,.$$

The first term in this is equal to $2W_{ik}\,\delta x^i \delta x^k\,\bar{d}t$, from (22). However, due to the definition of the second fundamental form h_{ik} ([3a]), the latter is just $-2B\,h_{ik}\,\delta x^i \delta x^k\,\bar{d}t$. Since W_{ik}, as well as h_{ik} lies in $\mathfrak{B}$, that will imply that:

$$W_{ik} = -B\,h_{ik}\,. \tag{24}$$

That is the stated relation.

That will immediately imply the following theorem ([4]):

If a space moves transversally without strain then it will be geodetic in the tube that is swept out.

The proof is immediate from the theorem on pp. 13, which demands that $W_{ik} = 0$, and the relation $h_{ik} = 0$, which is true for geodetic hypersurfaces.

10. Conditions for bending without stretching. – We imagine a one-parameter family of spaces and pose the question of whether they can be mapped isometrically to each other; i.e., when they can be regarded as a series of positions of a space that moves without stretching. For that to be true, it is necessary and sufficient that a representation of the family of spaces must exist:

$$x^\lambda = x^\lambda\,(x^i,\,t) \qquad \begin{array}{l} \lambda = \bar{1},\ldots,\bar{m}, \\ i = 1,\ldots,n \end{array} \tag{25}$$

for which $\partial_t\,a_{ik} = 0$. The rate of strain tensor allows us to formulate this problem precisely.

Different representations (25) represent different coordinate systems. If there exists one of them for which $\partial_t\,a_{ik} = 0$ then we will consider the corresponding value of a_i and call w_i a strong vector that has the components $-\alpha_i$ in this distinguished coordinate system. Due to the fact that $\partial_t\,a_{ik} = 0$, from (21), we will have:

$$2W_{ik} = -\nabla_k\,\alpha_i - \nabla_i\,\alpha_k = \nabla_k\,w_i + \nabla_i\,w_k$$

([3a]) Cf., e.g., **Duschek-Mayer**, *Lehrbuch der Differentialgeometrie*, Teubner, 1930, Bd. I, pp. 126, (13). The notations there are somewhat different.

([4]) For a different approach to hypersurfaces in a **Riemann**ian space, confer **A. Pantini**, "Sur la déformation le long de trajectoires orthogonal," Bull. Soc. St. Cluj **6** (1931) and Mathematica **5** (1931).

in that coordinate system. That relation between tensors must always exist when it exists in a special coordinate system. The condition for the existence of a vector w_i for which:

$$\nabla_k w_i + \nabla_i w_k = 2W_{ik} \tag{26}$$

is then necessary for the isometry of the family of surfaces. The fact that it is sufficient is implied by the converse argument. If such a vector w_i exists then we can choose a coordinate system in which $\alpha_i = - w_i$. (That is certainly possible!) However, the relation (26) will assume just the form $\partial_t a_{ik} = 0$ in that coordinates.

The equation (26) strongly recalls the **Killing** equation ([5]) for a rigid deformation and will go over to it as long as $W_{ik} = 0$, so when a rigid orthogonal deformation exists. One can infer even more conclusions from that equation, namely, ones that relate to the infinitesimal isometry and the presentation of all possible isometries ([6]).

11. Strongly-covariant commutation condition. – As is known, one says that two displacements dx^i, $\bar{d}x^i$ *commute* when:

$$d\,\bar{d}x^i = \bar{d}\,dx^i. \tag{27}$$

In a rheonomic space, one must include:

$$d\,\bar{d}t = \bar{d}\,dt.$$

The condition (27) is indeed an invariant relation, but its individual terms are obviously not vectors. We pose the problem of rewriting it in a strongly-invariant way.

To that end, we consider the expression:

$$\delta\bar{\delta}x^i - \bar{\delta}\,\delta x^i = d\,\bar{\delta}x^i + \Gamma^i_{hj}\,\bar{\delta}x^h dx^j + \Gamma^i_{\ h}\,\bar{\delta}x^h dt - d\,\bar{\delta}x^i - \Gamma^i_{hj}\,\delta x^h \bar{d}x^j + \Gamma^i_{\ h}\,\delta x^h \bar{d}t\,.$$

Since $\delta\bar{\delta}x^i - \bar{\delta}\,\delta x^i$ is a strong tensor, so is the latter expression. However, if we choose a normal coordinate system (i.e., $\alpha^i = 0$) then, since $\bar{\delta}x^i = \bar{d}x^i$, $\delta x^i = dx^i$, it will be identical to:

$$\Gamma^i_{\ h}(\bar{\delta}x^h\,dt - \delta x^h\,\bar{d}t) = \tfrac{1}{2}a^{ik}\partial_t a_{kh}\,(\bar{\delta}x^h\,dt - \delta x^h\,\bar{d}t) = W^i_h(\bar{\delta}x^h\,dt - \delta x^h\,\bar{d}t)\,.$$

That is the desired formula. We can define the rate of strain tensor by means of this formula, and we shall actually pursue that path when we extend it (pp. 23).

([5]) Cf., e.g., *Ricci-Kalkul*, pp. 212, (271).

([6]) Cf., **A. Wundheiler**, "Conditions pour une surface flexible inextensible," C R. Acad. Sci. Paris **193** (1931).

II.

Strong invariants of an inhomogeneous quadratic differential form and a Pfaffian system. – Rheononholonomic geometry.

12. Concept of rheononholonomic geometry. – In order to justify our applications to mechanics, we must adapt our concepts and results to non-holonomic and simultaneously rheonomic spaces. We shall first give a general guideline. If a surface moves in space according to the equations:

$$x^i = x^i (x^\alpha, t) \qquad i = 1, 2, 3; \ \alpha = \underline{1}, \underline{2},$$

$$dx^i = b^i_\alpha \, dx^\alpha + v^i \, dt, \qquad b^i_\alpha = \partial_\alpha x^i, \ v^i = \partial_t x^i$$

then a surface element and a guiding velocity v^i will exist at each moment at each point of the surface. They are coupled to each other as derivatives of well-defined functions by certain integrability conditions, so they cannot be chosen freely.

We now renounce those conditions (and this is the fundamental step) and choose the b^i_α and v^i to be completely *independent of each other*. We will then obtain a structure that consists of a *time-dependent m*-dimensional direction element and a vector at each point of the space. We call it a *non-holonomic, rheonomic subspace*.

However, since such an element-vector pair is assumed at each point of space, we will not have the analogue of a moving surface here, but a family of moving surfaces. If n is the dimension number of the subspace and m is that of the direction element then we will get a family of ∞^{n-m} "surfaces" in the holonomic case. One must probably keep that in mind when one would like to visualize non-holonomic geometry correctly. Failing to observe that fact has led various authors to make errors in several instances (commuting displacements!).

We will assume that a rheononholonomic space is given by the equations:

$$dx^i = b^i_\alpha \, dx^\alpha + v^i \, dt, \qquad \alpha = \underline{1}, \ldots, \underline{m}. \tag{29}$$

It is easy to see that non-holonomic geometry must be the invariant theory of the groups:

$$dx^i = a^i_I \, dx^I + \omega^i \, dt, \qquad I = \bar{1}, \ldots, \bar{n}, \tag{30}$$

$$dx^\alpha = b^\alpha_\lambda \, dx^\lambda + \varpi^\alpha \, dt, \qquad \lambda = \underline{1}', \ldots, \underline{m}'. \tag{31}$$

In fact, not only is a subspace specified by (29), but also a *coordinate system*. If one performs a linear transformation (31) on the dx^α then one will get another representation of that subspace:

$$dx^i = b^i_\lambda \, dx^\lambda + \tilde{v}^i \, dt$$

that is just as good as the previous one. The transformation (31) must then be invariant with respect to properties that depend upon the coordinate system.

There might be a distinguished coordinate system (e.g., when (29) is actually holonomic) in which the apparent anholonomity is only due to a clumsy choice of the dx^α. One must then distinguish sharply between non-holonomic subspaces and holonomic spaces in a non-holonomic representation. A criterion for the apparent holonomity must be unconditionally invariant under (31); we will give that criterion later.

We then clarify that *rheononholonomic geometry is the invariant theory of the group:*

$$dx^i = a_I^i\, dx^I + \omega^i\, dt, \qquad\qquad dx^\alpha = b_\lambda^\alpha\, dx^\lambda + \varpi^\alpha\, dt,$$

and an *inhomogeneous quadratic differential form.* The concept of that geometry must be included as a special case of the one that was introduced up to now. However, that shows that the increased complexity in comparison to the holonomic case is meaningless. That explains the fact that the properties of the first order of differentiation that are independent of the integrability conditions (and those are indeed the most important ones) obviously read the same in both cases.

For $m = n$, the subspace will be identical to the ambient space, and (31) will be simply a coordinate transformation. We will check the meaning of our concepts in this case as an illustration.

We will denote rheononholonomic spaces by [$\mathfrak{B}$].

Now, our problem consists of generalizing the concepts that were introduced in I. to non-holonomic subspaces. One deals with, *inter alia*, the fundamental tensor, the longitudinal velocity, the strongly-covariant differential, the rate of strain tensor, etc. All of those structures must go over to the usual ones in the holonomic case.

13. Projection into the virtual subspace. – Let the rheonomic, non-holonomic subspace be given by the equations:

$$dx^i = b_\alpha^i\, dx^\alpha + v^i\, dt\,. \tag{32}$$

They determine the *virtual subspace* $\mathfrak{B}$. A vector v^i *lies in that subspace* when it can be represented in the form:

$$v^i = b_\alpha^i\, v^\alpha\,.$$

A vector is *orthogonal to* $\mathfrak{B}$ when it is orthogonal to every vector that lies in $\mathfrak{B}$. We adapt that concept to arbitrary tensors by referring them to a well-defined index.

