Schubert Collection XXXIV

Line geometry

with applications

by

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Foreword

A systematic line geometry that also considers the analytical methods has not appeared in the German language since Plücker's original work *Neue Geometrie des Raumes, gegründet auf die Betrachtung der geraden Linie als Raumelement* (I, 1868, II, 1869) (Sturm's *Liniengeometrie*, 3 vols., 1892, 1893, 1896 is purely synthetic); in other languages, there are only monographs on individual, generally extended, parts of line geometry (namely, Koenigs, *Géométrie réglée*, 1895). Thus, I welcomed the challenge of writing a book on line geometry, since this problem seemed worthwhile to me.

Corresponding to the aims of the "Schubert Collection," the first two chapters are kept at a completely elementary level; i.e., no use will be made here of line coordinates and only a very general use of projective geometry. In the latter chapter, the demands on the reader increase somewhat; at that point, the presentation is more accessible to students in advanced courses.

The principal topic in the present first volume is defined by linear complexes, congruences, and linear manifolds of such complexes, along with the applications that this part of line geometry admits. The higher-degree structures that thus appear will likewise serve as a preparation for the second volume, which will primarily treat algebraic line structures of degree higher than one and infinitesimal line geometry. Since university instruction nowadays devotes more attention to applied mathematics, I have drawn the circle of applications as widely as possible; e.g., I have considered the relations between line geometry and graphical statics.

In order to make the presentation as independent as possible of other books, I have interpolated a chapter on "Imaginary elements," in order for the beginner to acquire some facility with this now-indispensable theory and expressly to acquire, not merely a familiarity, but an insight into its power. I felt moved by similar considerations to include § 80. Generally, I have restricted myself to the theory of imaginaries in its narrowest sense, in order to attain a relatively complete presentation, namely, the laws of meets and joins, but I believe that some simplifications can be achieved.

I have sought to make all of the structures as intuitive as possible. The intuitive appeal depends less on whether the analytical or synthetic method was employed (when the abstract generality of projective geometry is just as non-intuitive as analytic geometry), as much as on whether it is possible to create the structures in a metrically distinguished manner. This is clearly attained only in the elementary context; one should then desire more such progress. Conspicuously less care has been devoted to – e.g. – how a net of rays or real focal lines actually "look." Theorems 105 to 110 and Fig. 47 will then fill this gap.

New material is found in §§ 43, 54, 55, 59, 60, 68, 83, B. In addition, many new proofs of old theorems are given, and the details of known things are enlarged: Thus, the geometric meaning of tetrahedral line pointers is still not discussed in its entirety (Theorem 47) and the discussion of imaginary tangents of the second type for a surface of second order (§ 72) is not given explicitly. The examination of congruences of axes of net complexes will be given with consideration for all of the special cases, which did not happen in either Plücker or Ball (*The Theory of Screws*, 1900), but is necessary when one wishes to address the applications to mechanics and the evaluation of each individual case of three degrees of freedom of motion on that basis.

I have used the convenient Grassmann expression of “pointers,” instead of “coordinates” extensively (and correspondingly, “system of pointers,” “line pointers,” etc.), and have likewise absorbed a few convenient notations of Sturm. I have the taken the opportunity of making mention of the further literature in the exercises for the reader when there was no reason to mention it in the text. Other volumes of the “Schubert Collection” are cited by S.S.

The need for the synthetic or the analytic method will alternate. The “purity of method” then seems to pay off only in the fundamental disciplines. Later on, it becomes tedious and should be superseded by those tools that deliver the most rapid and natural progress.

Innsbruck, on 26 February 1902.

Konrad Zindler.

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Introduction

The goal of line geometry.

The manifold of lines in space is four-fold; i.e., the individual line depends upon four constants or parameters. If we – e.g. – determine a line through its two intersection points S, S' with two fixed planes then we will obtain each line only once, and each of the points S, S' will then need only two pointers (coordinates) in order to determine it in its plane. If we base the analytical representation:

$$x = az + \alpha, \qquad y = bz + \beta$$

on a system of parallel pointers with the running pointers x, y, z then we can consider a, b, α, β to be the four parameters of the line. Inside of this four-fold domain (*) there are only one-fold, two-fold, and three-fold domains, analogous to the way that inside of a three-fold extended point space there are one-fold and two-fold domains (viz., curves and surfaces). The one-fold domains of straight lines are called *ruled surfaces*, the two-fold domains are *ray systems* – or frequently, *ray congruences* (and also *line congruences* or, more casually, *congruences*) – and three-fold domains are *line complexes* or *complexes*. For example, the totality of all normals to a surface defines a ray congruence, and all of the tangents to it define a complex (of a very special type). From the analytic representation above, we will also obtain complexes when we restrict the choice of four parameters by imposing a condition:

$$f(a, b, \alpha, \beta) = 0.$$

Two such conditions specify a congruence and three of them specify a ruled surface. Four independent conditions specify only a discrete number of straight lines, analogously to the way that three surfaces generally intersect in a finite number of points.

The examination of the one-fold, two-fold, and three-fold line manifolds now defines the goal of line geometry in a narrower sense; an extension of these things will be made in the conclusion of § 43. In addition, line geometry admits applications to mechanics, graphical statics, and the study of motion, with which we shall also be concerned.

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*) Instead of "manifold" we will frequently use the shorter expression "domain."

Chapter I.

The null system and the ray twist.

§ 1. The screwing motion.

Let P be a point of a circular cylinder with a radius r, let N be its perpendicular projection onto the axis a of the cylinder, and let P' be its perpendicular projection onto a plane ε that is perpendicular to a (Fig. 1). If P moves in such a way that the circular arc described by P' and the line segment that is described by N always have the same relationship through all of time then P will describe a *helix*. It is simplest to think of the motion of N, and therefore, that of P', as well, as being uniform. Let τ be the velocity of P, and let ω be the angular velocity of the point P', so $r\omega$ will be its absolute velocity. If one regards the plane ε from the side to which τ is directed then the rotation of P' can result in a positive or negative sense, so ω can have two possible signs. In the former case, the helix will be called *right-wound*, or *right-going* (when regarded from the outside of the cylinder, it will proceed to the right), while in the latter case, it will be *left-wound* or *left-going*. (*) This convention is independent of the sense in which the helix s is traversed; if we rotate the cylinder in Fig. 1 in such a way that its base comes to lie above then s will remain right-wound. Let P_0 be the intersection of s and ε, and let τ be the time that has elapsed since the point P_0 was at P. We can unwind the triangle $P_0P'P$ that lies in the cylinder onto a plane and obtain a right-angled triangle with the smaller sides:

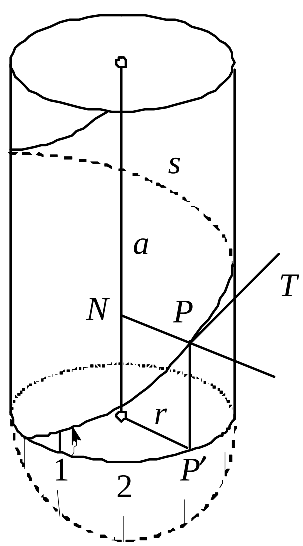

Figure 1.

$$P_0P' = r\omega t, \qquad P'P = \tau t;$$

the *pitch* of the helix will be then determined from:

$$\tan \vartheta = \frac{\tau}{r\omega}. \tag{1}$$

Since ω is positive for right-wound helices and negative for left-wound ones, ϑ will be acute in the former case and obtuse in the latter; i.e., we will always understand ϑ to mean the angle that the velocity vector of the point P_0 defines with the tangent to the circle under a positive rotation. If P' describes a complete circle then one will have $P_0P' = 2r\pi$, thus, $2r\pi = r\omega t$, and the associated time will be:

(*) Thus, e.g., the corkscrew and the screw that is used most generally are "right-wound." This manner of speaking is common in the study of machines; in theoretical geometry, the terminology is occasionally inverted.

$$t = \frac{2\pi}{\omega}.$$

$P'P$ is equal to the altitude h; one therefore has:

(2) $$h = \frac{2\pi\tau}{\omega}.$$

s is completely determined by the axis a and the tangent T to a point on it. If s is right-wound then the helix that is determined by T as its axis and a as its tangent, will also be so. Thus, we will call a *pair of skew lines* that are not perpendicular to each other right-wound or left-wound according to whether one or the other kind of helix is determined when one takes one line to be the axis and the other one to be the tangent.

If a geometric structure rotates with a uniform angular velocity ω and simultaneously proceeds along a with the uniform velocity τ then one will say that the structure executes a uniform *screw motioning* or a *screw* with axis a. Since one can think of each point of space as being tightly coupled with the moving structure, one will arrive at the notion of all of the moving space $\mathfrak{R}$ executing a screw in a second space that is at rest. Each point P of $\mathfrak{R}$ will thus describe a helix whose altitude, from (2), will be independent of its position, and whose pitch, from (1), will decrease with increasing r; merely the points of a describe a itself. We will call the entire screw right-wound or left-wound according to whether its individual helices have that character, respectively. Thus, the paths of the points will depend, not upon the absolute values τ and ω, but only upon their ratio:

(3) $$\frac{\tau}{\omega} = \mathfrak{k}$$

and the position of the axis; $\mathfrak{k}$ will be called the *parameter* or the *slope* of the screw. For $r = 1$, one will have $\tan \vartheta = \mathfrak{k}$, so:

Theorem 1: *The slope of the screw is equal to the tangent of the pitch of those helices that lie on a cylinder of radius one.*

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§ 2. The null system.

We think of a space that contains a screw around a and fix our attention on a definite time point; we call the normal plane to the path that a point P has at this time its *null plane* ν, on the grounds of things that we shall discuss later. Thus, each point of space is associated with a plane that goes through it. If we think of the normal plane as being carried along during the screw motion then for each point of time it will remain the null plane of the point P. Any association will then be independent of the chosen time point. Conversely, when ν is given, we will wish to find a point P whose null plane is ν. When

$\nu \perp a$, its intersection point N with a will be the desired point. In the other cases, we will draw the line $g \perp a$ in ν through N; we will only have to look for P on g. Its distance r from N must fulfill condition (1), where ϑ is known from the normal to ν. If we temporarily consider only the absolute values on both sides of the equation then r will be determined uniquely by them. However, there are two points g at this distance that are symmetric to N; one of them will be the desired one, whose normal to ν in the direction of a will wind the same way as the given screw. P is called the *null point* of ν. The association is thus uniquely reciprocal; only the planes that are parallel to a will contain no null point. This gap will be closed shortly.

Theorem 2: *Let each point of a screw be associated with the normal plane to its path. The geometric relationship that is thus defined is uniquely reciprocal and is called a null system.*

The axis a of the screw is also called the *axis* of the null system, and the parameter $\mathfrak{k}$ of the screw is also the *parameter* (*slope*) of the null system. The null system depends upon the position of the axis and $\mathfrak{k}$; thus, since there are ∞^4 lines, there will be a five-fold infinitude of null systems. From the way that a null system $\mathfrak{N}$ comes about, it emerges that it will go to itself under a rotation around a. That is, if one rotates a point along with its null plane around a then the plane will always remain the null plane of the point. One says that $\mathfrak{N}$ "admits" a rotation around a. Likewise, it will admit a displacement along a, and therefore a screw, as well, which is composed of such rotations and displacements, and therefore not only those screws that defined it, but:

Theorem 3: *A null system admits an arbitrary screw around an axis.*

Therefore, one can imagine the way that the points and planes of space are associated with each other by way of $\mathfrak{N}$. One can thus restrict oneself to the points P of a line g that cut a perpendicularly at N. Every other point of space can then be taken to a point of g by a suitable screw. For the sake of ease of notation, we think of a as being vertical, write equation (1) in the form:

$$\tan \vartheta = \frac{\mathfrak{k}}{r}, \tag{1a}$$

and learn from this that: When r increases from zero to ∞, P will move along g from N to infinity; thus, the associated helix of the screw will always get flatter, while the null plane will always get steeper with respect to the horizontal plane.

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§ 3. On the motion of a line.

Suppose that a line g moves in space in some way, and let the velocity vector a at one of its points A be perpendicular to that line. Let β be the velocity vector of a second point B on g, let g_1 be a neighboring position of g, let g' be the parallel to g_1 that is drawn through A, and finally, let ε be the normal plane to g at B. We can then take g to g_1 in such a way that we first bring g to g' by just a rotation around A, and then to g_1 by a parallel displacement; after the first step, the velocity vector of B will lie in ε, after the second step, it will be parallel to a, and therefore it will likewise be perpendicular to g. One composes β from these two vectors, so it will also lie in ε when it does not vanish. We would like to extend this argument by an analytical proof; i.e., to show that:

Theorem 4: *If the path tangent to a moving line is perpendicular to the line at one of its points then the same thing will be true for all of its points.* (*)

We locate the origin of a rectangular system of pointers at A and the z-axis along the line g; a second point B on g has the pointer $z = r$. During the motion of g, A will describe a curve:

(4) $$x = \varphi(t), \qquad y = \psi(t), \qquad z = \chi(t),$$

with the tangent α at A, and B will describe a curve:

(5) $$x_1 = \varphi_1(t), \qquad y_1 = \psi_1(t), \qquad z_1 = \chi_1(t) + r,$$

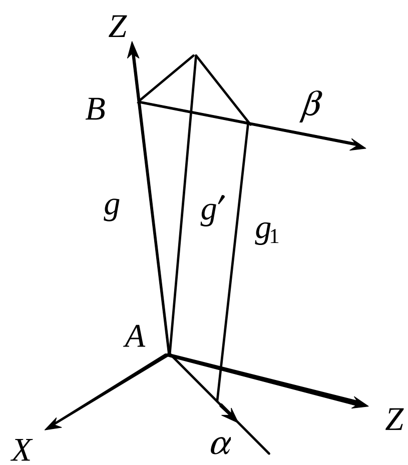

Figure 2.

with the tangent β at B. Thus, it will follow from the position of the pointer system that all six functions φ, ψ, χ must vanish at $t = 0$, if we measure the time t from the beginning of the motion. However, one must also have:

$$\chi'(0) = 0$$

if the direction cosines of α are proportional to $\varphi'(t)$, $\psi'(t)$, $\chi'(t)$, and, by assumption, one has $\cos(\alpha, Z) = 0$. What one must then prove is that dz_1 / dt will vanish for $t = 0$. Because the points A and B preserve their separation r during the motion, one will have:

$$(\varphi_1 - \varphi)^2 + (\psi_1 - \psi)^2 + (r + \chi_1 - \chi)^2 = r^2;$$

after differentiation by t, one will obtain:

$$(\varphi_1 - \varphi)(\varphi_1' - \varphi') + (\psi_1 - \psi)(\psi_1' - \psi') + (r + \chi_1 - \chi)(\chi_1' - \chi') = 0;$$

(*) This theorem may have been first expressed by Chasles ["Propr. geom.. rel. au mouv. inf. petit…." Comptes R., **16** (1843)].

if we set $t = 0$ in this then we will get:

$$r[\chi_1'(0) - \chi'(0)] = 0,$$

so:

$$\chi_1'(0) = 0.$$

———————

§ 4. The ray twist.

During a motion in space, a point P will have ∞^1 path normals at a definite moment, and indeed, if ν is the normal plane to its path then they will all be rays of the pencil (P, ν). We define:

The totality of the path normals of points of space under a screw motion is called a *ray twist* (Strahlengewinde), or simply a *twist* (Gewinde).

Thus, from Theorem 2, a twist is likewise defined by any null system; namely, if we consider all rays at each point P of space that go through that point and lie in its normal plane then we will obtain a twist. We ask whether some ray s of the twist, besides ν, can go through P. Such a line must lie in the null plane of one of its points. The path tangent at this point Q would then be perpendicular to s, so (*), from Theorem 4, it would be perpendicular to the path tangent at P, as well. However, this is possible only when s lies in ν. Likewise, it follows from Theorem 4 that the path tangent to each point Q of a twist ray s is perpendicular to s; thus, s is contained in the null plane ν of each of its points. If Q moves on s then ν will rotate around s, while, from Theorem 2, the possibility that ν can return to the same position that Q had earlier will be excluded. There are ∞^3 points in space; ∞^1 rays of the twist go through each of them, which then define a plane pencil. On the other hand, each ray appears as a ray of the associated pencil, along with each of its points. There are thus only ∞^3 rays in the twist.

Theorem 5: *The rays of a twist define a triply-infinite manifold. A pencil of lines therefore goes through each point of space and lies in each plane of space.*

We must prove the last part of this theorem as follows: If the plane ε is parallel to the axis a to begin with then we will choose a line g in it that is parallel to a. Since the null system admits a displacement along g, all points of g will have parallel null planes. This plane must be the null plane for each its points, since a second ray of the twist in ε goes through it. If ε is not parallel to a then we will know a pencil of rays of the twist in ε whose vertex will be the null point that is associated with it as in § 2; on the same grounds as before, no other ray of ε can belong to the twist. Thus, Theorem 5 will be proved completely, and we can now also associate the planes that are parallel to a – and thus, from § 2, contain no null points – with one of them. Namely, for each such plane ε, since the vertex of the pencil of rays of the twist, which is likewise the null point, points

(*) The case that was provided for in § 3, in which β vanishes, cannot enter into the argument here, since no point whatsoever will remain at rest under a uniform screw motion.

to infinity in a certain direction, we will have to consider the point V that is at infinity in this direction to be the null point of ε. There is no other ray of the twist through V than it; if we then draw a plane ε' through such an ε then ε' will contain two parallel rays of the twist – namely, s, and the line of intersection line with ε – which is impossible, since ε' has its null point at infinity. From § 2, it will emerge that one will now also have, conversely, a null plane that is associated with each point at infinity (for the moment, with the exception of the U that are axis directions). However, if a plane that is always parallel to an arbitrary starting point points to infinity then its null point will always point to infinity in the axis direction; we thus have to consider the point U to be the null point of the plane at infinity. This is consistent with the fact that U is the only point at infinity through which no ray of the twist at a finite point can go. It is therefore the points and planes of space without exception (except, of course, the elements at infinity) that are now reciprocally associated with each other into a null system. A twist ray will also be called a *guide line* of the associated null system.

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§ 5. Polar pairs of lines.

If we focus on two points P, P' of a line g (Fig. 3) that does not belong to the twist then its null planes will not go through g, and will thus intersect in a line g' that is skew to g. All rays of the pencils (P, g'), like the pencils (P', g'), will be rays of the twist, especially QP and QP', when Q is an arbitrary point of g'. The locus of the twist rays through Q is a plane pencil of lines that is determined completely by the two rays QP and QP'; thus, Qg is the null plane of Q, and likewise, $Q'g$ is the null plane of Q', such that, conversely, g is also determined by the intersection of the null planes of arbitrary points Q, Q' of g'. Two such mutually-associated lines will be called *polars* of the null system or twist. Each ray through Q that intersects g will belong to the twist. Since this is true for each point Q of g, one will have:

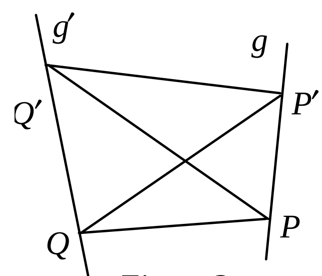

Figure 3.

Theorem 6: *All rays that intersect two associated polars belong to the twist.*

The null plane of a point on one of the two lines is therefore the connecting plane with the other one. We summarize this along with a result from § 4 into:

Theorem 7: *If a point describes a line g then its null plane will rotate around a line g′ that agrees with g or is skew to it according to whether g is or is not a ray of the twist that is bound to the null system, respectively; the roles of g and g′ are interchangeable.*

As one sees immediately from the way that one generates null systems by screws, the null planes of two points P, P' will be parallel only when the connecting line g of P and P' is parallel to the axis a. g' will then point to infinity and will be represented by the locus of the pencil of parallel planes that is defined by g; g will then be called a *diameter* of the null system. Theorem 7 thus also preserves its tangible content in this case, except that a parallel displacement appears in place of an actual rotation. If, by comparison, a point describes an line at infinity g' then, in a real sense, the null plane will rotate around the polar diameter g. Only for the lines at infinity of the parallel planes to a, which are to be regarded as rays of the twist, does the rotation enter in place of a parallel translation. The axis is the only diameter for which the associated locus is perpendicular to it.

Theorem 8: *If a ray of the twist intersects a line g then it will also intersect its polar g'.*

With g', it will then likewise lie in the null plane of its point of intersection with g. Let h, h' be a second polar pair, where h may cut either g or g'. A ray s that intersects g, g' will be a ray of the twist; if it also intersects h then, from Theorem 8, it will also intersect h'. Now, the rays s that intersect three given ones g, g', h will define a second-order family of rulings $\mathfrak{R}$ (cf., Reye, *Geom. d. Lage, I*, Vortr. 10). Since all of the h' will be intersected, h' will be a ray of the guiding family $\mathfrak{R}'$, which g, g', h will also belong to. One says that four lines have *hyperbolic position* when they belong to the same second-order family of rulings.

When g and h intersect each other (Fig. 4), g', as well as h', must lie in the null plane σ of the point of intersection S; thus, they will also intersect in a point T whose null plane τ will include g, as well as h. The four lines are arranged such that they each intersect two others, so the connecting line of the intersection point will be identical with the line of intersection of the connecting planes. We would like to treat this situation under the name of "hyperbolic position." (If necessary, we shall distinguish between the "special hyperbolic position" and the "general" kind). The totality of all lines that intersect g, h, g' will split into the two pencils of rays (S, σ) and (T, τ). Thus, we can say:

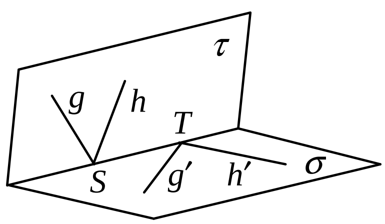

Figure 4.

Theorem 9: *Two arbitrary pairs of associated polars of a null system will have hyperbolic position.*

We now let h coincide with a; if a line s intersects h, as well as h', then that will say that it intersects h perpendicularly. $\mathfrak{R}'$ will now include a line at infinity h', and will therefore be a hyperbolic paraboloid $\mathfrak{P}$, one of whose guide lines will be perpendicular to the axis and the other of which will be parallel to it. (*) $\mathfrak{P}$ will therefore be equililateral, and each of the two principal generators will be intersected perpendicularly by all of the lines of the other family. The axis is the one principal generator; the other one, since it is

(*) For the reader that is familiar with the geometry of position, we remark that the very important Theorem 10 will be proved once more in § 8.

intersected perpendicularly by g, g', will include the shortest distance between these two lines, and therefore:

Theorem 10: *The shortest distance between two polars of a null system intersects the axis perpendicularly.*

Three rays determine a family of rulings. If all three of them belong to a twist then we will focus on a line g of the guiding family. Then, from Theorem 8, its polar g will also belong to the family. It follows from this, in connection with Theorem 6, that:

Theorem 11: *If three rays of a family of rulings $\mathfrak{R}$ belong to a twist then each ray of $\mathfrak{R}$ will belong to the twist.*

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§ 6. The null system as a reciprocal relationship.

From Theorem 7 and the end of § 4, one deduces that each "spatial element" – namely, a point, line, or plane – is always again associated with a spatial element through a null system, and indeed with a plane, line, or point, resp., and in such a way that if the point ranges over a line (the plane rotates around a line, resp.) then the corresponding plane will rotate around the corresponding line (the associated point will describe the associated line, resp.). If the point Q lies in the null plane ν of the point P then QP will be a ray of the associated twist; thus, conversely, the null plane of Q will also go through P. The points of the field ν will thus correspond to the planes of the pencil P, and furthermore, the lines of the field will correspond to the lines of the pencil. Thus, when g lies in ν, g' will go through P. Finally, if two lines g, h intersect each other then the corresponding ones will also intersect (§ 5).

If a point lies in a plane, a point lies in a line, a line lies in a plane, or finally, if two lines intersect then one will say that in all of these cases the two spatial elements are *incident*. By using this terminology, we can summarize the aforementioned individual theorems into one theorem:

Theorem 12: *If two spatial elements are incident then the ones that correspond to them in a null system will be incident.*

Which sort of incidence one is dealing with will be determined by the type of spatial element itself. The null system will thus belong to what are called in projective geometry "reciprocal relationships" or "correlations;" we will come back to this later (§ 46).

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§ 7. Analytical representation of the null system.

We immediately recall § 2, and we wish to be able to write down the equation of the null plane of a point P when its pointers x, y, z are given. We let the Z-axis of a rectangular system of pointers of the first type coincide with the axis a of a null system. The equation of a plane through P will have the form:

$$A(\xi - x) + B(\eta - y) + C(\zeta - z) = 0, \tag{6}$$

where ξ, η, ζ are its running pointers. Should it be the null plane of P then A, B, C would have to be proportional to the direction cosines of the path tangent at P under the screw; we must then compute these direction cosines. Let P come to the position $P_1 \equiv (x_1, y_1, z_1)$ at the time t as a result of the screw. Let $P' \equiv (x, y, 0)$ and $P_1' \equiv (x_1, y_1, 0)$ be the projections of P and P_1 onto the XY-plane. P_1 will then be obtained from P' by rotating in the XY-plane through the angle $\beta = \omega t$ (§ 1). Thus, if r, α are the polar pointers for P' in the XY-plane (Fig. 5) and r_1, α_1 are those of P_1' then one will have:

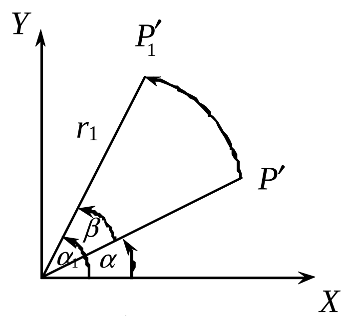

Figure 5.

$$r_1 = r, \qquad \alpha_1 = \alpha + \beta,$$
$$x_1 = r_1 \cos \alpha_1 = r\,(\cos \alpha \cos \beta - \sin \alpha \sin \beta),$$
$$y_1 = r_1 \sin \alpha_1 = r\,(\sin \alpha \cos \beta + \cos \alpha \sin \beta),$$

or:

$$x_1 = x \cos \beta - y \sin \beta, \qquad y_1 = x \sin \beta + y \sin \beta;$$

in addition, the altitude of P above the XY-plane will increase by $\tau\, t$. The connection between the pointers for P and P_1 will then be represented by:

$$\begin{aligned} x_1 &= x \cos \omega t - y \sin \omega t, \\ y_1 &= x \sin \omega t + y \cos \omega t, \\ z_1 &= z + \tau t. \end{aligned} \tag{7}$$

These equations represent the path curve of the point P, in which x_1, y_1, z_1 are the running pointers and t is the parameter. For $t = 0$, one will have $P_1 \equiv P$, as it should be. If we would then like to know three quantities that are proportional to the direction cosines of the path tangent at P then we would have to differentiate with respect to t and then set $t = 0$. This would yield:

$$\left(\frac{dx_1}{dt}\right)_0 = -\,y\;\omega, \qquad \left(\frac{dy_1}{dt}\right)_0 = x\;w, \qquad \frac{dx_1}{dt} = t.$$

We have substituted these three quantities for A, B, C in equation (6) and thus obtained, while taking equation (3) into account, the equation of the null plane at P:

$$x\,\eta - \xi\, y + \mathfrak{k}\,(\zeta - z) = 0. \tag{8}$$

We now consider τ and ω to be quantities endowed with signs; however, one confirms that the null system will depend upon only their ratio $\mathfrak{k}$. When one changes both of their signs simultaneously, a screw will come about in the opposite sense, but with the same path as before. A positive $\mathfrak{k}$ corresponds to a right-wound screw, while a negative $\mathfrak{k}$ corresponds to a left-wound one.

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§ 8. The position of polar pairs.

