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Non-holonomic Relativity and the Unitary Theory of Einstein and Mayer

By

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VRĂNCEAUNU [1] has recently established a non-holonomic unitary theory of fields in the schema of his theory of non-holonomic spaces [2].

The present note has the goal of making the relationship between this non-holonomic theory of relativity and the unitary theory of Einstein and MAYER [3] as clear as possible.

In order to establish the non-holonomic unitary theory of fields one may, as the author has already pointed out [4], start with the following hypotheses:

(A) A Riemannian space V_5 admits an infinitesimal point translation:

$$x^{\lambda *} = x^\lambda + \phi^\lambda\, dt, \qquad (\lambda, \mu, \nu \ldots = 1, 2, 3, 4, 5) \tag{1}$$

that leaves invariant the fundamental quadratic form:

$$d\sigma^2 = G_{\lambda\mu}\, dx^\lambda dx^\mu \tag{2}$$

in which $\phi^\lambda = \delta_5^\lambda$ is a unit vector.

(B) Spacetime is identified with a non-holonomic space V_5^4 that is defined in V_5 by:

$$\phi_\lambda\, dx^\lambda = 0, \tag{3}$$

in which $\phi_\lambda = G_{\lambda\mu}\, \phi^\mu$.

If one considers only coordinate transformations of the form:

$$\begin{cases} \bar{x}^i = \bar{x}^i(x^j) \\ \bar{x}^5 = \bar{x}^5 + f(x^j) \end{cases}, \qquad (i, j, k, \ldots = 1, 2, 3, 4), \tag{4}$$

then one concludes from these hypotheses that the components $G_{\lambda\mu}$ of the fundamental tensor in any coordinate system are independent of the fifth coordinate x^5 and $G_{5\lambda} = \phi_\lambda$, $G_{55} = 1$.

Moreover, one may write (2) in the form:

$$d\sigma^2 = g_{ij}\, dx^i\, dx^j + (\phi_i\, dx^i + dx^5)^2, \tag{5}$$

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in which $g_{ij} = \mathrm{G}_{ij} - \phi_i\, \phi_j$.

Since the g_{ij} transform under the coordinate transformation (4) like the components of a tensor in a space V_4 that is described by x^i, they constitute ten functions that represent the gravitational field, whereas the ϕ_i are the components of a vector in V_4 that represents the electromagnetic field.

The non-holonomic space V_4^5 thus defined is totally geodesic. Since a geodesic in V_4^5, i.e., a geodesic that satisfies the equation:

$$\phi_\lambda\, dx^\lambda = 0,$$

is given by [5]:

(6)
$$\frac{d^2x^\lambda}{ds^2} + (\Gamma^\lambda_{\mu\nu} + \phi^\lambda \phi_{\mu,\nu})\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0,$$

in which:

(7)
$$\Gamma^\lambda_{\mu\nu} = \tfrac{1}{2}\mathrm{G}^{\lambda\omega}\left(\frac{\partial \mathrm{G}_{\omega\mu}}{\partial x^\nu} + \frac{\partial \mathrm{G}_{\omega\nu}}{\partial x^\mu} - \frac{\partial \mathrm{G}_{\mu\nu}}{\partial x^\omega}\right),$$

$$\phi_{\mu,\nu} = \frac{\partial \phi_\mu}{\partial x^\nu} - \phi_\lambda \Gamma^\lambda_{\mu\nu}$$
$$= \frac{1}{2}\left(\frac{\partial \phi_\mu}{\partial x^\nu} - \frac{\partial \phi_\nu}{\partial x^\mu}\right),$$

as a consequence (60) reduces to:

$$\frac{d^2x^\lambda}{ds^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} = 0;$$

however, these are the equations for geodesics in V_5.

Thus V_4^5 is totally geodesic.

We define a tensor $g_{\lambda\mu}$ in V_5 by:

(8)
$$g_{\lambda\mu} = G_{\lambda\mu} - \phi_\lambda \phi_\mu.$$

Since this tensor satisfies the identity:

$$g_{5\lambda} = g_{\lambda 5} = 0,$$

one may consider that $g_{\lambda i}$ transforms under a coordinate transformation (4) like the components of a tensor in V_5 with respect to the index λ, and like the components of a vector in V_4 with respect to i.