A vector can be decomposed into two summands, one of which lies in $\mathfrak{B}$, and the other of which is, by contrast, orthogonal to $\mathfrak{B}$. We call the former summands the *longitudinal components*, or the *projection of the vector onto* $\mathfrak{B}$, while the latter are the *transverse components.*

One easily verifies the following theorems: If we set:

$$b_{\alpha\beta} = a_{ik}\, b^i_\alpha\, b^k_\beta\,, \qquad b^{\alpha\gamma}\, b_{\gamma\beta} = b^\alpha_\beta = \begin{cases} 1 & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases}, \qquad b^\alpha_i = b^{\alpha\beta}\, b_{\beta l} = b^{\alpha\beta}\, b_{ik}\, b^k_\beta\,,$$

$$b^i_k = b^i_\alpha\, b^\alpha_k \qquad \text{(one always has } bb = b\text{!)}$$

then the displacements:

$$\underline{v} = bv$$

will always give the projections of vectors v onto $\mathfrak{B}$, and indeed with different coordinates: *Projection is equivalent to multiplication by b*. We let b denote the *unit tensor* on $\mathfrak{B}$ and consider the quantities (33) to be its various representations in terms of components. Obviously, one always (i.e., for every component) has, symbolically:

$$bb = b.$$

If a tensor has the index a in some position then one can replace it with the more general index i by using the formula:

$$T^i = b^i_\alpha\, T^\alpha \qquad (T_i = b^\alpha_i\, T_\alpha\,, \text{ resp.}).$$

If we set:

$$c = a - b$$

symbolically then c will be the unit tensor on the space $\mathfrak{C}$ that is orthogonal to $\mathfrak{B}$.

The concepts of projection and of lying in $\mathfrak{B}$ will also carry over to tensors by relativizing them with respect to one or more indices. If a tensor lies in $\mathfrak{B}$ with respect to one index then it will be orthogonal to $\mathfrak{C}$ relative to that index, and conversely.

14. Non-holonomic fundamental quantities and canonical form. (Cf., § **5**) – We first have:

(34) $$dx^i = b^i_\alpha\, dx^\alpha + (\alpha^i + v^i)\, dt, \quad \delta v^i = b^i_\alpha\, \dot{x}^\alpha + (\alpha^i + v^i)\,.$$

Just as in theorem (7) on pp. 6, we will prove that $\partial v^i / \partial \dot{x}^\alpha = b^i_\alpha$ is a strong vector relative to the index i. However, v^i is a scalar relative to α, so $\partial v^i / \partial \dot{x}^\alpha = b^i_\alpha$ is a strongly-covariant vector relative to α.

In order find the fundamental form for the subspace, we introduce (34) into the expression ds^2. That will give:

$$ds^2 = b_{\alpha\beta}\, dx^\alpha\, dx^\beta + 2\beta_\alpha\, dx^\alpha\, dt + \mathrm{B}\, dt^2.$$

One has:

$$\beta_\alpha = b^i_\alpha (\alpha^i + v^i)$$

in this. As on pp. 7, it follows from this that:

$$b_{\alpha\beta} = a_{ik}\, b^i_\alpha\, b^k_\beta$$

is a strongly-covariant tensor in regard to α, β. Likewise, as in § **5**, it follows further that:

$$\underline{v}_\alpha = b_{\alpha\beta}\, \dot{x}^\beta + \beta_\alpha , \qquad \underline{\delta}\, x_\alpha = b_{\alpha\beta}\, dx^\beta + \beta_\alpha\, dt ,$$

$$\underline{v}^\alpha = \dot{x}^\beta + \beta^\alpha , \qquad \underline{\delta}\, x^\alpha = dx^\alpha + \beta^\alpha\, dt$$

are strong vectors. The underline below the symbols shall refer to the subspace. We call these constructions the *longitudinal velocities* and the *absolute elementary displacements in* [$\mathfrak{B}$].

We can now rewrite equation (34) into the form:

$$\delta x^i = b^i_\alpha\, \underline{\delta} x^\alpha + (\alpha^i + v^i - b^i_\alpha\, \beta^\alpha)\, dt ,$$

from which it will now follow that:

$$B^i = \alpha^i + v^i - b^i_\alpha\, \beta^\alpha$$

is a strong vector. We call it the *transverse velocity* and write:

(35) $$\delta x^i = b^i_\alpha\, \underline{\delta} x^\alpha + B^i\, dt, \qquad v^i = b^i_\alpha\, \underline{v}^\alpha + B^i.$$

That is a *canonical and completely-invariant form* for the equations of [$\mathfrak{B}$]. We shall give its geometric interpretation.

We see immediately that B^i is orthogonal to $\mathfrak{B}$. In fact:

$$b^\alpha_i\, B^i = b^\alpha_i\, (\alpha^i + v^i) - b^\alpha_i\, b^a_\beta\, \beta^\beta = b^\alpha_i\, (\alpha^i + v^i) - \beta^\alpha = 0.$$

It then follows that $b^i_\alpha\, v^\alpha$ lies in $\mathfrak{B}$, and therefore v^α, as well.

It follows from equations (35) that each vector that belongs to [$\mathfrak{B}$] has components B^i that are orthogonal to $\mathfrak{B}$. We can write that as:

(36) $$c\ \delta x = B\ dt, \qquad cv = B,$$

if we write the unit tensor on the orthogonal space $\mathfrak{C}$ by c. The geometric interpretation for $m = 2$, $n = 3$ is very intuitive. The endpoints of all vectors that belong to [$\mathfrak{B}$] that issue from the same point lie on the same plane, which is parallel to the virtual plane $\mathfrak{B}$.

We further have the invariant:

$$\mathcal{B} = \mathsf{B} - \beta_\alpha \beta^\alpha,$$

which is analogous to $\mathcal{A}$. However, we have:

$$ds^2 = \delta x_i\, \delta x^i + \mathcal{A}\, dt^2 \,. \tag{37}$$

If we now substitute (35) in formula (37) then we will get:

$$ds^2 = \underline{\delta} x_\alpha\, \underline{\delta} x^\alpha + (\mathcal{A} + B_i\, B^i)\, dt^2 \equiv \underline{\delta} x_\alpha\, \underline{\delta} x^\alpha + \mathcal{B}\, dt^2,$$

from which, it will follow that:

$$\mathcal{B} = \mathcal{A} + B_i\, B^i.$$

That is the *relationship between the transverse viv vivas of the spaces* $\mathfrak{A}$ *and* $\mathfrak{B}$.

15. The strongly-covariant differential in $\mathfrak{B}$. – Let $\mathfrak{B}$ be the virtual subspace of a rheonomic space [$\mathfrak{B}$], and let b be its unit tensor. If v^i is a vector field that lies in $\mathfrak{B}$ then dv^i will no longer lie in $\mathfrak{B}$, in general. We must take care to find a strongly-covariant differential that satisfies the conditions (8), and itself lies in $\mathfrak{B}$ for quantities that lie in $\mathfrak{B}$, in addition.

We shall call the expression:

$$\underline{\delta} v = b\, \delta v$$

the *differential that is induced in* $\mathfrak{B}$, or simply, the $\mathfrak{B}$-*differential*, which represents the *projection of the ordinary covariant differential onto* $\mathfrak{B}$.

If T is a tensor of higher rank then we will get its $\mathfrak{B}$-differential when we project each index of the ordinary differential onto $\mathfrak{B}$.

It is obvious that the $\mathfrak{B}$-differential fulfills the conditions 1, 2, 3, 4 of § **6**. We shall then verify 5. In fact, if v lies in $\mathfrak{B}$ then we will have:

$$v_\alpha = b_{\alpha\beta}\, v^\beta,$$

so

$$\underline{\delta} v = v\, \underline{\delta} b + b\, \underline{\delta} v = v\, \underline{\delta} b + \underline{\delta} v$$

since $\underline{\delta} v$, which lies in 𝔅, will not be altered by displacement by b. We then have:

$$v\,\underline{\delta} b = 0$$

for an arbitrary vector v in 𝔅, and since $\underline{\delta} b$ likewise lies in 𝔅, we will get $\underline{\delta} b = 0$, as claimed.

Hence, the fundamental tensor b of 𝔅 is likewise "constant" with respect to the 𝔅-differential, just as the fundamental tensor a is constant with respect to the 𝔅-differential. A complete analogy exists here. It is clear that the induced differential coincides with the ordinary differential in the holonomic case, since it is determined completely by the five conditions then. If 𝔅 coincides with 𝔄 then the other two differential operations will also coincide.

One can give a differential form to $\underline{\delta} u$:

$$\underline{\delta} u^{\alpha} = du^{\alpha} + \Gamma^{\alpha}_{\beta i}\, u^{\beta} dx^{i} + \Gamma^{\alpha}_{\beta}\, u^{\beta} dt\,,$$

$$\underline{\delta} u_{\beta} = du_{\beta} - \Gamma^{\alpha}_{\beta i}\, u_{\beta}\, dx^{i} - \Gamma^{\alpha}_{\beta}\, u_{\alpha}\, dt$$

that is similar to that of the 𝔄-differential. However, that is hardly interesting, since we will not use that explicit form, except in § **19**.

16. Induced curvature. – We began with the remark that the 𝔄-differential of 𝔅-quantities does not lie in 𝔅, so it will be different from the 𝔅-differential. We then calculate the difference of two quantities. We have:

$$\delta u = b\;\delta u + u\;\delta b = \underline{\delta} u + u\;\delta b$$

or [cf., (19)]:

$$\delta u^{i} - \underline{\delta} u^{i} = u^{k}\,\delta b^{i}_{k} = u^{k}\,\nabla_{j}\, b^{i}_{k}\,\delta x^{j} + u^{k}\,\nabla_{t}\, b^{i}_{k}\, dt\,.$$

When δx lies in 𝔅, that form will temporarily determine the projections of the coefficients of 𝔅 relative to k. We get two tensors for the induced curvature ([7]):

$$H^{\cdot\cdot i}_{\beta\alpha} = b^{k}_{\underline{\alpha}}\, b^{j}_{\underline{\beta}}\, \nabla_{\underline{j}}\, b^{i}_{\underline{k}}\,, \qquad H^{\cdot\cdot i}_{\alpha} = b^{k}_{\underline{\alpha}}\, \nabla_{t}\, b^{i}_{\underline{k}}\,. \tag{38}$$

([7]) Introduced by **Schouten**, *Ricci-Kalkül*, pp. 158, (197), pp. 162, (82), in which one will also find details about it. For non-holonomic space: Math. Zeit. **30**, footnote ([1]). Proof of the holonomity condition: **Schouten-Kampen**, pp. 776.

