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The stability criteria of elastomechanics

By **H. Ziegler**

Translated by D. H. Delphenich

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1. Introduction. – In kinetics, the equilibrium configuration of a system is referred to as *stable* when it will move only slightly from that configuration under the influence of any sufficiently-small, but otherwise arbitrary, perturbation (and therefore perform a small oscillation about it). That definition is the basis for the processes of small motions, which assume only the linearizability of the differential equations of motion, and in the case of constant coefficients along the path of the exponential Ansatz, the roots of the characteristic equations, as well as the **Routh-Hurwitz** criterion ([1]), allow one to ascribe stability to a system.

The stability problem in elastomechanics consists of determining the critical load at which an originally-stable equilibrium configuration will become unstable. In order to solve it, one can call upon the definition of stability directly, and in that way obtain the following criteria, which are valid with no further assumptions:

> *Kinetic stability criterion:* The critical value is the smallest load under which a suitable perturbation will lead to a motion that does not take place in the immediate neighborhood of the equilibrium configuration.

In practice, this kinetic criterion was mostly employed before by **L. Euler** in his handling of the essentially-simpler criterion (but not equivalent to it with no further assumptions):

> *Static stability criterion:* The critical value is the smallest load under which a further (non-trivial) equilibrium configuration will first exist along with the original (trivial) one,

and in more recent times, it is the:

> *Energetic stability criterion:* The critical value is the smallest load under which the total potential energy of the body will no longer be positive-definitive

that is preferred.

([1]) **E. J. Routh**, *Advanced Rigid Dynamics*, 6th ed., London, 1930, pp. 228. **A. Hurwitz**, Math. Ann. **46** (1895), pp. 273.

The results that have been achieved by the last-mentioned criterion have pushed the question of their legitimacy into the background and led to a purely-static conception of the intrinsically-kinetic stability problem. The relative infrequency of non-conservative problems, for which that conception of things would lead to errors, explains the fact that (to cite just one major example) **S. Timoshenko** employed the last two criteria exclusively in his celebrated standard textbook ([1]), and without giving any kinetic basis.

In **A. Pflüger**'s book on stability problems in elastostatics ([2]), he referred to the fact that only the first criterion comes into question for non-conservative systems [of course, without proof and without inferring any consequences from that remark ([3])]. Since then, the author ([4]) has proved that, for example, the buckling problem and the rotational speed problem of the shaft that is torsionally strained by an axial moment are non-conservative and that the stability criteria that were employed up to now will break down.

In light of that state of affairs, one raises the question of whether it might not be possible to legitimize the energetic criterion for non-conservative systems by reformulating it. The goal of this article is to answer that question, and in fact, in the negative sense.

2. The kinetic criterion. – A typical [one-dimensional, non-gyroscopic ([5]), and conservative] stability problem is **Euler**'s buckling problem for the (arbitrary, say, flexibly mounted at both ends) rod with mass $\mu(x)$ per unit length and bending stiffness $\alpha(x)$. His energy of motion was:

$$T = \tfrac{1}{2}\int_0^l \mu\,\dot{y}^2\,dx\,,$$

and he composed the potential energy from the deformation energy and the potential of the axial compression according to:

$$V = U - P\,W_0\,, \qquad U = \tfrac{1}{2}\int_0^l \alpha\,y''^2\,dx\,, \qquad W_0 = \tfrac{1}{2}\int_0^l y'^2\,dx\,. \tag{2.1}$$

If φ_k is the k^{th} normalized eigenfunction that belongs to the eigenfrequency σ_k of the unloaded, laterally-oscillating rod then one can represent the vibrations under the compression P in the form:

$$y(x, t) = \sum_{k=1}^{\infty} q_k(t)\,\varphi_k(x)\,,$$

([1]) **S. Timoshenko**, *Theory of Elastic Stability*, New York and London 1936.
([2]) **A. Pflüger**, *Stabilitätsprobleme der Elastostatik*, Berlin, Göttingen, Heidelberg, 1950, pp. 67.
([3]) Despite his remark on pp. 67, he treated a non-conservative problem with the static criterion on pp. 217 (also cited in **L. Collatz**, *Eigenwertaufgaben mit technischen Anwendungen*, Leipzig 1949, pp. 41).
([4]) **H. Ziegler**, Z. angew. Math. Phys. **2** (1951), pp. 265.
([5]) Cf., **H. Ziegler**, "Zum Begriff des konservativen System," Elem. der Math., to appear.