Since we can bring about a situation where the shortest distance from a line g to a falls along the X-axis by a suitable rotation of the null system around the a axis and a displacement along it, we can restrict ourselves to the lines g that intersect the X-axis perpendicularly. Such a line is determined by the X-pointer c of its point of intersection S and its pitch angle ν in the XY-plane. For the numerical evaluation of this angle, we assume the following: A positive sense of rotation in the YZ-plane is fixed (and in each of those that are parallel to it) by the positive side of the X-axis. Now, if g_1 is the projection of g onto the YZ-plane (Fig. 6) then we will understand the pitch angle ν of g with respect to the XY-plane to be the angle (Y, g_1). If c and ν are given then we will pose the problem of finding the polar g'. The equations of the lines g are:

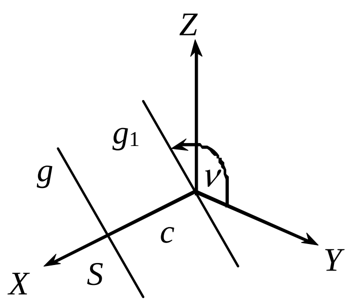

Figure 6.

$$x = c, \quad \frac{z}{y} = \tan\,\nu.$$

The pointers to a point P will then be:

$$c,\ y,\ y \cdot \tan\,\nu,$$

and will depend upon only the single variable y. If we substitute them into equation (8) then we will get the null plane of P in the form:

$$c\eta + \mathfrak{k}\zeta - y\,(\xi + \mathfrak{k}\tan\,\nu) = 0. \tag{9}$$

From this equation, we can, by a choice of y, compute the equation of the null plane of an arbitrary point of g; in particular, we would like to find the null plane ε of S and the null plane ε' of the infinitely-distant point U of g. We set $y = 0$ and obtain:

(10) $$\frac{\zeta}{\eta} = -\frac{c}{\mathfrak{k}}$$

for the equation of ε. In order to find the equation of ε', we next write equation (9) in the form:

$$\frac{c\eta + \mathfrak{k}\zeta}{y} - (\xi + \mathfrak{k}\tan\nu) = 0.$$

If we now let y become infinite then the plane will approach the limiting position:

(11) $$\xi = -\mathfrak{k}\tan\nu;$$

viz., the null plane of U. The plane (10) includes the X-axis, as it must, while the plane (11) is perpendicular to it. The line of intersection of both of them is, from § 5, the desired polar g'. Moreover, one finds, from the form of equation (9) that the common points of the planes (10) and (11) lie in the plane (9) for each value y, because their pointers make the components $c\eta + \mathfrak{k}\zeta$ and $\xi + \mathfrak{k}\tan\nu$ go to zero; thus, this plane will rotate around the line of intersection of the other two. Thus, the first part of Theorem 7 is proved once more, and independently of § 3, *et seq*., and likewise for Theorem 10. Equations (10) and (11) then represent a line g' that, like g, intersects the X-axis perpendicularly; if c', ν' are its analogous determining parameters then one will have:

$$\frac{\zeta}{\eta} = \tan\nu', \qquad x = c',$$

and if we compare this with (10) and (11) then we will find that:

$$\tan\nu' = -\frac{c}{\mathfrak{k}}, \qquad c' = -\mathfrak{k}\tan\nu',$$

or, more symmetrically:

(12) $$\begin{aligned} c &= -\mathfrak{k}\tan\nu', \\ c' &= -\mathfrak{k}\tan\nu, \end{aligned}$$

from which the relation between the four determining parameters of the lines g, g' can be expressed. We then deduce from this that:

(13) $$c : c' = \tan\nu' : \tan\nu.$$

We ask when g' will coincide with g. For this, one must have $c' = c$, so, from equation (12):

$$c = -\mathfrak{k}\tan\nu.$$

For the twist rays s, whose determining parameters we indicate by the index s, one will then have:

$$\tan \nu_s = -\frac{c_s}{ŧ}. \tag{14}$$

If we compare the right-hand sides of this equation and equation (1) then we will find (taking into account the change in sign) that the product is – 1. In fact, ν will be larger than ϑ by $\pi / 2$ when g comes to the position s.

We would like to provide an intuitive glimpse of the distribution of polar pairs that lets us describe the line g of the pencil (S, ε') (Fig. 7) and follow the motion of g'. Since ε, U are the corresponding elements to S, ε', we can infer from Theorem 12 that g' describes the pencil of rays (ε, U); i.e., it moves in the null plane of S parallel to the position s of g. We can also deduce this situation and the quantitative properties of the motion from equation (12). Besides s, we then consider the two distinguished positions, in which g is parallel to the Y or Z axis, resp., and call them p and q, resp. According to whether ŧ is positive or negative, g will be the rotation in the positive sense that begins at the position p and proceeds to the position q or s, resp.; one gather this from equation (14) or immediately from the geometric meaning of the sign of ŧ (§ 7). In order to fix the presentation, we think of ŧ as being positive. From equation (12), c' decreases from 0 to $-\infty$ while ν increases from 0 to $\pi / 2$. Also, under a further rotation, the subsequent intervals will correspond to the following table:

ν	c'
0 to $\pi / 2$	0 to $-\infty$
$\pi / 2$ to ν_s	$+\infty$ to c
ν_s to π	c to 0

This situation is made apparent in Fig. 7, where the distinguished positions of g and the corresponding ones g' are sketched in. Furthermore, the angle that ν describes is indicated by arcs that carry the same numerals as the corresponding line segments that range through the point of intersection (g', X); q' lies at infinity.

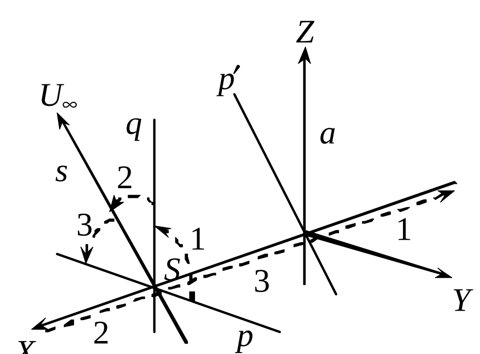

Figure 7.

If the shortest distance from g to g' intersects these lines at N, N', and the axis at M, then we can deduce from Figure 7 or equation (13) that:

Theorem 13: *Depending upon whether the projections of g, g' onto the YZ-plane lie in different quadrants or the same one, M will lie inside or outside of NN', resp.*

§ 9. The pencil of rays.

Up to now, we have always thought in terms of real screws. Thus, if the displacement component τ vanishes then we will merely have a rotation around a. It is immediately apparent that that the path normals in this case will consist of all spatial points of all the ∞^3 lines that intersect a. One calls their totality a "special twist" or a *bush of rays* and a is its *carrier*. It no longer corresponds to an actual null system. The null points of the points of the axis would then be indeterminate, and all of the other planes would intersect the axis. The reciprocal uniqueness of the association would then break down. However, this exceptional null system will always be represented by equation (8) ($\mathfrak{k} = 0$).

On the other hand, if the rotational component ω vanishes then $\mathfrak{k}$ will be infinite. If one previously divides equation (8) by $\mathfrak{k}$ then one will obtain the equation $\zeta - z = 0$ in this case; in fact, it will always represent the normal plane to the path at each point. One is then left with a displacement in the direction of the Z-axis. The twist now consists of all ∞^3 rays that are parallel to the XY-plane, and can be regarded as a bush of rays whose carrier is the line at infinity in this plane.

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§ 10. Ways of determining a twist or null system.

A twist is determined uniquely:

a) *By the axis a and a ray s* (*which must be skew to a*).

We then choose the shortest distant from a to s to be the X-axis of a system of pointers that is arranged as in § 8, in which we can fix the positive direction arbitrarily such that v_s and c_s are then given, and one can compute $\mathfrak{k}$ from equation (14).

If a polar pair g, g' (with the shortest distance NN') of a null system is given then a must intersect the line segment NN' perpendicularly. In addition, the condition (13) must be fulfilled. One can therefore choose the direction of a inside of a pencil arbitrarily, from which, the plane will also be determined, and the angle will result from that. Thus, M is determined uniquely from equation (13), such that one must divide NN' in the ratio $\tan \nu' : \tan \nu$ (cf., also Theorem 13). If one adds a connecting line for g, g' then one will be in the realm of case a). A null system is then also determined uniquely:

b) *By a polar pair and the direction of a, which must be deduced from the pencil of directions g, g′.*

There are thus ∞^1 twists in which g, g' are polar. If one chooses a third line h (that is skew to g and g') then, by Theorem 9, h' will be restricted to the family of rulings that g, g', h belong to, so it can therefore likewise assume just ∞^1 positions. One will thus be able to choose h' arbitrarily inside of the family of rulings. However, a twist will then be determined. Thus, if d is the shortest distance between g, g', and d_1 is that between h, h' then, from Theorem 10, a must be the shortest distance between d, d_1. If d, d_1 intersect then a will be the common perpendicular at the point of intersection. A null system is then also determined uniquely:

c) *By two polar pairs in hyperbolic position.*

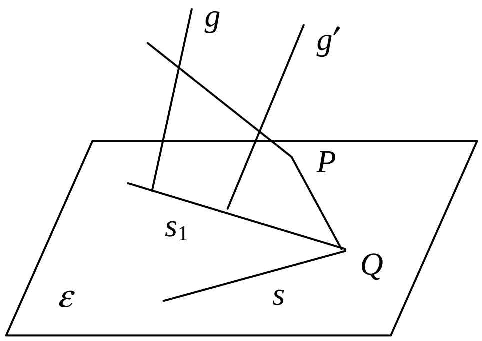

Figure 8.

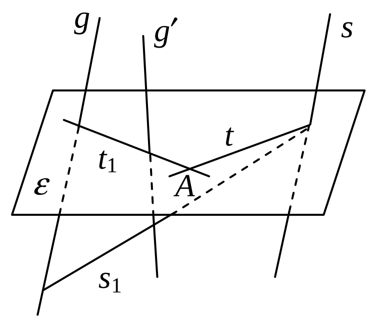

Figure 9.

In particular, one can choose h' to be identical with h. The twist will then give a ray s, in addition to g, g', and one can construct the null plane at each point P of space. One can, in fact, next find (Fig. 8) the null plane Q of the plane $(P, s) = \varepsilon$ when one knows, besides s, the ray of the twist s_1 in this plane that is determined by the point of intersection of g, g'. One then knows the ray PG of the twist by way of P, as well as the ones that intersect g, g'. Dually, one can construct the null point A to any plane ε in space (Fig. 9). Namely, one can next find the null plane of its point of intersection S with s when one knows the ray of the twist s_1 through S that intersects g, g', in addition to s. One obtains a ray of the twist t in ε as the intersection with the plane (s, s_1) and a second one t_1 through the point of intersection g, g' with ε. A null system is then also determined by:

d) *A polar pair and a guide line.*

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§ 11. Arrangement of the rays of the twist.

It is important for us to provide a picture of the rays of a twist that is as intuitive as possible. However, since a manifold of ∞^3 indistinguishable lines is difficult to draw, it will be necessary for us to decompose it into submanifolds. We would like to propose an especially intuitive decomposition as follows: For the sake of ease of expression, we think of the axis a as horizontal; if $MN = c$ is the shortest distance from a ray s of a twist to a, where M lies in a and N lies on s, then the pitch of s relative to the horizontal plane χ will be determined by equation (14). If one describes a circular cylinder around a with a radius c then s will contact this cylinder, and all of the tangents to the cylinder that have the same pitch relative to c and are wound (§ 1) the same way as s relative to a will likewise belong to the twist. From equation (14), this pitch will increase with increasing cylinder radius. The ∞^3 rays of the twist can then be arranged into the ∞^2 tangents to ∞^1 circular cylinders; each point of a cylinder will contact a ray of a twist and will be perpendicular to the path tangent to the point that is associated with it by means of the screw. The ∞^2 tangents rays of the twist that contact the same cylinder can then be again arranged into the ∞^1 tangents to ∞^1 helices, which will intersect the path helices of the cylinder points perpendicularly everywhere, and will therefore be wound oppositely to them. One say that a helix (or any curve, for that matter) is *contained in a twist* when its tangents belong to the twist. Any helix that is contained a twist can itself be called right-wound or left-wound, so:

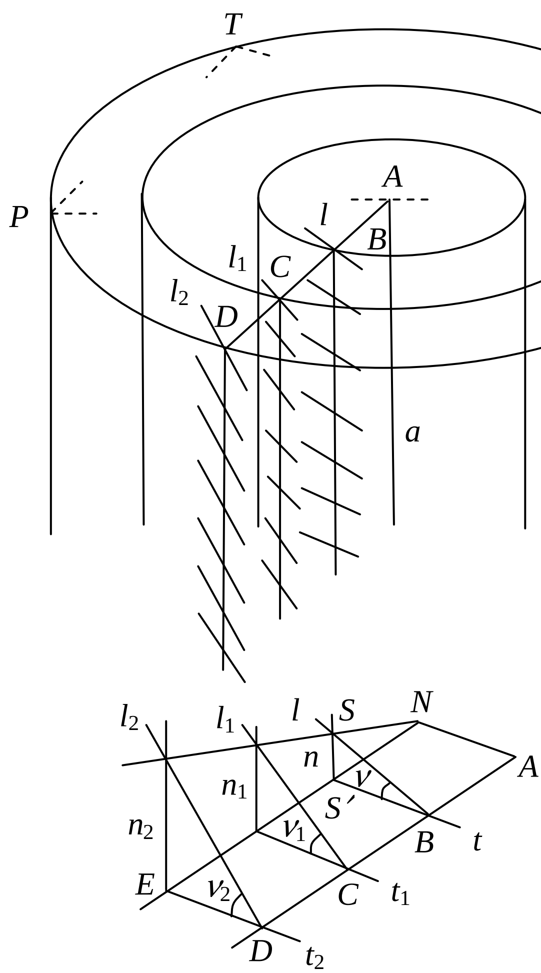

Figure 10.

Theorem 14: *A ray twist is wound oppositely to the screw that defines it.*

In Fig. 10, we have represented three coaxial cylindrical surfaces that intersect a plane through *a* and the pencil of parallel rays of the twist that contact along the resulting cylinder generators, which is suggested by short lines. If one rotates this system of ∞^2 lines around *a* then one will obtain all of the rays of the twist. Since, from equation (3), a positive $\mathfrak{k}$ corresponds to a right-wound helix, equation (8) will represent a left-wound twist for $\mathfrak{k} > 0$.

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§12. The moment of two rods and two lines.

One calls the general line segment a *vector* when only its length, direction, and sense are in question. Vectors are free to move parallel to themselves without changing their meaning. By comparison, a force – e.g., one that acts on upon a rigid body – is linked with a definite line of action. We, along with H. Grassmann, Jr. (cf., Ges. W. I, b, pp. 438), call a line segment whose length, the line that it lies on, and the sense of progress along it is at issue a *rod* (Stab). A rod is thus freely translatable only along this line – viz., its "carrier" – without losing its meaning.

b) We will understand the phrase *the moment of two rods a, b* to mean the product of their lengths, their shortest distance, and the sine of the angle of inclination between their carriers. The moment will then be zero when the carriers intersect (or are parallel) to each other. However, in order to be able to define the sign, we will arbitrarily assume that the line of shortest distance AB (where A lies on a, and B and b) has a positive direction, from which, a positive sense of rotation ([*]) will be fixed on the planes that are perpendicular to it, as well as the angle (a, b) (cf., § 8). We then understand the *moment of two rods*, when their measures are a and b, to mean:

$$M(a, b) = - a \cdot b \cdot AB \cdot \sin(a, b). \tag{15}$$

The basis for the minus sign will be made clear in Theorem 15. The moment is independent of how the positive direction was fixed in the shortest distance; the last two

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([*]) We consider a positive sense of rotation in a plane to mean the one that is opposite to clockwise when it is viewed from the positive side of its normal, and will thus agree with the sense of rotation of the Earth around its axis, when it is seen from the North Pole, and furthermore, with the sense of the motion of the Earth in the ecliptic, when it is viewed from the northern side of the ecliptic, which is how most civilized countries define a complete year. One now understands the angle (a, b) between two directions (i.e., half-rays) in a plane to mean the angle that is subtended when one rotates a half-ray from the direction a into the direction b in the positive sense of rotation; one will then have:

$$\measuredangle(b, a) = 2\pi - \measuredangle(a, b).$$

When only trigonometric functions come under consideration, one can set:

$$\measuredangle(b, a) = - \measuredangle(a, b),$$

and one will be dealing with angles between zero and 2π (excluding the upper limit). It is not always necessary to introduce a positive sense of rotation, or – what amounts to the same thing – to observe the sequence of legs in the angle, because many results will be independent of that; e.g., all of the ones into which only the cosine of the angle enters. We then speak of the *absolute* angle measure. It only needs to extend from zero to π (excluding both limits). We will get an example of its use shortly (Theorem 15).

factors will change sign under a change in that positive direction. On the same basis, one will have:

$$M(b, a) = M(a, b).$$

By comparison, if a rod is turned into the opposite direction then the moment would change its sign, because the angle (a, b) would change by π. Up to now, we have thought of the positive direction in the carrier of the rod by this very means, such that the numbers a, b were positive. Any other possible convention would have no effect on the moment. If the positive direction for the carrier of a changed then the sign of a, as well as that of $\sin(a, b)$, would change, too. From Fig. 11, one sees (*) that the moment of the rods a, b will remain positive as long as the sense of rotation that the rod b determines relative to a is positive (**). When we carry over the notion of the twist of two lines (§ 1) to rods, we can also deduce the last part of the following theorem from Fig. 11:

Theorem 15: *The moment of two rods agrees in sign with the sense of rotation that one rod determines relative to the other; the moment of two right-wound rods will be positive when they define an absolute acute angle.*

The remaining cases can be obtained from this one.

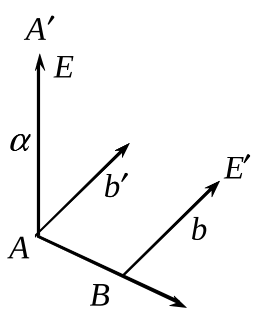

Figure 11.

c) By the term *the moment of two lines*, we mean the moment of two unit rods on these lines; in order for the sign to be determined, as well, the positive direction on the lines must be fixed. We intend the term *the moment of a rod relative to an axis* to mean the moment of the rod relative to a unit rod on the axis; by the term *the moment of a rod ST relative to a point P*, we mean the moment of the rod relative to an axis that lies through P and is perpendicular to the connecting line PST; it will thus be equal to twice the area of the triangle PST. These definitions are adapted naturally from the demands of mechanics.

d) We regard the volume of a tetrahedron $ABCD$ as being positive or negative according to whether the triangle ABC, when viewed from D, yields a positive or negative sense of traversal for an interior point, resp. (***) It will therefore change sign when one exchanges two neighboring symbols, or even two arbitrary ones (cf., Balzer,

(*) In Fig. 11, the line segment AB is thought of as going downwards to the right. Thus, $\measuredangle(a, b)$ will lie in the fourth quadrant when $AB > 0$. Thus, $M(a, b) > 0$. The rods will be right-wound.

(**) One thinks of there being a force in b that rotates a body around the axis a. According to whether the sense of this rotation is positive or negative, we will say that the rod b determines a positive or negative sense of rotation, resp., relative to the rod a. The sense of rotation that the rod a determines relative to b will have the same sign.

(***) *Möbius* first introduced (1827) a sign for a tetrahedral volume; cf., the Baryc. Calc., § 19 (Ges. W., Bd. I) and "Über die Bestimmung des Inh. eines Polyeders," § 18 (Ges. W., Bd. II). He took the sense of rotation of a clock hand to be positive, but regarded the surface BCD from A; nevertheless, for us and for *Möbius*, every tetrahedron $ABCD$ will have the same sign. The convention that is used here says, besides Theorem 16, that the tetrahedron $ABCD$ will be positive when A is the origin and B, C, D lie in the X, Y, and Z axes, resp., of a system of pointers of the first type.

Determin., § 1), and likewise when one cyclically permutes the four symbols. By comparison, one has:

$$ABCD = CDAB.$$

If one denotes the rods AB, CD by m, n and expresses the tetrahedral volume that they determine by concatenation then the sign will also be:

$$mn = nm.$$

If one displaces an edge of a tetrahedron along itself without changing its length (Fig. 12; $B'C' = BC$) then its volume will not change; therefore, either the height of D or the magnitude of the associated base surface will have changed. If one displaces two opposite sides AA', BB' of a tetrahedron (Fig. 11) until their starting points lie on the base points of their shortest distance then one will see that the six-fold tetrahedral volume $6V$ is equal (up to absolute value) to the moment of the rods AA', BB'. $6V$ is also equal to the volume of a parallelepiped whose base surface is the parallelogram that is determined by a, b' ($b' \nparallel b$ goes through A) and whose height is AB. However, this equality will also be true for the signs; from the definition of the sign of the tetrahedral volume it will then emerge that it agrees with the sense of the rotation that the rod AA' determines relative to the axis BB'. If we compare this with Theorem 15 then we will find:

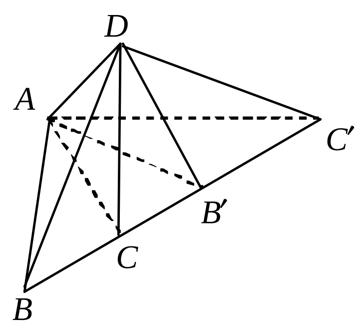

Figure 12.

Theorem 16: *The moment of the rods AA', BB' is also equal to six times the volume of the tetrahedron $AA'BB'$, including the sign.*

e) We can derive an analytical expression for the moment of two lines g_1, g_2 from this, when g_1 is given by one of its points $P_1 \equiv (\xi_1, \eta_1, \zeta_1)$ and its direction angles α_1, β_1, γ_1, and analogously g_2 is given by $P_2 \equiv (\xi_2, \eta_2, \zeta_2)$ and its direction angles α_2, β_2, γ_2. If two arbitrary rods P_1P_1' and P_2P_2' lie on the line then the six-fold tetrahedral volume will be given by (cf., Balzer, *Determin.*, § 15):

$$6P_1P_1'P_2P_2' = \begin{vmatrix} 1 & \xi_1 & \eta_1 & \zeta_1 \\ 1 & \xi_1' & \eta_1' & \zeta_1' \\ 1 & \xi_2 & \eta_2 & \zeta_2 \\ 1 & \xi_2' & \eta_2' & \zeta_2' \end{vmatrix}. \tag{16}$$

One can recognize the fact that this equation also has the correct sign most simply by a special case: If one assumes that P_1 is the origin and that P_1', P_2, P_2' lie along the X, Y, Z-axis, resp., of a rectangular system of the first type at a distance of one from the origin then $6P_1P_1'P_2P_2' = +1$; on the other hand, the determinant will also be:

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{vmatrix} = +1.$$

Since the tetrahedral volume does not change its sign as long as a vertex never intrudes on the plane of the other three during a motion of the vertices that preserves their sequence, and the determinant never vanishes under this assumption, as well, the agreement of the signs must always be true if it is true in a special case. (*) One will then have (Theorem 16):

$$M(P_1 P_1' P_2 P_2') = \begin{vmatrix} 1 & \xi_1 & \eta_1 & \zeta_1 \\ 0 & \xi_1' - \xi_1 & \eta_1' - \eta_1 & \zeta_1' - \zeta_1 \\ 0 & \xi_2 - \xi_2 & \eta_2 - \eta_1 & \zeta_2 - \zeta_1 \\ 0 & \xi_2' - \xi_2 & \eta_2' - \eta_2 & \zeta_2' - \zeta_2 \end{vmatrix}.$$

In particular, if $P_1 P_1'$ and $P_2 P_2'$ are two unit rods then one will have:

$$\xi_1' - \xi_1 = \cos \alpha_1, \qquad \xi_2' - \xi_2 = \cos \alpha_2, \qquad \text{etc.;}$$

thus:

$$M(g_1, g_2) = \begin{vmatrix} \xi_1 - \xi_2 & \cos \alpha_1 & \cos \alpha_1 \\ \eta_1 - \eta_2 & \cos \beta_1 & \cos \beta_2 \\ \zeta_1 - \zeta_2 & \cos \gamma_1 & \cos \gamma_2 \end{vmatrix}, \tag{17}$$

where $(\xi_1, \eta_1, \gamma_1)$ and $(\xi_2, \eta_2, \gamma_2)$ are any arbitrary points of g_1 and g_2, resp.

Practice problems:

1. What is the actual content of Theorem 7 for the case that was expressed explicitly in which g was the line at infinity of a plane that was parallel to the axis.

2. How does one calculate the null point of a given plane $A\xi + B\eta + C\zeta + D = 0$ with the help of equation (8)?

(*) One must then observe that a sequence of ones must be written in the first column (or row), while the rest of the rows (or columns) correspond to the sequence of vertices.

3. As a result of the properties of null systems, equation (8) cannot change when one rotates the pointer system around the *Z*-axis or displaces it along it; this is confirmed immediately.

4. Draw the figure that is analogous to Fig. 7 when $\mathfrak{k}$ is negative.

5. In connection with that, construct g' for a given position of g.

6. What happens to the manner of determination *b*) in § 10 when one chooses the direction of the axis to be perpendicular or parallel to the *g*?

7. In the manner of determination *e*) in § 10, it is immediate how one can construct the null plane to any point, and conversely, from *d*).

8. How can the manner of determination *a*) in § 10 be regarded as a special case of d)?

9. How does one find the axis in § 10, *d*) most quickly?

10. How does it help one when d, d_1 coincide in § 10, c)?

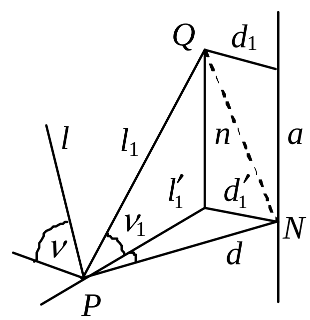

Figure 13.

11. If a is the axis (thought of as vertical) of a null system (Fig. 13), d is the shortest distance to a ray l of the twist, and ν is its inclination with respect to the horizontal plane then (l, d) will be the null plane of P. From equation (14), one will have the relation: $d \cdot \cos \nu = -\mathfrak{k}$. If d_1, ν_1 have the corresponding meanings for another ray l_1 of the twist that goes through P then one will also have $d_1 \cot \nu_1 = -\mathfrak{k}$. One will then have:

$$d_1 \cot \nu_1 = d \cdot \cot \nu.$$

This is verified immediately.

12. If one intersects a tetrahedron with a plane along a parallelogram (How?), and then displaces an edge that is parallel to E along itself (§ 12) then the parallelogram will displace congruent to itself in E; one can also infer the invariance of the volume of the tetrahedron from this (Cavalieri's principle).

13. The quotient of the cross-section of the screw cylinder and the height of the screw twist will be constant for all helices that are contained in a null system. (*Silldorf*, Zeitschr. f. Math. u. Phys., Bd. 20).

13a) Three points A, B, C and three planes α, β, γ that go through them, respectively, will determine a null system when the plane (A, B, C) goes through the point (α, β, γ).

Chapter II

Applications to the theory of motion, mechanics, and graphical statics.

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§ 13. Moments of forces.

If a rigid body is in motion around an axis *a*, and a force *k* is applied to it along a line of action that circles *a* in a perpendicular plane at a distance $AB = d$ (the notations and conventions are the same as in § 12) then, as is known (*) in the theory of rotations of bodies, the product *kd* will play a role that is analogous to that of force in the theory of moving mass points; it is called the *moment* of the force *k* with respect to the axis *a*. If a positive direction for *a* is fixed then a positive sense of rotation around *a* will also be given, and one will then call the moment positive or negative depending upon whether the rotation that the force brings about has a positive or negative sense, respectively. If one traces a unit line segment *AE* in the positive direction along *a* then the moment (as well as its sign) equals six times the volume of the tetrahedron $AEBB'$ (if $k = BB'$), because it will equal the volume of a parallelepiped with height *AE* and base $AB \cdot BB'$.

If *k* lies arbitrarily with respect to *a* then the only component of *k* that will come under consideration for the rotation around *a* will consist of its projection onto a plane that is perpendicular to *a*, and one understands the moment of *k* on *a* to mean the moment of this component. The volume of the tetrahedron will not change under projection since the height will remain unchanged when one considers *ABE* to be its base (Fig. 11). Thus, the moment of *k* relative to *a* will still be equal to $6AEBB'$. From this, in conjunction with theorem 16, it will follow that if *g* is the line of action of the force then one will have:

Theorem 17: *The moment of a force relative to an axis a is equal to either the moment of the force rod and a unit rod on a or the moment of a, g multiplied by the magnitude of the force.*

We understand the term "moment of a force system relative to *a*" to mean the algebraic sum of the moments of all of the individual forces relative to *a*. In particular, if the force system is a force couple then its moment relative to a normal to its plane is known to be independent of the location of this normal, and for that reason one will call it, by abuse of terminology, the moment of the force couple. It can be made more intuitive by drawing a line along such a normal whose length is equal to the magnitude of the moment and is directed in such a way that it makes the sign of the force couple positive when it points in the direction of the positive normal to the plane. It is known that since the plane of the force couple can be translated parallel to itself arbitrarily, the line segment that represents this moment will have the character of a vector (see the

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(*) In § 13, we will recall certain theorems of mechanics, although we will assume that the simplest ones are already known.

beginning of § 12). It is also known that moment vectors can be added geometrically, as well as the force rods whose carriers they intersect (viz., the parallelogram of forces). In the latter case, the resultant force relative to each axis in space also has the same moment as the system of components.