The symmetry of the tensor $H_{\alpha\beta}^{\cdot\cdot i}$ in the indices β and α is equivalent to the holonomity of $\mathfrak{B}$. That is an important theorem that was found by **Schouten** ([7]). Along with $H_{\beta\alpha}^{\cdot\cdot i}$, we will often employ the equivalent tensor:

$$H_{jk}^{\cdot\cdot i} = b_j^\beta\, b_k^\alpha\, H_{\beta\alpha}^i ,$$

which is symmetric (antisymmetric, resp.) at the same time as $H_{\beta\alpha}^{\cdot\cdot i}$.

17. Centrifugal vector for non-holonomic spaces. – We now introduce a vector S_α by means of the invariant form (cf., pp. 8):

$$\tfrac{1}{2}\delta\mathcal{B}\,\bar{d}t \cdots \bar{\delta}(B_i\,\delta x^i) = S_\alpha\,\delta x^\alpha\,\bar{d}t . \tag{39}$$

Here, we have $\bar{\delta} x^i = B^i\,\bar{d}t$, so δx^i lies in $\mathfrak{B}$. $\bar{\delta}$ is the purely-temporal, so to speak, with respect to [$\mathfrak{B}$], and δ is purely spatial. In addition, we assume that δ and $\bar{\delta}$ commute. In order to prevent any misunderstandings, we expressly remark that δx^i will no longer belong to $\mathfrak{B}$ after we apply the displacement $\bar{\delta}$, such that $\bar{\delta}(B_i\,\delta x^i)$ must be non-zero.

Due to the commutation relation (28), one has:

$$\bar{\delta}\,\delta x^i = \bar{\delta}\,\delta x^i + W_j^{\,i}\,\delta x^j\,\bar{d}t = \delta(B^i\,\bar{d}t) + W_j^{\,i}\,\delta x^j\,\bar{d}t ,$$

so (39) is actually a form in δx^α. It is easy to calculate S_α explicitly. One will get:

$$S_\alpha = b_\alpha^i\,(\tfrac{1}{2}\partial_i\mathcal{A} - \nabla_t B_i - B^k\,\nabla_k B_i - W_{ik}B^k) . \tag{39*}$$

S_α plays a fundamental role in mechanical applications, and because of that, it will be called the *absolute centrifugal force.* In addition, one will meet up with it in the holonomity conditions.

18. Rate of strain tensor for non-holonomic spaces. – As we asserted on pp. 16, we will now introduce the rate of strain tensor for non-holonomic spaces by starting with the commutation condition. Here, we will encounter the following complication: If two fields δx^i and $\bar{\delta} x^i$ lie in [$\mathfrak{B}$] then they will not commute, in general. We shall then modify the process in the sense that we will require only commutation along a curve that lies in [$\mathfrak{B}$], which can always be achieved. By repeated differentiation of:

$$\delta x^i = \underline{\delta} x^i + B^i\,dt,$$

we will obtain:

$$\bar\delta\,\delta x^i = \bar\delta\,\underline{\delta} x^i + \bar\delta\, B^i\, dt + B^i\, \bar\delta dt = \bar\delta\,\delta x^i + \nabla_k\, B^i\, \bar\delta x^k\, dt + \nabla_t\, B^i\, \bar{d}t\, dt + B^i\, \bar\delta dt\,.$$

Projecting onto B will yield:

$$b_i^\alpha\, \bar\delta\,\delta x^i = \bar\delta\,\underline{\delta} x^\alpha + b_i^\alpha\, \nabla_k\, B^i\, \bar\delta x^k\, dt + b_i^\alpha\, \nabla_t\, B^i\, \bar{d}t\, dt\,.$$

Similarly:

$$b_i^\alpha\, \delta\,\bar\delta x^i = \bar\delta\,\underline{\delta} x^\alpha + b_i^\alpha\, \nabla_k\, B^i\, \delta x^k\, \bar{d}t + b_i^\alpha\, \nabla_t\, B^i\, \bar{d}t\, dt\,.$$

We subtract corresponding terms:

$$(\bar\delta\,\underline{\delta} x^\alpha - \underline{\delta}\,\underline{\bar\delta} x^\alpha) = b_i^\alpha\, (\bar\delta\,\delta x^i - \delta\,\bar\delta x^i) + b_i^\alpha \nabla_k\, B^i\, (\delta x^k\, \bar{d}t - \bar\delta x^k\, dt)\,.$$

If δ and $\bar\delta$ commute along [$\mathfrak{B}$]-curve in $\mathfrak{A}$ then we can apply (28) to the points of that curve and get:

$$(\bar\delta\,\underline{\delta} - \underline{\delta}\,\underline{\bar\delta})\, x^\alpha = b_i^\alpha\, (W^i_{\ k} + \nabla_k\, B^i)(\delta x^k\, \bar{d}t - \bar\delta x^k\, dt)\,.$$

However, since:

$$\delta x^k\, \bar{d}t - \bar\delta x^k\, dt = \underline{\delta} x^k\, \bar{d}t - \underline{\bar\delta} x^k\, dt\,,$$

we will have:

$$(\underline{\bar\delta}\,\underline{\delta} - \underline{\bar\delta}\,\underline{\delta})\, x^\alpha = b_i^\alpha\, (W^i_{\ k} + \nabla_k\, B^i)(\delta x^k\, \bar{d}t - \underline{\bar\delta} x^k\, dt)\,,$$

and since δ and $\bar\delta$ lie in $\mathfrak{B}$, that will give:

$$(\underline{\bar\delta}\,\underline{\delta} - \underline{\delta}\,\underline{\bar\delta})\, x^\alpha = b_i^\alpha\, b_\beta^k\, (W^i_{\ k} + \nabla_k\, B^i)(\underline{\delta} x^\beta\, \bar{d}t - \underline{\bar\delta} x^\beta\, dt)\,.$$

Finally, we have:

$$b_i^\alpha\, b_\beta^k\, \nabla_k\, B^i = b_i^\alpha\, b_i^j\, b_\beta^k\, \nabla_k\, B^i = -\, B^i\, b_i^\alpha\, b_\beta^k\, \nabla_k b_i^j = -\, B^i\, H_{\beta\cdot i}^{\cdot\alpha}\,,$$

from (38). Thus, we ultimately have:

$$(\underline{\bar\delta}\,\underline{\delta} - \underline{\delta}\,\underline{\bar\delta})\, x^\alpha = (b_i^\alpha\, b_\beta^k\, W^i_{\ k} - B_i\, H_{\beta\alpha}^{\cdot\cdot i})(\underline{\delta} x^\beta\, \bar{d}t - \underline{\bar\delta} x^\beta\, dt)\,.$$

We call δx^i and $\underline{\bar\delta} x^i$, which are the projections of commuting differentials onto $\mathfrak{B}$, *quasi-commuting differentials*. If [$\mathfrak{B}$] is holonomic then the quasi-commuting differentials will almost commute, and the formula will determine the rate of strain tensor of the holonomic space $\mathfrak{A}$.

We then set:

$$(40) \qquad \underline{W}_{\alpha\beta} = b_i^\alpha\, b_\beta^k\, W^i_{\ k} - B_i\, H_{\beta\alpha}^{\cdot\cdot i}\,,$$

by definition.

It will then follow from this formula that when the subspace is either geodetic or homogeneous ($B^i = 0$), the rate of strain tensor for the ambient space will be the same as the one for the subspace. Due to the holonomity condition (pp. 23), we will have immediately that:

The rate of strain tensor $W_{\alpha\beta}$ will be symmetric if and only if the subspace is holonomic.

19. Curvature tensor. – If we start with the expression for the "cyclic differential":

$$\Delta u^\alpha = (\bar{\delta}\,\delta - \delta\,\bar{\delta})\,u^\alpha$$

then we will get:

$$(\bar{\delta}\,\delta - \delta\,\bar{\delta})\,u^\alpha = R^{\alpha}_{\cdot\beta ik}\,\delta x^i\,\bar{\delta} x^k + R^{\alpha}_{\cdot\beta i}\,u^\beta(\delta x^i\,\bar{d}t - \bar{\delta} x^i dt) \tag{41}$$

after a well-known calculation in Riemannian geometry. Now, we must introduce the absolute displacements δx^i, $\bar{\delta} x^i$ on the right-hand side in place of dx^i, $\bar{d}x^i$, in order to have a completely strongly-invariant form.

For the strong curvature tensors $R^{\alpha}_{\cdot\beta ik}$ and $R^{\alpha}_{\cdot\beta i}$, we have:

$$R^{\alpha}_{\cdot\beta ik} = \partial_k\,\Gamma^\alpha_{\beta i} - \partial_i\,\Gamma^\alpha_{\beta k} + \Gamma^\alpha_{\gamma k}\,\Gamma^\gamma_{\beta i} - \Gamma^\alpha_{\gamma i}\,\Gamma^\gamma_{\beta k}\,,$$

$$R^{\alpha}_{\cdot\beta i} = \partial_t\,\Gamma^\alpha_{\beta i} - \partial_i\,\Gamma^\alpha_{\beta} + \Gamma^\alpha_{\gamma}\,\Gamma^\gamma_{\beta i} - \Gamma^\alpha_{\gamma i}\,\Gamma^\gamma_{\beta k} - \alpha^k R^\alpha_{\beta ik}\,.$$

However, we must make a few remarks. In the 𝔅-differential, the differentiated vector u indeed lies in 𝔅, but the displacement along which one calculates the differential is entirely arbitrary. For that reason, in the expression for the 𝔅-differential:

$$\underline{\delta} u^\alpha = du^\alpha + \Gamma^\alpha_{\beta i}\,u^\beta\,dx^i + \Gamma^\alpha_{\beta}\,u^\beta\,dt\,,$$

the third "differential" index i in the $\Gamma^\alpha_{\beta i}$ can be coupled with an arbitrary quantity. However, that implies that the first two ("vector") indices of the curvature tensors have arbitrary positions in 𝔅, while the last two (i.e., the "differential") indices have arbitrary positions in the ambient space. The introduction of such quantities insures a greater flexibility in the devices used to treat problems in curvature; e.g., the variational equations (§ **29**).