and interpret the q_k as position coordinates, which are normal coordinates for $P = 0$. The three components of the total energy are expressed in terms of them in the form:

$$T = \tfrac{1}{2}\sum_{k=1}^{\infty} \dot{q}_k^2\,, \qquad U = \tfrac{1}{2}\sum_{k=1}^{\infty} \sigma_k^2\, q_k^2\,, \qquad W_0 = \tfrac{1}{2}\sum_{i,k=1}^{\infty} q_i\, q_k \int_0^l \varphi_i' \varphi_k'\, dx\,.$$

The kinetic potential is then:

$$L = \tfrac{1}{2}\sum_{k=1}^{\infty} (\dot{q}_k^2 - \sigma_k^2\, q_k^2) + \tfrac{1}{2} P \sum_{i,k=1}^{\infty} q_i\, q_k \int_0^l \varphi_i' \varphi_k'\, dx\,,$$

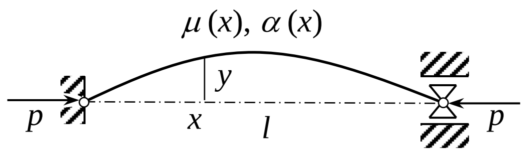

Figure 1. Conservative problem: Axially compressed two-sided pin-jointed rod.

and the Lagrange relations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \quad (k = 1, 2, \ldots) \tag{2.2}$$

will lead to the differential equations of motion:

$$\ddot{q}_k + \sigma_k^2\, q_k - P \sum_{i,k=1}^{\infty} q_i \int_0^l \varphi_i' \varphi_k'\, dx = 0 \qquad (k = 1, 2, \ldots). \tag{2.3}$$

In the case of a homogeneous prismatic rod that is pin-jointed at both ends, one has:

$$\sigma_k^2 = \frac{k^4 \pi^4}{l^4} \frac{\alpha}{\mu}\,, \qquad \varphi_k(x) = \sqrt{\frac{2}{\mu\, l}} \sin \frac{k \pi x}{l}\,,$$

and since the eigenfunctions φ_k, as well as their derivatives φ_k', are orthogonal in the interval 0, …, l, the simultaneous system (2.3) will decompose into the independent differential equations:

$$\ddot{q}_k + \left(\frac{k^4 \pi^4}{l^4} \frac{\alpha}{\mu} - \frac{k^2 \pi^2}{\mu\, l^2} P\right) q_k = 0 \qquad (k = 1, 2, \ldots). \tag{2.4}$$

The eigenfrequencies under the load P are then given by:

$$\sigma_k'^2 = \frac{k^2 \pi^2}{\mu\, l^2} \left(\frac{k^2 \pi^2\, \alpha}{l^2} - P\right) \qquad (k = 1, 2, \ldots),$$

and the demand that they should all be real will lead to **Euler**’s buckling load:

$$P_k = \frac{\pi^2 \alpha}{l^2} . \tag{2.5}$$

The kinetic stability criterion is true with no further assumptions since it is based upon the definition of stability directly. In fact, it is easy to see that the procedure that was employed here can be adapted to higher-dimensional problems with no further discussion, and in such a way that one extends the right-hand sides of the **Lagrange** equations (2.2) by means of generalized forces Q_k, so it can also be extended to gyroscopic and non-conservative problems. However, since the simultaneous system (2.3) decomposes only in exceptional cases, the method is generally more involved.

3. The energetic criterion. – If the position coordinates of a conservative elastic system are measured from the trivial equilibrium position (i.e., the position of the unloaded system), and the potential energy in that position is set to zero then one will have the relations:

$$V(0, \ldots) = 0 , \qquad \frac{\partial V}{\partial q_k}(0,\ldots) = 0 \qquad (k = 1, 2, \ldots). \tag{3.1}$$

From the law of conservation of energy, the energy of motion T will decrease with increasing distance from the equilibrium position with certainty, or at best in special situations, according to whether $V(q_1, q_2, \ldots)$ is or is not positive-definite, respectively. For conservative systems, the kinetic criterion can then be replaced with the energetic one.