One understands the phrase "the moment of a force k relative to the point P" to mean the moment of k relative to a normal to the plane (P, k) that goes through P.

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§ 14. Normal form for a force system.

Let a force k be given, along with $l = AA' \perp k$ (Fig. 14) and $g \parallel k$, which goes through A'. If we let two opposite, but equal, forces k_1, $-k_1$ ($k_1 = k$) act at A' then the entire system will be equivalent to the single force k. k and $-k_1$ define a rotational moment whose vector m is directed under the plane; k, l, m thus lie (in alphabetical order) like the axes of a system of pointers of the first type. If we denote the lengths of these rods and vectors with the same symbols (*) then we will have $m = kl$; one can express this result as follows:

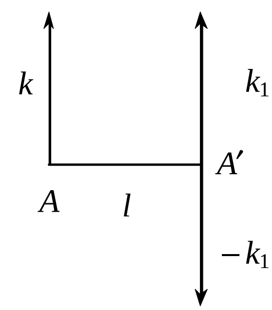

Figure 14.

Theorem 18: *One may displace a force rod k in a direction that is perpendicular to it when one adds a moment m of magnitude kl, such that k, l, m point like the axes of a pointer system of the first type.*

Since the force rod is displaced arbitrarily along its carrier, one will replace l with an arbitrary vector that points from A to any point of g; however, the special situation in the theorem is sufficient. The moment relative to an arbitrary axis does not change under the transformation of theorem 18; as an algebraic sum, it is then independent of the order and composition of the arrangement. Thus, the moment of a moment vector m relative to an axis a is equal to the component of m that falls along a.

If a force system is now given then we can "reduce" it at an arbitrary point P of space – i.e., all of its forces can act at P if one adds a suitable rotational moment, so all of the forces can be summarized as a single force and all of the rotational moments, as a single moment. Thus, if a force couple were present in the system then, with no further assumptions, we could displace its moment vector to P. The reduction to P is unique and the force system can now be represented by a single force rod k and a single moment vector M, which can likewise go through P. From Theorem 18, it now follows that:

Theorem 19: *The length and direction of k are independent of the location of the point P.*

(*) A misunderstanding can therefore not come about since we do not employ the Grassmann symbolism (except in § 16).

We ask whether one can further reduce at another point O in such a way that the plane of the rotational moment becomes perpendicular to the line of action of the force, so the carriers of k and M will coincide. We decompose M into two components $m_1 \perp k$ and $m \parallel k$ (Fig. 15). We then reduce at a point O that is at a distance l from the half of the normal to the plane (k, M) for which the angle k, m_1 appears to be 90° (not 270°). We must then (Theorem 18) add a moment vector m' that has a magnitude of kl, and is directed oppositely to m_1. Thus, if we choose $l = m_1 / k$ then m_1 will be directly cancelled by m', and what will remain after the reduction at O is only a force of magnitude k and the moment m. An arbitrary point of its new common carrier a (the "axis" of the force system) can play the role of O. However, aside from this, the reduction to this "normal form" for the force system is unique. The reduction from P to O is indeed unique; if one were to go from another point P to a point O' on a line a' then, from Theorem 19, one would at least have $a \parallel a'$, and a reduction of O to O' would not change the direction of the moment vectors, which, from Theorem 18, would be impossible.

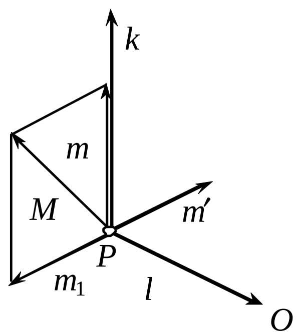

Figure 15.

We call the aggregate of a force and its rotational moment whose plane is perpendicular to the force a *dyname*. Due to its force part, it has the mobility of a rod that displaces along its carrier, and we do not regard it as different from one.

Theorem 20: *Any spatial force system may be reduced to a unique dyname.*

Naturally, it is not out of the question that the force or moment part of the dyname might vanish in some special cases.

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§ 15. Simplest form for a force system.

As such, we consider the forms in which the force system is the smallest aggregate possible, namely, two components that contain either a), a force and a moment, or b), two forces.

a) We now start with the dyname k, m with the carrier a, and displace (Fig. 16) k to k' along the line segment $OP = l$, while adding the moment $m' = kl$, which we compose with m at P into a single moment M. Since we can consider k, l, m' to be the axes of a reference system of the first type, the angle (m', k), as seen from P, will always be a right angle. If M defines an angle ϑ with the plane l, m' (for the angle convention, cf., § 8), moreover, then one will have:

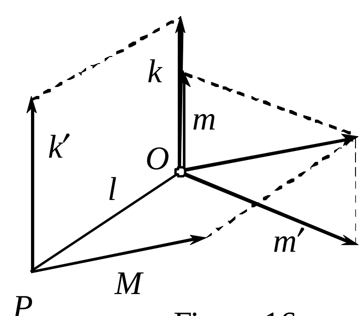

Figure 16.

1) $$\tan \vartheta = \frac{m}{m'} = \frac{m}{kl},$$

where m is positive or negative according to whether it has the same or the opposite direction to k, respectively. As a result of the orientation of our system of pointers itself, k and l will always be positive. If we compare equation 1) with equation (1.a) in Chapter I, § 2 then we will see that both equations will be identical, up to the sign of the distance along the axis, if we set:

2) $$\mathfrak{k} = \frac{m}{k}.$$

Thus, the direction of M agrees with the tangent of a screwing motion $\mathfrak{S}$ around a with parameter m / k. Since the construction preserves its validity under a displacement along a or a rotation around a this will be true for not only the point P, but also for each point in space. This screwing motion will have the same sense as that of the screw $\mathfrak{S}'$ that confers a homogeneous, coaxial, circular cylinder to the dyname. (*) Since the plane of the moment, and also the null plane of the screw, are perpendicular to M we can summarize the most important results and broaden them as follows:

Theorem 21: *If we reduce a dyname k, m to the form k′, M at all points of space in succession then a moment plane will be associated with each point P. This association will define a null system with the parameter* $\mathfrak{k} = m : k$. *The screw that belongs to this null system will have a left or right winding according to whether k, m has the same or opposite direction, resp. We then say that the screw itself, as well as the force system, has a right or left winding, resp. The magnitude of M is, by our reduction,* $\sqrt{m^2 + k^2 r^2}$, *where* $r = (P -\!| \, a)$. (**)

Since the vector M points into the first or fourth quadrant of the plane (M, k') according to whether the dyname has a left or right winding, respectively, one will obtain the intuitive rule:

Theorem 22: *The axis of a dyname* $\mathfrak{D}$ *lies on that side from which the angle* (k', M) *appears to be concave.* $\mathfrak{D}$ *is wound right or left according to whether k′, M subtends an absolutely* (***) *acute or obtuse angle, resp.*

We look for all axes a that go through a point P and relative to which, a given force system has a null moment. If we reduce the system at P, where it takes the form k', M,

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(*) Instead of the cylinder, one can take an arbitrary body whose principal axis of inertia is along a. However, one may believe that $\mathfrak{S}$ and $\mathfrak{S}'$ are identical. The motion that the body exhibits under the influence of the dyname will then depend upon not only the dyname, but also on the mass and mass distribution (i.e., moment of inertia) of the body.

(**) We employ the notation of Rohn and Papperitz (*Darst. Geom.*) for the distance from a point to a line.

(***) Cf., the first remark in § 12.

then k' will always have a null moment relative to an axis through P, but only when the associated moment plane E goes through a. The "null axis" through P thus defines the plane pencil (P, E). (*) From Theorems 21 and 14, and § 4, it now follows immediately that:

Theorem 23: *The null axes of a force system $\mathfrak{S}$ define a screw of lines that is wound oppositely to $\mathfrak{S}$.*

b) We call a system of two rods a *rod cross* (Stabkreuz), in the style of Buddes "vector cross (Vectorkreuz)" (mechanics), or, when forces are present on the rods, also a *force cross*. We now examine how a force system $\mathfrak{S}$, which we will think of as being given in the form of a dyname $\mathfrak{D}$, can be replaced with a force cross k_1, k_2. In order for the force cross to be equivalent to $\mathfrak{D}$, it must have the same moment as $\mathfrak{D}$ relative to every axis and a null moment about the same axes as $\mathfrak{D}$. Now, the null axes define a screw (the associated null system is denoted $\mathfrak{N}$). If a ray s cuts the aforementioned k_1 then the moment of k_1 relative to s will be null. Thus, the moment of k_2 relative to s must also be null; i.e., s must also intersect k_2. Thus (§ 5), in any case, we have to look for the carrier of a rod cross that is equivalent to k_1, k_2 only among the polar pairs of $\mathfrak{N}$.

We ask whether, conversely, each polar pair g_1, g_2 can be the carrier of such a cross of rods. We choose an arbitrary line g_1; let P be the foot on it of the shortest distance from the axis a of the force system, and let ε be the plane $\parallel a$ through g_1. If we reduce $\mathfrak{S}$ at P then k', as well as M, will lie in ε. The plane μ of the moment M will be perpendicular to the vector M, as well as to the plane ε that is depicted in Fig. 17, in which its trace σ is found. Now, one can arbitrarily choose the magnitude of the force of a couple, through which one would like to represent a rotational moment, and thus determine the distances to the lines of action. We draw one of the forces of the couple by which we would like to represent M through P and choose it to be large enough that its resultant with k' will fall along g_1. We will then find the force PT when we draw $QS \parallel \sigma$ and $ST \parallel k'$. We

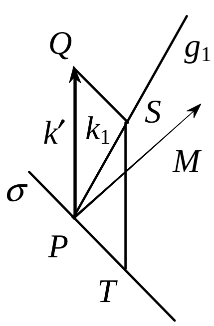

Figure 17.

(*) This explains the name "null system." Since the null axes that go through a point fill up a plane, Möbius (*Statik*, § 84; Ges. W., Bd. III), the discoverer of null systems, called this plane the "null plane" of the point, and conversely, the point was its "null point" (1837). Then, v. Staudt (*Geom. d. Lage*, Art. 321; 1847) called this arrangement a "null system." In recent times, Giorgini ("Intorno alle propr. geom.. dei movim. di una sist. di punti di forma invar." Mem. di mat. e di fis. della soc. it. delle sc. res. in Modena, Tomo XXI, submitted 1830, printed 1836) is occasionally called the discoverer of null systems. In the reference cited, he treated various problems in the composition of and decomposition of motions; however, he did not find the actual characteristic association between points and planes that constitutes a null system. In art. 32, he made some incorrect assertions about, for instance, the problem of decomposing a given motion along three axes being soluble (*in general*), while the conditions cease to apply for seven axes (cf., *loc. cit.*, § 85).

The present theorems on null systems were known to Möbius, for the most part; in addition to *der Statik*, cf. also "Über eine bes. Art dualer Verh…" Journ. f. Math., Bd. 10 (1833).

have now arrived at the fact that $\mathfrak{S}$ can be replaced with a force $k_1 = PS$ that points in the direction g_1 and the other force k_2 of a force couple that, from what was said earlier, must lie along g_2 .

The construction breaks down only: First, when g_1 coincides with k' (then the lever arm of the force couple would then become infinite and one would be in case *a*), and second, when g_1 coincides with σ. g_1 would then become the guide line of $\mathfrak{N}$, so g_2 would coincide with g_1, and it would therefore be conceivable that the problem would no longer be soluble. We would like to make it more intuitive how the rod cross appears when g_1 rotates around P and comes into the vicinity of the position σ. Then, from § 8, g_2 would likewise move in the vicinity of the position σ, and PT would simultaneously grow unbounded. The two rods of the cross would thus become oppositely directed while their carriers would move ever closer, and both of them would become infinitely large. Once we know that a polar pair g_1, g_2 (if g_1 and g_2 are different and both of them lie at finite points) can always be carrier of an equivalent force cross in a real sense, we can find the rods of the cross in a simpler way when we let k' point in the directions g_1, g_2 . We summarize:

Theorem 24: *A given dyname k, m can be replaced by a force cross in* ∞^4 *ways. One can choose the line of action* g_1 *of the one force (excluding the guide lines and diameter of the associated null system* $\mathfrak{N}$*) arbitrarily, while the other one is its polar* g_2 *in* $\mathfrak{N}$*. One obtains the magnitude and sense of both forces by decomposing the vector k in the directions* g_1, g_2 .

Conversely, if a force cross k_1, k_2 is given then we would like to find the equivalent dyname *k*, *m* in the simplest way. Indeed, one can also apply the general process of § 14 here, and one then clearly arrives at the point: When we add the vectors k_1, k_2 geometrically (PC # QB), we will obtain the magnitude and direction of k in k' (Fig. 18), from which we will know, in addition (Theorem 10), that it intersects the shortest path PQ to the rods of the cross perpendicularly. One must then treat only the determination of this intersection point S. From Chap. I, equation (13), one must have:

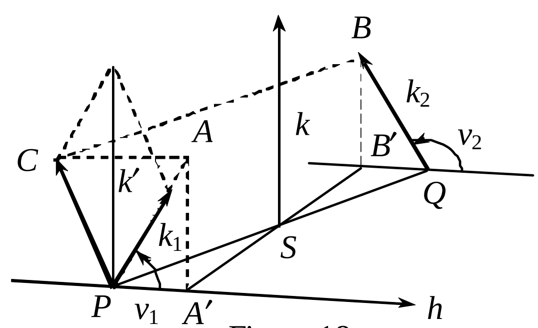

Figure 18

$$SP : SQ = \tan v_2 : \tan v_1 = \cot v_1 : \cot v_2 ,$$

where the angles v_1, v_2 are measured from one of the two directions h, as well as k', which is perpendicular to PQ; they lie in the normal plane $\mathfrak{H}$ of k', which we would like to think of as horizontal. We thus choose two points A and B on k_1 and k_2, resp., by the construction of S, which are at the same level above $\mathfrak{H}$ ($CA \parallel h$), project them onto $\mathfrak{H}$ from A' and B', and let PQ intersect $A'B'$ at S. Space will be divided into two halves by

$\mathfrak{H}$, and likewise by the connecting plane $\mathfrak{B}$ of k and k'. k_1 and k_2 will always point to different half-spaces relative to the division by $\mathfrak{B}$, but, relative to the division by $\mathfrak{H}$, they can point to the same or different half-spaces. In the latter case, PA' and QB' will be in the same half-space relative to $\mathfrak{B}$, and S will lie outside of PQ. Since two opposite corners of a parallelogram have the same distance from a diagonal, the projections of k_1 and k_2 onto $\mathfrak{H}$ will be equal and opposite, and will thus define the moment m of a dyname, and it will follow that:

Theorem 25: *If one adds the vectors of a given force cross k_1, k_2 geometrically then one will obtain a direction k. The rod of the equivalent dyname $\mathfrak{D}$ will be found by composing the components of k_1, k_2 that lie in the direction of k, while the components that lie in the normal plane of k will define the rotational moment of $\mathfrak{D}$.*

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§ 16. A theorem of Chasles.

If $k_1 = KL$, k_2, and k_1', k_2' are two equivalent force crosses then they will have the same moment relative to any axis. If $e = EE'$ is a unit rod on A then the moment of k_1 relative to A will be equal to six times the volume of the tetrahedron $EE'KL$. We can also denote this volume briefly by ek_1, where we thus think of each rod symbol as being replaced with the two symbols of its endpoints. We can now express the facts above by the equation:

$$ek_1 + ek_2 = ek_1' + ek_2',$$

which will, however, also be true when we replace e by the symbol of an arbitrary rod a on A, because that will come down to the same thing as multiplication by a numerical factor. A, and therefore a, as well, can be chosen arbitrarily. If we now let a coincide with each of the four rods k in sequence then each tetrahedron will always vanish, and we will obtain the equations:

$$k_1\,k_2 = k_1k_1' + k_1k_2', \qquad k_2\,k_1 = k_2k_1' + k_2k_2',$$
$$k_1k_1' + k_2k_1' = k_1'k_2', \qquad k_1k_2' + k_2k_2' = k_2'k_1'.$$

By adding these four equations, one will obtain:

$$k_1\,k_2 = k_1'k_2'.$$

For two arbitrary s, s', one will always have, in fact:

$$ss' = s's,$$

because the volume of a tetrahedron *ABCD* does not change in sign when one switches the pairs *AB* and *CD* as wholes (§ 12, d). Thus:

Theorem 26: *The volume of the tetrahedron that is determined by a force cross is the same for all equivalent force crosses.*

The symbolism that is employed here agrees, in fact, with that of Grassmann; the proof goes back to Möbius [Ges. W., Bd. III, pp. 503 or Crelles J., Bd. 4 (1829), pp. 179, *et seq*.].

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§ 17. Composition of velocities.

If a rigid body moves then at a definite time point each point P will have a certain velocity that can be visualized by means of a line segment through P. One calls such a system of ∞^3 line segments, each of which is applied to a definite point, a *vector field*. (*) The velocity distribution of the points of a rigid body is thus represented by a vector field, in such a way that we can think of it as being extended over the entire infinite space, because we can think of each point of space as being tightly coupled with the rigid body (§ 1).

One now cares to speak of the connection between velocities and motion in a rigid body in two senses of the term:

a) A rigid body K_1 moves in a space that is thought of being as at rest (e.g., the Earth in the universe when the Sun is thought of as fixed); this motion corresponds to a vector field V_1. A second rigid body K_2 moves relative to K_1 (e.g., a projectile on the Earth). This motion corresponds (when one thinks of the Earth as at rest) to the vector field V_2. The motion of K_2 in a space at rest will then correspond to a vector field V. It will be found when one adds the vector fields V_1 and V_2 geometrically at each individual point of the line segment. One calls these operations the *composition* of the vector fields V_1 and V_2. Since K_2 also moves as a rigid body relative to the space at rest, it clear from the outset that under composition a vector field will again come about, whose velocity distribution will correspond to a possible motion of a rigid body.

b) A rigid motion can perform various motions in a rest space. We attach two motions to each time point and compose the two vector fields V_1, V_2 that correspond to these time points into a field V. In this way of looking at things (**), it is not clear from

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(*) The name is not entirely appropriate, insofar as we think of the word "vector" as meaning a line segment that is free to move in all of space while preserving its direction, while the line segments of the vector field differ from the rods by the fact that that their initial points are actually also fixed. The terminology may thus be all the more easily retained, as there are only ∞^3 vectors in space to begin with (∞^2 directions and ∞^1 lengths in each of them), so one hardly has any reason to speak of a manifold of ∞^3 vectors in a real sense.

(**) It finds its place in physics (in general, mostly for accelerations, instead of velocities) where one knows the motions that the body individually takes on as a result of two simultaneously acting causes. Here, we have only a rigid body (e.g., a freely falling magnet) and always the same reference body (the Earth). One knows the accelerations that the magnet assumes as a result of the weight alone and the magnetic field of the Earth alone. By contrast, this way of looking at things will also enter into velocities the outset that *V* represents such a velocity distribution that corresponds to a possible motion of a rigid body. However, since we already know this from *a*), it will also follow here; the operation of composition itself does differ at all in the two cases. On this basis, it not always necessary to specify from which viewpoint a definite composition of vector fields originated.

In special cases, a vector field is capable of a simpler geometric representation: If a body rotates around an axis *a* with the angular velocity ω then this state will be characterized completely when one applies the segment ω to *a* in the direction that appears to be positive when the positive side of *a* is assumed to have the given sense of rotation. The line segment will thus have the character of a rod, and will replace the vector field that corresponds to the rotation for us; we will call it a *rotating rod*, and *a* will be its *carrier*. Angular velocities whose axes intersect will, it is known, be composed by adding the corresponding rotational moment geometrically.

If a body executes a displacement (i.e., a translation) then this state can be characterized by a displacement vector that replaces the vector field that corresponds to the displacement. The composition of displacements will result from the geometric addition of the corresponding vectors.

§ 18. Duality between forces and velocities.

Two rotational rods whose carriers are parallel, and whose line segments are thus equal and opposite (a rotational couple) are, it is known ([*]), equivalent to a displacement vector whose magnitude and direction are determined by the rods in the same way as the moment vector when the rods refer to a force couple. For that reason, the construction that led up to Theorem 18 will also be valid for the motions that result when one replaces each force rod with a rotation rod and each moment vector with a displacement vector. In the case of motions, Theorem 18 corresponds to:

Theorem 27: *One may perform a displacement l of a rotation rod d in a direction that is perpendicular to it if one adds a displacement vector s of magnitude ld such that d, l, s are directed like the axes of a pointer system of the first type.* ([**])

Everything up to now has pointed to the following complete analogy between the forces on a rigid body and velocities, which one might care to refer to as a duality between forces and velocities: Forces and rotational velocities have the character of rods and, as such, when their carriers intersect, they will be added geometrically. Rotational moments and displacement velocities have the character of vectors and, as such, will be

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when one treats an impulsive force. Thus, one must remark that the two vector fields *V* that one obtains for accelerations by analogous definitions in the cases a) and b) are no longer identical (Coriolis theorem).

([*]) Cf., say, Schell, *Th. d. Beweg. u. d. Kräfte*, *I.*, 2nd ed., pp. 168.

([**]) One remarks that the alphabetical sequence of symbols *k*, *l*, *m* in Theorem 18 is also preserved here for *d*, *l*, *s*.

added geometrically. A force couple is equivalent to a rotational moment, and a rotational couple is equivalent to a displacement velocity. From the rule of Theorem 18, forces can be displaced, and from Theorem 27, rotations can be displaced. (The vector l is separate from the duality.)

One can therefore immediately carry over those theorems and constructions about force systems that are derived from merely the aforementioned theorems and purely geometrical properties of the null system to the composition and decomposition of velocities when one considers the rods and vectors that enter into the theorems and constructions as having a different interpretation, namely: The rods become rotational rods, and the vectors become displacement vectors. We thus state the following theorem, whose meaning we will explain directly in detail:

Theorem 28: *In the theorems on the composition and decomposition of forces and velocities for rigid bodies, one can equivalently exchange any two of the juxtaposed concepts:*

Force	Rotational velocity
Rotational moment (Force couple)	Displacement velocity.

Thus, the rotational velocity always belongs to a definite axis. One must observe that the assumed properties of forces are true only for rigid bodies; for that reason, Budde aptly said that duality itself is a property of rigid bodies. Since the dislocations of a definite starting time t that begin the velocities at the time t become increasingly more precise as the time increment grows smaller, the theorems on the composition of velocities will also become more precise for the composition of smaller motions; for that reason, one often speaks of a "composition of infinitely-small motions," which one more precisely refers to as *the composition of velocities*, when one thinks of the transition from difference quotients to differential ones as having been performed.

We call a screw motion that proceeds with a definite velocity a *winding*. By duality, from Theorem 28, it will correspond to a dyname. A winding (τ, ω), as well as a dyname (m, k), belongs to a certain null system $\mathfrak{N}$ (§ 2 and 15). However, conversely, infinitely many windings and dynames will correspond to the same $\mathfrak{N}$, which come about by multiplying both components of one of them by a numerical factor. In both cases, the parameter $\mathfrak{k}$ of $\mathfrak{N}$ depends only upon the ratio $\tau : \omega\,(m : k)$ (§ 1 and 15). (*)

For many investigations of dynames and windings, only the null systems that they define come under consideration. Once again, we can think of this as being completely represented by a helix on a cylinder of radius one and pitch tan $\vartheta = \mathfrak{k}$ (§ 1). We casually refer to such helices on a cylinder of radius one as *screws*. (**) Null systems and screws are oriented equally and oppositely to each other. With Ball (*Theory of Screws*), we say that a winding or a dyname "lies on a screw," or also, "the screw is the carrier of a winding or a dyname," when the null system of screw is identical with the null system of

(*) In order to give this the correct sense, one draws one's attention to the fact that $\mathfrak{k}$ is a linear quantity.

(**) An extension of this concept will be presented in § 36.

the dyname or that of the winding. Only the winding, but not the dyname, has an immediate intuitive connection with the screw on which it lies. Above all, it must be emphasized that a dyname and a winding, even when they lie on the same screw, have no causal connection in and of themselves (cf., the first remark in § 15), but it is only under their composition with others of their kind that they play the same role and correspond to each other under duality.

We can now also determine a winding by its parameter (viz., its pitch) $\mathfrak{k}$, its axis, and its velocity ω, which we will causally refer to as not only the velocity of the rotational part, but also the velocity of the winding. One will then have:

$$\tau = \mathfrak{k}\ \omega.$$

Analogously, one can determine a dyname $\mathfrak{k}$, its axis, and its "intensity" k. One will then have:

$$m = \mathfrak{k}\ k.$$

We need the expression "pitch" for $\mathfrak{k}$ (in Ball's terminology), and, by way of analogy, also for dynames, regardless of whether it has no obvious mechanical meaning here. For $k = 0$, one has the "unit dyname," and for $\omega = 1$, one has the "unit winding."

Poinsot (*Theor. nouv. de la rot. des corps*, 1834) first suggested the duality between forces and rotational velocities.

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§ 19. Psychological remarks.

As far as the theory of cognition is concerned, the composition of velocities is simpler than that of forces because velocity is a purely geometrical concept, while force is a physical notion. Nevertheless, the theory of the composition of forces is more intuitive and generally simpler in all of the other regards than the corresponding one for velocities: First of all, force, whether a compression or a tension, takes on a form that is just as tangible and perceptible in its manifestation (*) as motion, although it is a notion in which a process of differentiation does not seem intrinsic (as it is with velocity). Furthermore, several forces can act on a rigid body simultaneously (e.g., in the form of elastic strings or coil springs in a definite state of stress), while a body cannot have several velocity states at the same time; moreover, a decomposition of this state is merely a way of imagining things. Finally, the conceptualization of a force system in its simplest form is, so to speak, exhaustible (two forces for a force cross, three, for a force along with a force couple), while an individual rotating rod already represents infinitely many points of the body, and will substitute for the entire infinite vector field. Thus, a force rod represents the many rods that immediately follow as a rotating rod. On this basis, we have presented the theory of forces as the simplest one.

(*) It is neither desirable nor possible to extend this tangible manifestation to the foundation of mechanics.

§ 20. The instantaneous axis.

The position of a body *K* in space is determined completely by the position of three of its points that define a real triangle, because the position of a plane in the body is determined by them. If we first assume a point *M* of *K* to be fixed then we can choose the other two *A*, *B* to be points of a spherical surface that has *M* for its center. The position of *K* will then be determined by the position of *A*, *B* alone, which can go to a nearby position *A′*, *B′* under a motion. One can arrive at the same dislocation when one first takes *A* to *A′* along the largest circle of a rotation, for which *B* must go to *B′* by a rotation around *MA′*, and one can think of the transition as be so contrived that one tightly couples *MA′* with a second rigid body *K′*, which experiences only the first rotation relative to the space *R*, which is thought of being at rest. Now, the motions of *K′* relative to *R* and of *K* relative to *K′* can also be performed simultaneously in many ways; e.g., uniformly. If we pass to the limit in which we let the neighboring position *A′B′* go back to *AB* then at the time point in question the motion of *K′* relative to *R* will be characterized by a rotating rod that is perpendicular to *MA* and the motion of *K* relative to *K′*, by a rotating rod whose carrier coincides with *MA*. The vector field of the actual motion of *K* will come about (§ 17a) by composing the two fields that correspond to the rotations; however, we know that this will again give a rotation. Thus:

Theorem 29: *Under a motion of a rigid body, a point of which is fixed, all points will have the same velocities as when the body rotates around a definite axis with a definite velocity.*

We now consider a general motion of a body *K* under which three points *A*, *B*, *C* might correspond to the nearby position *A′B′C′*. We can then think of the dislocation as being so contrived that we can think of the point *C* of *K* as being tightly coupled with another body *K′* that merely experiences a displacement along the vector *CC′*, while *K* moves freely around *C* relative to *K′*. By a passage to the limit that is analogous the one before, we recognize that the vector field for the actual motion of *K* will consist of the composition of two vector fields, one of which will correspond to a displacement velocity τ' whose magnitude and direction coincides with the velocity of *C*, while the other one, to the rotation ω' that *K* experiences relative to *K′* (Theorem 29). If the motions of *K′* relative to *R* and of *K* relative to *K′* are assumed to be arbitrary then what will always result is a possible motion of *K*. Thus, τ' and ω' will bee *independent of each other*, in and of themselves, but under a definite motion of *K* they will also be determined completely by a choice of the point *C*. We thus arrive at the fact that the velocity vector field of a rigid body at a definite time point is known by the aggregate of a rotating rod ω' and a displacement vector τ', and this is the point at which the fact of the duality between forces and velocities can be of value: We can reduce τ' and ω' to an arbitrary point of space (exactly like a system of forces in § 14), corresponding to the arbitrary choice of *C*. We choose the reduction point in such a way that a displacement vector τ with the same direction as the rod ω' will appear in place of τ' (viz., the analogue of the dyname), where

ω' is displaced parallel to ω. τ and ω then define a winding (§ 18) with the parameter $\mathfrak{k}$ = τ / ω (§ 1).