If 𝔅 coincides with 𝔄 then all calculations will remain valid, and we will get simply the curvature tensors of the holonomic space 𝔄, which lie arbitrarily in all of their indices. We point out once more that the general quantities refer to vectors that indeed lie in 𝔄, but are displaced arbitrarily.

20. Holonomity condition. – As was mentioned above, the equations:

$$dx^i = b^i_\alpha\, dx^\alpha + v^i\, dt$$

will determine not only a subspace, but also a coordinate system in it. It might happen that the subspace is holonomic, but the coordinate system $\{\alpha\}$ is not. For that reason, the conditions that are usually given:

$$b^k_\beta\, \partial_k\, b^i_\alpha = b^k_\alpha\, \partial_k\, b^i_\beta$$

cannot be fulfilled at all, even for holonomic subspaces. The correct holonomity condition must be invariant under the transformation:

$$dx^\alpha = b^\alpha_\lambda\, dx^\lambda + \varpi^\alpha\, dt,$$

as well as:

$$dx^i = a^i_I\, dx^I + \omega^i\, dt.$$

This problem can be solved only within the language of rheononholomic geometry.

We shall now consider the rheonomic subspace [$\mathfrak{B}$]. Should it be holonomic then the virtual space $\mathfrak{B}$, as well as the subspace $\mathfrak{C}$ that arises from its motion, must be holonomic. However, that is not satisfied: The vector B^i, which belongs to our rheonomic subspace, must be precisely the "transverse velocity."

If $\mathfrak{B}$ is holonomic then, from the theorem that was stated on pp. 23, one must have:

(42) $$H_{\beta\alpha}^{\cdot\cdot l} = H_{\alpha\beta}^{\cdot\cdot l}.$$

Similarly, the holonomity of $\mathfrak{C}$ demands that:

(43) $$e^k_{\underline{k}}\, e^j_{\underline{j}}\, \nabla_{\underline{j}}\, e^i_{\underline{k}} = e^j_{\underline{j}}\, e^k_{\underline{k}}\, \nabla_{\underline{k}}\, e^i_{\underline{j}},$$

in which e^i_k means the unit tensor on $\mathfrak{C}$. We must express those conditions in terms of b and B^i. If we set:

$$B^i = B\, n^i,$$

in which n^i is a unit vector, then we will easily verify that:

(44) $$e^i_k = b^i_k + n^i\, n_k.$$

We now develop (43) on the basis of (44) and $b^i_h\, B^h = 0$:

$$e^k_{\underline{k}}\, e^j_{\underline{j}}\, \nabla_{\underline{j}}\, e^i_{\underline{k}} = b^h_{\underline{h}}\, b^k_{\underline{k}}\, \nabla_{\underline{k}}\, b^i_{\underline{h}} + b^h_{\underline{h}}\, b^k_{\underline{k}}\, \nabla_{\underline{k}}\, n^i\, n_{\underline{h}} + n^{\underline{h}}\, n^{\underline{k}}\, n_h\, n_k \nabla_{\underline{k}}\, e^i_{\underline{h}} + (b^h_{\underline{h}}\, n^{\underline{k}}\, n_k + b^k_{\underline{k}}\, n^{\underline{h}}\, n_h) \nabla_{\underline{k}}\, e^i_{\underline{h}}.$$

The symmetry of the third summand is obvious. That of the first follows from (42). The symmetry of the second one follows from the following calculation:

$$b_h^{\underline{h}}\, b_k^{\underline{k}}\, \nabla_{\underline{k}}\, n^i\, n_{\underline{h}} = b_h^{\underline{h}}\, b_k^{\underline{k}}\, n^i\, \nabla_{\underline{k}}\, n_{\underline{h}} = b_h^{\underline{h}}\, b_h^j\, b_j^{\underline{h}}\, n^i\, \nabla_{\underline{k}}\, n_{\underline{h}} = -\, n_{\underline{h}}\, n^i\, b_h^j\, b_k^{\underline{k}}\, \nabla_{\underline{k}}\, b_j^{\underline{h}} = -\, n_{\underline{h}}\, n^i\, H_{kh}^{\cdot\cdot\underline{h}}\,,$$

and due to (42), that expression is symmetric. What will remain is the expression:

$$(b_h^{\underline{h}}\, n^{\underline{k}}\, n_k + b_k^{\underline{k}}\, n^{\underline{h}}\, n_h)\nabla_{\underline{k}}\, e_{\underline{h}}^i\,.$$

If we introduce the notation:

$$\Phi_{[ik]} = \Phi_{ik} - \Phi_{ki}$$

then we can rewrite (43) in the form:

$$(b_h^{\underline{h}}\, n^{\underline{k}}\, n_k + b_k^{\underline{k}}\, n^{\underline{h}}\, n_h)\nabla_{[\underline{k}}\, e_{\underline{h}]}^i = 0.$$

We contract that with u^h, and then with u^k, and obtain the equivalent conditions:

$$b_h^{\underline{h}}\, u^{\underline{k}}\, \nabla_{[\underline{k}}\, e_{\underline{h}]}^i = 0, \qquad b_k^{\underline{k}}\, u^{\underline{h}}\, \nabla_{[\underline{k}}\, e_{\underline{h}]}^i = 0,$$

which we will shall ultimately write in the form:

$$b_h^{[\underline{h}}\, u^{\underline{k}]}\, \nabla_{\underline{k}}\, (b_{\underline{h}}^i + u^i\, u_{\underline{h}}) = 0. \tag{45}$$

What still remains is the condition for B^i. If [$\mathfrak{B}$] is holonomic then the displacement $\bar{\delta}$ in (39) of the element δx^i will obviously lead to another one that lies in $\mathfrak{B}$. Hence, one will then have not only $B_i\, \delta x^i = 0$, but also $\bar{\delta}\,(B_i\, \delta x^i) = 0$, and one will get:

$$S_\alpha = \tfrac{1}{2}\partial_\alpha \mathcal{B}\,. \tag{46}$$

The conditions (42), (45), and (46) are the desired necessary and sufficient holonomity conditions for rheonomic subspaces.

III.

The absolute equations of mechanics.

21. Overview of the application of tensor calculus to mechanics. – In the year 1900 ([8]), the creators of the absolute differential calculus, **Ricci** and **Levi-Civita**, had already written out the equations of motion for a scleronomic and holonomic mechanical

([8]) **Ricci** and **Levi-Civita**, “Méthodes du calcul différentel absolu et leurs applications,” Math. Ann. **54** (1900), pp. 179. [In Polish translation: Prace matematyczno-fizyczne **12** (1901).]

system in terms of the tensor symbolism, in which the covariant derivative was employed. That form includes quantities that are invariant under only the point-transformations:

(47) $$x^i = x^i (x^I).$$

In that way, the mechanical system is considered to be a point in a multi-dimensional **Riemann**ian space, namely, the *configuration space*, whose fundamental form is:

(48) $$ds^2 = 2T\,dt^2 = a_{ik}\,dx^i\,dx^k ,$$

in which $2T$ is the *vis viva* of the system. That suggests the possibility of finding cyclic coordinates by means of that multi-dimensional picture and giving the conditions for the existence of an integral that is linear in the generalized velocities.

The advantages that one can glean from that multi-dimensional representation have their roots in the fact that one can employ the intuitions that arise from three-dimensional space in order to envision known theorems (e.g., Hertz's principle of the straightest path), but also guess some new results with their help. One then, e.g., obtain variational equations for the paths of mechanical systems by generalizing the **Jacobi** equations for the geodetic deviation [§ **29**, (19)]. In the same way, we have obtained our theorems on reaction forces that are given in §§ **30**, **31** by generalizing the known intrinsic equations of motion. The same intrinsic equations make it very simple to infer a series of consequences in regard to the evolution of the paths of mechanical systems, the evolution of the motion, etc., that **Painlevé** had obtained in a different, more formal way ([9]).

The first systematic treatment of mechanics by means of the tensor calculus goes back to **J. L. Synge** (1926) ([10]). He employed two types of multi-dimensional pictures for mechanical systems. One of them is the one that was discussed before in terms of the "kinematical" ds^2 (48), while the other one is based upon the ds^2 of the action:

$$ds^2 = 2\,(h - V)\,T\,dt^2.$$

Synge likewise wrote the mechanical equations for scleronomic and non-holonomic systems by means of the covariant derivative, but as an application of it, he gave only some considerations in regard to the stability of motion that were based upon the variational equations that **Levi-Civita** had obtained before that generalized the **Jacobi** equations. He also gave criteria for the existence of $n - 1$ cyclic coordinates.

However, his main contribution was that he was the first to point out the important advantage that one can gain from the multi-dimensional pictures: Namely, the difference between holonomic and non-holonomic systems almost vanishes formally with that way of conceptualizing them, which one must naturally take *cum grano salis*. That conclusion is not so glaring in **Synge**'s symbolism. One can say the same thing for **Vrănceanu**'s equations for non-holonomic systems, which were dealt with in a series of

---

([9]) **P. Painlevé**, "Sur les trajectoires réelles," Bull. Soc. Math. de France (1894).

([10]) **J. L. Synge**, "On the Geometry of Dynamics," Phil. Trans. Roy. Soc. A**226** (1926). – "Geodesics in non-holonomic Geometry," Math. Ann. **99** (1928). – "Hodographs of General Dynamical Systems," Trans. Roy. Soc. of Canada **25** (1931).

notes by that author in the Accademia dei Lincei ([11]). He calculated with the orthogonal congruences that were characteristic of the older Italian school, and the transparency of his methods was eclipsed by what could be attained by applying **Schouten**'s symbolism. That was first done in 1928 by **Horák** ([12]). **Vrănceanu**'s applications, like those of **Synge**, are restricted to the stability of conservative systems.