The potential energy in the neighborhood of the equilibrium position can often be represented by the truncated power series:

$$V(q_1, q_2, \ldots) = V(0, \ldots) + \sum_{k=1}^{\infty} \frac{\partial V}{\partial q_k}(0,\ldots)\, q_k + \frac{1}{2} \sum_{i,k=1}^{\infty} \frac{\partial^2 V}{\partial q_i\, \partial q_k}(0,\ldots)\, q_i\, q_k , \tag{3.2}$$

and since (3.1) implies that here, not only does the first term vanish, but so does the second one (namely, the first variation of V), so the energetic criterion can also be formulated by saying that the third term, which is referred to as the second variation of V (up to the factor of 1/2), must be positive-definite if the system is to be stable.

In the example of Sec. **2**, the potential energy $V = U - P\, W_0$ is positive-definite for only sufficiently-small values of P. For a homogeneous prismatic rod that is pin-jointed at both ends, it will follow from (2.4) that the q_k are also normal coordinates for the loaded rod, and the condition that:

$$V = \frac{1}{2} \sum_{k=1}^{\infty} \sigma_k'^2\, q_k^2$$

must be positive-definite, so all of the $\sigma_k'^2$ are greater than zero, will once more lead to the **Euler** buckling load (2.5).

The domain of validity of the energetic criterion is obviously narrower than that of the kinetic one, insofar as it assumes the existence of a stationary, single-valued potential energy. Meanwhile, it is still conceivable that for non-conservative systems, it can be replaced with another argument of a static nature, e.g., one based upon work.

4. The static criterion. – If the potential energy of a conservative system admits the truncated development (3.2) then it can be represented by one of the form:

$$V = U - \lambda W_0 = \tfrac{1}{2}\sum_{i,k=1}^{\infty} c_{ik}(\lambda)\, q_i\, q_k\,, \tag{4.1}$$

when one generalizes the representation (2.1) by introducing a parameter λ that is characteristic of the load and calls upon (3.1), as well as (3.2). When one now transforms the quadratic forms U and W_0 to principal axes, instead of T and V, one will obtain new normal coordinates p_k in place of the q_k, in which U and W_0 will be expressed by:

$$U = \tfrac{1}{2}\sum_{k=1}^{\infty} p_k^2\,, \qquad W_0 = \tfrac{1}{2}\sum_{k=1}^{\infty} a_k\, p_k^2\,. \tag{4.2}$$

One then has:

$$V = \tfrac{1}{2}\sum_{k=1}^{\infty} (1-\lambda\, a_k)\, p_k^2\,, \qquad \frac{\partial V}{\partial q_k} = (1-\lambda\, a_k)\, p_k \qquad (k = 1, 2, \ldots).$$

As long as λ is sufficiently small, the bracketed expressions in those relations will both be positive, and since the relations:

$$\frac{\partial V}{\partial q_k} = (1-\lambda\, a_k)\, p_k = 0 \qquad (k = 1, 2, \ldots) \tag{4.3}$$

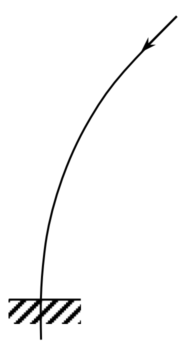

Figure 2. Non-conservative problem: Tangentially-compressed rod, clamped at one end.

represent the equilibrium conditions for the system, there will exist no non-trivial equilibrium position. On the other hand, if one of the bracketed expressions becomes zero for the first time with increasing λ then V will lose its positive-definite character. However, at the same time, (4.3) will admit a non-trivial solution.

For the homogeneous prismatic rod that is pin-jointed at both ends in Section **2**, the normal coordinates p_k will coincide with the q_k . The non-trivial equilibrium positions are characterized by the vanishing of the brackets (2.4), and upon setting the smallest one equal to zero, one will again obtain the **Euler** buckling load (2.5).

As a result of those arguments, the static stability criterion will be true for conservative systems whose potential energy can be represented by (4.1) [(4.2), resp.].

However, it remains conceivable that its domain of validity can be extended, and even to non-conservative systems in some situations.

5. Non-conservative problems. – Systems that are capable of vibration will already become non-conservative when one considers the damping that is always present and depends upon the state of the system. In general, one returns to a conservative system that can be treated by each of the three criteria by neglecting the damping. Meanwhile, it can happen (and this is the only case that will be examined) that an elastic system can lose its conservative character under loading due to the fact that the load, which is independent of the state of motion, cannot be derived from a stationary, single-valued potential. That includes the rod that **A. Pfluger** cited ([1]), which carried a continuous load that fell along the tangent to the elastic line and remained that way under deformation, as well as the one that the free shaft that the author treated ([2]), which was loaded by an axial moment vector. The basis for that abnormal behavior lies in the fact that neither a force with constant magnitude that is rigidly coupled with the body (and therefore takes part in its motion) nor a constant moment vector is conservative ([3]).