Theorem 30: *Under an arbitrary motion of a rigid body, at each moment, all of its points will have the same* velocity *as if the body had experienced a definite winding with a definite velocity. The axis of this winding is called the* "instantaneous axis," *and its parameter is called the* "instantaneous parameter" *or the* "instantaneous pitch." *The velocity vector field at that instant is determined completely, up to a numerical factor, by the instantaneous axis and instantaneous parameter, which together determine the* "instantaneous winding."

It is not out of the question that τ or ω might vanish. In the former case, a pure rotation would appear in place of the winding, while in the latter case, it would be a pure translation; one can regard the latter as if the instantaneous axis were the line at infinity of all planes that are perpendicular to the displacement. τ and ω can also vanish simultaneously; at such a time point, all points of the body will have a zero velocity, and the instantaneous axis will either be completely indeterminate or will be defined to be the limiting position of the neighboring instantaneous axes. If, e.g., a body begins a uniform, accelerated winding at the time *t* then one will also obtain the axis of the winding at *t* as the instantaneous axis in this way.

The discoverer of the instantaneous axis was Giulio Mozzi (*Discorso mat. sopra il rotamento mom. dei corpi, Napoli*, 1763). (*)

§ 21. The decomposition of a winding in its simplest form.

We can connect this section immediately with § 18; now, however, we know from Theorem 30 that the results that we derived for the velocity vector field of a winding will remain valid for each instant of an arbitrary motion.

We next carry over the most important theorems of § 15 on the law of duality to the case of a winding. If $\mathfrak{k} = \tau : \omega$ is the pitch of the winding then one can displace the rotating rod ω of the winding parallel to itself in an arbitrary direction that is perpendicular to it along the line segment *r*, when one replaces the displacement vector τ by another one *T* of absolute value $\sqrt{\tau^2 + \omega^2 r^2}$ that lies in the same way as the vector *M* of § 15a). Furthermore:

Theorem 31: *A winding* (τ, ω) *can be represented in* ∞^4 *ways as the resultant of two rotational velocities. One can choose the axis g of one rotation arbitrarily (except for the*

(*) The following simple consequence of corollary IX in this paper is still worthy of mention: When arbitrarily many forces are given, one can always find two of them that are equivalent, one of which lies in an arbitrary plane *E* and the other of which is perpendicular to it. One can then decompose any force into a component in *E* and one that is perpendicular to *E*.

guide lines and diameter of the null system $\mathfrak{N}$ that is associated with the winding). The other axis is the polar g' of g in $\mathfrak{N}$. One obtains the angular velocity of both rotations by decomposing ω in the directions g, g'. Conversely, the construction in Fig. 18 that served to define the composition of two rotations will remain unchanged.

Thus, we must dispose of an objection: The derivation of Theorem 24 made use of the concept of moment of a force system relative to an axis, which has no natural analogue in the theory of motion. Is one therefore justified in transferring the results in the name of duality? One knows that if one replaces a result on rotating rods with one on force rods and the displacement vectors with moment vectors, and then performs the reduction at a point of the axis then the same dyname will come about. Thus, while preserving the original meaning, the given winding must result from the same reduction, because the theorems on composition and decomposition of rods and vectors in § 18, especially Theorem 27, are independent of whether one interprets them as forces or velocities. By this argument, one can once more subsequently convince oneself that, in fact, all of the results of § 15 can be transferred to the realm of the study of motion, even if the derivation of these results cannot always be immediately transferred.

As an application of Theorem 31, we treat a question that has no analogue in the theory of forces: Under an arbitrary motion, a plane E in a rigid body determines an intersection line g_1 with a neighboring position. One seeks the limiting position of g_1 (relative to the composition of the neighboring lines with the original one), or the "characteristic" (for Chasles) of the plane E. If E is perpendicular to the instantaneous axis a then the characteristic will be the line at infinity of E; if $E \parallel a$ then it will be any parallel to a that lies in E and has the shortest distance to a. We now exclude these special cases.

An instantaneous axis and an instantaneous parameter determine an instantaneous winding and an associated null system. In order to get a glimpse of the distribution of velocity vectors at all of the points of E, we erect a normal g' to E at the null point N. The polar g of g' lies in E (Theorem 12). If we replace the instantaneous winding with two rotational velocities that have axes g, g' then the rotation g will confer each point of E with a velocity component that is perpendicular to E and the rotation g' will confer one that lies in E. Only for the points of g does the first component vanish, while the second component vanishes for N. One can now (§ 17) think of the winding as being performed (Fig. 19) in such a way that E rotates around g, while g simultaneously rotates around g' and thus comes into the neighboring position g_1', which also lies in E. The moving line g_1' will constantly be the intersection of E with its neighboring position. For that reason, the limiting position g of g_1' will be the characteristic of the plane. Thus:

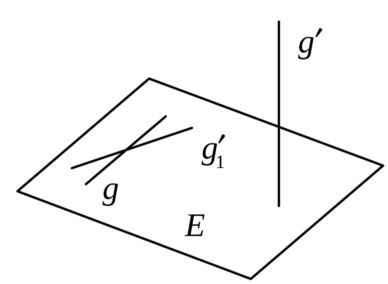

Figure 19.

Theorem 32: *Under a winding, the null point N of a plane E will be the only point of E whose velocity is perpendicular to E. The location of the points of E whose velocity* *falls in E is a line g that is the polar to the normal to E at N; g is likewise the characteristic of E.*

In order to learn more about the position of g we set:

$$\tan \nu = -\cot \nu'$$

in equation (12) of § 8 (since the angle itself differs from 90°) and obtain:

$$c\,c' = -k^2,$$

i.e.:

Theorem 33: *The null point and characteristic of a plane lie on opposite sides of the axis a at distances from a such that parameter of the null system is proportional to the geometric mean of the absolute values of these distances.*

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§ 22. The instantaneous winding of a body, five points of which are constrained to remain on five surfaces.

If a rigid, planar system moves in its own plane then the motion will be determined completely when the paths are prescribed for two points, since the motion of the connecting line segment will then be determined, and this will direct the entire system uniquely. Analogously, we can prescribe the motion of a rigid body by a number of its points $P_1, P_2, \ldots, P_k$ and surfaces $F_1, F_2, \ldots, F_k$, upon which they must remain during the entire motion. It will be shown that for $k = 5$ the motion of the body is determined uniquely, in general. Whether we do or do not know the direction that P_i is given at the beginning of the motion in F_i, we do still know that the normal n_i to the surface F_i at P_i must also be the path normal for P_i, and thus the guide line of the null system that is linked with the instantaneous winding. Therefore, if a twist is determined completely by the five rays $n_1, \ldots, n_5$ then it will also be the instantaneous winding of the motion that the body can exhibit from the given position, and we pose the problem: *Determine the axis and the parameter of a twist when one is given five rays* $n_1, \ldots, n_5$.

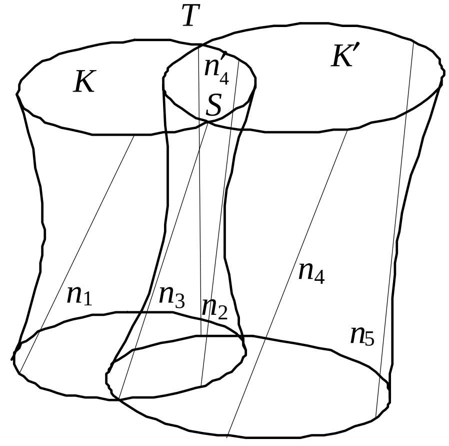

Figure 20.

We determine the two common transversals g, g' of $n_1, \ldots, n_4$, which, from Theorem 8, must be polar to each other in the twist. From § 10, d), the twist will then be determined; one constructs g, g' when one intersects the hyperboloid H that is determined by n_1, n_2, n_3 with n_4. g, g' go through the intersection points as the rays of the guiding family of n_1, n_2, n_3. If the intersection points are not real then one can enforce a real construction of the following kind (Fig. 20): One considers the hyperboloid $(n_3, n_4, n_5) \equiv H'$, in addition to H. If one cuts both of them with a plane E then one will get two conic sections K, K' in them that have the point $(E, n_3) \equiv S$ in common, and thus, at least one other point T. A ray n_4' of the family of rulings (n_3, n_4, n_5) will go through T, which, from Theorem 11, will also be a ray of the twist. If we replace n_4 with n_4' then g, g' will become real. We provide ourselves with yet a second polar pair h, h', with the use of another quadruple from the five rays n and then construct the instantaneous axis a from Theorem 10. The path tangent t_i at P_i is that tangent to F_i at P_i that is perpendicular to the intersection of F_i with the null plane of P_i. Now, from § 1, the instantaneous winding is determined by the axis and one of the tangents t_i; therefore:

Theorem 34: *If five points of a rigid body are constrained to move on a surface then the instantaneous winding that the body can execute will be determined uniquely (up to its velocities) by that data in all cases in which a twist is determined uniquely by five surface normals.*

If P_1 and P_2 coincide (but not F_1 and F_2), and likewise P_3 and P_4, then one will be dealing with the case in which two points of a body are constrained to stay on two curves (viz., the intersection curves of some pair of surfaces), and a third point is constrained to remain on a surface, as well. Here, determining the instantaneous winding will be a special case of the problem that was just solved (which we will find a *linear* solution for in § 47, *d*), moreover), which is why not all of the special cases that were present were considered. Later on (§ 70), we will see that the absence of intersection points does not affect the solution.

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§ 23. Planar frameworks and associated reciprocal force planes.

We imagine ourselves as being in the plane of a system of material rods that are linked at their ends in such a way that the system can move only as a whole. For the sake of simplicity, we further assume that the rods do not cross anywhere, so the polygon in the plane that they define is simply covered. Such systems of rods are special cases of "planar frameworks," and come about with iron bridges, roofs, etc. If the external conditions (e.g., the loads on the bridge, the wind pressure on the roof, etc.) are given then in order to construct these objects one must know the effect of tension or compression on each individual rod in order to be able to determine the rigidity that one gives it. The points at which several rods join together are called *nodes.* One now makes the assumption that the external forces act only on the nodes of the framework; i.e., that they can be distributed over the nodes (viz., a "statically-determinate" framework) according to the laws of statics (including the dead weight of the rods), resp. One further considers only the tensile or compressive stresses in the rods (but not the effects of shear or torsion), as if the rods were linked with each other at the nodes in an articulated way. This arrangement has proved to be adequate for the applications, when it also does not deviate too far from reality, since the links are usually defined by numerous rivets.

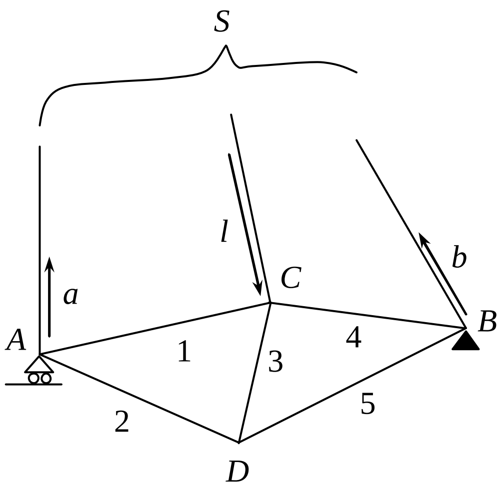

Figure 21.

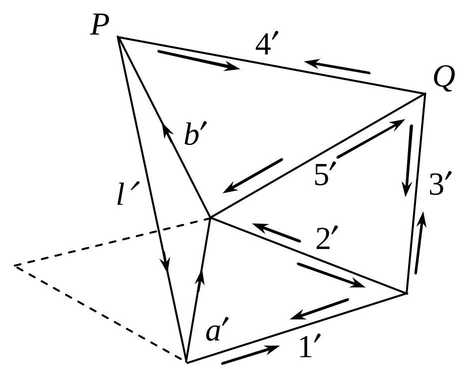

Figure 22.

We shall show that one can average out the stresses in a planar framework by the simplest possible example: Let *A*, *B*, *C*, *D* (Fig. 21) be the nodes of a framework that consists of five rods 1, …, 5, which represents a carrier of the resistance that the pressure *l* on the node *C*, which is given in magnitude and direction, must experience. The carrier itself is supported by a fixed support at *B* and a moving one at *A* in order to be able to yield to temperature effects. If, for the sake of simplicity, we ignore the dead weight of the rods then *l* will provoke certain support reactions *a* and *b* at *A* and *B*, resp., where *a* must be normal to the base of the moving support; if we ignore the friction then moving surfaces can support only a pressure that is normal to each of them. Since the external forces *a*, *l*, *b* that act on the carrier are in equilibrium, their line of action must go through a point *S* that is determined by the known lines of action of *a* and *l*. In order to find the magnitudes of *a* and *b*, one imagines that arbitrarily many forces that go through a point are in equilibrium if and only if their vectors form a *closed* force polygon when added together according to the rules of geometric addition. One then draws (Fig. 22) $l' \parallel l$, $b' \parallel b$, and $a' \parallel a$ through the endpoints of that segment; one thus obtains the lengths and senses of *a* and *b*, since the sense of traversal over the force polygon is given by the sense of l'. In order to determine the stresses in the rods 1 and 2, we think of the node *A* as being divided by a cut that meets the rods 1 and 2 (Fig. 23). In order for *A* to remain in equilibrium, we must replace the stresses in the rods with forces that are directed towards *A* or away from *A* according to whether tension or compression prevails on the rod in question. We experience the sense and magnitude of these forces when we draw a triangle whose sides a', $1'$, $2'$ are parallel to the directions *a*, 1, 2, while a' also agrees with *a* in magnitude. One can likewise connect this with Fig. 22, in order to not repeat anything. The arrow at a' gives a sense of traversal that the arrows inside the triangle also follow. This therefore

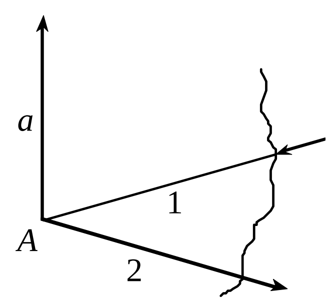

Figure 23.

gives the sense of the forces that must be applied to the intersection surface in order to replace the actual stresses. On then sees that 1 is in a state of tension and 2 is compressed. Now, one can go on to the rod stresses at a neighboring node: A cut through 2, 3, 5 divides *D*. The stress in a rod through this node is known already. One can then find the other two stresses by the methods that were just applied when one connects the triangle 2′, 3′, 5′ to the two triangles that were referred to already in Fig. 22. The arrows inside of this triangle will again give the sense of the forces in the intersection surface. Therefore, the arrows on both sides of 2′ must naturally have the opposite sense, since the two triangles to which 2′ belongs refer to two nodes that lie at the ends of 2, and both of them are drawn through the middle of the rod, and thus, in opposite directions. A sense of traversal is also determined in the triangle 2′, 3′, 5′ by 2′, which establishes the direction of the other two arrows. One then sees that 3 is now in a state of tension and 5 is compressed. One then goes on to even newer neighboring nodes. In our example, however, the determination of the stress in rod 4, which we can consider to belong to *B*, is now unnecessary. We then connect *b*′ with the triangle *b*′, 5′, 4′, of which, the side 5′ has already been referred to; since the triangle must close when we draw the parallel to 4 through *P*, the connecting line *PQ* must be parallel to 4, in its own right, which will give one a check.

One can arrange the arrows on both sides of each rod in such a way that one can learn from the force plane by itself, regardless of whether the rod in question is in a state of tension or compression; in the former case, one draws the arrows in such a way that points are closer to each other, and in the latter case, the origins are closer.

Fig. 22 defines a *force plane*; i.e., a schema of lines that represent the stresses that act in the rods of the framework in length and direction. One sees that we have succeeded here in arranging a force plane in such a way that each rod is referred to a parallel regardless of whether it is met by two nodes. Any node of the framework corresponds to a closed polygon of the force plane; this is also true at *C*, where the lines *l*, 1, 3, 4 meet. In fact, *l*′, 1′, 3′, 4′ define a closed rectangle. Since the node *C* was not employed in its construction, one cannot force this situation when it does not occur in its own right. The sense of traversal that delivers the force through *C* will be denoted by arrows on the external boundary of this rectangle. Naturally, it is only in the triangle that corresponds to the external forces that the arrows are denoted (along its sides) in the actual sense that correspond to these forces. However, conversely, the lines that define a polygon in the framework (e.g., *a*, 2, 5, *b*) also correspond to lines in the framework that go through a point. Due to these properties, the force plane is called *reciprocal*.

If one wishes to arrive at a reciprocal force plane then the process of construction will be determined uniquely as long as the first triangle is denoted by *a*′, *b*′, *l*′. Indeed, one can just as well employ the two dotted lines in place of the lines 1′ and 2′ when one is merely dealing with ascertaining the stresses in the rods 1 and 2; however, since *a*, *l*, 1 define a triangle in Fig. 21, *a*′, *l*′, 1′ must go through a point. It is thus no longer in doubt through which endpoint of *a*′ one has to draw the parallel to 1. The possibility of reciprocal force planes is in no way obvious; the theory of null systems is conducive to the proof of its existence for certain kinds of frameworks. For that, we must therefore first fall back on the theory of polyhedra.

§ 24. From the theory of polyhedra.

If one traverses any face of an everywhere-convex polyhedron $\mathfrak{P}$ that lies entirely at finite points in the same sense (e.g., considered to have the positive sense from the outside) then any edge will then be crossed twice in the opposite senses. This "Möbius edge law" can also be expressed in a dual form: If one rolls a plane around any edge of $\mathfrak{P}$ in the same sense (which can never cut the polyhedron, but always have an edge or a face in common with it) then each edge will then appear twice as the axis of opposite rotations.

If one then begins to traverse a face in an arbitrary sense then one can also, without knowing whether one finds oneself on the outside or inside of $\mathfrak{P}$, go on to a neighboring face when one always observes that the same edges must be traversed in opposite senses, since they are associated with two different faces. Any face will then obtain a definite sense of traversal *independently of the type of the transfer*.

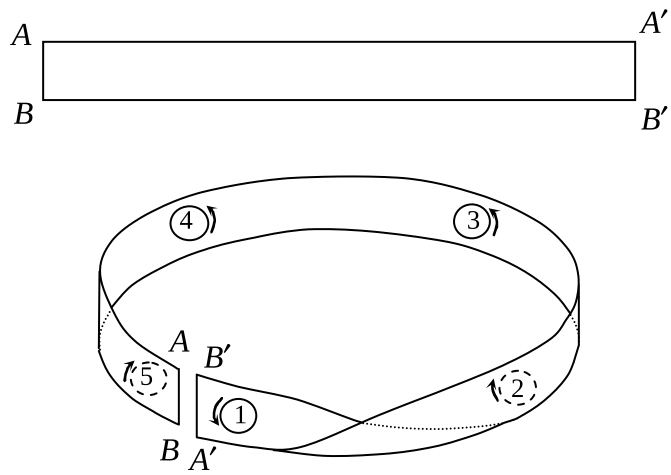

Figure 24.

However, this is not possible for all polyhedra: In order to get an intuitive picture of a "one-sided zone," one bends a rectangular strip $AA'B'B$ (Fig. 24) such that A' goes to B and B' goes to A. The boundary of the strip can now be traversed completely in a single circuit. If one displaces a small closed curve, which shall be denoted by, say, 1 on the visible side, together with a definite sense of rotation, into the positions 2, …, 5, until it arrives at 1 again then the curve will come to the other side of position 1 and the sense of traversal will have been converted to the opposite one; a half-ray that is constantly normal to the surface will serve the same purpose as the curve. One can go from any location on the surface to the other side of the location in question without crossing over the boundary or breaking through the surface. Indeed, one cannot distinguish two different sides of the strip as a whole at the individual locations on the strip. *Möbius* (*)

(*) And simultaneously *Listing* (cf., *Stäckel*, Math. Annal., Bd. **52**).

first discovered this situation (1858); according to him, such surfaces are called *one-sided* (*).

In order to have an example of a *closed*, one-sided surface, one needs only to eliminate the boundary of the strip, perhaps in such a way that one connects an arbitrary point of space with it by a conic surface. Thereby, self-intersections of the surface will be unavoidable, and if one would like to go from one side to a place on the other side then one will generally need to break through the surface, but never through the part of it along which one actually moves.

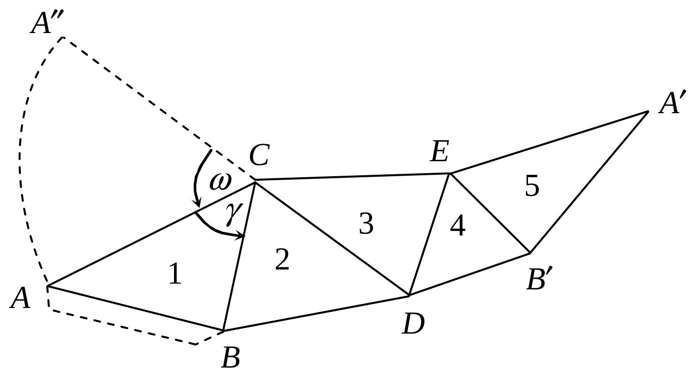

Figure 25.

Analogously, one can define closed polyhedra when one (in order to give a simplest possible example) perhaps breaks a net of five triangles that are connected in sequence along each side (Fig. 25; $A'B' = AB$) along those common triangle sides in such a way that A' again coincides with B and B', with A. In order for this to be practicable, one might not depict the net arbitrarily (the reader takes it to be similar to Figure 25, and allows a boundary along AB in order to attach it along $B'A'$ or to fasten it with a pin), but one can choose the first four triangles arbitrarily only within certain limits. One then fixes the middle triangle 3, so the geometric locus of A is a conic surface of radius AC (with the exception of certain components of the sphere surface), while the geometric locus of B' is merely a circle. One will then rotate B' far enough that it is at a distance of AC from C, whereupon one can bring A into coincidence with B' by a suitable choice of the lengths of the sides of the triangle. The final triangle 5 is then determined completely. The strips will have only *one* boundary: $A, C, E, A' \equiv B, D, B' \equiv A$. In order to get a *closed*, one-sided polyhedron, one can eliminate the boundary by connecting it with an arbitrary space point through five triangles, but not when one connects a vertex on the boundary itself with the opposite edges by three triangles; cf., Brückner, *Vielecke und Vielflache, Theorie und Geschichte*, 1900, art. **55**.

The edge law is not valid for the one-sided polyhedra. However, it is restricted to everywhere-convex polyhedra $\mathfrak{P}$, except that $\mathfrak{P}$ must only be two-sided; i.e., when one starts with a well-defined location on the outer surface, one must always come back to the same side of that location, as one might also come back to the same location while

(*) The terminology "double surface" of function theory means one-sided surface. One finds the basis for this in the following picture: If one finds oneself on a definite side of a two-sided surface then only that one side will be accessible, while for the one-sided surfaces the two sides will be accessible from the location in question, whereby the location in question will have "doubled," so to speak.

moving on the surface. Thus, $\mathfrak{P}$ can be "extraordinary"; i.e., possess a surface whose perimeter intersects itself; Cf., *Brückner*, *loc. cit.*, art. **60**.

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§ 25. Reciprocal polyhedra.

For each element (vertex, edge, face) of a polyhedron $\mathfrak{P}$ in a null system, whose axis a we think of as vertical, we seek the corresponding element (null plane of the vertex, polar of the edge, null point of the face, respectively). If the edges k_1, k_2, …, k_n go through the vertex E then its polars k_1', k_2', …, k_n' will lie in the null plane ε of E (Theorem 12), which will also appear among the corresponding elements. Conversely, if the edges κ_1, κ_2, …, κ_n lie on a face φ of $\mathfrak{P}$ then the polars κ_1', κ_2', …, κ_m' will go through a point F of φ that also appears among the corresponding elements. However, in order for us to convince ourselves that all corresponding elements actually comprise the faces, edges, and vertices of a second polyhedron $\mathfrak{P}'$, and how that happens, we will pose the following argument, in which, for sake of intuitive appeal, we will restrict ourselves to the case in which $\mathfrak{P}$ is an everywhere-convex polyhedron that lies entirely at finite points: We assume that no face of $\mathfrak{P}$ is parallel to a and project $\mathfrak{P}$ onto a horizontal plane. We also refer to the edges of $\mathfrak{P}$, which therefore provide the periphery of the projection, as *peripheral edges;* their totality defines the "true periphery" of $\mathfrak{P}$. If a plane $\mathfrak{E}$ rotates around an edge k of $\mathfrak{P}$ then it will come to two positions in which it will include faces of the polyhedron, and these faces will cut out the null point on the polar k'. When we think of a line as connected at infinity, we say that k' will be divided into two parts by the two null points. If $\mathfrak{E} \parallel a$ then the null point will go to infinity. Thus, if a plane $\mathfrak{E}$ is rolled around a vertex A of $\mathfrak{P}$ then its null point will describe a closed polygon a that either has two infinite sides or lies at finite points entirely, according to whether A does or not belong to the true periphery of $\mathfrak{P}$, resp. However, in the first case, we will change the law of rotation for the rolling in such a way that we will establish that for the peripheral edges, and only for them, $\mathfrak{E}$ shall rotate around those edges of a polyhedron to the next ones in such a way that they describe a wedge in which the polyhedron itself lies. A peripheral edge k will now, as before, appear twice as an axis of opposite rotations, since each of the previous two (established in the previous §) rotations would be associated with their opposites. Therefore, the null point of the plane will traverse the corresponding (now, *finite*) piece of the polar k' two more times in the opposite senses. We thus have the:

Theorem 35: *If $\mathfrak{P}$ is an everywhere-convex polyhedron that lies at finite points entirely, no face of which is parallel to the axis a of a null system, then the finite pieces of* *the polars of all edges of $\mathfrak{P}$ will close together into a polyhedron $\mathfrak{P}'$ that will likewise obey the law of edges.*

Namely, if one thinks of the edge law for $\mathfrak{P}$ as being expressed in the dual form (§ 24, beginning) then it will follow for $\mathfrak{P}'$ in the original form. One can easily say more: If $\mathfrak{E}$ rolls around A in the just-altered way then it can only come to the same position (in which it links the two peripheral edges) twice when A belongs to the periphery. Therefore, this will overlap the finite pieces of the sides once in the corresponding polygon a' of $\mathfrak{P}'$, except when those peripheral edges meet the vertex. a' will then emerge from a in such a way that the two possible sides at infinity will be replaced with their finite extensions.