All of those investigations were basically carried out for only scleronomic systems. Indeed, **Horák** also wrote his equations for rheonomic systems, but as far as their clarity was concerned, they were not distinguished by anything that could make them supersede the older explicit equations (e.g., **Woronetz**, **Tzénoff**, **Hamel**). The same thing is true to a even greater degree for **Vrănceanu**, who addressed the problem once more a year ago ([13]) and wrote out the equations for rheonomic and non-holonomic systems in terms of his symbolism.

22. Absolute mechanics. – The basis for all of that inconvenience is the following: The theory of rheonomic systems will prove to be simple only when it is constructed using the right terminology. However, in this case, "right" means that only those terms that are independent of the admissible coordinate systems can have an intrinsic meaning. Now it is clear (and this is the crux of the matter) that for a rheonomic system that is referred to the parameters x^i, all parameter systems that are coupled by a time-dependent transformation:

$$x^i = x^i(x^I, t) \tag{49}$$

are completely equivalent and cannot be distinguished. If one imagines a point on a deformable surface then that will be clear with no further discussion. One must consider mechanical quantities, in the true sense, to be systems that are invariant under these *kinematical transformations* (49), and not only the "geometric" ones (47), and thus, ones that behave tensorially. By recasting the known term "absolute differential calculus," we will refer to the representation of mechanics in such "strongly-invariant" terms as *absolute mechanics,* and call those strongly-invariant quantities *absolute mechanical quantities.* Formally, it will be identical in many aspects with the rheonomic geometry that was developed above or also with the "strong" tensor calculus.

We must go a step further in order to justify the non-holonomic systems. If such a thing is given by the condition equations:

$$dx^i = b^i_\alpha \, dx^\alpha + v^i \, dt$$

then simultaneously-independent "non-holonomic" parameters will be introduced for that system. As was already discussed on pp. 17 for a similar situation, all other representations that one can obtain by applying the parameter transformations:

([11]) **G. Vrănceanu**, "Sopra le equazioni del moto di una sistema anomolo," Rend. Lincei **4** (1926). – "Sopra la stabilità geodetica," *ibid.* **5** (1927). – "Stabilità geodetica. Applicazioni ai sistemi conservati della Meccanica" *ibid.* **5** (1927). – "Sullo scostamento geodetico nelle varietà anolome," *ibid.* **6** (1928). – "Sopra i sistemi anolonomi a legami dipendenti dal tempo," *ibid.* **13** (1931).

([12]) **Z. Horák**, "Sur les systèmes non holonomes," Bull. Int. Acad. Tchéque **24** (1928).

([13]) Cf., the last of the cited notes by **Vrănceanu**.

(50) $$dx^\alpha = b^\alpha_\lambda\, dx^\lambda + \varpi^\alpha\, dt$$

will be entirely equivalent and undistinguishable. Therefore, if a quantity that is referred to a non-holonomic system is to have an intrinsic meaning – so it will be an "absolute mechanical quantity" – then it must behave invariantly under the transformations (50). It is only in that picture that one can hit the nail on the head when one would like to construct an adequate theory of rheonomic systems, and it is only with those quantities that the explicit equations of the general systems can prove to be simple. We then ultimately establish that:

"Absolute mechanics" refers to the invariant theory of the non-holonomic and rheonomic transformations (50) *and the quadratic form for kinetic energy:*

$$2T = a_{ik}\, \dot{x}^i\, \dot{x}^k + 2\alpha_i\, \dot{x}^i + \mathrm{A}\,.$$

The applications that we shall give later will presumably seem to justify that viewpoint. However, we expressly emphasize that absolute mechanics is only in setting for true rheonomic and non-holonomic systems: Naturally, invariance under ordinary point-transformations is achieved completely for a scleronomic system, in which there is a distinguished coordinate system.

23. Mechanical interpretation of the "strong" quantities. – Next, let a holonomic mechanical system be referred to the parameters x^i. Let its *vis viva* be:

(51) $$2T = a_{ik}\, \dot{x}^i\, \dot{x}^k + 2\alpha_i\, \dot{x}^i + \mathrm{A}\,.$$

In § **5**, we called the quantities α_i "longitudinal guideline." If one imagines, e.g., a point that moves on a surface in three-dimensional space:

$$x^\lambda = x^\lambda\,(x^i, t) \qquad \begin{array}{l} \lambda = \bar{1}, \bar{2}, \bar{3} \\ i = 1, 2 \end{array}$$

then one can easily calculate, as on pp. 5 (4), that α_i is just the projection of the guiding velocity onto the surface. However, A will be the square of the guiding velocity, and therefore the "guiding *vis viva*." Naturally, those two quantities depend upon the chosen coordinate system for the surface, so upon the "identity" of the point on the surface, and are thus not absolute mechanical quantities. However, in return, the "transverse *vis viva*":

$$\mathcal{A} = \mathrm{A} - \alpha_i\, \alpha^i$$

(i.e., in our example, the square of the guiding *vis viva* in the absolute direction that is orthogonal to the surface) is a strong quantity.

Similarly, x^i is not a strong quantity, since it likewise depends upon chosen "identity" of the point on the surface. By contrast, if we take the quantities:

$$v^i = \dot{x}^i + \alpha^i,$$

then we can verify in our illustrative example that they are the projections of the absolute velocity of a point that moves with the surface onto that surface. However, that must already be an absolute quantity, as we proved on pp. 6. We shall call it the *longitudinal velocity*.

Let us now go on to non-holonomic systems. First of all, what does the equation:

$$dx^i = b^i_\alpha \, dx^\alpha + v^i \, dt$$

mean? We see that it couples the system, and thus, the representative point of mass 1 in the n-dimensional rheonomic space with the fundamental form (51), in such a way that its "velocity" cannot be arbitrary. It must be composed of a *relative* velocity $\dot{x}^i$ that lies in the virtual space and an induced *guiding velocity*. However, neither of them is invariant, which was pointed out above. By contrast, if we go over to the canonical strongly-invariant form [cf., (35), (36)]:

$$v^i = \underline{v}^i + B^i, \qquad c^i_k \, v^k = B^i$$

then that will show that the "transverse" components of the "total" velocity are determined completely; that is what we mean by *transverse guidance*. One can interpret that by saying that the endpoints of the possible velocities of material points that are found at the location M must lie in an m-dimensional "plane" that is parallel to the virtual plane $\mathfrak{B}$ at the distance B^i. If B^i is equal to zero then that will simplify to a restriction on the possible velocities only in regard to their directions, without restricting their magnitudes. However, in the general case, the magnitude of the velocity will be coupled with its direction.

$B^2 = B_i \, B^i$ will be the *relative transverse vis viva*. By contrast:

$$\mathcal{B} = \mathsf{A} + B_i \, B^i$$

will be the "total" guiding *vis viva*. If the constraints are actually holonomic then all of those quantities will go over to the aforementioned quantities v^i and A.

24. Absolute equations for a holonomic system. – We are now dealing with a form of the equations of motion that are not only invariant under the rheonomic transformations (since one of them is already Lagrange's equation), but also consist of nothing but absolute mechanical quantities. That is no longer the case for the Lagrangian equations. Indeed, $v^i = \dfrac{\partial T}{\partial \dot{x}^i}$ is a strong quantity, but that is no longer true for $\dfrac{d}{dt}\dfrac{\partial T}{\partial \dot{x}^i}$ and $\dfrac{\partial T}{\partial x^i}$.

We start from Hamilton's principle in the form:

$$\int (\bar{\delta} T + Q_i \, \bar{\delta} x^i) \, dt = 0 \tag{52}$$

and calculate with strong covariants from the outset. δ denotes the strong differential along the path, and $\bar{\delta}$ means a "strongly-covariant variation" that commutes with it. We have ($\bar{\delta} t = 0$!):

$$\delta 2T = \delta(v^2 + \mathcal{A}) = \delta(v_i \, v^i + \mathcal{A}) = 2v_i \, \delta v^i + \delta \mathcal{A}.$$

Furthermore:

$$v_i \, \bar{\delta} v^i \, dt = v_i \, \bar{\delta}(v^i \, dt) = v_i \, \bar{\delta} \, \delta x^i ,$$

and from the commutation relations, this is:

$$= v_i \, \delta \bar{\delta} x^i - v_i W^{i}_{\cdot j} \, \bar{\delta} x^j \, dt = \delta(v_i \, \bar{\delta} x^i) - \delta v_i \, \bar{\delta} x^i - v_i W^{i}_{\cdot j} \, \bar{\delta} x^j \, dt .$$

The complete differential integrates out in (52), and what will remain is:

$$-\left(\frac{\delta v_i}{dt} + W^{j}_{\cdot i} \, v_j\right) \bar{\delta} x^i \, dt .$$

We now write:

$$\bar{\delta} \mathcal{A} = \partial_i \mathcal{A} \; \bar{\delta} x^i = 2 \, S_i \, \delta x^i,$$

from (39*). We then get the equations of motion in the form:

$$\frac{\delta v_i}{dt} + W^{j}_{\cdot i} \, v_j = S_i + Q_i . \tag{53}$$

The reader will easily verify that, e.g., in the case of a rotating plane, $\mathcal{A}$ will be the potential of the centrifugal force and S_i will be the centrifugal force itself. Therefore, the term *absolute centrifugal vector* is appropriate. $W_{ij} \, v^j$ recalls the Coriolis force, but it is something quite different, since the Coriolis force is not an absolute quantity, and it will vanish in a suitable coordinate system as only a relative, fictitious force. One sees directly the sort of simplifications that will enter when $W = 0$.