As a prime example of this class of problems that is indeed not very broad is scope, but still has practical significance, one can consider the homogeneous and prismatic rod that is clamped at one end and loaded with a tangential unit force P at the free end (Fig. 2). The work done by P under the displacement to the (trivial) equilibrium position depends upon the way that this displacement was performed; the load P is therefore not conservative. The energetic criterion breaks down in the form that was given Sec. **1**, and since one can show effortlessly that no non-trivial equilibrium position can exist, one must conclude from the static criterion that there is unlimited stability to the rod.

The open question in Secs. **3** and **4** of whether that static or energetic criterion can be modified in such a way that it would also be applicable to non-conservative systems can be answered with a simple example, such as a chain of linked rods, which can serve as a simplified model of the elastic rod (Fig. 2) according to **K. Marguerre** ([4]).

6. A model. – Fig. 3 shows a double pendulum with the angles of rotation φ_1, φ_2 (which are assumed to be small) that consists of two rigid rods of length l, and whose masses m_1, m_2 are thought to be concentrated at the distances a_1, a_2 from the links. Two vertical forces G_1, G_2 act at m_1, m_2, which can be interpreted as weights, the axial force P acts at the free end, and restoring moments $c\,\varphi_1$, $c\,(\varphi_2 - \varphi_1)$, as well as the damping moments $b\,\dot{\varphi}_1$, $b\,(\dot{\varphi}_2 - \dot{\varphi}_1)$, act in the joints.

The total energy of motion is equal to:

$$T = \tfrac{1}{2}[(m_1\,a_1^2 + m_2\,l^2)\,\dot{\varphi}_1^2 + 2\,m_2\,l\,a_2\,\dot{\varphi}_1\,\dot{\varphi}_2 + m_1\,a_2^2\,\dot{\varphi}_2^2]\ ,$$

([1]) **A. Pflüger**, *loc. cit.*, pp. 41.
([2]) **H. Ziegler**, Zeit. angew. Math. Phys. **2** (1951), pp. 279.
([3]) Cf., **H. Ziegler**, "Zum Begriff des Konservativen System," *loc. cit.*
([4]) **K. Marguerre**, *Neuere Festigkeitsprobleme des Ingenieurs*, Berlin, Göttingen, Heidelberg, 1950, pp. 211.

up to second-order quantities. The potential energy of the forces G_1, G_2, as well as the restoring moments, is:

$$V = \tfrac{1}{2}[(G_1 a_1 + G_2 l + 2c)\varphi_1^2 - 2c\varphi_1\varphi_2 + (G_2 a_2 + c)\varphi_2^2] ,$$

and the generalized non-conservative forces (that are due to P and the damping moments) are calculated to be:

$$Q_1 = P\,l(\varphi_1 - \varphi_2) - b(2\dot\varphi_1 - \dot\varphi_2) .$$
$$Q_2 = b(\dot\varphi_1 - \dot\varphi_2) .$$

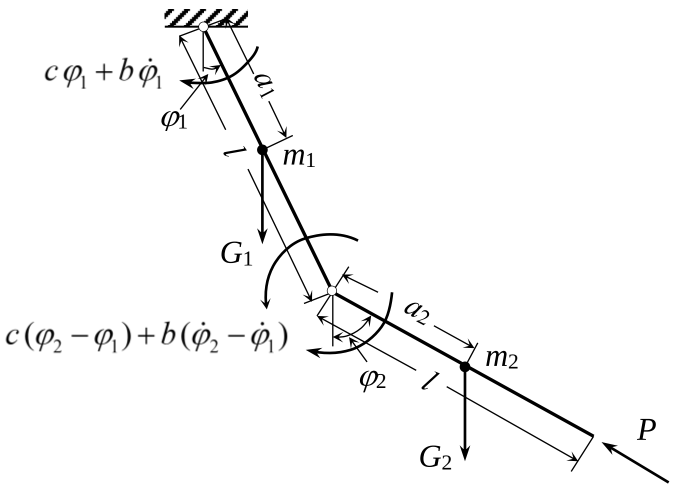

Figure 3. Double pendulum as a two-component model of the elastic rod in Fig. 2.