If we then let the vertex A be described by an edge k that successively sweeps out all of the edge angles of the vertex then the edge will not come to the same position twice, except when the shell cuts the vertex itself. In addition, if A is everywhere convex then k will never meet the extension of another side of the vertex, as it does in the one in which it actually moves. Therefore, k' will rotate around the vertices of the polygon a in such a way that it will not come to the same position twice and will never go through another vertex of a except the one that it actually rotates around; i.e., it will roll around the outer sides of an everywhere-convex polygon, which we assume either lies at finite points entirely or has the type of Fig. 26. We conclude and summarize, when we again replace the polygon a that appears in $\mathfrak{P}'$ with a':

Theorem 36: *If A is a vertex of $\mathfrak{P}$ that belongs to the true periphery then the two peripheral edges that go through A might or might not be divided by ordinary edges; in the first case, the corresponding polygon a' of $\mathfrak{P}'$ will consist of two intersecting line segments* (AB, $A'B'$, Fig. 27) *and two line paths that link its endpoints and return to the double point of its convex side (as long as it contains more than one segment, to begin with); in the second case, only the part on one side of the double point will remain. If A does not belong to the periphery then a' will be everywhere convex.*

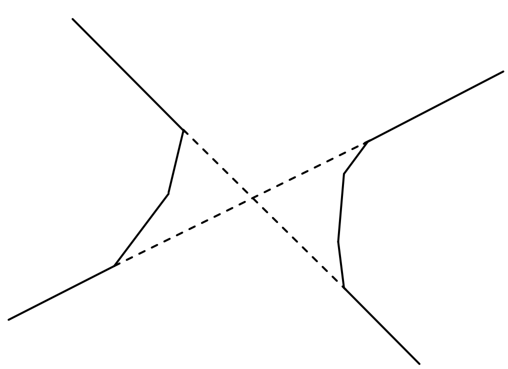

Figure 26.

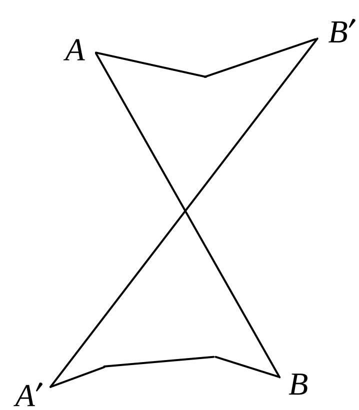

Figure 27.

One then sees that when one considers the finite pieces of the polars k' (which is necessary for our purposes), $\mathfrak{P}'$ will be an extraordinary polyhedron, in general; for that reason, it was necessary to be careful in proving Theorem 35. (One will find examples in the practice problems.) It follows further from Theorem 10 that:

Theorem 37: *The projections of the edges of reciprocal polyhedra onto H are parallel.*

The edges and faces of $\mathfrak{P}$ are incident with the faces and edges of $\mathfrak{P}'$; however, the latter fact is inessential for our present purpose. By contrast, Theorems 35 and 37 will remain true when one replaces $\mathfrak{P}'$ with another polyhedron $\mathfrak{P}_1'$ that emerges from $\mathfrak{P}'$ by an arbitrary parallel displacement and similarity transformation. If we now project $\mathfrak{P}$ and $\mathfrak{P}'$ onto H orthogonally then two figures $\mathfrak{N}$, $\mathfrak{N}'$ will arise that we would like to call *reciprocal nets.* Theorems 35 ands 37 immediately give:

Theorem 38: *If one projects a convex polyhedron that lies at finite points entirely perpendicularly onto a plane that is not perpendicular to any face of the polyhedron then a net* $\mathfrak{N}$ *will arise for which there will always exist a reciprocal net* $\mathfrak{N}'$; *i.e., a system of nothing but finite segments, each of which is parallel to a segment of* $\mathfrak{N}$, *with the property that all segments in one net that emanate from a point will correspond to segments in the other one that define a closed polygon. All of these polygons in* $\mathfrak{N}'$ *can also be traversed in such a way that every segment in the two polygons to which they belong will be traversed in opposite senses.*

§ 26. The existence of reciprocal force planes.

We assume that all of the lines of action k of the external forces (including the support reactions) on a planar framework can be regarded as the projections of an everywhere-convex, spatial polyhedron $\mathfrak{P}$; finally, should all lines k go through a single point S, then all of the rods s of framework would define a net $\mathfrak{N}$, for which we will construct a reciprocal one $\mathfrak{N}'$. The point S will then correspond to a polygon s that we traverse in the sense that corresponds to the actual directions of the forces through S. We indicate this sense of traversal by arrows on the sides k' of the polygon (cf., Fig. 22), with which the sense of traversal of the neighboring polygon is also determined. Any other segment s' of $\mathfrak{N}'$ will contain two arrows, according to whether it can be regarded as a segment of one or the other polygon, and in fact we can thus always go on to neighboring polygons and determine the sense of traversal of each of them, whereby we know from Theorem 38 that the result will be independent of the choice of intermediate polygons, even when the individual polygons overlap. If we now let the force that is determined by the corresponding polygon act upon each node of the framework then the framework will be in equilibrium, in any case. However, since many choices can be made arbitrarily in the construction of $\mathfrak{N}'$ with the help of a null system, there will be numerous reciprocal nets $\mathfrak{N}'$ (*) to the same net $\mathfrak{N}$, and one must show that among them there is one for which the segments k' are equal to the given external forces. We direct our attention to the vertex of $\mathfrak{P}$ whose projection is S and then prove the:

Theorem 39: *A spatial vertex can always be mapped by a null system in such a way that the sides of its corresponding polygon in $\mathfrak{N}'$ will have arbitrarily-given lengths when only the geometric sum of these lengths is zero.*

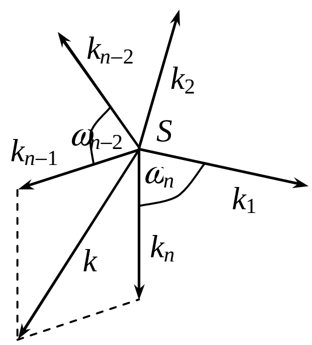

Figure 28 a).

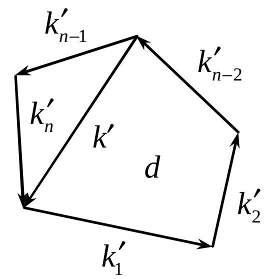

Figure 28 b).

We assume that this theorem is true for vertices with $n - 1$ edges and then prove it for n-edged vertices: Let the given lengths be carried by the lines k_ν through S, and let them be denoted by k_ν, in their own right. We replace two of them – say, k_{n-1} and k_n – with their geometric sum k (Fig. 28a). In space, this construction corresponds to an extension of those sides of the n-edge that correspond to angles ω_{n-2} , ω_n in space until they cut the new edge k_0 . (It should create no misunderstanding when we denote the elements in space and their projections by the same symbols.). The n-edge can be chosen in manifold ways, such that the projection of k_0 falls along the diagonal k. Theorem 39 is true for the vertex that is defined by the $n - 1$ edges $k_1, k_2, \ldots, k_{n-2}$. By mapping in a null system (parallel translation, similarity transformation) and projection, one can then obtain a polygon σ (Fig. 28 b) for which each of the sides k'_1, k'_2, …, k'_{n-2}, k' are not only parallel, but also equal to the corresponding segment k_ν . If we now consider the plane that rolls around the edges through S, according to § 25, and instead of taking it from the position ω_{n-2} to the position ω_n immediately by a rotation around k, we first rotate it around k_{n-1} into the position (k_{n-1}, k_n) and then around k_n into the position ω_n then its point in $\mathfrak{N}'$ that corresponds to its null point, instead of describing the segment k' whose two endpoints join k'_{n-1}, k'_n on the circuit, will become the segments that are not only

(*) In fact, one can distribute the resultant R of the external forces at F on the same lines of action through the same nodes in various way, even when one does not change R , and therefore the support reactions, as well, so one will also get various stress states.

parallel to the segments k_{n-1}, k_n , but also equal to them, since a segment can be decomposed into two given directions in only one way. Theorem 39 is then true in general, since it is self-explanatory for all of the three-edged vertices up to now.

Now, since σ correctly represents the external forces, the net $\mathfrak{N}'$ will be, in fact, a reciprocal force plane of the framework. Then, on the one hand, the stress state will be determined uniquely by the external forces, and on the other hand, as we saw before, $\mathfrak{N}'$ will represent a possible equilibrium state of the framework; it must then represent the actual stress state under the action of the given external forces, as long as only they are represented correctly by $\mathfrak{N}'$. We then have the:

Theorem 40: *If a planar framework with two supports, together with the lines of action of the external forces (which might go through the same point S), can be regarded as the projection of a convex polyhedron then there will exist a reciprocal force plane.*

Naturally, one will not return to the null system now for the actual construction of the force plane, but apply the process in § 23, which must then produce, at the least, the reciprocal force plane when one always finds the neighboring nodes by performing the process, by which, the stress state is still unknown in merely two rods. However, one cannot deduce the existence of the force plane from the possibility of performing the process.

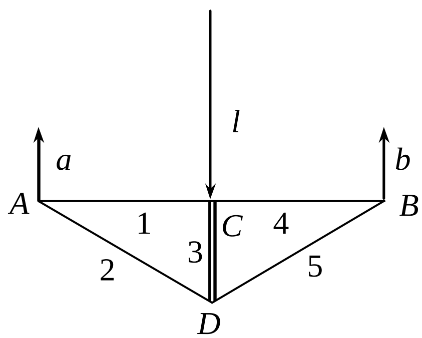

Figure 29.

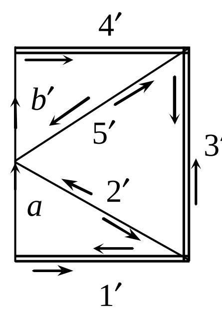

Figure 30.

Now, nothing will be affected if the point *S* goes to infinity; σ will then coincide with a line, but in such a way that its sides keep definite lengths. For example, if *S* in Fig. 21 goes to infinity then Fig. 29 will come about. (At the same time, we let symmetry enter in, and denote the moved rod by double lines.) The reciprocal force plane assumes the form of Fig. 30. One easily sees that the assumptions in Theorem 40 will be fulfilled for all carriers of the type in Fig. 31: in it, *S* lies at infinity. One thinks of the broken line path *AMB* as being in the reference plane, while one considers the line *AB* to be the projection of a broken line that arches a distance *AB* in a plane that is normal to the reference surface. The parallel lines of action of the forces will then be edges of a prismatic shell, of which, the one that goes through *A* and *B* will lie in the reference plane. One also recognizes that the process in § 23 suffices for the actual construction of the reciprocal force plane (cf., Appendix I, problem 23).

In order to not stray too far and give a picture of the connection between the theory of null systems and graphical statics, we have made very simply-restricted assumptions that were not all essential, but must be proved in a more rigorous theory, such as one finds in the following papers: *Cremona-Migiotti*, "Die reziproken Figuren in der graph. Statik," (Zeitschr. d. österr. Ingenieur- u. Architektenver., 1873); *F. Schur*, "Üb. d. recipr. Fig. d. graph. Statik," ([*]), and especially: *F. Schur*, "Über ebene einfache Fachwerke," (Math. Annalen, Bd. 48).

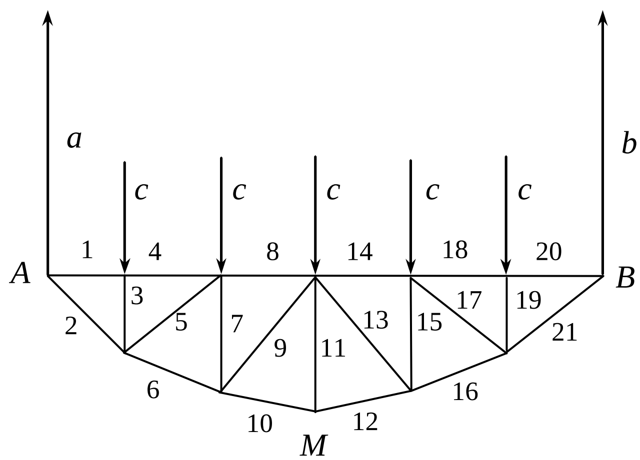

Figure 31.

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§ 27. The polar system of a paraboloid of rotation.

In this §, we assume a knowledge of the general polar properties of second-degree surfaces (cf., S. S., Bd. XXV, § 4).

Suppose that a meridian parabola of a paraboloid of rotation *P* lies in the image plane *B* (Fig. 32), g_1 is the perpendicular projection of a line $g \parallel B$, d_1 is the diameter of the meridian parabola that is conjugate to g_1, and *G* and *D* are the planes through g_1 (d_1, resp.) that are perpendicular to *B*. *D* will then be the diametral plane for the chords of *P* that are parallel to g_1; i. e., the polar plane of the points at infinity of these chords. Therefore, if a line *g* is parallel to g_1 then its polar g' will lie in *D*. If one projects *g* and g' onto a plane that is perpendicular to the rotational axis then the projected planes will be parallel and perpendicular to *B*, respectively, and thus perpendicular to each other. It will follow from this that:

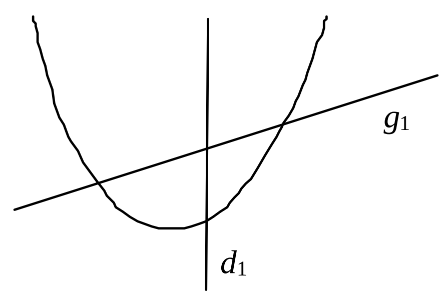

Figure 32.

Theorem 41: *Two polar lines of a paraboloid of rotation will produce two mutually-perpendicular lines when projected onto a plane that is perpendicular to the axis.*

([*]) (Zeitschr. f. Math. u. Phys., Bd. 40).

The polar system, just like the null system, is a special case of the general spatial correlation. If one seeks the polyhedron that corresponds to a polyhedron in a polar system, then projects the two polyhedra (perhaps, onto the tangential plane to the vertex of P), and finally rotates the one through 90° then one will obtain two nets that have the same properties as the ones that were considered in § 25. This is the way by which *Maxwell* first exploited the theory of correlations in graphical statics; cf., *Hauck*, "Über die Beziehung des Nulls. u. lin. Strahlenkompl. zum Polarsyst. des Rotationsparaboloides (Zeitschr. f. Math. u. Phys., Bd. 31, 1886).

Practice problems:

14. a) Relative to which of all the axes that go through a point P does a force system have the greatest moment?

b) The moments relative to all axes that go through P are proportional to the lengths that are determined on these axes by a certain spherical surface.

c) If the moments are given relative to three axes through P then find the moment relative to a fourth axis through P.

15. a) The moments relative to the axes in a plane E are proportional to the distances from these axes to the null point of E.

b) Given the moments relative to three axes in E, construct the moment relative to a fourth axis in E.

16. Amongst all possible decompositions of a dyname into a force cross, find the ones for which the two forces:

a) Have an equal magnitude,

b) Are perpendicular to each other,

c) Are both present at the same time.

17. For a general motion of a rigid body, at each instant:

a) The normal planes to the paths of all points in a plane will go through a certain point of this plane.

b) The normal planes to the paths of all points of a line g will again go through a line g'. If g and g' are distinct then the velocity vectors of all points of g will be the same as if g were rotating around g'; if g is identical to g' then the vectors of all points of g will be perpendicular to g.

18. We direct our attention to a line g in a moving rigid body.

a) Which points of it have the smallest velocity?

b) In which limiting position does the base point have the shortest distance from a neighboring position?

19. The perpendicular projections of two polars onto a plane E intersect on the characteristic of E (*Mannheim*, *Géométrie cinématique*, pp. 105).

20. How does the search for the instantaneous winding of § 22 simplify for the special case that was mentioned at the conclusion of this § ?

21. Ascertain the spherical regions that are attainable for the point A in Fig. 25.

22. In a null system with axis A, ascertain the reciprocal polyhedra to:

a) A tetrahedron whose altitude falls upon A.

b) α), an octahedron and β), a cube, two opposing edges of which fall upon A. One should pursue the arguments and theorems of § 25, *in concreto*, for these examples.

23. Construct the reciprocal force plane to the framework that is represented in Fig. 31 when equally large, vertical forces act upon all nodes of the horizontal segment AB.

24. If one rotates a polar system of a paraboloid of revolution with axis a and parameter $2p$ through 90° around a, and then reflects it relative to the tangential plane to the vertex then it will define a null system with parameter $\mathfrak{k} = p$ with the original position.

Chapter III

Line pointers, rod pointers, and equations in them ([1])

§ 28. The concept of a line pointer.

The lines in space define a four-fold manifold (see Introduction). One thus needs (at least) four numbers in order to specify a single line, which one can call the *pointers* (coordinates) of the line. These characteristic numbers of the line can be chosen in very different ways: For example, one can refer to the representation of the lines in a parallel system:

$$x = r\,z + \rho, \qquad y = s\,z + \sigma \tag{1}$$

(x, y, z are the running pointers), and define the four numbers r, s, ρ, σ, which determine the line completely, and conversely are associated with one uniquely, to be the pointers.

However, this pointer system has a great drawback: If one were to perform a transformation of it then equations (1) would go to:

$$x' = r'z' + \rho', \quad y' = s'z' + \sigma', \tag{2}$$

and one would have to be able to calculate the new pointers r', s', ρ', σ' from the old ones if the orientation of the new pointer system were given in terms of the old one. One would then find that the formulas that represented the new line pointers as functions of the old ones (or conversely) would be *nonlinear* in those pointers. In order to convince oneself of that, it will suffice to consider a special case: We imagine the original system of parallels to be rectangular and perform a cyclic permutation of the pointer axes, so:

$$x = y', \;\; y = z', \;\; z = x'.$$

Equations (1) will then go to:

$$y' = r\,x' + \rho, \quad z' = s\,x' + \sigma.$$

We must now bring these equations into the form (2):

$$x' = \frac{z'}{s} - \frac{\sigma}{s}, \quad y' = \frac{r}{s}z' + \frac{s\rho - r\sigma}{s}$$

by solving for x', y'. If we compare this with (2) then we will find that:

([1]) As a preliminary to this chapter, one can read Chapter III in S. S. IX (*Analyt. Geom. des Raumes*, Part I), in which rectangular, homogeneous, line pointers were presented independently of the tetrahedral pointers, and appeared as a special case of them.

$$r' = \frac{1}{s}, \qquad \rho' = -\frac{\sigma}{s},$$
$$s' = \frac{r}{s}, \qquad \sigma' = \frac{s\sigma - r\sigma}{s}.$$

An algebraic equation in line pointers would thus change its degree under a pointer transformation. While the degree is thus something that is characteristic of an algebraic geometric structure for the usual pointer systems, here, the degree would depend upon not only the structure itself, but also upon the pointer system. For that reason, we will define *other* quantities to be line pointers that we will introduce in connection with the tetrahedral point and plane pointers.

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§ 29. Homogeneous point and plane pointers.

In this and the following paragraphs, we will summarize the most important properties of homogeneous point and plane pointers, in order to be able to discuss them more easily. Since the proofs will not be included here, we shall refer to the thorough presentation in *Killing's Lehrbuch der analytischen Geometrie,* 1901 (II. Teil).

a) **Definition:** We choose a "pointer tetrahedron" – or "basic tetrahedron" – and a positive side for each of its planes, and denote the distances from an arbitrary point *P* to these planes by d_1, d_2, d_3, d_4 . We will define four numbers to be the pointers of that point *P*, which will behave like those distances when each of them is measured with an arbitrary unit of measurement. If one measures d_i with the unit e_i then $d_i : e_i$ will be the new measurement. If:

$$\frac{1}{e_i} = \kappa_i$$

then:

(3) $$\rho\, x_i = \kappa_i\, d_i \qquad (i = 1, 2, 3, 4)$$

can serve as the defining equation of the pointers x_i . The four numbers κ_i are fixed for all points of space; ρ is an arbitrary proportionality factor.

Let *E* be a plane for which we distinguish a positive and a negative side, let δ_1, δ_2, δ_3, δ_4 be the distances from the four tetrahedral vertices to it, and let $\mathfrak{e}_i$ be the units of measurement that they are measured with. One then defines the pointers u_1, u_2, u_3, u_4 of the plane *E* analogously by:

$$\rho\, u_i = \frac{\delta_i}{\mathfrak{e}_i},$$

or, when $1 : \mathfrak{e}_i = \lambda_i$, by:

(4) $$\rho\, u_i = \lambda_i\, \delta_i \qquad (i = 1, 2, 3, 4).$$

Only the *ratios* of the pointers of a point or plane will then come under consideration, among which, one will find three independent ones. Conversely, a quadruple of numbers x_i (u_i, resp.) will be assigned to a point (plane, resp.) uniquely.

The surfaces and corners of the basic tetrahedron that have the same indices lie opposite to each other.

b) One can choose the constants κ_i arbitrarily, and determine the ratios of the λ_i arbitrarily from them in such a way that the condition for the incidence of P and E will be as simple as possible, namely:

$$\sum_{i=1}^{4} u_i x_i = 0. \tag{5}$$

c) If one goes from one homogeneous pointer system x_i, u_i to another x_i', u_i' then the old pointers will be *linear, homogeneous* functions of the new ones with the same names, and likewise for the new as functions of the old, namely:

$$\begin{cases} \rho x_i = a_{i1}x_1' + a_{i2}x_2' + a_{i3}x_3' + a_{i4}x_4', \\ \sigma u_i = A_{i1}u_1' + A_{i2}u_2' + A_{i3}u_3' + A_{i4}u_4', \end{cases} \tag{6}$$

in which the A_{ik} are the adjoints of the determinants $|\, a_{ik} \,|$. Solving this for the new pointers will give:

$$\begin{cases} \rho' x_i' = A_{i1}x_1 + A_{i2}x_2 + A_{i3}x_3 + A_{i4}x_4, \\ \sigma' u_i' = a_{i1}u_1 + a_{i2}u_2 + a_{i3}u_3 + a_{i4}u_4. \end{cases} \tag{7}$$

Conversely, each such system of linear transformations can be regarded as a pointer transformation. If one sets the right-hand side of equations (6) equal to zero then one will obtain the equations for the faces and vertices of the old basic tetrahedron relative to the new ones, and analogously, (7) will give the equations of the elements of the new tetrahedron relative to the old system.

d) Under the transition from a rectangular system to a homogeneous one, the homogeneous pointers will be *entire*, linear functions of the rectangular ones x, y, z, which will be linear, homogeneous, *fractional* functions (with common denominators) of the homogeneous pointers, namely:

$$\rho\, x_i = a_{i1}\, x + a_{i2}\, y + a_{i3}\, z + a_{i1}\, x \qquad (i = 1, \ldots, 4) \tag{8}$$

$$x = \frac{\sum_{i=1}^{4} A_{i1}x_i}{\sum_{i=1}^{4} A_{i4}x_i}, \qquad y = \frac{\sum_{i=1}^{4} A_{i2}x_i}{\sum_{i=1}^{4} A_{i4}x_i}, \qquad z = \frac{\sum_{i=1}^{4} A_{i3}x_i}{\sum_{i=1}^{4} A_{i4}x_i}. \tag{9}$$

e) The points at infinity find an actual representation in tetrahedral points: A quadruple of numbers x_i that makes the denominator of equations (9) zero means an point at infinity whose representative direction one will find when one seeks the ratios of the numbers.

f) If i, j, k, m are the numbers 1, 2, 3, 4 in any sequence, P_i are the vertices, and E_i are the opposite planes of the basic tetrahedron then x_k, x_l, x_m will also be the tri-metric pointers of the point of intersection of $P_i\,P$ with the planes E_i in that plane relative to P_k, P_l, P_m as the basic triangle. u_k, u_l, u_m will also be the tri-metric pointers of the line of intersection $E_i\,E_i$ in the plane E_i relative to the same basic triangle.

g) The point whose four pointers are equal to each other (e.g., equal to one) is called the *unit point e*, and the planes whose four pointers are all equal is called the *unit plane* $\mathfrak{E}$. If the constants κ and λ are chosen as in *b*) then the following relationship will exist between *e* and $\mathfrak{E}$: If *k* is a tetrahedral edge, and *S* is the point of intersection of the opposite edge with $\mathfrak{E}$ then the two tetrahedral planes that go through *k* will be harmonically separated by *e* and *S*, and if $\mathfrak{B}$ is the connecting plane of the opposite edge with *e* then the two tetrahedral vertices will be harmonically separated by $\mathfrak{E}$ and $\mathfrak{B}$ on *k*.

---

§ 30. Geometric representation of the pointer ratios by double ratios.

We choose a plane *E* with the pointers u_i and consider a well-defined pointer ratio; e.g., $u_2 : u_1$. Let *S* be the point of intersection of the edge P_1P_2 of the basic tetrahedron with *E*, and let $\mathfrak{S}$ be its intersection with $\mathfrak{G}$ (Fig. 33). From § 29 *a*), one will then have:

$$\frac{u_2}{u_1} = \frac{\delta_1}{\delta_2} : \frac{\mathfrak{e}_2}{\mathfrak{e}_1}.$$

Now, one will have:

$$\frac{\delta_2}{\delta_1} = \frac{P_2S}{P_1S},$$

since the former ratio emerges from the latter by projective onto the normal direction of *E*. Analogously, one will have:

$$\frac{\mathfrak{e}_2}{\mathfrak{e}_1} = \frac{P_2\mathfrak{S}}{P_1\mathfrak{S}},$$

so (*):

(*) We employ the notation:

$$(ABCD) = \frac{AC}{CB} : \frac{AD}{DB}$$

(10) $$\frac{u_2}{u_1} = (P_2\, P_2\, S\, \mathfrak{S}).$$

We choose a point P with the pointers x_i and consider the ratio:

$$\frac{x_2}{x_1} = \frac{d_2}{d_1} : \frac{e_2}{e_1}.$$

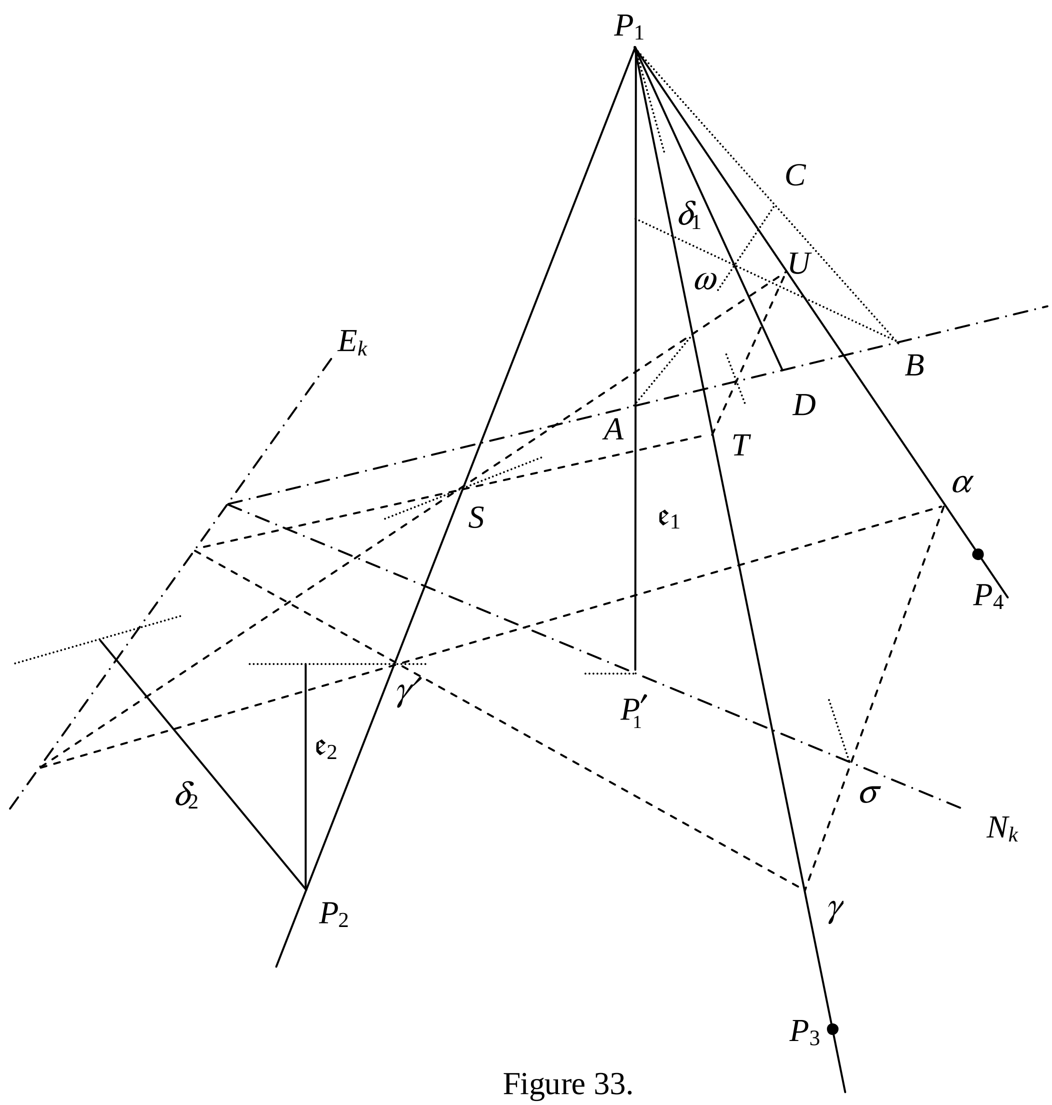

Figure 33.