It is, perhaps, interesting to consider the known theory of relative motion from that standpoint. In that theory, we have a point in ordinary space, so a strongly-scleronomic system. The equations of equations reduce to the "Newtonian" form here:

$$\frac{\delta v_i}{dt} = Q^i$$

If the coordinate system is rheonomic (which is just the case for moving axes) then one can specify the equation as follows:

$$\frac{dv^i}{dt} + \Gamma^i_{hj} v^h v^j + \Gamma^i_h v^h = Q_i + S_i \,.$$

If the rheonomic coordinate system is defined by moving axes then it will be easy to see that $\Gamma^i_{hj} = 0$, and that Γ^i_h are the known rotation structures, such that $\Gamma^i_h v^h$ represents simply the fictitious Coriolis force. However, $\mathcal{A}$ will then be the potential of the centrifugal guiding forces. In the absolute treatment of the situation, they are all hidden in the δv^i, and have no absolute mechanical sense. Moreover, that should be clear, since they can be transformed away.

25. Equations for non-holonomic systems. – For non-holonomic systems with the supplementary conditions in the canonical form:

$$v^i = \underline{v}^i + B^i,$$

we have:

(54) $$\frac{\delta v^i}{dt} + W^i_{\cdot j} v^i = S^i + Q^i + R^i,$$

in which R^i means the reaction force that is normal to the virtual space. We project onto the virtual space $\mathfrak{B}$ and introduce the (underlined) quantities everywhere, which are referred to that space:

$$\frac{\delta v^i}{dt} = \frac{\delta \underline{v}^i}{dt} + \frac{\delta B^i}{dt} = \frac{\delta \underline{v}^i}{dt} + \frac{\nabla_k B^i \delta x^k + \nabla_t B^i dt}{dt} = \frac{\delta \underline{v}^i}{dt} + \nabla_k B^i \cdot \underline{v}^k + B^k \nabla_k B^i + \nabla_t B^i \,.$$

We substitute that in (54) and project onto $\mathfrak{B}$, whereby $b_i^\alpha R^i$ will vanish:

$$\frac{\delta \underline{v}^\alpha}{dt} + b_i^\alpha (W^i_{\ k} + \nabla_k B^i) b^k_\beta \underline{v}^k + b_i^\alpha (-S^i + B^k \nabla_k B^i + \nabla_t B^i + W^i_{\ k} B^k) = b_i^\alpha Q^i \,.$$

However, from formulas (39*) and (40):

(55) $$\frac{\delta \underline{v}}{dt} + \underline{W}^\alpha_\beta \underline{v}^\beta = \underline{S}^\alpha + \underline{Q}^\alpha \,.$$

That is the absolute form for rheononholonomic systems that was announced ([14]). It will lead directly to a correct and meaningful classification of dynamical systems.

26. Classification of mechanical systems. – An adequate, meaningful classification of mechanical systems must start from the properties that are inherent to the systems themselves, and not the chosen system of parameters. One cannot refer to the system as rheonomic when its kinetic energy depends upon time, since any scleronomic system that is evaluated with respect to moving axes would also be rheonomic. That same thing is true for the usual definition of holonomity that we wrote down above (pp. 26). An adequate classification must come about from the absolute mechanical quantities.

We summarize the holonomity conditions (42), (45), (46) once more at this point:

$$H_{\beta\alpha}^{\cdot\cdot i} = H_{\alpha\beta}^{\cdot\cdot i},$$

$$b_k^{[h} n^{k]} \nabla_{\underline{k}} (b_{\underline{h}}^i + n^i n_{\underline{h}}) = 0,$$

$$S_\alpha = \frac{1}{2} \frac{\partial \mathcal{B}}{\partial x^\alpha}.$$

Scleronomity. – We have already proved (pp. 14) that a holonomic system is strongly-sceleronomic, that is, it has a homogeneous kinetic energy that is completely independent of time in a suitable coordinate system, so when we have both:

$$W = 0, \qquad \mathcal{A} = 0.$$

However, we can also point to a system for which one has only:

$$W = 0$$

as something that is especially simple. Indeed, that will not necessarily be a system in which the kinetic energy is homogeneous, but probably one in which its quadratic part does not depend upon time. We shall then simply call the system *scleronomic*.

For non-holonomic systems, one can imagine the classification that the equations of motion (55) lead to as follows:

	$\underline{W}_{\alpha\beta} = 0$	semi-scleronomic
(56)	$\underline{W}_{\alpha\beta} = 0, \quad \underline{S} = 0$	quasi-scleronomic

([14]) For scleronomic systems, $W = 0$, $S = 0$ give those equations are given by all authors that employed the geometric representation. Cf., e.g., **Schouten**, Math. Zeit. **30**, pp. 171, (116), and the cited paper by **Horák**.

$$W_{ik} = 0, \quad \partial_i \mathcal{A} = 0, \quad \nabla_t b_{\alpha\beta} = 0, \quad B^i = 0 \qquad \text{scleronomic.}$$

27. The natural equations of motion. – In ordinary point mechanics, one naturally writes the equations that represent the projections of the equations of motion onto the tangent and normal for the path:

$$m\frac{dv}{dt} = F_t\,, \qquad m\frac{v^2}{\rho} = F_n\,.$$

We would now like to present similar equations for arbitrary dynamical systems, including non-holonomic ones. That does not happen without complications: For example, one can imagine that one can hardly speak of *paths* in a rheonomic space, since a rheonomic transformation will change the "identity" of the point, and therefore every curve, as well. However, we have one certain guidepost: If we refer ordinary space (e.g., simple three-dimensional **Riemann** space) to a rheonomic coordinate system (e.g., moving, rectangular axes) then we can also write down all quantities in that coordinate system, such as tangent, normal, curvature, etc. They can be expressed in a well-defined, easy to state way in terms of our strong quantities. However, that will show us the way: We construct the same expressions in the general case. The reader will easily find the following representation by heeding that suggestion.

Let a "curve":

$$x^i = x^i(t)$$

be given in a rheonomic subspace [$\mathfrak{B}$]. By that notion, one actually imagines a structure that assigns a certain curve to every (rheonomic) coordinate system: viz., something similar to a vector, *mutatis mutandis*. In general, we set:

$$d\sigma^2 = b_{ik}\,\delta x^i\,\delta x^k$$

(in which we drop the inconvenient underlines on the $\mathfrak{B}$-quantities, for simplicity). That is our *arc-length element*. Obviously, we have:

$$v = \frac{d\sigma}{dt} \qquad (v^2 = b_{ik}\,v^i\,v^k).$$

We also have:

$$1 = b_{ik}\frac{\delta x^i}{d\sigma}\frac{\delta x^k}{d\sigma}.$$

We call $u^i = \delta x^i / ds$ the *unit tangent* or simply the *tangent* to the curve. We further introduce the strong vector (δ is the strong $\mathfrak{B}$-differential!):

$$k^i = \frac{\delta u^i}{d\sigma}$$

as the *curvature* of our curve. The absolute value of that vector:

$$k = \sqrt{b_{ik}\frac{\delta u^i}{d\sigma}\frac{\delta u^k}{d\sigma}} \qquad (k^i = k\,n^i)$$

is the *scalar curvature*. Obviously, one has:

$$u_i \frac{\delta u^i}{d\sigma} = 0. \tag{57}$$

It is probably pointless to remark that in the strongly-holonomic case, all of those quantities will agree with the ones that were known before (in the usual sense of the word).

We now write:

$$\frac{\delta v^i}{dt} = v\frac{\delta u^i}{dt} + u^i\frac{dv}{dt} = v^2\frac{\delta u^i}{d\sigma} + u^i\frac{dv}{dt}.$$

That is the well-known decomposition into tangential and normal accelerations. If we substitute that into the equations of motion and multiply them by u_i in one case and by k_i in the other, while observing (57), then we will get:

$$\frac{dv}{dt} = (S^i + Q^i)\,u_i\,,$$

$$v^2\,k = (S^i + Q^i)\,n_i\,,$$

resp.

These are the natural equations that were announced before. One can say that the first of them determines the type of evolution for a given *path*, and therefore the velocity. That will be true verbatim in the strongly-scleronomic case ($W = 0$, $S = 0$). The equations will then assume the form:

$$\frac{dv}{dt} = Q^i\,u_i\,, \qquad v^2\,k = Q^i\,n_i\,,$$

and the curvature will naturally be explicitly independent of time. One can infer a number of **Painlevé**'s conclusions ([9]) from those equations, which **Franck** also reached ([15]), to some extent, but on the basis of different equations that were less simple and intuitive, and which were not written out in precisely geometric terms.

([15]) **Ph. Franck** and **L. Berwald**, "Über eine kovariante Gestalt der Differentialgleichungen der Bahnkurven allgemeiner mechanischer Systeme," Math. Zeit. **21** (1924). **Ph. Frank**, "Die geometrische Deutung von **Painlevé**'s Theorie der reellen Bahnen allgemeiner mechanischer Systeme," Proc. 1st Inter. Congress for Applied Mechanics, Delft, 1924.

28. Energy integral for rheonomic and non-holonomic systems. – We shall now give a case in which an integral exists that is analogous to the energy integral for scleronomic systems. We will say "an energy integral," in the broader sense, to mean an integral of the form:

$$T = h' - V'(x^i, t),$$

which we can also write in the form:

$$v^2 = h - 2V(x^i, t), \tag{58}$$

since

$$v^2 + \mathcal{A} = 2T.$$

The problem can be posed more generally when one looks for integrals that agree with kinetic energy only in their quadratic terms, but can differ in their linear terms. They can then have the absolute form:

$$v^2 = A_i\, v^i - 2V(x^i, t) + h,$$

in which A_i is a strong vector and V is a strong scalar. We find such an integral in the known case of **Painlevé** ([15a]). Our absolute equations also allow us to answer that question, but they do not explain the criteria that they imply, since they require the integrability of certain partial differential equations. We then restrict ourselves to the case in which an integral (58) exists and introduce a new concept that is the rheonomic counterpart to a potential.

We call V an ***absolute potential*** *for the vector field X_i when the strongly-invariant conditions:*

$$X_i = -\partial_i V, \qquad \nabla_t V = 0$$

are fulfilled.