The **Lagrange** relations:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot\varphi_k}\right) - \frac{\partial L}{\partial\varphi_k} = Q_k ,$$

$$(L = T - V, \quad k = 1, 2)$$

lead to the differential equations of motion:

$$(m_1 a_1^2 + m_2 l^2)\ddot\varphi_1 + m_2 l a_2 \ddot\varphi_2 + b(2\dot\varphi_1 - \dot\varphi_2) + [G_1 a_1 + (G_2 - P)l + 2c]\varphi_1 + (Pl - c)\varphi_2 = 0 ,$$
$$m_2 l a_2 \ddot\varphi_1 + m_2 l a_2 \ddot\varphi_2 - b(\dot\varphi_1 - \dot\varphi_2) \qquad - c\varphi_1 + (G_2 a_2 + c)\varphi_2 = 0 ,$$

and the exponential Ansatz:

$$\varphi_k = A_k e^{\lambda t} \qquad (k = 1, 2) \tag{6.1}$$

leads to the characteristic equation:

$$p_0\lambda^4 + p_1\lambda^3 + p_2\lambda^2 + p_3\lambda + p_4 = 0 , \tag{6.2}$$

whose coefficients are given by:

$$\left.\begin{aligned} p_0 &= m_1 m_2 a_1^2 a_2^2 ,\\ p_1 &= [m_1 a_1^2 + m_2(l^2 + 2l a_2 + 2a_2^2)]b ,\\ p_2 &= (m_1 a_1^2 + m_2 l^2)(G_2 a_2 + c) + m_2 a_2^2 [G_1 a_1 + (G_2 - P)l + 2c] - m_2 l a_2 (Pl - 2c) + b^2 ,\\ p_3 &= [G_1 a_1 + G_2(l + 2a_2) + 2c]\,b ,\\ p_4 &= [G_1 a_1 + (G_2 - P)l + 2c](G_2 a_2 + c) + (Pl - c)c . \end{aligned}\right\} \tag{6.3}$$

The double pendulum is stable as long as all of the roots $\lambda_1, \ldots, \lambda_4$ of the characteristic equation have non-positive real parts. Then and only then does the most general motion, which is composed of four solutions of the form (6.1), remain bounded.

If (as will be true many times in what follows) any two roots are equal and opposite (say $\lambda_2 = -\lambda_1$, $\lambda_4 = -\lambda_3$) then the oscillator will be stable as long as the values $\lambda_1^2 = \lambda_2^2$, $\lambda_3^2 = \lambda_4^2$ are negative or zero.

If one of the roots vanishes then, according to (6.1), a solution will exist with constant roots φ_1, φ_2, that do not vanish simultaneously, i.e., a non-trivial equilibrium position. Conversely, the existence of a non-trivial equilibrium position demands the vanishing of one of those roots.

7. Consequences:

a) If one sets $b = 0$, $m_1 = 2\,m$, $m_2 = m$, $a_1 = a_2 = l$, $G_1 = G_2 = 0$ then the pendulum in Fig. 3 can be regarded as a two-component model for the rod in Fig. 2 (rotated by π), in which the mass is concentrated at the modes, and the proper weight can be neglected, along with damping. With:

$$p_0 = 2m^2 l^4, \quad p_2 = m l^2 (7c - 2Pl), \quad p_4 = c^2, \tag{7.1}$$

the characteristic equation will then go to the biquadratic equation:

$$p_0 \lambda^4 + p_2 \lambda^2 + p_4 = 0, \tag{7.2}$$

whose discriminant is:

$$\Delta = p_2^2 - 4 p_0 p_4 = m^2 l^4 (41c^2 - 28Plc + 4P^2 l^2). \tag{7.3}$$

For $P = 0$, the roots $\lambda_1^2 = \lambda_2^2$, $\lambda_3^2 = \lambda_4^2$ of the characteristic equation are less than zero (Fig. 4), since along with the discriminant (7.3), all of its coefficients (7.1) are positive. The most general motion is then bounded and can be composed of the two normal oscillations in the known way. If one lets P increase then Δ will decrease and change sign for $p = (7/2 - \sqrt{2})c/l = P_k$. The roots λ_1^2, λ_3^2 will then approach each other with increasing P and coincide for $P = P_k$, and since they will become complex conjugate for larger values of P:

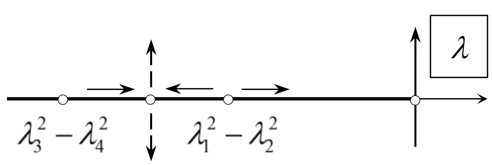

Figure 4. Displacement of the roots of the characteristic equation with increasing load.

$$P_k = (7/2 - \sqrt{2})\frac{c}{l} = 2.086\frac{c}{l} \tag{7.4}$$

will be the critical load.