One can now replace the points P and e, which involve the distance ratio of the planes E_1, E_2, with arbitrary points of the connecting planes PP_3P_4 and eP_3P_4, resp. – e.g., with their points of intersection S' and $\mathfrak{S}'$ with the edge P_1P_2 . (Fig. 34. $P_3P_4 \equiv y$ is the line of intersection of E_1E_2 . Only two vertices P_1, P_2, two faces E_1, E_2, and two edges k, y of the

for the double ratio, in which we regard AB as the divided segment, and C, D as the dividing points.

tetrahedron are indicated.) Thus, if d_1', d_2' have the same meaning for S' and e_1', e_2' have the same meaning for $\mathfrak{S}'$ that the unprimed quantities do for P and e then one will have:

$$d_1 : d_2 = d_1' : d_2', \qquad e_1 : e_2 = e_1' : e_2',$$
$$d_1' : e_1' = P_2S' : P_2\mathfrak{S}', \qquad d_2' : e_2' = P_1S' : P_1\mathfrak{S}',$$

so

(11) $$\frac{x_2}{x_1} = (P_1\, P_2\, S'\mathfrak{S}') .$$

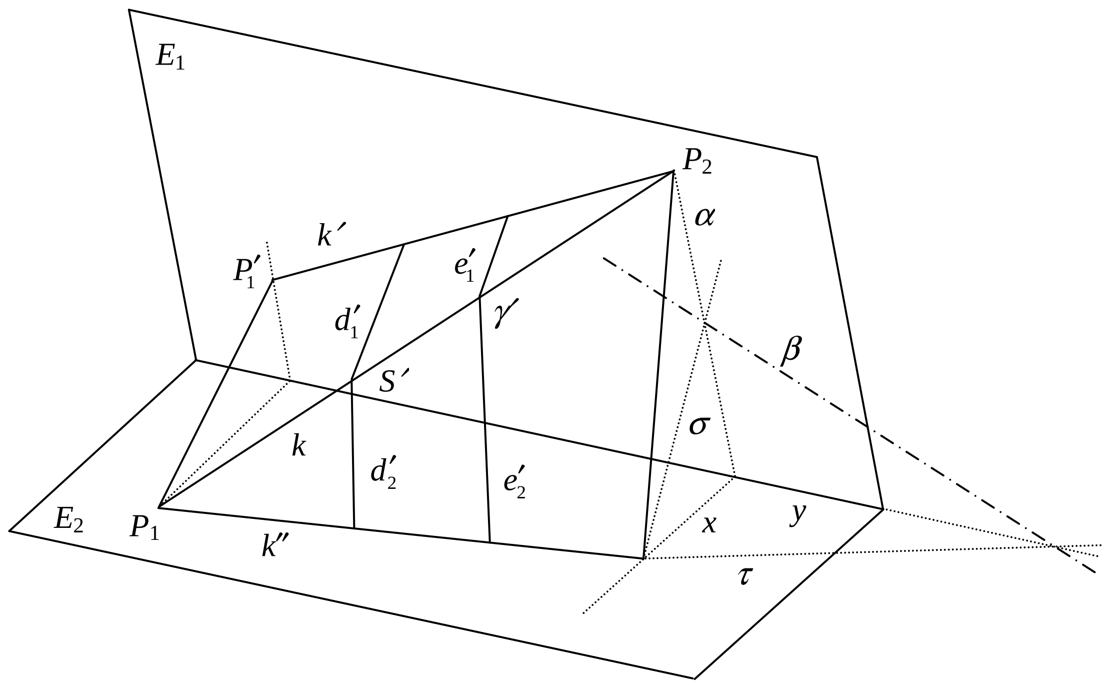

Figure 34.

The elements P_2, P_1, S, $\mathfrak{S}$ in equation (10) correspond dually to four planes E_2, E_1, (P, y), (e, y). The latter would then be made to intersect with k in S_2, S_1, S', $\mathfrak{S}'$. Now, $S_2 \equiv P_1$, $S_1 \equiv P_2$, and this is the basis for the fact that the points P_1, P_2 have changed places in equation (11) when compared to equation (10).

Theorem 42: *If $P_i\, P_k$ is a tetrahedral edge k, and y is the opposite edge then the pointer ratios $u_i : u_k$ for a plane E and the pointer ratios $x_i : x_k$ for a point P can be represented by double ratios on k, and in fact: If S, $\mathfrak{S}$ are the points of intersection of k with E and the unit plane, respectively, in the first case, while S', $\mathfrak{S}'$ are the intersections of k with the planes (P, y), (e, y), resp., in the second case, then one will have:*

$$\frac{u_i}{u_k} = (P_i\, P_k\, S\, \mathfrak{S}), \qquad \frac{x_i}{x_k} = (P_k\, P_i\, S'\mathfrak{S}').$$

This geometric interpretation of the pointer ratios goes back to Fiedler (*Darst. Geom.*)

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§ 31. The parallel pointer as a special case of the tetrahedral pointer.

We take P_1 to be the origin of a parallel system whose positive semi-axes X, Y, Z are directed towards the other vertices P_2, P_3, P_4, resp., of the basic tetrahedron, and we would like to look for the connection between the parallel pointers x, y, z of a point P and its tetrahedral pointers x_i, under the assumption that we can shift the tetrahedral plane E_1 to infinity. The X-axis will then play the role that the edge k did in the previous paragraph.

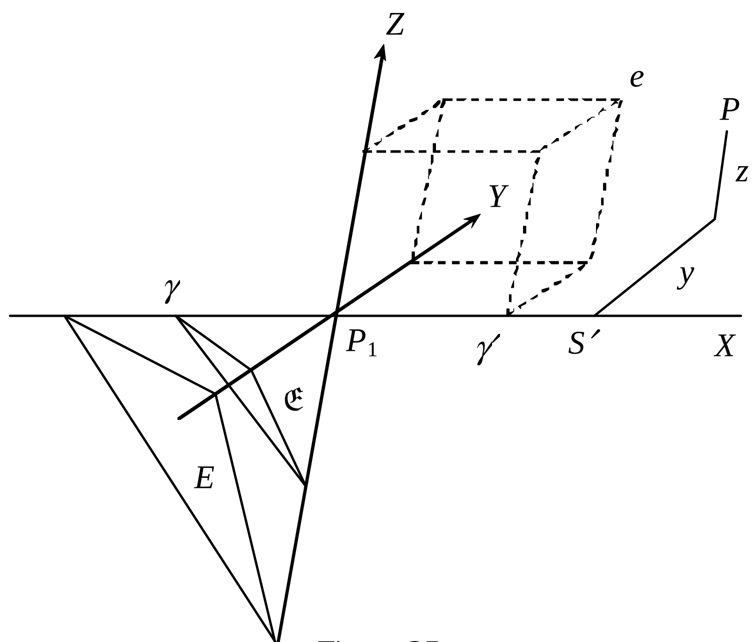

Figure 35.

The opposite edge is the line at infinity in the YZ-plane, so $P_1S' = x$ (Fig. 35) and:

$$\frac{x_2}{x_1} = \frac{P_1S'}{S'P_2} : \frac{P_1\mathfrak{S}'}{\mathfrak{S}'P_2} = \frac{P_1S'}{P_1\mathfrak{S}'} : \frac{S'P_2}{\mathfrak{S}'P_2}.$$

Now, one will have:

$$\lim \frac{S'P_2}{\mathfrak{S}'P_2} = 1$$

when E_1 goes to infinity. Thus, $x_2 : x_1$ will be precisely equal to x in the event that we choose $P_1\mathfrak{S}' = 1$; analogous statements will be true for $x_3 : x_1$ and $x_4 : x_1$. We thus choose the unit point e to be the vertex that is opposite to P_1 in a parallelepiped that is determined by the three unit segments on the positive semi-axes. Under this assumption, one will have:

$$\frac{x_2}{x_1} = x, \qquad \frac{x_3}{x_1} = y, \qquad \frac{x_4}{x_1} = z.$$

As the unit plane, $\mathfrak{E}$ must now be chosen to be the connecting plane for the three negative unit segments that emanate from P_1 . Furthermore, if u, v, w are the parallel pointers of a plane E and u_i are its tetrahedral pointers then one will have:

$$u = -\frac{1}{P_1 S},$$

so:

$$\frac{u_2}{u_1} = \frac{P_2 S}{SP_1} : \frac{P_2 \mathfrak{S}}{\mathfrak{S} P_1} = \frac{P_2 S}{P_2 \mathfrak{S}} : \frac{P_1 \mathfrak{S}}{P_1 S},$$

and:

$$\lim \frac{P_2 S}{P_2 \mathfrak{S}} = 1,$$

so

$$\lim \frac{u_2}{u_1} = \frac{P_1 \mathfrak{S}}{P_1 S},$$

so it will be equal to u precisely, since $P_1\mathfrak{S} = -1$. Since the position of the unit point and unit plane were established once and for all (§ 29, *g*), we can say that:

Theorem 43: *If we locate the origin of a parallel system at the vertex P_1 of the basic tetrahedron and the positive semi-axes X, Y, Z in the directions of P_2, P_3, P_4 then the unit plane will cut out lengths of* – 1 *from the semi-axes, and if the tetrahedral plane E_1 goes to infinity then* $\frac{x_2}{x_1}, \frac{x_3}{x_1}, \frac{x_4}{x_1}; \frac{u_2}{u_1}, \frac{u_3}{u_1}, \frac{u_4}{u_1}$ *will go to x, y, z; u, v, w, resp.*

Thus, when the equation G of a structure is given in homogeneous point or plane pointers then one can derive the equation G' of that structure in parallel pointers from it in such a way that one replaces x_1, x_2, x_3, x_4 ; u_1, u_2, u_3, u_4 with $1, x, y, z$; $1, u, v, w$, resp. Indeed, this will be true when the coefficients of G are numerical only when the two pointer systems are oriented with respect to each other as was specified in Theorem 43, so when they are general, unrestricted, arbitrary numbers, for an arbitrary orientation of the systems with respect to each other (§ 29, *c*), in the sense that the totality of the structures that are represented by G is identical with the totality of ones that are represented by G'.

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§ 32. General (tetrahedral), homogeneous, line pointers.

a) The quantities r, s, ρ, σ that were employed temporarily in § 28 as line pointers were likewise *coefficients* in the equations of two planes that went through the line, and indeed, two distinguished (projecting) planes. Analogously, if a line g is given then we would like to consider the distinguished planes ε_i through g and the vertices P_i of the basic tetrahedron by employing tetrahedral point and plane pointers, namely, to calculate the coefficients of their equations, or – what amounts to the same thing – their *pointers*. Two such planes would, in general, suffice for the determination of g. In order to not show any preference then, we would like to consider all four of them. We think of g as being given by the pointers v_i, w_i of two planes ε_v, ε_w that go through g. All four planes will belong to the pencil of planes (ε_v, ε_w). The pointers of ε_1 must then be obtained in the form:

(12) $$u_i = \lambda\, v_i + \mu\, w_i \qquad (i = 1, 2, 3, 4).$$

For the determination of the ratio $\lambda : \mu$, one can appeal to the fact that $u_1 = 0$ for a plane that goes through P_1 . Thus, equation (12) will give:

$$\frac{\lambda}{\mu} = -\frac{w_1}{v_1}, \qquad \text{e.g.,} \quad \lambda = -w_1, \qquad \mu = v_1 .$$

If we substitute this into equation (12) for i = 1, 2, 3, 4 then we will get:

$$0, \quad v_1\, w_2 - v_2\, w_1\,, \quad v_1\, w_2 - v_2\, w_1\,, \quad v_1\, w_2 - v_2\, w_1$$

for the pointers of the plane ε_1 . One computes the pointers of ε_2 , ε_3 , ε_4 analogously. We summarize them in the following table, in which we have set:

$$v_i\, w_k - v_k\, w_i = p_{ik}\,;$$

one then has:

(13) $$p_{ki} = -\,p_{ik}\,.$$

Table of the pointers of the planes through g and the vertices of the basic tetrahedron:

(14)
$$\begin{array}{c|cccc} \varepsilon_1 & 0 & p_{12} & p_{13} & p_{14} \\ \varepsilon_2 & p_{21} & 0 & p_{23} & p_{24} \\ \varepsilon_3 & p_{31} & p_{32} & 0 & p_{34} \\ \varepsilon_4 & p_{41} & p_{42} & p_{43} & 0 \end{array}$$

Due to equation (13), one can write the numbers in this table in such a way that only the six quantities:

$$p_{12}\,,\; p_{13}\,,\, p_{14}\,,\, p_{23}\,,\, p_{34}\,,\, p_{42}$$

appear in it (*). Since the v and w by which the p are defined are already determined only up to a common factor, only the ratios of the p will come under consideration, as well.

(*) One sees that the two indices are written in their natural sequence everywhere, *except for* p_{42} . The fact that p_{42}, not p_{24}, was singled out from the six pointers has its roots in the fact that the formulas would be clearest then. For example, a minus sign would appear in equation (16) otherwise.

We now define six numbers that behave like these six quantities p to be the line pointers of the line g, where (with the inclusion of a proportionality factor ρ):

$$\rho\, p_{ik} = v_i\, w_k - v_k\, w_i \tag{15}$$

will be preserved as the ultimate definition of the quantities p. Since only the ratios of the six quantities p come under consideration, they will be equivalent to five numbers. There must then exist a relation between them, since we already know (Intro. and § 28) that a line can have only four mutually-independent pointers. One will find this relation when one develops the determinant:

$$\Delta = \begin{vmatrix} v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \\ v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \end{vmatrix}$$

in the elements of the first two rows (Laplace's determination theorem; cf., Pascal, *Determ.*, § 4). Thus (when we omit a factor of 2):

$$p_{12}\, p_{34} + p_{13}\, p_{42} + p_{14}\, p_{23} = 0. \tag{16}$$

Any six pointers of a line will fulfill this equation, which will be an identity as long one goes back to the v and w. A line is then, in general, already determined by five of its the pointers (whose ratios are equivalent to the four quantities). Indeed, the fact that g is already determined by two of the planes ε_i corresponds to the fact that any two rows of the table (14) contain five p quantities. Nevertheless, one will preserve all six pointers, due to symmetry, and yet another reason that will be illuminated in § 39, *a*).

Conversely, any six numbers that fulfill condition (16) can be considered to be pointers of a line. If one then builds the table (14) with the six numbers then one can write down the equations of four planes ε_i corresponding to that table, which one can show will all go through one and the same line. This will be the case when two of the four triples of ε_i have a common line of intersection. It is once more characteristic of this that all determinants of order three of the three-rowed matrices of these two triples must vanish. One can then verify that the adjoints of two rows of the table (14) will vanish as long as the equation (16) is fulfilled. This table is [due to equation (13)] skew-symmetric. Thus, the adjoints of the principal diagonal terms will vanish as skew-symmetric determinants of odd order (cf., Pascal, *Determ.* § 16). The adjoints of two elements that are symmetric to the principal diagonal will differ only by the factor of $(-1)^3$. One then has to carry out the calculation only for the adjoints of five elements. For example, the adjoint of p_{12} is:

$$p_{43}\, (p_{14}\, p_{23} + p_{13}\, p_{42} + p_{12}\, p_{34}).$$

It will then vanish because of equation (16); the remaining adjoints will likewise contain the left-hand side of equation (16) as a factor.

Theorem 45: *The lines in space and the ratios of six numbers that satisfy equation* (16) *are in one-to-one correspondence with each other.*

The lines at infinity are also included in this; the pencil ε_v, ε_w would then become only a pencil of parallel planes, and the conclusions would remain valid.

b) In a), we thought of a line as the carrier of a pencil of planes, and calculated the pointers of those planes of the pencil that went through the vertices P_i of the basic tetrahedron. From the principle of duality, we must acknowledge that the following process is justified: We think of g as the carrier of a point sequence through two of its points P_y, P_z, whose pointers y_i, z_i are given, and calculate the point of intersection S_i with the tetrahedral faces E_i . If P is any point of the sequence g with the pointers x_i then one will have:

$$x_i = \lambda\, y_i + \mu\, z_i \qquad (i = 1, 2, 3, 4).$$

Should P coincide with S_1 then one would need to have $x_1 = 0$; we can thus set $\lambda = -z_1$, $\mu = y_1$. If σ is an arbitrary proportionality factor then we will set:

$$\sigma\,\pi_{ik} = y_i\, z_k - y_k\, z_i \qquad (\text{so } \pi_{ki} = -\pi_{ik}) \tag{17}$$

and get:

$$0, \quad \pi_{12}, \quad \pi_{13}, \quad \pi_{14}$$

for the pointers of S_1 . One already sees that one can carry over all of the essential results of *a*). Namely, the six numbers:

$$\pi_{12}, \quad \pi_{13}, \quad \pi_{14}, \quad \pi_{23}, \quad \pi_{34}, \quad \pi_{42}$$

fulfill the relation:

$$\pi_{12}\, \pi_{13} + \pi_{13}\, \pi_{42} + \pi_{14}\, \pi_{23} = 0, \tag{16a}$$

and under this assumption their ratios will be in one-to-one correspondence with the lines in space.

e) Since the determination of a line is achieved from either the system of p or the system of π, one must be able to determine the one system from the other one. In order to find that connection, we next write the table (14) and the corresponding one for π next to each other:

$$\begin{matrix} 0 & p_{12} & p_{13} & p_{14} \\ p_{21} & 0 & p_{23} & p_{24} \\ p_{31} & p_{32} & 0 & p_{34} \\ p_{41} & p_{42} & p_{43} & 0 \end{matrix} \qquad \begin{matrix} 0 & \pi_{12} & \pi_{13} & \pi_{14} \\ \pi_{21} & 0 & \pi_{23} & \pi_{24} \\ \pi_{31} & \pi_{32} & 0 & \pi_{34} \\ \pi_{41} & \pi_{42} & \pi_{43} & 0 \end{matrix} \tag{18}$$

A row on the left represents the pointer of connecting plane (g, P_i), while a row on the right represents an intersection point (g, E_i). Since each of the four points S_i lies in each of the planes ε_i, resp., one must get zero when one combines a row on the left with a row on the right in such a way that one forms the sum of the products of just as many elements (§ 29, *b*). This gives sixteen relations, four of which – viz., the ones that arise from combining the same number of rows – will be three-parameter relations, while the other ones will be two-parameter relations. We write down those of the latter that arise from employing the first row on the left:

$$p_{13}\,\pi_{23} + p_{14}\,\pi_{24} = 0,$$
$$p_{12}\,\pi_{32} + p_{14}\,\pi_{34} = 0,$$
$$p_{12}\,\pi_{42} + p_{13}\,\pi_{43} = 0,$$

which we can also write:

(19)
$$\frac{p_{12}}{\pi_{34}} = \frac{p_{13}}{\pi_{42}} = \frac{p_{14}}{\pi_{23}}.$$

This already follows from any two of the relations. Thus, each of them will be a consequence of the other two. Analogously, one will obtain:

$$\frac{p_{12}}{\pi_{34}} = \frac{p_{23}}{\pi_{14}} = \frac{p_{42}}{\pi_{34}}$$

by applying the second row on the left. We still lack a relation for p_{34}. One obtains it, e.g., from the third row on the left and the second one on the right:

$$\frac{p_{34}}{\pi_{12}} = \frac{p_{13}}{\pi_{42}}.$$

One sees that the chain of equations (19) and the two following ones have a term in common. One can thus combine them into a single one:

(20)
$$\frac{p_{12}}{\pi_{34}} = \frac{p_{13}}{\pi_{42}} = \frac{p_{14}}{\pi_{23}} = \frac{p_{23}}{\pi_{14}} = \frac{p_{34}}{\pi_{12}} = \frac{p_{42}}{\pi_{13}}.$$

On the sequence of indices, cf., the rem. in *a*).

One refers to the p as *axial pointers* and the π as *ray pointers*, since g can be thought of as the axis of a pencil of planes by the definition of p and the carrier of a point sequence by the definition of the π. The axial pointers are thus identical with the ray pointers, up to the sequence (*). If one denotes:

(*) Since it superfluous to keep both kinds of pointers in most investigations, one must then prefer one of them, at least, for its *notation*. There is no point in introducing a common symbol r_{ik}, as *Koenigs* did (*Géom. reglée*, pp. 8). The equality is therefore only apparently justified. One must then decide whether r_{ik} should be identical to p_{ik} or π_{ik}, and when Koenigs chose the former, he also actually gave preference to the axial pointers. Many authors conversely denote the ray pointers by p_{ik} and the axial pointers by π_{ik} or q_{ik}. That fact is, in and of itself, unimportant, but the following situation speaks for the denoting of the axis pointers by p_{ik} (with *Koenigs*): It is generally customary to keep the symbol p as the line pointers where a difference between axis and ray pointers is not necessary. By our choice, p_{ik} has a simple connection to the moment of the line g relative to the tetrahedral edge $P_i\,P_k$ (§ 34). Generally, the same thing will be true for

$$P = p_{12}\, p_{34} + p_{13}\, p_{42} + p_{14}\, p_{23} ,$$
$$\Pi = \pi_{12}\, \pi_{34} + \pi_{13}\, \pi_{42} + \pi_{14}\, \pi_{23}$$

then one can also briefly write the connection as:

$$\rho\, p_{ik} = \frac{d\Pi}{d\pi_{ik}}, \qquad \sigma\, \pi_{ik} = \frac{dP}{dp_{ik}}. \tag{21}$$

§ 33. Rectangular, homogeneous and inhomogeneous, line pointers.

The line pointers will keep their well-defined meanings for an arbitrary choice of basic tetrahedron, even when one of its planes is shifted to infinity. We would like to see how the connection between the line pointers of g and the point and plane pointers of the elements by which we think of g as being given will be specialized when E_1 coincides with the plane at infinity and, at the same time, the other three planes define a rectangular vertex. We prepared for this in § 31 and now assume, as we did there, that the unit point and unit plane are such that $x_2 : x_1$ goes to x, etc. (Theorem 43). One had:

$$\rho\, p_{ik} = v_i\, w_k - w_i\, v_k ,$$
$$\sigma\, \pi_{ik} = y_i\, z_k - z_i\, y_k .$$

Since we are dealing with two planes and two points here, we would like to distinguish their rectangular pointers in such a way that we put primes on the plane w_i and the point $(z_i) \equiv P'$. From Theorem 43, one will then have to replace:

$$\frac{v_2}{v_1}, \frac{v_3}{v_1}, \frac{v_4}{v_1}; \quad \frac{w_2}{w_1}, \frac{w_3}{w_1}, \frac{w_4}{w_1}; \quad \frac{y_2}{y_1}, \frac{y_3}{y_1}, \frac{y_4}{y_1}; \quad \frac{z_2}{z_1}, \frac{z_3}{z_1}, \frac{z_4}{z_1}$$

with the symbols below:

$$u, \quad v, \quad w; \quad u', \quad v', \quad w'; \quad x, \quad y, \quad z; \quad x', \quad y', \quad z',$$

resp. If we then set $\rho = \rho'\, v_1\, w_1$, $\sigma = \sigma'\, y_1\, z_1$, and then once more write ρ, σ, instead of ρ', σ', then we will get:

$$(22) \qquad \begin{aligned} \rho\, p_{12} &= u' - u, & \rho\, p_{23} &= uv' - u'w, \\ \rho\, p_{13} &= v' - v, & \rho\, p_{34} &= vw' - v'w, \\ \rho\, p_{14} &= w' - w, & \rho\, p_{42} &= wu' - w'u, \end{aligned}$$

π_{ik} when one considers the moment relative to the edge (E_i, E_k). However, it is more convenient to regard the edge as the join of two points than the intersection of two planes.

$$\text{(23)}\qquad \begin{aligned} \sigma\pi_{12} &= x' - x, & \sigma\pi_{23} &= xy' - x'y, \\ \sigma\pi_{13} &= y' - y, & \sigma\pi_{34} &= yz' - y'z, \\ \sigma\pi_{14} &= z' - z, & \sigma\pi_{42} &= zx' - z'x. \end{aligned}$$

We set $\sigma = 1$ and denote:

$$\text{(24)}\qquad \begin{aligned} x' - x &= q_1, & yz' - y'z &= q_4, \\ y' - y &= q_2, & zx' - z'x &= q_5, \\ z' - z &= q_3, & xy' - x'y &= q_6. \end{aligned}$$

We call these six quantities q the *homogeneous, rectangular pointers* of the line PP'. Once again, only their ratios will come under consideration, since P, P' are arbitrary on it, but with the restriction that the sign of the pointers has a geometric interpretation. Namely, q_1, q_2, q_3 are the cosines of the direction of P, which are proportional to P'. We summarize the connection between the quantities p, p, q in the following table:

$$\text{(25)}\qquad \begin{array}{cccccc} \pi_{12} & \pi_{13} & \pi_{14} & \pi_{34} & \pi_{42} & \pi_{23} \\ q_1 & q_2 & q_3 & q_4 & q_5 & q_6 \\ p_{34} & p_{42} & p_{23} & p_{12} & p_{13} & p_{14}\,. \end{array}$$

The relation (16) will then go to:

$$\text{(26)}\qquad \sum_{i=1}^{3} q_i\, q_{i+3} = 0.$$

Conversely, if six quantities q that fulfill (26) are given then we already know from § 32 (Theorem 45) that they will be the pointers of a line. In order to find two points on it, one can choose – say, z – arbitrarily and calculate the other pointers from equations (24).

q_4, q_5, q_6 also have a simple geometric meaning. We next ascertain them under the assumption that the absolute values of the q are fixed by:

$$\text{(27)}\qquad q_1^2 + q_2^2 + q_3^2 = 1.$$

This can be achieved by multiplying by the factor $1/\sqrt{q_1^2 + q_2^2 + q_3^2}$ in two ways, due to the double-valuedness of the root (for the case of $q_1 = q_2 = q_3 = 0$, cf., § 36). Now, one has:

$$\text{(28)}\qquad PP' = 1.$$

When this condition is fulfilled, we will call the six pointers *normal pointers* and denote them by q'. If P_1 is the origin of the pointer system then the equation of the plane $P_1PP' \equiv E$ will have the form:

$$Ax + By + Cz = 0$$

and will be fulfilled by the pointers of P and P'. One will then have:

$$A : B : C = yz' - y'z : \ldots = q_4 : q_5 : q_6 ;$$

i.e., if n is the normal to E then q_4, q_5, q_6 will be proportional to the cosines of the direction n, and one will fix the direction of n to be the direction from which the triangle P_1PP' is seen to be positive. One convinces oneself of the latter most simply by choosing the segment PP' to be special (e.g., in the XY-plane). The same thing must then be true in general, since such relations are invariant under a pointer transformation (cf., § 12, *e*). In order to ascertain the meaning of the absolute values of q_4', q_5', q_6', we lay a segment s on n in the direction that was just fixed with a length that is equal to the shortest distance from the line g to P_1 . Due to equation (28), s will also be the moment of the rod PP' relative to P_1 (cf., § 12, *c*). Thus, if s_1, s_2, s_3 are the projections of s onto the pointer axes then they will represent the components of that moment, and the faces will be the projections of the triangle P_1PP'; i.e., the quantities $yz' - y'z$ will be numerically equal (both of them will be multiplied with the cosine of the same angle); s_1, s_2, s_3 will then be identical with q_4', q_5', q_6', respectively [see *Drach*: "Zur Theorie der Raumgeraden and der lin. Kompl.," Math. Ann. II (1869)].