One immediately verifies that the equation:

$$X_i\, \delta x^i = -dV$$

is true, which is, in turn, an invariant counterpart to the equation of elementary work.

We return to the energy integral. We scalar multiply the equations of motion:

$$\frac{\delta v_i}{dt} + W_{ik}\, v^k = S_i + Q_i$$

(we have omitted the underline that refers to the subspace) by the velocity:

$$v^i \frac{\delta v_i}{dt} + W_{ik}\, v^i\, v^k = (S_i + Q_i)\, v^i.$$

The covariant derivative of a scalar is identical with the ordinary one, so:

([15a]) Cf., e.g., **P. Appell**, *Traité de Mécanique rationelle*, t. **2**, 4th ed., § 448, pp. 329, or **P. Painlevé**, *Leçons sur l'intégration des équations de la dynamique*, Paris, Hermann, 1895, pp. 89.

$$\frac{1}{2}\frac{dv^2}{dt} + W_{ik}\, v^i\, v^k = (S_i + Q_i)\, v^i.$$

If equation (58) is true then no terms that are quadratic in v^i can appear on the right-hand side, and the rate of strain tensor must be skew-symmetric then. If we assume that the sum $S_i + Q_i$ possesses an absolute potential:

$$S_i + Q_i = -\,\partial_i\, V, \qquad \nabla_t\, V = 0$$

then we can write:

$$(S_i + Q_i)\, v^i = -\,\frac{dV}{dt}.$$

If:

$$W_{ik} = -\,W_{ki}\,, \quad S_i + Q_i = -\,\partial_i\, V, \qquad \nabla_t\, V = 0$$

then it will possess the energy integral:

$$v^2 = h - 2V.$$

Those conditions will assume an interesting form for a holonomic system. Since the rate of strain tensor is symmetric then, one must have simply $W = 0$, so the system must be semi-scleronomic. If we further assume that there is a potential – U in the usual sense and set:

$$V = \tfrac{1}{2}\mathcal{A} - U$$

then V will be the absolute potential of $S_i + Q_i$ when the condition $\nabla_t\, V = 0$ is fulfilled, which requires the independent of V from t, to some extent. The energy integral will then assume the form:

(*) $$v^2 = h + \mathcal{A} + 2U.$$

If the conditions:

$$W_{ik} = 0, \qquad \nabla_t\, (\mathcal{A} + 2U) = 0$$

are fulfilled for a holonomic system with the potential – U then it will possess an energy integral (*).

We expressly point out that this case is different from the **Painlevé** case ([16]). The conditions that are given by our theorems are also necessary (*cum grano salis*) in the sense that $W = 0$ must be true in any case. The conditions for the existence of the potential can be weakened.

([16]) **Vrănceanu** sought to generalize just the **Painlevé** integral in the aforementioned notes, which can make sense only for a non-absolute treatment of rheonomic systems, since being independent of time is not an invariant condition. The **Painlevé** integral is, so to speak, an accidental phenomenon that does not correspond to any mechanical fact.

29. Variational equations for curves in [$\mathfrak{B}$]. – We shall give certain equations that are fundamental to the problem of the stability of rheonomic systems as an application of this. Those would be the variational equations for arbitrary curves in a rheononholonomic space. Naturally, we understand that to mean that every element of the curve lies in [$\mathfrak{B}$]. The canonical condition equations:

(59) $$c_k^i\,\delta x^k = B^i\,dt$$

will then be fulfilled [cf., (36)].

We first define the deviation for two infinitely-close curves C and C', of which we will assume that their directions will also differ by infinitely little at infinitely-close points. We relate the points of both curves (each of which has a well-defined time coordinate) to each other arbitrarily, under which corresponding points are naturally infinitely close. We let $\bar{\delta}$ denote the displacement that takes a point of C to the corresponding point of C'.

We shall call the vector:

$$p^i = \bar{\delta} x^i$$

the *deviation* of the curves C and C'. If we denote the elementary displacement along the curves C and C' by δ then we will have:

(60) $$\bar{d}\,dt = d\,\bar{d}t\,, \qquad \bar{\delta}\,\delta x^i - \delta\,\bar{\delta}\,x^i = W_j^i(\delta x^i\,\bar{d}t - \bar{\delta}\,x^i\,dt)\,,$$

since the two displacements obviously commute. We now seek the differential equations that p^i must satisfy for *every* pair of curves in [$\mathfrak{B}$].

From (59), we have:

(61) $$\bar{\delta} c_k^i\,\delta x^k + c_k^i\,\delta\,\bar{\delta}\,x^k = \bar{\delta}\,B^i dt + B^i\,\bar{\delta} dt\,.$$

From formula (19):

$$\bar{\delta} = p^j\,\nabla_j + \bar{d}t\,\nabla_t\,,$$

and due to (60), we will get some rather complicated equations from (61) after some calculations that will determine the transverse components of the differential δp^k.

However, those equations are quite unnecessarily complicated in practice, since an *isochronous* variation, for which $\bar{d}t = 0$, will suffice for most applications. We will have $\delta\,\bar{d}t = 0$, as well, and equations (61) will become homogeneous in p^i:

$$c_k^i\,\delta p^k + [\nabla_j\,c_k^i\,\delta x^k - (\nabla_j B^i + c_k^i\,W_j^k)\,dt]\,p^j = 0.$$

If [$\mathfrak{B}$] is scleronomic then $W = \mathcal{B} = 0$, and we will get simply:

$$c_k^i\,\delta p^k + \nabla_j\,c_k^i\,\delta x^k\,p^j = 0.$$

Those equations are true for arbitrary curves in [$\mathfrak{B}$]. However, we can exhibit some new equations when we are dealing with a special class of them. For example, if C and C' represent the motions of a dynamical system then we will have its equations of motion. We first remark that:

$$\bar{\delta} v^i = \bar{\delta}\frac{\delta x^i}{dt} = \frac{dt\,\bar{\delta}\,\delta x^i - \delta x^i \bar{\delta}\,dt}{dt^2} = \frac{\bar{\delta}\,\delta x^i}{dt} - \frac{\delta x^i}{dt}\frac{\bar{\delta}\,dt}{dt}.$$

Due to (28), we have:

$$\frac{\bar{\delta}\,\delta x^i}{dt} = \frac{\delta p^i}{dt} + W^i_j\left(\frac{\delta x^i}{dt}\bar{d}t - \frac{p^i dt}{dt}\right) = \frac{\delta p^i}{dt} + W^i_j\left(v^j\,\bar{d}t - p^j\right).$$

Hence:

$$\bar{\delta} v^i = \frac{\delta p^i}{dt} + W^i_j\left(v^j\,\bar{d}t - p^j\right) - v^i\,\mu,$$

in which:

$$\mu = \frac{\bar{\delta}\,dt}{dt}$$

is infinitely small. If we now start from the equation for the *cyclic differential:*

$$(\underline{\bar{\delta}}\,\underline{\delta} - \underline{\delta}\,\underline{\bar{\delta}})\,v^\alpha = R^\alpha_{\beta ik}v^\beta\,\bar{\delta}x^i\,\delta x^k$$

then the remaining equations of deviation are not difficult to construct, which would be quite complicated in the general case, but for a scleronomic system and an isochronous variation, they will assume the form ([17]):

$$\frac{\delta}{dt}\frac{\delta p^i}{dt} - (\nabla_j\,Q^i + R^\alpha_{hik}v^h\,v^k)\,p^j = 0.$$

In the special case of $Q = 0$, we will get **Levi-Civita**'s generalization of the **Jacobi** equations ([18]).

A characteristic of the method that was followed here is the application of the *strongly-covariant variation* ([19]), which was encountered already in the derivation of the absolute equations from **Hamilton**'s principle.

([17]) One finds this formula written out non-invariantly in **Synge**, *loc. cit.*, pp. 79.

([18]) **Levi-Civita**, "Sur l'écart géodésique," Math. Ann. **97** (1926). Moreover, that problem was treated previously, and in more generality, by **Synge** in "Geometry of Dynamics," which is, unfortunately, not well-known.

([19]) I applied the covariant variation to the deviation problem in the note "Une simple démonstration de la formula de l'écart géodésique," Rend. Lincei **12** (1930), which also includes dynamical applications. The implementation and generalization to non-holonomic systems, as well as a certain method for rheonomic ones, is included in the author's paper: "Über die Variationsgleichungen für affine geodätische Linie und nichtholonome, nichtkonservative dynamische Systeme," Prace Matem. Fizyczne **37** (1931).

IV.

Theory of reaction forces.

30. The fundamental law of reaction forces. – We will now solve some fundamental problems in regard to reaction forces ([20]) in full generality by means of our multi-dimensional representation and the equations of motion that were obtained. The theorems that we shall prove relate to not only reactions that replace all of the constraints, but also to reactions whose introduction will replace only some of the constraints. We will call those reaction forces *partial*.

Perhaps the explanation below for the mechanical sense that corresponds to the concept of a multi-dimensional reaction would not be superfluous. Such a multi-dimensional reaction is basically a complex of generalized forces that replace the constraints dynamically and are calculated for the parameters of the free system that one would get by removing those constraints. For example, for a system that consists of a finite number of points, the "multi-dimensional reaction" will be the totality of components of the reactions that act upon each individual point, and one's knowledge of those reactions will insure one's knowledge of each of the individual reactions. By contrast, for a rigid body, the multi-dimensional reaction will give only the resultant and moment of the reaction forces (in the usual sense) that act upon the various points of the body. Those reactions will not be determined by the multi-dimensional reaction, in general.

We pose the following problems:

1. Calculate the equivalent reaction explicitly for smooth constraints.

2. How does one compose the reactions? That is: How does the reaction that replaces several constraints depend upon the reactions that replace those individual constraints?

3. How does a reaction change when one strengthens the constraints; i.e., by adding new ones?

4. How do reactions in real motions differ from reactions in virtual motions?

We shall obtain the answers to all of those questions as simple consequences of a fundamental theorem that we will prove shortly. That theorem, which one can consider to be a rather distant generalization of **Meusnier**'s theorem, is concerned with the change

---

The problem was not addressed by an absolute treatment, and the rheonomic systems were interpreted in an $(n + 1)$-dimensional space.