Since (7.1) says that $p_4 > 0$, the roots of (7.2) will be non-zero for arbitrary values of P. There will then be no non-trivial equilibrium position, as one can also confirm by considering the action of the forces in Fig. 3. It will then follow already from this simple example that:

Theorem 1:

The static stability criterion can break down for non-conservative systems.

However, one might suspect that it will break down only in the absence of a non-trivial equilibrium position.

b) If one modifies the example by considering the proper weight by setting $G_1 = 2\,G$, $G_2 = G$ now, instead of $G_1 = G_2 = 0$ then one will get:

$$p_0 = 2m^2 l^4 , \quad p_2 = m l^2 (7c - 2P l + 6G l) , \quad p_4 = c^2 + 5c G l - P G l^2 + 3G^2 l^2 \tag{7.5}$$

instead of (7.1) and:

$$\Delta = m^2 l^4 (41c^2 - 4c(7P - 11G) l + 4P^2 l^2 - 16 P G l^2 + 12 G^2 l^2] \tag{7.6}$$

in place of (7.3).

For $P = 0$, the roots $\lambda_1^2 = \lambda_2^2$, $\lambda_3^2 = \lambda_4^2$ of (7.2) are again negative, so the pendulum is stable, as was to be expected. If one lets P increase then the stability will be lost as soon as (Fig. 4) either $\lambda_1^2 = 0$ or $\lambda_1^2 = \lambda_3^2$, because at least λ_1^2 will leave the negative real axis then. The critical load is then determined from the sharper of the two requirements:

$$p_4 = 0 , \qquad \Delta = 0 , \tag{7.7}$$

the first of which is, at the same time, the condition for the existence of a non-trivial equilibrium position. If one implements (7.7) by means of (7.5) and (7.6) then one will get the critical load:

$$P_h' = 5\frac{c}{l} + 3G + \frac{c^2}{l^2} \cdot \frac{1}{G}$$

on the basis of the static stability condition alone, but the kinetic one will imply:

$$P_k = \frac{7}{2}\frac{c}{l} + 2G - \sqrt{2\frac{c^2}{l^2} + 3\frac{c}{l}G + G^2} ,$$

in addition.

Since $P_h' > 0$, there is a load for which a non-trivial equilibrium position exists. Meanwhile, since $P_k < P_h'$ and P_h' grows without limit as $G \to 0$, the critical load will be P_k, and not the load

P_h' that the static criterion implies, which can be considerably greater under some circumstances. One will then have:

Theorem 2:

The static stability criterion can also break down in the presence of a non-trivial equilibrium position (and imply a critical load that is much too high in some situations).

That theorem proves the unusability of the static criterion from Sec. **1** for non-conservative systems. Meanwhile, the question of whether it is possible to determine P_k in another, purely static, way (e.g., by considering the work done) still remains open.

c) If one neglects the damping and proper weight by setting $b = 0$, $G_1 = G_2 = 0$ then with $m_1 = m_2 = m$, $a_1 = a_2 = l / 2$, one will get a two-component model of the rod in Fig. 2 (rotated through π) that is equivalent to the one that was discussed in (a) and differs from it only by the fact that the mass is now thought to be concentrated at the midpoints of the rods, and not the nodes. The characteristic equation here is also (7.2), but in that way, (7.1) and (7.3) will be replaced with:

$$p_0 = \tfrac{1}{16} m^2 l^4, \quad p_2 = \tfrac{1}{2} m l^2 (11c - 3Pl), \qquad p_4 = c^2,$$

$$\Delta = \tfrac{3}{16} m^2 l^4 (39c^2 - 22Pl\,c + 3P^2 l^2).$$

Since one again has $p_4 > 0$, one will get the critical load by setting Δ equal to zero, and indeed with:

$$P_k = 3\frac{c}{l},$$

it will be an essentially higher value than was obtained from (7.4) in the case (a). One will then have:

Theorem 3:

For non-conservative systems, the critical load depends upon the otherwise-equal ratios (and very strong in some situations) of the mass distribution.