Theorem 46: *Of the six normal pointers for a line g,* q_1', q_2', q_3' *are the direction cosines of the line and* q_4', q_5', q_6' *are the projections of a vector onto the axes whose length is equal to the shortest distance from the origin to the line and is directed along a normal to the connecting plane* (g, U) *on the same side from which the sense of rotation is positive when the direction of g is determined on U; i.e.,* q_4', q_5', q_6' *are the moments of g relative to the X, Y, Z-axes.*

One can confirm the last part of the theorem immediately when one – e.g. – lays a unit rod UC along the Z-axis and calculates the moment M_z of the rods PP', UC *as in* § 12, *e*). Since C has the pointers 1, 0, 0, one will find that:

$$M_z = \begin{vmatrix} 1 & 1 & 1 & 1 \\ x & x' & 0 & 0 \\ y & y' & 0 & 0 \\ z & z' & 0 & 1 \end{vmatrix} = xy' - x'y = q_6 .$$

Thus, $M_z = q_6'$ when PP' is a unit rod.

We would like to ascertain the connection between the pointers q and the quantities r, s, ρ, σ of § 28: If the points $P \equiv (x, y, z)$, $P' \equiv (x', y', z')$ lie on the line:

$$\xi = r\,\zeta + \rho, \qquad \eta = s\,\zeta + \sigma$$

then the following equations will be fulfilled:

$$\begin{aligned} x &= r\,z + \rho, & y &= s\,z + \sigma, \\ x' &= r\,z' + \rho, & y' &= s\,z' + \sigma . \end{aligned}$$

If one eliminates r, s, ρ, σ, in sequence and then defines $xy' - x'y$ from them then if one recalls equations (24), one will find that:

$$
\begin{aligned}
r &= \frac{q_1}{q_3}, \qquad & s &= \frac{q_2}{q_3}, \\
\rho &= -\frac{q_5}{q_3}, \qquad & \sigma &= \frac{q_4}{q_3}, \\
& r\,\sigma - s\,\rho = -\frac{q_6}{q_3}. \qquad (29)
\end{aligned}
$$

If one denotes $r\,\sigma - s\,\rho$ by η then the relation (26) will go to the identity:

$$r\,\sigma - s\,\rho - \eta = 0.$$

The five quantities r, s, ρ, σ, η are called *inhomogeneous, rectangular line pointers.*

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§ 34. Geometric meaning of the tetrahedral line pointers.

We have seen that three of the rectangular normal pointers of a line g are equal to the moments of g with respect to the three finite edges of the basic tetrahedron, and we ask whether a similar relationship can also exist for general tetrahedral pointers. We assume that the basic tetrahedron $P_1\,P_2\,P_3\,P_4$ has a positive volume. The moment of any two opposite edges $P_1\,P_2$, $P_3\,P_4$; $P_1\,P_3$, $P_4\,P_2$; $P_1\,P_4$, $P_2\,P_3$ will be positive then, although one must observe that the indices are written in the same sequence as the indices in the pointers of a line or in the relation (16). We draw a plane $E \perp P_3\,P_4$ through an edge $P_1\,P_2$ and consider the point of intersection O to be the origin of a rectangular system (Fig. 36) of the first kind, in which the X-axis might fall along OP_1 and the Z-axis might fall in the direction $P_3\,P_4$. As a result of our assumptions, the angle $P_1OP_2 = \omega$ will always be acute. We determine a line g (along with its positive direction) by a unit rod PQ on it; its projection $P'Q'$ onto the reference plane E appears in Fig 37. Let the pointers of P be ξ_1, η_1, ζ_1 in the rectangular system and y_i in the tetrahedral one; one will have analogous pointers ξ, η, ζ, z_i for Q. Let the distances to the points P, Q from the reference plane E_2 be d_2, δ_2; one will have analogous distance d_1, δ_1 for E_1. We would like to calculate the moment M_{34} of the rod PQ relative to the Z-axis from Chap. I, equation (17). We can shift ξ_2, η_2, ζ_2 to O, set $\cos\gamma_2 = 1$, $\cos\alpha_2 = \cos\beta_2 = 0$, and obtain:

$$M_{34} = \xi_1 \cos\beta_1 - \eta_1 \cos\alpha_1,$$

or since:

$$\cos\alpha_1 = \xi - \xi_1, \qquad \cos\beta_1 = \eta - \eta_1,$$

one will get:

$$M_{34} = \xi_1\eta - \eta_1\xi, \qquad (30)$$

which one usually writes down immediately, since the moment must be twice the area of the triangle $OP' Q'$. We would like to convert this expression such that only the tetrahedral pointers of the points P, Q enter into it. One can set (§ 29; the unit point lies inside the tetrahedron):

$$\frac{y_2}{\kappa_2} = d_2 = \eta_1 , \qquad \frac{z_2}{\kappa_2} = \delta_2 = \eta . \tag{31}$$

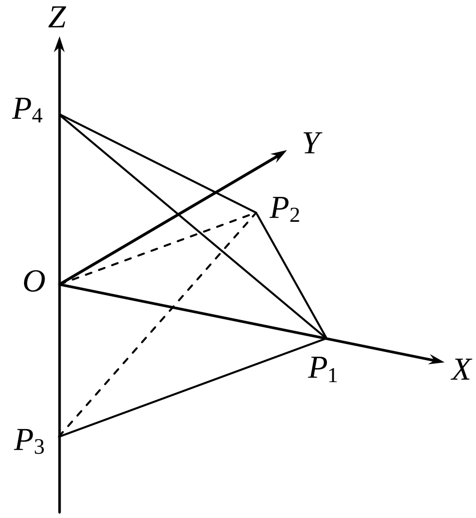

Figure 36.

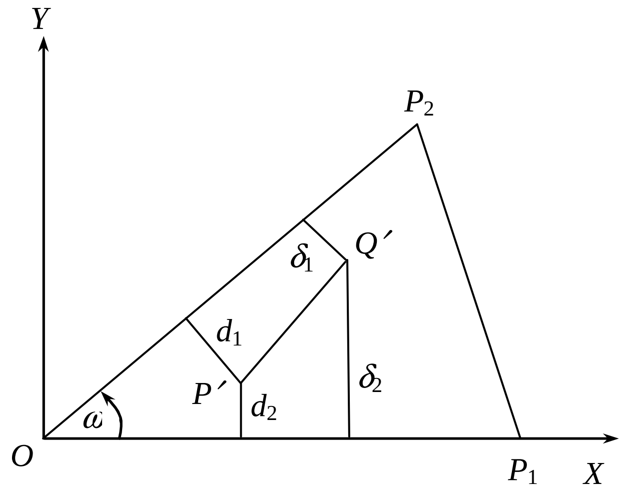

Figure 37.

If we rotate the rectangular system through the angle ω and denote the new pointers with primes then the positive side of the pointer planes will be carried along in such a way that they will arrive outside of the tetrahedron. For that reason, one will have:

$$\frac{y_1}{\kappa_1} = d_1 = -\eta_1', \qquad \frac{z_1}{\kappa_1} = \delta_1 = -\eta, \tag{32}$$

and furthermore:

$$\xi = \xi' \cos\omega - \eta' \sin\omega,$$
$$\eta = \xi' \sin\omega + \eta' \cos\omega,$$

so

$$\eta' = -\xi \sin\omega + \eta \cos\omega,$$

and if one recalls equations (30) and (31) then:

$$\xi \sin\omega = \frac{z_1}{\kappa_1} + \frac{z_2}{\kappa_2}\cos\omega,$$

$$\xi_1 \sin\omega = \frac{y_1}{\kappa_1} + \frac{y_2}{\kappa_2}\cos\omega.$$

With that, one will have:

$$(\xi_1\,\eta - \eta_1\,\xi)\sin\omega = \frac{y_1 z_2 - y_2 z_1}{\kappa_1 \kappa_2}.$$

If we set the factors ρ and σ equal to + 1 in equations (3), (4), (15), and (17) then we will get:

(33) $$\kappa_1\ \kappa_2 \sin \omega_{34} \cdot M_{34} = \pi_{12} = \pi_{34}\ .$$

From the remark that was made at the beginning of this paragraph, the analogous formulas for all remaining pointers will also be always correct (up to sign) when ω_{ik} means the internal face angle at the tetrahedral edge $P_i\ P_k$ in question everywhere. If one lets the unit point and a line g pass through a tetrahedral plane E_l by fixing the tetrahedron then the sign of π_{ik} will change when l is one of the indices i, k. By contrast, when one switches the notations of two tetrahedral vertices, with which, the moment of two opposite edges will become negative, then one will increase $\angle\ P_1OP_2$. If i, k, l, m are always the four indices 1, 2, 3, 4 in any sequence then it will follow from this that:

Theorem 47: *Let* $Y \equiv (y_i)$ *and* $Z \equiv (z_i)$ *be two points at a unit distance, let* ω_{ik} *be the absolute value of the face angle at the edge* $P_i\ P_k$ *of the basic tetrahedron, and let* M_{ik} *be the moment of the rod* YZ *relative to a unit rod that lies in the edge* $P_i\ P_k$ *and coincides with its direction. Moreover, one will set* $\rho = \sigma = 1$ *in equations* (3), (4), (15), (17), *such that the line pointers* π *and the* p *(which, from equation* (20) *are set equal to them) will also be fixed (up to absolute value) by:*

(34) $$\pi_{lm} = y_l\ z_m - y_m\ z_l\ .$$

One will then have:

(35) $$p_{ik} = c_{ik}\ M_{ik}\ ,$$

where:

(36) $$c_{ik} = \pm\ \kappa_l\ \kappa_m \sin \omega_{ik}\ .$$

In this, the upper sign will be valid in any of the six equations for a basic tetrahedron with a positive volume when the unit point lies in the same wedge at the edge $P_i\ P_k$ *in which the tetrahedron itself lies, or when it lies in the opposite wedge; otherwise, the lower sign will be true. The opposite thing will be true for a basic tetrahedron with a negative volume. The (always positive) constants* κ *have the same meaning that they did in* § 29.

If one drops the assumption on the absolute values of the line pointers, or – what amounts to the same thing – the length of the rod YZ, then one can always still say: The line pointers p behave like the moments of the line g relative to the unit rod on the same-named tetrahedral edges when each moment is multiplied with a certain constant (that is independent of the position of the line). The ratios of these constants will then be determined by equation (36) alone.

---

§ 35. Rod pointers.

If we fix – say – P of the points P, P' in § 33 and displace P' along the line PP' then the absolute values of the rectangular pointers of the line (24) will change proportional to the distance:

$$PP' = d,$$

as would emerge from the derivation of the geometric meaning of q_4', q_5', q_6', and can, by the way, be confirmed arithmetically from:

$$q_1 = x' - x = d \cos(g, x) = dq_1',$$
$$q_4 = yz' - y'z = y(z + dq_3') - (y + dq_2')\, z = d(yq_3' - zq_2'),$$

etc., where the factor of d is constant. The absolute values of the pointers of a line will thus take on a meaning as soon as one considers, not only the line, but also a length on it – i.e., a *rod* (§ 12, *a*). The numbers q that are defined by equations (24) will become the *pointers of a rod* in that way. There is a *five-fold* infinitude of rods in space. In connection with Theorem 46, it follows that:

Theorem 48: *Any six numbers q that satisfy the relation* (26) *are rectangular pointers of a rod of length $d = \sqrt{q_1^2 + q_2^2 + q_3^2}$ that has the moments q_4, q_5, q_6 with respect to the axes; the direction cosines of its carrier are proportional to q_1, q_2, q_3. In order to find a point on the carrier, one can choose perhaps z' arbitrarily and then calculate the remaining pointers of the origin from equations* (24).

Something analogous is true for tetrahedral pointers: If we fix the point Y in Theorem 47 and move Z along g then we can set:

$$z_i = y_i + \zeta_i,$$

where the various values ζ_i will be proportional to the distance d from the point Z to Y for the same i, since the distances from a point to one and the same pointer plane will also be proportional to the same-named pointers in tetrahedral pointers. Thus, $\zeta_i = c_i\, d$, and:

$$\pi_{lm} = y_l\,(y_m + z_m) - y_m\,(y_l + z_l) = d\,(y_l l\, c_m - y_m\, c_l),$$

where the factor next to d is independent of the position of the point Z; thus:

Theorem 49: *If one fixes the absolute values of the pointers of a line as in Theorem 47 then they can be considered to be the pointers of a rod YZ. Conversely, if the pointers of a rod are given then one will find two points that represent the rod when one sets e.g., $z_1 = 0$, chooses y_1 arbitrarily, and calculates the remaining six quantities z_i, y_i from them using equations* (34).

---

§ 36. Pointers for a field (rotational moment). The ultimate concept of screw.

If $q_1 = q_2 = q_3 = 0$ then equation (26) will always be fulfilled. However, we will still not get a rod in the proper sense. In order to recognize whether, and in which sense, q_4, q_5, q_6 can, in turn, be regarded as the pointers of a spatial structure, we link this case to the regular one by a consideration of the limit: We let g go to infinity in the plane E of § 33 in an arbitrary way, and simultaneously reduce to rod on g such that its rotational moment relative to the origin P remains constant. q_4, q_5, q_6 will then suffer no change during this (Theorem 48), while q_1, q_2, q_3 will diminish proportional to the rod length and vanish. The triangle surface that is determined by P_1 and the rod will be the one that remains unchanged during the entire process, and the position of its plane, its magnitude, and the sense of traversal will come under consideration on it.

This argument will be coupled to another one and completed: If the rod $P'P''$ means a force then, with the inclusion of the rotational moment that it exerts relative to the origin, we can let it go through the origin (§ 14), and thus decompose rotational moments, as well as forces, into three components. In that way, we will come to the six pointers q_1, …, q_6 precisely, which can then be regarded as the *pointers of a force*. If q_i' is a second force whose carrier might cut that of q_i then both forces will have a resultant whose pointers p_i one will find by algebraic addition of the same-named pointers of the individual forces:

$$p_i = q_i + q_i' .$$

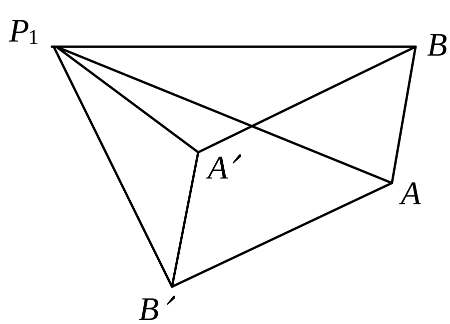
Figure 38.

If q_i and then q_i' define a force-couple then one will get $p_1 = p_2 = p_3 = 0$ from the same process. On the other hand, p_6 means the algebraic sum of the XY-projections of the double triangle that the two rods determine with the origin. Now, the vector components of the rod are equal and opposite, so (Fig. 38):

$$2ABP_1 + 2A'B'P_1 = 2ABP_1 - 2B'A'P_1 = ABA'B' ;$$

i.e.: the origin drops out completely, and p_6 means a parallelogram in the XY-plane, and only its magnitude and sense of traversal will come under consideration. On the other hand, one knows that the essence of a force-couple lies only in its rotational moment, which is characterized by a surface patch in the plane in which it acts (or one that is parallel to it). If we ignore the mechanical meaning then we will now consider:

$$0, 0, 0, p_4, p_5, p_6$$

to be the *pointers of a surface patch, and only its magnitude, its sense of traversal, and the orientation of its plane* will come under consideration, which we can also think of as being arbitrarily (and also curvilinearly) bounded. The three pointers of the surface patch are its three projections onto the three pointer planes. According to *Grassmann, Jr.*, such a surface patch is called a *field*, while it will be called a *plate* when the position (not just the orientation) of its plane also comes under consideration (cf., *Grassmann*, Ges. W., Bd. I, *b*, pp. 438).

We have seen how the rod will go to a field when it is moved to infinity and how the force that it perhaps represents will simultaneously be replaced with a rotational moment. We would like to interpret the analogous passage to the limit in the event that the rod s means the unit rod of the axis of a rotation, and σ denotes a line that remains fixed under the motion of the body. If s is moved to infinity then the field f that thus emerges will then mean a plane whose orientation will remain fixed during the motion, and the rotation will go to a displacement T that is perpendicular to the field. The moment of a force k with respect to σ will be measured by the six-fold tetrahedral volume that the two rods determine. k and f also determine a volume V when one makes f a cylindrical body whose generator has the length k. The component of k that comes under consideration for the motion in the direction T is now, in fact, proportional to V, and indeed equal to V precisely, in the event that f is a *unit field.* We will therefore also call V the moment of k relative to f, here, and say "the moment of a force system" to mean "the algebraic sum of the individual moments."

Theorem 50: *Should two force systems be equivalent then their moments relative to an arbitrary axis and an arbitrary field would have be equal to each other.*

The second part of the theorem says nothing but the fact that its components must be equal in an arbitrary direction; however, the consideration above makes this appear to be a special case of the first part.

The moment of the force-couple f relative to k will also be represented by the same volume V, if conversely f represents a force-couple and k, a *unit rod*. The moment of a field (i.e., a force-couple) relative to another one will now be defined completely by the foregoing passage to the limit, but it will always be zero.

A field is an adequate geometric representation of a rotational moment (which we have represented by a vector up to now according to § 13), and can likewise serve as the representation of a translational velocity (in the form of a rotational-couple) by means of duality. From now on, we will then also frequently think of a dyname and a twist as represented by a rod s and a field f that is perpendicular to it, instead of by a rod and a vector. This aggregate $\mathfrak{A}$ of a rod and a field is the purely geometric concept that remains when one abstracts, on the one hand, from the mechanical meaning of the dyname, and on the other hand, from the kinematical one of the twist. $\mathfrak{A}$ also determines completely the null system N (with the pitch $\mathfrak{k} = f : s$) that belongs to the dyname or twist. In § 18, we thought of N as being represented by a certain helix, namely, a *screw*. We can now just as well think of N as being represented by $\mathfrak{A}$ and carry over the term "screw" to an *aggregate of a rod and a field that is perpendicular to it.* We can do this, since only the rules of composition and decomposition of such aggregates are necessary for that concept, but not the way that we choose to illustrate it; in an equally-justified way, it can either (in closer connection with the twist) happen for a helix or (in connection with the dyname) with an aggregate of a rod and a field. We have thus generally enlarged the concept, to some extent, since the absolute values of s and f also come under consideration now, which were not represented for the helix, in its own right. Where it is not expressly given, it will emerge from the context whether we are using the concept of "screw" in its narrow (i.e., non-metric) meaning or the broader (i.e., metric) meaning. Dynames, screws, and twists will then behave conceptually like forces, rods, and rotational velocities (around a definite axis), respectively; the last three are limiting cases of the first three.

For $s = 1$, we obtain a *unit screw* (cf., the analogous concept at the conclusion of § 18).

---

§ 37. Moment of two rods and the shortest distance between two lines.

a) Let $P'P'' = p$ and $Q'Q'' = q$ be two rods with rectangular pointers p_i and q_i ; their endpoints might have the pointers $x', y', z', x'', y'', z''; \xi', \eta', \zeta'; \xi'', \eta'', \zeta''$. The six-fold volume of the tetrahedron is then (§ 12, *e*):

$$6 \cdot P'P''Q'Q'' = \begin{vmatrix} 1 & x' & y' & z' \\ 1 & x'' & y'' & z'' \\ 1 & \xi' & \eta' & \zeta' \\ 1 & \xi'' & \eta'' & \zeta'' \end{vmatrix}.$$

From Theorem 16, the moment M of the two rods will be just as large. If one develops the determinant along the first two rows (cf., § 32, *a*) then one will obtain, if one recalls equations (24):

$$M = p_1\, q_4 + p_2\, q_5 + p_3\, q_6 + p_4\, q_1 + p_5\, q_2 + p_6\, q_3$$

or

$$M = \sum_{i=1}^{6} p_i\, q_{i+3}\,,$$

in which the indices come under consideration only modulo 6. It emerges from Theorem 48 that one will get the moment of the two carriers when one divides by $p \cdot q$, with which:

$$p = \left|\sqrt{p_1^2 + p_2^2 + p_3^2}\right|, \quad q = \left|\sqrt{q_1^2 + q_2^2 + q_3^2}\right|.$$

b) If two lines p, q, along with their positive directions, are given by two rods that lie in them then let A be the foot of their shortest distance on p and let B be the one on q. We regard the segment:

$$d = AB$$

(all sign conventions are as in § 12) as the rod and call it the *rod of the distance from p to q*, or more briefly, *the distance rod* (p, q); we would like to calculate its pointers d_i .

If we always take the sense of p to q – i.e., from A to B – to be positive on d then all three quantities p, q, d will be positive. If we further denote the angle (p, q) by ω then (Theorem 48) we will have:

$$\cos\,\omega = \frac{p_1q_1 + p_2q_2 + p_3q_3}{pq}.$$

If we substitute the up-to-now known quantities into § 12, equation (15) then we will get:

$$\sum p_i\, q_{i+3} = -\,p\;q\;d \cdot \sin\,\omega.$$

We can deduce the sign from sin ω, but that expression, in its own right, can be deduced from:

$$\sin^2\,\omega = 1 - \cos^2\,\omega,$$

$$= \frac{(p_2q_3 - p_3q_2)^2 + (p_3q_1 - p_1q_3)^2 + (p_1q_2 - p_2q_1)^2}{p^2 \cdot q^2}.$$

We can see from the sign of cos ω whether the rods define an absolutely acute or obtuse angle, and from that, using Theorem 15, whether they are right-wound or left-wound. If we let w denote the square root of the numerator in the last formula, provided with the sign of sin ω, then we will get:

(37)
$$d = -\frac{\sum_{i=1}^{3} p_i q_{i+3}}{w} = -\frac{M}{w},$$

where d becomes positive, in its own right.

In order to calculate the pointers of d, we consider that d, as well as p, cuts q perpendicularly, so:

$$\sum_{i=1}^{3} p_i\, d_i = 0, \qquad \sum_{i=1}^{3} q_i\, d_i = 0, \qquad \sum_{i=1}^{3} d_i^2 = d^2,$$

(37.a)
$$d_1 = -\frac{M}{w^2}(p_2\,q_3 - p_3\,q_2), \qquad d_2 = -\frac{M}{w^2}(p_3\,q_1 - p_1\,q_3),$$
$$d_3 = -\frac{M}{w^2}(p_1\,q_2 - p_2\,q_1).$$

The parentheses in these expressions would define the direction of d from which the rotational sense of p through the concave angle to q would seem positive (cf., the derivation of Theorem 46). As one convinces oneself in any special case, the quantities d_1, d_2, d_3, and also their signs, will then be given correctly by these equations.

We find the other three pointers from the equations:

$$\sum_{i=1}^{3} p_i\, d_{i+3} = -\sum_{i=1}^{3} p_{i+3}\, d_i\,,$$
$$\sum_{i=1}^{3} q_i\, d_{i+3} = -\sum_{i=1}^{3} q_{i+3}\, d_i\,,$$

$$\sum_{i=1}^{3} d_i \, d_{i+3} = 0.$$

Their determinant is – M. One finds:

(37.b) $$d_4 = \frac{M}{w^4}\left[p_1 q^2 P + q_1 p^2 Q - (q_1 P + p_1 Q)\sum_{i=1}^{3} p_i q_i \right],$$

where:

$$P = \begin{vmatrix} p_4 & p_5 & p_6 \\ p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{vmatrix}, \qquad Q = \begin{vmatrix} q_4 & q_5 & q_6 \\ p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{vmatrix},$$

and d_5, d_6 are obtained by cyclic permutations (Cayley, *Coll. Pap.*, Vol. X, Sec. 682, 1878).

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§ 38. Incidence of a line with a point or a plane.

Should the line π_{ik} lie in the plane u_i then the pointers of the latter would have to fulfill the equations of the point of intersection of π_{ik} with the tetrahedral faces; one can deduce these equations from Table (18). One must then have:

(38) $$\begin{array}{llll} & \pi_{12}u_2 & +\pi_{13}u_3 & +\pi_{14}u_4 = 0, \\ \pi_{21}u_1 & & +\pi_{23}u_3 & +\pi_{24}u_4 = 0, \\ \pi_{31}u_1 & +\pi_{32}u_2 & & +\pi_{34}u_4 = 0, \\ \pi_{41}u_1 & +\pi_{42}u_2 & +\pi_{43}u_3 & \qquad = 0. \end{array}$$

If any two of these four equations are fulfilled then the other one will be, as well; firstly, upon geometric grounds, and secondly, because all determinants of third order from the table of p will vanish (32.a). From now on, we will frequently replace the symbols for the tetrahedral line pointers that have two indices with ones that have one index, according to table (25), except that we will now write p_i, instead of q_i . For example, we would write equations (38) in the new notation as:

(39) $$\begin{array}{llll} & p_1u_2 & +p_2u_3 & +p_3u_4 = 0, \\ -p_1u_1 & & +p_6u_3 & -p_5u_4 = 0, \\ -p_2u_1 & -p_6u_2 & & +p_4u_4 = 0, \\ -p_3u_1 & +p_5u_2 & -p_4u_3 & \qquad = 0. \end{array}$$

The conditions for the dual situation can be obtained from the left-hand side of Table (18). However, Table (25) shows that in the result (39) we only have to replace the u

with the x and simultaneously change each index of a p by 3 in order to obtain the condition for the point x to lie on the line p:

$$
\begin{array}{rrrrl}
 & p_4 x_2 & +p_5 x_3 & +p_6 x_4 & =0, \\
-p_4 x_1 & & +p_3 x_3 & -p_2 x_4 & =0, \\
-p_5 x_1 & -p_3 x_2 & & +p_1 x_4 & =0, \\
-p_6 x_1 & +p_2 x_2 & -p_1 x_3 & & =0.
\end{array}
\tag{40}
$$

Using (25), one can once more write down this condition in axial or ray pointers, as required.