Thus, the two quantities:

(9)
$$g^i_\lambda = g^{ij} g_{j\lambda} \qquad (g^{ij}\, g_{ij} = \delta^i_k),$$

and:

(10)
$$g^\lambda_i = G^{\lambda\mu}\, g_{\mu i} \qquad (G^{\lambda\mu}\, G_{\mu\nu} = \delta^\lambda_\nu),$$

are mixed tensors.

As one may easily verify, the $G^{\lambda\mu}$ satisfy the equations:

$$(11) \qquad \begin{cases} G^{ij} = g^{ij}, \\ G^{i5} = -g^{ij}\phi_j, \\ G^{55} = 1 + g^{ij}\phi_i\phi_j, \end{cases}$$

and therefore g^i_λ and g^λ_i have the following explicit forms:

$$(12) \qquad \begin{cases} g^\lambda_i = \delta^\lambda_i - \phi^\lambda\phi_i, \\ g^i_\lambda = \delta^i_\lambda, \end{cases}$$

from which one easily obtains the identities:

$$(13) \qquad g_{ij} = g^\lambda_i g^\mu_j \; G_{\lambda\mu},$$

$$(14) \qquad G_{\lambda\mu} = g^i_\lambda g^j_\mu g_{ij} + \phi_\lambda \phi_\mu,$$

$$(15) \qquad g^i_\lambda g^j_\mu = \delta^j_i,$$

$$(16) \qquad g^\lambda_i g^i_\mu = \delta^\lambda_\mu - \phi^\lambda \phi_\mu,$$

$$(17) \qquad \phi_\lambda \; g^\lambda_i = 0,$$

g^λ_i, g^i_λ, and ϕ_λ correspond to γ^λ_i, γ^j_λ, and A_λ, respectively, in the previously-cited memoir of EINSTEIN and MAYER.

When a vector V^λ in V_5 satisfies:

$$\phi_\lambda V^\lambda = 0,$$

one says that it is in V^5_4. A covariant vector W_λ is in V^5_4 when it satisfies:

$$W_\lambda \phi^\lambda = 0.$$

These definitions may be extended to affinors. For example, $g_{\lambda\mu}$, as defined by (8), is certainly in V^5_4, since:

$$g_{\lambda\mu}\phi^\lambda \phi^\mu = g_{55} = 0.$$

When one is given a vector v_i in V_4, i.e., a space framed by the coordinates x^1, x^2, x^3, and x^4, one may form a vector in V_5 by $g^\lambda_i v^i$, and this vector is in V^5_4:

$$\phi_\lambda(g^\lambda_i v^i) = 0.$$

If one is given a vector v^i in V_4 and a scalar ρ then one may form a vector V^λ that is not in V^5_4 from:

(18) $$g_i^\lambda v^i + \rho\phi^\lambda.$$

We shall consider the covariant derivative to be geometrically of this type.

A connection in the space V_5 is well defined by the CHRISTOFFEL symbols that one forms from $G_{\lambda\mu}$:

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} G^{\lambda\omega}\left(\frac{\partial G_{\omega\mu}}{\partial x^\nu} + \frac{\partial G_{\omega\nu}}{\partial x^\mu} - \frac{\partial G_{\mu\nu}}{\partial x^\omega}\right),$$

which have the following values:

(19) $$\begin{cases} \Gamma^\lambda_\mu = \Gamma^\lambda_{5\mu} = \phi^\lambda_\mu, \\ \Gamma^i_{jk} = \begin{Bmatrix} i \\ jk \end{Bmatrix} + \phi^i_j\phi_k + \phi^i_k\phi_j, \\ \Gamma^5_{jk} = -\phi_i\Gamma^i_{jk} + \frac{1}{2}\left(\frac{\partial\phi_j}{\partial x^k} + \frac{\partial\phi_k}{\partial x^j}\right), \end{cases}$$

in which:

(20) $$\begin{cases} \phi_{\lambda\mu} = \frac{1}{2}\left(\frac{\partial\phi_\lambda}{\partial x^\mu} - \frac{\partial\phi_\mu}{\partial x^\lambda}\right), \\ \phi^\lambda_\mu = G^{\lambda\nu}\phi_{\nu\mu}, \\ \phi^i_j = g^{ik}\phi_{kj}, \\ \begin{Bmatrix} i \\ jk \end{Bmatrix} = \frac{1}{2} g^{ih}\left(\frac{\partial g_{hj}}{\partial x^k} + \frac{\partial g_{hk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^h}\right). \end{cases}$$

Take a vector V^μ whose components are independent of the variable x^5 and consider a displacement in V_4^5:

$$\delta V^\nu = \frac{\partial V^\nu}{\partial x^\mu} dx^\mu + \Gamma^\nu_{\lambda\mu} V^\lambda dx^\mu$$

in which:

$$\frac{\partial V^\nu}{\partial x^\mu} dx^\mu = \frac{\partial V^\nu}{\partial x^i} dx^i,$$

and:

$$dx^5 = -\phi_i\, dx^i,$$

since the vector must be in V_4^5, which is defined by $\phi_\lambda\, dx^\lambda = 0$.

One therefore has:

(21) $$\delta V^\nu = \left(\frac{\partial V^\nu}{\partial x^i} + \Pi^\nu_{\lambda i} V^\lambda\right) dx^i,$$

in which:

(22) $$\Pi^{\nu}_{\lambda i} = \Gamma^{\nu}_{\lambda i} - \Gamma^{\nu}_{\lambda 5}\phi_i \;.$$

In the sequel, we let ";" denote the covariant derivative with respect to the connection parameters $\Pi^{\nu}_{\lambda i}$. For a mixed tensor, one defines the covariant derivative as follows:

(22) $$T^{\nu}_{\lambda i;j} = \frac{\partial T^{\lambda}_{\mu i}}{\partial x^j} + T^{\lambda}_{\nu i}\,\Pi^{\nu}_{\mu j} - T^{\nu}_{\mu i}\,\Pi^{\lambda}_{\nu j} - T^{\lambda}_{\mu k}\begin{Bmatrix} k \\ ij \end{Bmatrix}.$$

Since:

$$\Pi^{\lambda}_{\mu i} = \Gamma^{\lambda}_{\mu i} - \Gamma^{\lambda}_{\mu 5}\phi_i,$$

one has, for $G_{\lambda\mu}$:

$$\begin{aligned} \mathrm{G}_{\lambda\mu\, i} &= \frac{\partial G_{\lambda\mu}}{\partial x^i} + G_{\nu\mu}\Pi^{\nu}_{\lambda i} - G_{\lambda\nu}\Pi^{\nu}_{\mu i} \\ &= \left(\frac{\partial G_{\lambda\mu}}{\partial x^i} + G_{\nu\mu}\Gamma^{\nu}_{\lambda i} - G_{\lambda\nu}\Gamma^{\nu}_{\mu i}\right) + (\phi_{\lambda\mu} + \phi_{\mu\lambda})\phi_i. \end{aligned}$$

Both of the two terms in the right-hand side are annulled, due to the definitions, just like $\Gamma^{\lambda}_{\mu\nu}$ and $\phi_{\lambda\mu}$, and one thus obtains:

(23) $$\mathrm{G}_{\lambda\mu\, i} = 0.$$

This is the first condition that was posed by Einstein and Mayer (loc. cit., pp. 548 (I)), which geometrically signifies that the length of a vector does not change when it is parallel displaced with respect to $\Pi^{\lambda}_{\mu i}$.

Having said this, we calculate the covariant derivative $g^{\lambda}_{i;j}$ of g^{λ}_{i}. Before doing this, we write down the relationship between $\Gamma^{\lambda}_{\mu\nu}$ and $\begin{Bmatrix} a \\ bc \end{Bmatrix}$:

(24) $$\Gamma^{\lambda}_{\mu\nu} = g^{\lambda}_{a} g^{b}_{\mu} g^{c}_{\nu}\begin{Bmatrix} a \\ bc \end{Bmatrix} + \frac{1}{2}\phi^{\lambda}\left(\frac{\partial\phi_\mu}{\partial x^\nu} + \frac{\partial\phi_\nu}{\partial x^\mu}\right) + \phi^{\lambda}_{\mu}\phi_\nu + \phi^{\lambda}_{\nu}\phi_\mu,$$

from which, one obtains:

(25) $$\Pi^{\lambda}_{\mu i} = g^{\lambda}_{a} g^{b}_{\mu}\begin{Bmatrix} a \\ bi \end{Bmatrix} + \frac{1}{2}\phi^{\lambda}\left(\frac{\partial\phi_\mu}{\partial x^i} + \frac{\partial\phi_i}{\partial x^\mu}\right) + \phi^{\lambda}_{i}\phi_\nu \;.$$

Now:

$$g^{\lambda}_{i;j} = \frac{\partial g^{\lambda}_{i}}{\partial x^j} + g^{\nu}_{i}\Pi^{\lambda}_{\nu j} - g^{\lambda}_{k}\begin{Bmatrix} k \\ ij \end{Bmatrix},$$

but, on the other hand:

$$\frac{\partial g^{\lambda}_{i}}{\partial x^j} = \frac{\partial}{\partial x^j}(\delta^{\lambda}_{i} - \phi^{\lambda}\phi_i)$$

$$= -\phi^{\lambda}\frac{\partial \phi_i}{\partial x^j},$$

$$g_i^{\nu}\Pi_{\nu j}^{\lambda} = g_a^{\lambda}\begin{Bmatrix} a \\ ij \end{Bmatrix} + \frac{1}{2}\phi^{\lambda}\left(\frac{\partial \phi_i}{\partial x^j} + \frac{\partial \phi_j}{\partial x^i}\right),$$

so therefore one has:

(26) $$g_{i;j}^{\lambda} = -\phi^{\lambda}\,\phi_{ij}\,.$$

The geometrical significance of (26) is the following:

If v^i is parallel displaced in V_4, i.e.:

$$\delta v^i = 0,$$

then the corresponding vector $g_i^{\lambda}v^i$ in V_4^5, after being displaced in V_4^5, satisfies the equation:

$$\delta(\,g_i^{\lambda}v^i\,) = (\,\delta g_i^{\lambda}\,)\,v^i$$
$$= -\,\phi^{\lambda}\,(\phi_{ij}\,v^i\,dx^j).$$

Therefore, the covariant derivative $\delta(\,g_i^{\lambda}v^i\,)$ is normal to V_4^5.

This is the second condition of Einstein and Mayer (loc. cit., pp. 549 (II)).

Furthermore, ϕ_{ij} is an antisymmetric tensor in (26), so (26) also expresses the fact that:

If v^i is parallel displaced in its proper direction in V_4 then the corresponding vector $g_i^{\lambda}v^i$ is also parallel displaced in V_4^5, since:

$$\delta(\,g_i^{\lambda}v^i\,) = (\,\delta g_i^{\lambda}\,)\,v^i$$
$$= -\,\phi^{\lambda}\,\phi_{ij}\,v^i\,v^j\,dt$$
$$= 0.$$

This is the third condition of EINSTEIN and MAYER (loc. cit., pp. 549 (III)).

Moreover, one has:

$$\phi_{;i}^{\lambda} = \frac{\partial \phi^{\lambda}}{\partial x^i} + \phi^{\nu}\Pi_{\nu i}^{\lambda}$$

(29) $$= g_j^{\lambda}\phi_i^{j},$$

which geometrically signifies that the covariant derivative of ϕ^{λ} is always found in V_4^5.

The equations of the trajectories will then be obtained in exactly the same manner as those of EINSTEIN and MAYER.

In spacetime, a curve is given by:

$$x^i(s),$$

but when one is dealing with a trajectory for an electrically charged particle one must also specify a scalar $\rho = e/m$, where e is the charge and m is the mass of that particle.