([20]) Several papers by **E. Gugino** were dedicated to the problem of the reaction forces in recent times. We cite: "Sur la détermination des forces de réaction dans le mouvement d'un systèmes matériel," C. R. Acad. Sc. Paris **191**, pp. 1118, and "Sul problema dinamico di un quasivoglia sistema vincolato ridotto all'analogo problema relativo ad un sistema libro," Rend. Lincei **12**, pp. 307). We will also find a paper with the same title by **A. Quarleri**, Boll. Un. Mat. It. **10** (1931). Those authors proved that the reaction forces depend upon only the state and indicated a path to calculating them in terms of the state, but gave no explicit formulas for doing that.

in the reaction forces when one strengthens the constraints. **Meusnier**'s theorem is concerned with curves on a surface that has a common direction at the same point, and says something about the projection of the curvature onto the surface normals. Our theorem is concerned with motions that are compatible with constraints and will go through the same position with the same velocity, as well as saying something about the component of the reactions in the direction that is transverse to the virtual space. One sees immediately how that corresponds to the theorems. In order to be able to express things conveniently, we define:

Two motions are said to ***contact*** *when they go through the same position (configuration, point) with the same velocity.*

Here, we must speak in terms of virtual space, since we are considering the general rheonomic case. We denote an entirely *arbitrary* system by [$\mathfrak{A}$], as well as the subspace that it corresponds to. Our theorem reads:

(62) *The reaction force that replaces the constraints* [$\mathfrak{B}$] *on the system* [$\mathfrak{A}$] *will have the same projection onto the direction that lies in* $\mathfrak{A}$ *and is orthogonal to* $\mathfrak{B}$ *for all tangent motions of the systems* [$\mathfrak{A}$] *that are compatible with the constraints* [$\mathfrak{B}$].

Proof: The equations of the virtual motion of the system [$\mathfrak{A}$] read:

$$\frac{\delta v}{dt} + W\, v = S + Q + R,$$

in which R denotes the reaction that corresponds to the virtual motion. The other notations need no explanation: For the sake of simplicity, we have omitted the underlines. We have also omitted the indices, which will probably not lead to any misunderstanding. We immediately take the constraints [$\mathfrak{B}$] in the canonical form (36):

$$c\, v = B,$$

in which c is the unit tensor in the space $\mathfrak{C}$ that is completely orthogonal to $\mathfrak{B}$ in $\mathfrak{A}$.

We get the theorem immediately by projecting onto $\mathfrak{C}$; i.e., upon multiplying by c:

$$c\frac{\delta v}{dt} = -\, c\, W\, v + c\, S + c\, Q + c\, R\,.$$

We shall now transform the left-hand side of this equation so that we can show that it depends exclusively upon the position and velocity, and thus depends upon the state, but not upon the acceleration $\delta v / dt$. That is the key point:

$$c\frac{\delta v}{dt} = \frac{\delta c\, v}{dt} - v\frac{\delta c}{dt} = \frac{\delta B}{dt} - v\frac{\delta c}{dt}\,.$$

We then have:

(63)
$$\underset{T}{R} = c\,R = c\,W\,v - v\frac{\delta c}{dt} + \frac{\delta B}{dt} - c\,S - c\,Q$$

for the transverse component of the reaction. The right-hand side of this is a quadratic function of the v^i and the position, since the derivatives are taken along the motion, so they will be linear (due to the rheonomity, in general), but not homogeneous functions of the v^i, and therefore the $\dot{x}^i$, as well. The right-hand side will then depend exclusively upon the state, and not on the acceleration, and will then have the same value for all tangent motions that are compatible with [$\mathfrak{B}$]. We assume that [$\mathfrak{A}$] is Euclidian space, [$\mathfrak{B}$] is a surface in that space, $Q = 0$ and $t = s$ (arc length), so the reaction force will be equal to the curvature, and we will get **Meusnier**'s theorem.

Principle of least reaction:

The actual motion corresponds to a smaller reaction than any other possible motion.

That is the answer to question 4. One gets the proof immediately. For the actual motion, the reaction R will be normal to the virtual space, so it will coincide with its transverse components. For any other possible motion, from the theorem (62) above, the reaction will have the transverse component R, so it will be greater than its projection R.

We point out that this theorem is true for not only the total reaction force, but for every partial reaction force individually.

31. – We shall now answer question 2 of the previous §.

The reaction will be weakened by a component that is orthogonal to it when the conrate of straint is strengthened.

Proof: We return to the system [$\mathfrak{A}$] and assume that new constraints have been added to the constraints [$\mathfrak{B}$], such that the corresponding rheonomic subspace will contract to [$\mathfrak{B}$]. We let $\bar{c}$ denote the unit tensor in the space $\bar{\mathfrak{C}}$, which is orthogonal to the new virtual subspace $\bar{\mathfrak{B}}$ in $\mathfrak{A}$.

We can then write:

$$\bar{c} = c + (\bar{c} - c).$$

The motion that corresponds to the constraints [$\bar{\mathfrak{B}}$] is a possible motion for the constraints [$\mathfrak{B}$]. From the fundamental theorem, we will have:

$$\bar{R} = \bar{c}\,\bar{R} = c\,\bar{R} + (\bar{c} - c)\,\bar{R} = \underset{T}{R} + (\bar{c} - c)\,\bar{R},$$

if the reaction force for that motion is denoted by $\bar{R}$, since $\bar{R}$ lies in $\mathfrak{C}$. However, $-c$ is obviously the unit tensor in the subspace that lies in $\bar{\mathfrak{C}}$ and is transverse to $\mathfrak{C}$ in it. The vector $(\bar{c} - c)\,R$ lies in that space, so it will be orthogonal to $\mathfrak{C}$, and therefore to R, as well, which was claimed.

Scleronomic constraints. – We shall assume that the system [$\mathfrak{B}$] is scleronomic, so from (56), $W = 0$, $B = S = 0$, and $\nabla_t\, c = 0$. The formula for the normal component to the reaction will assume a remarkable form. We have:

$$\underset{T}{R}^i = -v^j\, v^k\, \nabla_k\, c^i_j - c^i_k\, Q^k .$$

Since ∇ means differentiation in $\mathfrak{A}$, we will have:

$$-\, v^j\, v^k\, \nabla_k\, c^i_j = v^j\, v^k\, \nabla_k\, b^i_j = b^k_\alpha\, b^j_\beta\, \nabla_k\, b^i_j\, v^\alpha\, v^\beta = H_{\alpha\beta}^{\ \ i}\, v^\alpha v^\beta ,$$

since v^i obviously lies in $\mathfrak{B}$. We finally get:

(64) $$\underset{T}{R}^i = H_{\alpha\beta}^{\cdot\cdot i}\, v^\alpha v^\beta - \underset{T}{Q}^i ,$$

in which we have denoted the transverse component of the generalized force by $\underset{T}{Q}^i$. The formula (64) is a generalization of the known intrinsic equations of motion on a surface:

$$\frac{m\,v^2}{\rho} = F_n + R_n .$$

Composition of reactions. – We turn to problem 2 of § **30** and formulate it as follows: We consider the systems [$\mathfrak{B}_a$], $a = 1, \ldots, k$, which arises from [$\mathfrak{A}$] by introducing new constraints, and let $\underset{a}{R}^i$ denote the reaction forces that replace those constraints for a well-defined state. How do we then obtain the reaction that replaces all of those constraints simultaneously for the same state? Naturally, we assume that the constraints are independent of each other.

We let [$\mathfrak{B}$] denote the system that will arise by the simultaneous introduction of all constraints, let $\mathfrak{C}$ denote space in $\mathfrak{A}$ that is transverse to its virtual subspace, and let $\mathfrak{C}_a$ denote the space that is transverse to $\mathfrak{B}_a$. Since $\mathfrak{B}$ is the common intersection of all $\mathfrak{B}_a$, $\mathfrak{C}$ will be, dually, the (smallest) union of all $\mathfrak{C}_a$. Hence:

(65) *The resultant reaction lies in the smallest space that contains all spaces that are transverse to the partial virtual spaces.*

The motion of the system [$\mathfrak{B}$] will be a possible motion of the system [$\mathfrak{B}_a$] in any case. We let R denote the resultant reaction and will then have:

$$\underset{a}{c}{}_k^i \, R^k = \underset{a}{R}{}^i ,$$

from the fundamental theorem (62), which implies the theorem:

(66) *The projection of the resultant reaction force onto the space that is orthogonal to a partial virtual space is equal to the corresponding partial reaction force.*

We assert that the theorems (65) and (66) determined the reaction completely. In fact, two vectors cannot exist in $\mathfrak{C}$ that have the same projections onto [$\mathfrak{C}_a$] [as theorem (65) would require], since their differentials would be vectors in $\mathfrak{C}$ whose projections onto each $\mathfrak{C}_a$ would be zero, and would then be perpendicular to them. However, such a vector does not exist, since each vector in $\mathfrak{C}$ is a linear combination of certain vectors in the $\mathfrak{C}_a$, so it cannot be simultaneously orthogonal to all $\mathfrak{C}_a$.

The meaning of the theorems that were proved is actually the following:

The resultant reaction does not depend directly upon the state, but is determined completely by the partial reactions and purely-geometric data.

That consequence is by no means obvious, since the partial reactions do not determine the state that they correspond to at all.

We will arrive at more definite results when the dimension of each $\mathfrak{B}_a$ is smaller than the dimension of $\mathfrak{A}$ by one. The normal spaces will be simple lines then. From (66) the vectors $\underset{a}{R}{}^i$ ($a = \underline{1}, \ldots, \underline{k}$) will lie on each lines, and the resultant reaction will be a vector that lies in the space that is spanned by the $\underset{a}{R}{}^i$ and its projections onto the direction of each of those vectors will have a have a length that is equal to its own. One can refer to the resulting reaction as an *orthogonal sum of the partial reactions.*