Since masses do not enter into static investigations, if will follow from Theorem 3 that:

Theorem 4:

Non-conservative stability problems cannot be solved by the static method.

That shows that the energetic criterion in Sec. **1** cannot be legitimized for non-conservative systems, either. With the convention that only the kinetic criterion remains applicable, however, the question of the influence of damping will come to the foreground.

d) With $G_1 = G_2 = 0$, $m_1 = 2m$, $m_2 = m$, $a_1 = a_2 = l$, one will obtain a two-component model of the rod in Fig. 2 in which the internal damping is also considered. The characteristic equation has the form (6.2), in which the coefficients are given by:

$$\left.\begin{array}{ll} p_0 = 2m^2 l^4, & p_1 = 7ml^2 b, \\ p_2 = ml^2(7c - 2Pl) + b^2, & p_3 = 2cb, \\ p_4 = c^2. & \end{array}\right\} \tag{7.8}$$

Since equation (6.2), in contrast to (7.2), is no longer quadratic, it is recommended that one should employ the **Routh-Hurwitz** criterion ([1]), and as a result of it, the system will be stable in any case as long as the expressions:

$$p_0\,,\qquad p_1\,,\qquad p_2 - \frac{p_0\,p_3}{p_1}\,,\qquad p_3 - \frac{p_1^2\,p_4}{p_1\,p_2 - p_0\,p_3}\,,\qquad p_4$$

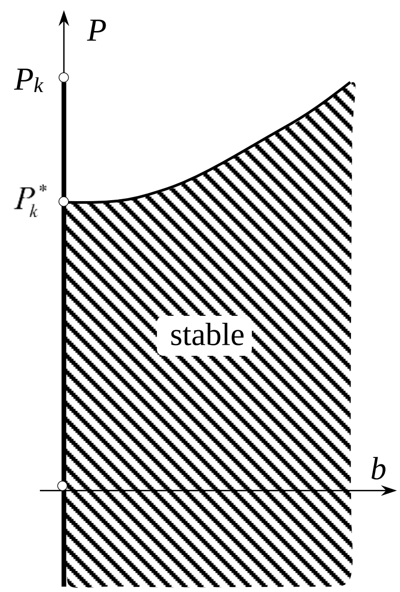

Figure 5. Stability domain of the model when one considers damping. (b = damping, P = load)

possess the same signs. If one substitutes the values (7.8) here then one will arrive at the stability conditions:

1. $b > 0$,

2. $P < \dfrac{45}{14}\dfrac{c}{l} + \dfrac{1}{2}\dfrac{b^2}{ml^3}$,

3. $P < \dfrac{41}{28}\dfrac{c}{l} + \dfrac{1}{2}\dfrac{b}{ml^3}$,

4. $c^2 > 0$,

the fourth of which will be fulfilled in all cases. One concludes from the first one that the system is unstable for negative damping (Fig. 5). The third one, which is sharper than the second, implies that for positive damping, the critical load is:

$$P_k = \frac{41}{28}\frac{c}{l} + \frac{1}{2}\frac{b}{ml^3}\,,$$

and as $b \to 0$, that will not go to (7.4), but to:

([1]) Cited in *loc.cit.*

$$P_k^* = 1.464\frac{c}{l}. \tag{7.9}$$

Mathematically, one can understand the discrepancy between P_k and P_k^* to mean that the **Routh-Hurwitz** criterion represents the conditions for the general solution to die away, so the stationary vibrations can already prove to be unstable in the sense of that criterion for loads $P_k^* < P < P_k$. That corresponds to the mechanical convention that whenever one calculates with damping (no matter how slight), and since it has a destabilizing effect in the interval $P_k^* < P < P_k$, one must always regard P_k^* as the critical load from the physical standpoint, and even for the undamped system, in practice. One will then have:

Theorem 5:

For non-conservative systems, damping can have a destabilizing effect,

as well as ultimately:

Theorem 6:

For non-conservative systems, the slightest damping can modify the critical load considerably.

Systems of the type considered here can then be treated with only the kinetic stability criterion. It is essentially more laborious than the other two in applications and will become even more cumbersome when one considers damping, which is known to be necessary. Meanwhile, the complications that it implies can hardly be avoided. They are probably intrinsic to the nature of the problem, and indeed based upon the fact that the energy loss in a non-conservative system can be very sensitive to slight variations in the course of motion.

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Professor's address: Professor Dr. **H. Ziegler**, Rüschlikon bei Zürich, Weierweg 6.