One may then define a vector in V_5 by:

(30) $$V^\lambda = g_i^\lambda \frac{dx^i}{ds} + \rho\phi^\lambda.$$

One poses the following conditions with respect to the trajectory:

The vector V^λ is parallel displaced along the curve in V_4, from which, one has:

$$V^\lambda_{;j}\frac{dx^j}{ds} = 0.$$

Therefore, one has:

(31) $$g^\lambda_{i;j}\frac{dx^i}{ds}\frac{dx^j}{ds} + g_i^\lambda\left(\frac{d^2x^i}{ds^2} + \begin{Bmatrix} i \\ jk \end{Bmatrix}\frac{dx^j}{ds}\frac{dx^k}{ds}\right) + \phi^\lambda\rho_{;j}\frac{dx^j}{ds} + \rho\phi^\lambda{}_{;j}\frac{dx^j}{ds} = 0.$$

However, on the other hand, one has:

$$g^\lambda_{i;j} = -\phi^\lambda\phi_{ij},$$

$$\begin{aligned}\rho_{;j}\frac{dx^j}{ds} &= (\phi_\lambda V^\lambda)_{;j}\frac{dx^j}{ds} = \phi_{\lambda;j}V^\lambda\frac{dx^j}{ds} \\ &= g^a_\lambda\phi_{aj}\left(g_i^\lambda\frac{dx^i}{ds} + \rho\phi^\lambda\right)\frac{dx^j}{ds} \\ &= \phi_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} \\ &= 0,\end{aligned}$$

so one finally has:

(32) $$\frac{d^2x^i}{ds^2} + \begin{Bmatrix} i \\ jk \end{Bmatrix}\frac{dx^j}{ds}\frac{dx^k}{ds} + \rho\phi^i_j\frac{dx^j}{ds} = 0,$$

$$\rho = \frac{e}{m} = \text{constant},$$

upon contracting (31) by g^h_λ.

In the memoir of EINSTEIN and MAYER, the [Lagrangian] density whose variation would give the field equations is nowhere to be found.

Here, one may take the variation:

(33) $$\delta\int Kg^{\frac{1}{2}}\,dx^1\,dx^2\,dx^3\,dx^4 = 0,$$

where:

$$g = |\,g_{ij}\,| = |\,G_{\lambda\mu}\,|,$$

$$K = G^{\lambda\mu}K_{\mu\lambda}, \qquad K_{\mu\lambda} = K_{\omega\mu\lambda}^{\cdots\;\omega},$$

$$K_{\omega\mu\lambda}^{\cdots\;\omega} = \frac{\partial\Gamma^\nu_{\lambda\omega}}{\partial x^\mu} - \frac{\partial\Gamma^\nu_{\lambda\mu}}{\partial x^\omega} + \Gamma^\kappa_{\lambda\omega}\Gamma^\nu_{\kappa\mu} - \Gamma^\kappa_{\lambda\mu}\Gamma^\nu_{\kappa\omega}.$$

(33) gives us [6]:

(34) $$\nabla_{\lambda\mu} - \phi_\lambda\,\phi_\mu\nabla = 0,$$

in which:

$$\nabla_{\lambda\mu} = K_{\lambda\mu} - \tfrac{1}{2}G_{\lambda\mu}K,$$

$$\nabla = G^{\lambda\mu}\,\nabla_{\lambda\mu}\,.$$

Upon contracting (34) by $g_i^\lambda g_j^\mu$, we have:

$$R_{ij} + 2\,\phi_i^k\,\phi_{kj} - \tfrac{1}{2}g_{ij}K = 0,$$

(35) $$R_{ij} - \tfrac{1}{2}g_{ij}K + 2\left[\phi_i^k\phi_{kj} + \tfrac{1}{4}\,g_{ij}\phi_h^k\phi_k^h\right] = 0,$$

in which R_{ij} and R are the Ricci tensor and the scalar curvature formed from g_{ij}, respectively.

In (34), if we contract with g_i^λ and set $\mu = 5$, then we obtain:

$$\phi_{i;\,j}^k = 0.$$

Finally, if we set $\lambda = \mu = 5$ then we have:

$$\nabla_{55} - \nabla = 0,$$

but:

$$\begin{aligned}\nabla &= G^{\lambda\mu}\,\nabla_{\lambda\mu} = G^{\lambda\mu}(K_{\lambda\mu} - \tfrac{1}{2}G_{\lambda\mu}\,K)\\ &= -\tfrac{3}{2}K,\\ \nabla_{55} &= K_{55} - \tfrac{1}{2}K\\ &= \phi_i^j\phi_j^i - \tfrac{1}{2}K,\end{aligned}$$

so:

$$\nabla_{55} - \nabla = K + \phi_i^j\phi_j^i\,.$$

As a consequence:

(36) $$R = 0.$$

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