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§ 39. Special positions of lines.

a) *Relative to the basic tetrahedron.*

Theorem 51: *If a line cuts the tetrahedral edge* $P_i\, P_k$ *then its pointer* p_{ik} *will vanish.*

This follows from Theorem 47; naturally, the parallel position is included in this. It also follows from the definition of π (Equation 17) that π_{ik} will vanish when the line cuts the edge $P_i\, P_k$. The reader can now easily infer what happens when the position of the line is even more special. If it ultimately coincides with $P_i\, P_k$ then p_{kl} will no longer be non-zero, except that the orientations of the pointer planes will enter in place of the tetrahedral edges $P_2\, P_3$, $P_3\, P_4$, $P_4\, P_2$.

b) *Incidence of two lines.*

Let p_{ik} and q_{ik} be the pointers for two lines p, q and:

$$p_{ik} = v_i\, w_k - v_k\, w_i\,, \qquad q_{ik} = v'_i\, w'_k - v'_k\, w'_i\,.$$

If the lines are incident (i.e., they intersect or are parallel) then the four planes v, w, v', w' will go through a point. One will then have:

$$\Delta = |\, v_i\ w_i\ v'_i\ \ w'_i\, | = 0 \qquad\qquad (i = 1, \ldots, 4).$$

On the other hand, if one develops Δ along the first two columns (cf., § 32, *a*) then one will obtain:

Theorem 52: *The condition of incidence between two lines p, q is:*

$$p_{12}\, q_{34} + p_{13}\, q_{42} + p_{14}\, q_{23} + p_{23}\, q_{14} + p_{34}\, q_{12} + p_{42}\, q_{13} = 0$$

or (25):

(41) $$\sum_{i=1}^{6} p_i\, q_{i+3} = 0.$$

If one denotes:

(42) $$\omega(p) = \sum_{i=1}^{6} p_i\, p_{i+3} = 2\sum_{i=1}^{3} p_i\, p_{i+3}\,,$$

(43) $$\omega(p, q) = \sum_{i=1}^{6} p_i\, q_{i+3} = \sum_{i=1}^{6} \frac{d\omega(p)}{dp_i} q_i = \sum_{i=1}^{6} \frac{d\omega(q)}{dq_i} p_i = \omega(q, p)$$

then the incidence condition (41) can be written:

(44) $$\omega(p, q) = 0,$$

and the condition for p to be the pointers of a line [equations (16) and (26)] will be:

(45) $$\omega(p) = 0.$$

If the condition (41) is fulfilled then that will pose the problem of calculating the pointers u_i of the connecting plane of the lines p, q. The system (39) must be fulfilled, and likewise:

(39.a) $$\begin{array}{rrrrl} & q_1u_2 & +q_2u_3 & +q_3u_4 & =0, \\ -q_1u_2 & & +q_6u_3 & -q_5u_4 & =0, \\ -q_2u_1 & -q_6u_2 & & +q_4u_4 & =0, \\ -q_3u_1 & +q_5u_2 & -q_4u_3 & & =0. \end{array}$$

If one employs the first equations in (39) and (39.a) and adds – perhaps – the second equation in (39) to this pair of equations G then one will find that:

$$u_1 : u_2 : u_3 : u_4$$

$$= \begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ 0 & p_6 & -p_5 \end{vmatrix} : - \begin{vmatrix} 0 & p_2 & p_3 \\ 0 & q_2 & q_3 \\ -p_1 & p_6 & -p_5 \end{vmatrix} : \begin{vmatrix} 0 & p_1 & p_3 \\ 0 & q_1 & q_3 \\ -p_1 & 0 & -p_5 \end{vmatrix} : \begin{vmatrix} 0 & p_1 & p_2 \\ 0 & q_1 & q_2 \\ -p_1 & 0 & -p_6 \end{vmatrix} .$$

If one replaces the first determinant $p_2\, p_5 + p_3\, p_6$ with $-p_1\, p_4$ after performing the calculation then the factor p_1 will appear in all determinants, and if one recalls (41) then one will find that the u will be proportional to the first row of the following table (46). One will obtain the same thing that one might also get by combining the third equation with the pair G. If one then takes another pair of just as many equations from the two quadruples, instead of G, then one will obtain the ratios of the u in another form, namely [when one sets $p_i\, q_k - p_k\, q_i = (i, k)$]:

Theory 53: *The pointers of the connecting plane of two incident lines p, q are proportional to the elements of an arbitrary column of the following table (whose two parts are thought of as adjacent)*:

$$\begin{array}{cc}
p_1\, q_4 + p_2\, q_5 + p_3\, q_6 & (2, 3) \\
(6, 5) & p_1\, q_4 + p_5\, q_2 + p_6\, q_3 \\
(4, 6) & (2, 4) \\
(5, 4) & (3, 4) \\[1ex]
(3, 1) & (1, 2) \\
(1, 5) & (1, 6) \\
p_2\, q_5 + p_4\, q_1 + p_6\, q_3 & (2, 6) \\
(3, 5) & p_3\, q_6 + p_4\, q_1 + p_5\, q_2\, .
\end{array} \tag{46}$$

We will obtain the pointers of the point of intersection of those lines p, q when we change all of the indices in this table by three (§ 38), or – what amounts to the same thing – switch the rows and columns. The indices of the p that appear in any column are:

$$\begin{array}{ccc}
1 & 2 & 3 \\
1 & 5 & 6 \\
4 & 2 & 6 \\
4 & 5 & 3
\end{array}$$

One confirms the legitimacy of generating the last three triples from the table:

$$\begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 & 6
\end{array}$$

For later use, we make note of the following situation: If the quantities p fulfill the relation (45), and likewise, the q, and both of them together fulfill the incidence relation (44) then the two systems of equations:

$$\sum_{m=1}^{4} p_{km} x_m = 0, \quad \sum_{m=1}^{4} q_{km} x_m = 0 \qquad (k = 1, \ldots, 4;\ p_{kk} = q_{kk} = 0) \tag{38.a}$$

which are dual to (38), will be fulfilled by one and only one common system $x_1 : x_2 : x_3 : x_4$. This is also independent of the geometric meaning of the quantities of an analytic entity that is therefore likewise true for complex numbers p, q, x. If one substitutes the values of x that were calculated in (38.a) in the equations then they will become identities.

c) *The rays of a pencil.*

Let a_i and b_i be the pointers of two lines a and b. If we define:

$$p_i = \lambda\, a_i + \mu\, b_i \qquad (i = 1, \ldots, 6) \tag{47}$$

then we will have:

$$\omega(p_i) = \omega(\lambda\, a_i + \mu\, b_i) = \sum_{i=1}^{6} (\lambda a_i + \mu b_i)\,(\lambda\, a_{i+3} + \mu\, b_{i+3})$$
$$= \lambda^2 \cdot \omega(a) + 2\,\lambda\,\mu \cdot \omega(a, b) + \mu^2 \cdot \omega(b).$$

We will have $\omega(a) = \omega(b) = 0$, since a_i and b_i are the pointers of a line. We now assume that a, b intersect. We will then have $\omega(a, b) = 0$, and $\omega(p_i)$ will be zero identically – i.e., the p_i will always be the pointers of a line, no matter how we choose λ and μ. Since only the ratios of these parameters come under consideration, (47) will represent only a simple manifold of lines, which we will prove is identical with the pencil of rays (a, b).

Namely, let c be a line that cuts a and b; we will then have:

$$\omega(a, c) = \omega(b, c) = 0$$
$$\omega(p, c) = \omega(\lambda\, a + \mu\, b, c) = \sum (\lambda\, a_i + \mu\, b_i)\, c_{i+3} \tag{48}$$
$$= \lambda \cdot \omega(a, c) + \mu \cdot \omega(b, c) = 0;$$

i.e., p and c will be incident for all values of λ, μ. If we now choose c arbitrarily in the connecting plane E of a¸ b then condition (48) will be fulfilled, and it will follow that p will also lie in E. However, if we draw c arbitrarily through the point of intersection S of a and b then it will likewise follow that p goes through S. Therefore, p will belong to the pencil of rays (S, E).

If, conversely, p is a certain line in the pencil, and c is a line that cuts p, but not a and b simultaneously, then one can determine $\lambda : \mu$ in such a way that:

$$\lambda \cdot \omega(a, c) + \mu \cdot \omega(b, c) = 0\,.$$

These values of λ, μ, when substituted in (47), will provide the pointers of a line p' that belongs to the pencil (a, b) and likewise cuts c, and must therefore be identical with p. Therefore, the representation (48) will actually subsume *all* rays of the pencil.

As a consideration of Table (25) would show, it is irrelevant whether we identify the a, b with the pointers p or the pointers π. When we again go to two indices, we can set:

$$a_\nu = x_i\, y_k - x_k\, y_i\,,$$

where ν are chosen from the sequence 1, …, 6 and i, k are chosen from the sequence 1, 2, 3, 4, corresponding to the table (25). If we assume that the x are pointers of S then we can set:

$$b_\nu = x_i y'_k - x_k y'_i\,,$$

so

$$b_\nu = \lambda\, a_\nu + \mu\, b_\nu = x_i(\lambda\, y_k + \mu\, y'_k) - x_k\,(\lambda\, y_i + \mu\, y'_i) = x_i\, z_k - x_k\, z_i$$

will be two of the four pointers of a point Z on the connecting line of the points y, y'. If we set $\mu : \lambda = \varphi$ equal to the four values $\varphi_1, \ldots, \varphi_4$, in sequence, then we will obtain four rays $\mathfrak{p}_1, \ldots, \mathfrak{p}_4$ of them pencil (a, b) whose double ratio will be equal to that of the corresponding points Z_1, …, Z_4 , resp., through which they go. However, the latter double ratio is (cf., Hesse, *Vorl. aus d. anal. Geom. d. geraden L.*, etc.):

$$\frac{Z_1Z_3}{Z_3Z_2} : \frac{Z_1Z_4}{Z_4Z_2} = \frac{\varphi_3-\varphi_1}{\varphi_3-\varphi_2} : \frac{\varphi_4-\varphi_1}{\varphi_4-\varphi_2} . \tag{49}$$

One will then obtain (*Voss*, Math. Ann. Bd. 8, pp. 57):

Theorem 54: *If a_i and b_i are two incident rays then the pencil that they determine will be represented by:*

$$p_i = \lambda\, a_i + \mu\, b_i \qquad (i = 1, \ldots, 6),$$

where the double ratio of four rays of the pencil will depend upon the parameters λ and μ in the same way that they do for point sequences and pencils of planes.

In particular, if we let $\mathfrak{p}_1$ coincide with a and $\mathfrak{p}_2$, with b, and write $\mathfrak{p}$, $\mathfrak{p}'$, instead of $\mathfrak{p}_3$, $\mathfrak{p}_4$, resp., then we will have:

$$\varphi_1 = 0, \qquad \varphi_2 = 0, \qquad \lim \frac{\varphi_4-\varphi_1}{\varphi_3-\varphi_1} = 1,$$

so

$$(a, b, \mathfrak{p}, \mathfrak{p}') = \frac{\varphi}{\varphi'} . \tag{49.a}$$

Theorem 55: *If a_i, b_i are two incident rays then the double ratio of the four rays a_i, b_i, $a_i + \varphi\, b_i$, $a_i + \varphi'\, b_i$ will be equal to $\varphi : \varphi'$; in particular, a_i, b_i will then be harmonically separated by $a_i + \varphi\, b_i$ and $a_i - \varphi\, b_i$.*

If we define a matrix of three rows and six columns from the pointers a_i, b_i, c_i of three rays of a pencil then, from (47), all of its three-rowed determinants will vanish. If, conversely, three rays a, b, p are given for which all three-rowed determinants of the matrix of their pointers vanish then we can determine multipliers λ, μ, ν that fulfill all six equations:

$$\lambda\, a_i + \mu\, b_i + \nu\, p_i = 0 \qquad (i = 1, \ldots, 6). \tag{50}$$

The ratios of those multipliers will then be determined uniquely by two of them (say, i = 1, 2). Due to the vanishing of the determinant:

$$\begin{vmatrix} a_1 & a_2 & a_i \\ b_1 & b_2 & b_i \\ p_1 & p_2 & p_i \end{vmatrix} ,$$

equation (50) must also be satisfied for an arbitrary i. If we choose $\nu = -1$ then we will get back (47). One says that a matrix has rank r when all of its determinants of order $r + 1$ vanish, but not all of them of order r. We can then express one part of Theorem 54 as:

Theorem 56: *The necessary and sufficient condition for three rays to belong to the same pencil is that the rank of the matrix of their pointers can be reduced to two.*

d) *Three rays of a sheaf or a field.*

Let three lines p, q, r be given, whose axial pointers are p_{ik}, q_{ik}, r_{ik}, resp., and whose ray pointers are π_{ik}, κ_{ik}, ρ_{ik} , resp. We assume that each one intersects the other one without belonging to the same pencil, which one can recognize from *b*) and *c*). They must then belong to either the same sheaf or the same field. In order to decide which of the two situations is present, we consider the determinant:

$$\Delta_i = \begin{vmatrix} \pi_{ik} & \pi_{il} & \pi_{im} \\ \kappa_{ik} & \kappa_{il} & \kappa_{im} \\ \rho_{ik} & \rho_{il} & \rho_{im} \end{vmatrix}.$$

From table (18) the three non-zero tetrahedral pointers of the point of intersection of one of the lines with the tetrahedral plane E_i is in each row, and thus also its tri-metric pointers in that plane (§ 29, *f*). If p, q, r belong to the same field then these three points of intersection must lie in a line, so Δ_i must vanish for all four values of i. Naturally, the vanishing of *one* of these determinants will generally suffice for one to be able to make that decision (*).

Theorem 57: *Three lines, each of which intersects the other ones, belong to the same sheaf or field according to whether those three-rowed determinants in the matrix of their ray pointers vanish whose elements lack one index completely or the ones in which one index appears in all nine elements. One can prove the converse by the use of axial pointers.*

e) *Hyperbolic position of four lines.*

Let four lines be given. Each four-rowed determinant in the matrix of their pointers a_i, b_i, c_i, p_i (i = 1, …, 6) might vanish. One can then find a representation [cf., the conclusion of *c*)]:

$$p_i = \lambda\, a_i + \mu\, b_i + \nu\, c_i \qquad (i = 1, \ldots, 6). \tag{51}$$

If q_i is any line then one will have:

(*) If one does not make the assumption that every line cuts the other one then the vanishing of all four Δ_i will merely say that the three points of intersection lie on a line g_i in each plane of the tetrahedron. This will also happen when p, q, r belong to the same family of rulings, along with two opposite edges. An analogous remark will be true in the dual case.

(52) $$\omega(p, q) = \lambda \cdot \omega(a, q) + \mu \cdot \omega(b, q) + \nu \cdot \omega(c, q),$$

and [cf., c)]:

(53) $$\omega(p) = \lambda^2 \cdot \omega(a) + \mu^2 \cdot \omega(b) + \nu^2 \cdot \omega(c) + 2\mu\nu \cdot \omega(b, c) + 2\nu\lambda \cdot \omega(c, a) + 2\lambda\mu \cdot \omega(a, b).$$

It follows from (52) that every line q that cuts a, b, c will also cut p.

The three lines a, b, c can then belong to the same sheaf or field; p will then also belong to that sheaf or field. Conversely, six numbers that can be calculated from (51) for an arbitrary choice of λ, μ, ν will be the pointers of a line (of the sheaf or field), since $\omega(p)$ will vanish identically from (53).

If we now pass over this special case then it will follow that when three a, b, c of four given lines a, b, c, p are mutually skew to each other, all four of them will (in general) have *hyperbolic position.* Conversely, if we calculate the p_i from (51) for an arbitrary choice of λ, μ, ν then we will obtain a fourth line of the family of rulings (a, b, c) when the p_i are, in fact, pointers of a line. The condition for this is that $\omega(p) = 0$ or:

(54) $$\mu\nu \cdot \omega(b, c) + \lambda\nu \cdot \omega(c, a) + \lambda\mu \cdot \omega(a, b) = 0,$$

in which the three quantities ω are constant numbers. They will then satisfy a simple infinitude of parameter ratios $\lambda : \mu : \nu$, corresponding to the ∞^1 lines of the family of rulings. We will come back to this later.

Therefore, if, say, a and b intersect in S (cf., Fig. 5, where one must replace g, h, g', with a, b, c, resp.) and their connecting plane τ intersects c at T then the plane (c, S) will be called σ. Should q cut the three lines a, b, c then it would have to belong to the pencil (S, σ) or (T, τ); since we can make the one choice just as well as the other, p must then belong to one of the pencils (S, τ), (T, σ). Conversely, the condition (54) will reduce to:

$$\mu\nu \cdot \omega(b, c) + \lambda\nu \cdot \omega(c, a) + \lambda\mu = 0$$

in this case. It can be fulfilled by either $\nu = 0$ [which gives the pencil (S, τ)] or by:

$$\frac{\lambda}{\mu} = -\frac{\omega(b,c)}{\omega(c,a)};$$

when one assigns arbitrary values to ν, this will yield the lines of the pencil (T, σ). If we then exclude the case in which three of the four given lines a, b, c, p belong to the same pencil, as was done in Theorem 56, then p will belong to the pencil T, σ, and we will be confronting the *special hyperbolic position* (§ 5).

Theorem 58: *If the rank of the matrix of the pointers of four lines is reduced to three (but no more) then they will have either general or special hyperbolic position, or they will belong to the same sheaf or field. The choice between these two cases will follow from* b) *and* d)*.*

(On the calculation of the double ratios of four hyperbolic lines, cf., prob. 82.)

f) *Rectangular pointers.*

All of the results that were derived in this paragraph also find an application in the special case of rectangular, homogeneous pointers, *mutatis mutandis*. Here, we further add: The condition that two lines p_i, q_i are perpendicular to each other is [§ 37, *b*)]:

$$p_1\, q_1 + p_2\, q_2 + p_3\, q_3 = 0.$$

We can first speak of special positions of more than four lines later on (Theorem 102).

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§ 40. Transformation of line pointers.

If a transformation of point pointers is established by:

$$\rho\, x_i = \sum_{\lambda=1}^{4} a_{i\lambda} x'_\lambda \qquad (i = 1, 2, 3, 4) \tag{55}$$

then we will wish that the ray pointers π_{ik} of a line relative to the old tetrahedron should be expressed in terms of the pointers π'_{ik} relative to the new tetrahedron. We can set:

$$\pi_{ik} = x_i\, y_k - x_k\, y_i\,, \qquad \rho\, y_k = \sum_{\mu=1}^{4} a_{k\mu} y'_\mu\,,$$

and obtain:

$$\rho\, \rho' \cdot \pi_{ik} = \sum_{\lambda=1}^{4} \sum_{\mu=1}^{4} a_{i\lambda} a_{k\mu} x'_\lambda y'_\mu - \sum_{\lambda=1}^{4} \sum_{\mu=1}^{4} a_{k\lambda} a_{i\mu} x'_\lambda y'_\mu\,.$$

In this, $x'_\lambda y'_\mu$ have the coefficients $a_{i\lambda}\, a_{k\mu} - a_{k\lambda}\, a_{i\mu}$; $x'_\mu y'_\lambda$ have the coefficients $a_{i\mu}\, a_{k\lambda} - a_{k\mu}\, a_{i\lambda}$, which differs from the previous expression only in sign. One will then have:

$$\rho\, \rho' \cdot \pi_{ik} = \sum (a_{i\lambda}\, a_{k\mu} - a_{k\lambda}\, a_{i\mu})\, \pi'_{\lambda\mu}\,, \tag{56}$$

in which, from now on, the sum will be extended over the six combinations of two Greek indices. The different choices of *i*, *k* will yield six such equations. Due to the connection with equation (20), one can express the old ray or axial pointers arbitrarily in terms of the new ray or axial pointers, respectively. However, the essential fact is:

Theorem 59: *A transformation of the tetrahedral line pointers will be mediated by a linear, homogeneous substitution of the line pointers.*

The $a_{i\lambda}$ (λ = 1, 2, 3, 4) were the pointers of the planes E_i of the old tetrahedron relative to the new ones (§ 29, *c*). Thus, from equation (15), the quantities:

(56.a) $$a_{i\lambda}\, a_{k\mu} - a_{k\lambda}\, a_{i\mu} = \mathfrak{p}_{ik,\,\lambda\mu}$$

will be the axial pointers of the edge (E_i, E_k) relative to the new tetrahedron. If we denote [cf., the second and third row of table (25)] the index combinations:

(I) 34, 42, 23, 12, 13, 14,

respectively, by the single symbols:

(II) 1, 2, 3, 4, 5, 6,

and thus set, e.g.:

$$\mathfrak{p}_{14,\,42} = \mathfrak{p}_{62}\ ,$$

then we can also write equation (56) as:

(57) $$\sigma\,\pi_n = \sum_{\nu=1}^{6} \mathfrak{p}_{n\nu}\pi'_\nu \qquad (n = 1, \ldots, 6).$$

From the geometric meaning of the coefficients, one can easily infer that not every linear substitution of the form (57) can be interpreted as a pointer transformation. Moreover, the $\mathfrak{p}$ in each row of pointers must, first of all, be a line, so they must fulfill the relation:

$$\sum_{\nu=1}^{6} \mathfrak{p}_{n\nu}\mathfrak{p}_{n,\nu+3} = 0;$$

secondly, from Theorem 57, certain three-rowed determinants must vanish: The tetrahedral edges, which intersect in P_i (e.g., in P_2), correspond to three such pairs in row (I), in which i (e.g., 2) is absent; if we go on to row (II) then, from Theorem 57, e.g., the determinant:

$$\begin{vmatrix} \mathfrak{p}_{1\alpha} & \mathfrak{p}_{1\beta} & \mathfrak{p}_{1\gamma} \\ \mathfrak{p}_{5\alpha} & \mathfrak{p}_{5\beta} & \mathfrak{p}_{5\gamma} \\ \mathfrak{p}_{6\alpha} & \mathfrak{p}_{6\beta} & \mathfrak{p}_{6\gamma} \end{vmatrix}$$

must vanish, as long as we set α, β, γ to be numbers from row (II) that lie under three pairs from row (I) that have common numerals. Since we also have four choices for the first indices, there will be 16 relations. We will obtain just as many when we express the idea that four times three edges of the old tetrahedron lie in a plane. In addition, one can write out the conditions that the edges of the tetrahedron intersect. Naturally, not all of these equations are independent of each other; moreover, we can enumerate the number of conditions as follows: If we introduce the pointers into the expression:

$$\omega(p) = \sum_{n=1}^{6} \pi_n \pi_{n+3}$$

by means of (57) then a quadratic form that has $6 + \binom{6}{2} = 21$ terms, in general, will arise, which must then reduce to:

$$\omega(\pi') = \sum_{\nu=1}^{6} \pi'_\nu \pi'_{\nu+3}\,,$$

up to a constant factor, if (57) is to represent a pointer transformation. Namely, whenever the π mean the pointers of a line, the same thing must be true for the π', and conversely. The equations $\omega(\pi) = 0$ and $\omega(\pi') = 0$ must then be completely equivalent and go to each other under the transformation. Therefore, eighteen coefficients must vanish in the new quadratic form, while the remaining three will remain equal to each other; that will give twenty conditions, which we shall not discuss in more detail (cf., *Klein*, "Über die Transf. der allg. Gl. des 2 Gr. zw. Linienkoord. auf eine kanon. Form," Math. Ann., Bd 23, pp. 546, *et seq*.).

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§ 41. Transformation of rectangular, homogeneous, rod pointers.

From equations (24), the rectangular, homogeneous pointers of a rod $P'P''$ were:

$$\text{(24)} \qquad \begin{aligned} q_1 &= x'' - x', & q_4 &= y'\,z'' - y''\,z', \\ q_2 &= y'' - y', & q_5 &= z'\,x'' - z''\,x', \\ q_3 &= z'' - z', & q_6 &= x'\,y'' - x''\,y'. \end{aligned}$$

We rotate the pointer system around the origin such that the direction cosines of the new axes ξ, η, ζ compared to the old ones x, y, z are given by the following table:

(58)

	ξ	η	ζ
x	a_1	a_2	a_3
y	b_1	b_2	b_3
z	c_1	c_2	c_3

The new pointers p of the rod $P'P''$ will then be:

$$\text{(24.b)} \qquad \begin{aligned} p_1 &= \xi'' - \xi', & p_4 &= \eta'\,\zeta'' - \eta''\,\zeta', \\ p_2 &= \eta'' - \eta', & p_5 &= \zeta'\,\xi'' - \eta''\,\xi', \\ p_3 &= \zeta'' - \zeta', & p_6 &= \xi'\,\eta'' - \xi''\,\eta'. \end{aligned}$$

We wish to express the old rod pointers q in terms of the new ones p. With the help of the table, one will find immediately that:

(59) $$q_1 = \sum_{i=1}^{3} a_i p_i , \qquad q_2 = \sum_{i=1}^{3} b_i p_i , \qquad q_3 = \sum_{i=1}^{3} c_i p_i ,$$

and furthermore:

$$q_4 = (b_1\, c_2 - b_2\, c_1)\, p_6 + (b_2\, c_3 - b_3\, c_2)\, p_4 + (b_3\, c_1 - b_1\, c_3)\, p_5 .$$

Now, any of the nine elements in (58) is equal to its adjoint (S. S., Bd. IX, § 18). Thus:

(60) $$q_4 = \sum_{i=1}^{3} a_i p_{i+3} , \qquad q_5 = \sum_{i=1}^{3} b_i p_{i+3} , \qquad q_6 = \sum_{i=1}^{3} c_i p_{i+3} ;$$

i.e., the last three pointers are expressed in terms of the corresponding new ones in the same way as the first three. Equations (59) and (60) together represent the desired transformation for an arbitrary rotation. In particular, if the rotation takes place around the X-axis of the old system through an angle of ω then the coefficients:

(58.b) $$\begin{array}{ccc} 1 & 0 & 0 \\ 0 & \cos\omega & -\sin\omega \\ 0 & \sin\omega & \cos\omega \end{array}$$

will enter in place of the table (58) in equations (59) and (60).

The first three pointers will not change under a subsequent parallel displacement of the system with the components $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$. As for the last three, when we denote the line pointers after the displacement by κ, we will find that:

(61) $$\begin{aligned} p_4 &= \mathfrak{y}\ \kappa_3 - \mathfrak{z}\ \kappa_2 + \kappa_4 , \\ p_5 &= \mathfrak{z}\ \kappa_1 - \mathfrak{x}\ \kappa_3 + \kappa_5 , \\ p_6 &= \mathfrak{x}\ \kappa_2 - \mathfrak{h}\ \kappa_1 + \kappa_6 . \end{aligned}$$

From (59), (60), (61), one can now easily summarize the general transformation formulas. One sees that:

Theorem 60: *Under a transformation of the rectangular, homogeneous, rod pointers, the old pointers can be expressed in terms of the new ones (and conversely) in a linear and homogeneous way.*

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§ 42. Equations in line pointers.

If F_1 is an entire, homogeneous function of degree n in the six pointers p_{ik} of a line then a condition will be imposed upon the line pointers by the equation:

(62) $$F_1\,(p_{12}, p_{13}, p_{14}, p_{34}, p_{42}, p_{23}) = 0,$$

which is equivalent to the equation:

(62) $$F_1(\pi_{34}, \pi_{42}, \pi_{23}, \pi_{12}, \pi_{13}, \pi_{14}) = 0.$$

The pointers of ∞^3 lines will then satisfy the equations; i.e., (62) or (63) will define a *line complex*. In fact, one can choose three of the five pointer ratios arbitrarily, and the other two from (62), and determine:

(16) $$P = p_{12}\, p_{34} + p_{13}\, p_{42} + p_{14}\, p_{23} = 0.$$

We also call the number n the *degree* of the complex. From Theorem 59, the degree is independent of the pointer system, and is therefore something that is characteristic of the complex. In order to ascertain its geometric meaning, we think of the line pointers in (62) and (63) as being replaced with their expressions in plane (point, resp.) pointers, and obtain:

(62.a) $$F_1(v_1\, w_2 - v_2\, w_1,\ v_1\, w_3 - v_3\, w_1,\ \ldots) = 0,$$

(63.a) $$F_1(y_3\ z_4 - y_4\ z_3,\ y_4\ z_2 - y_2\ z_4,\ \ldots) = 0.$$

Here, equation (16) will be fulfilled by the arguments identically. Now, if we fix the point y in (63.a) then that equation will represent a surface of degree n in the running coordinates z, which, from the nature of things, must be a conic surface with the vertex y. It must then be the locus of the points of space whose connecting line with y is a complex ray. Thus:

Theorem 61: *The lines of a complex of degree n that go through a well-defined point generally define* (*) *a conic surface of order n; viz., the "complex cone" of the point.*

It will follow from the dual consideration to the one that is linked to equation (62.a) that:

Theorem 62: *The rays of a complex of degree n that lie in a well-defined plane will generally envelop a curve of class n; viz., the "complex curve" of that plane.*

It follows from both theorems that:

Theorem 63: *The degree of a complex is equal to the number of its rays that belong to any pencil of rays in general position.*

In particular, if $n = 1$ (i.e., a *linear* complex) then all complex rays through a point will define a plane pencil of rays, just like all complex rays in a plane, as we encountered for a twist. In fact, it will be shown (§ 46) that the linear complex is identical to a twist (or to a bush of rays).

(*) We will discuss the exceptional points for which this is not the case in vol. II, along with an analogue of Theorem 62.

Two equations $F_1 = 0$, $F_2 = 0$ of the form (62) and of degree n, n', resp., will single out ∞^2 lines in space, and will thus define a *line congruence* $\mathfrak{C}$. Each individual equation will represent a complex, and $\mathfrak{C}$ will be the totality of rays that are common to both complexes C and C' (i.e., their *intersection*). The two curves that are defined in a plane by C and C' will have classes n and n', resp., and thus $n \cdot n'$ common tangents (*), which also belong to $\mathfrak{C}$. Likewise, nn' rays of $\mathfrak{C}$ will go through a point. One calls the number of rays in a congruence that go through a point its *order* and the number of rays that lie in a plane its *class;* when the order and class are equal, one will refer to the *degree* of the congruence. It then follows that:

Theorem 64: *The congruence that is the complete intersection of two complexes of degree n, n' has degree nn'.*

If we add yet a third equation $F_3 = 0$ of the form (62) and degree n'' then all three equations will generally define a simple manifold of lines – i.e., a *ruled surface* – that includes the rays common to all three complexes (which are defined by the individual equations) or also the common rays of the congruence that is defined by two equations and the complex that is defined by the third equation. It is also called the *intersection* of three complexes (or the congruence and the complex). In order to ascertain its degree, we consider how many of its lines p can cut a certain line p' in space. For this, one must satisfy the three equations $F_i = 0$, along with equation (16) and:

$$\sum_{i=1}^{6} p_i p'_{i+3} = 0.$$

These equations have the degree numbers:

$$n, n', n'', 2, 1$$

in the p, resp., and thus determine $2nn'n''$ pointer ratios; i.e.:

Theorem 65: *Three complexes of degree n, n', n'' generally* (**) *have a ruled surface of degree $2n\,n'n''$ in common.*

It likewise follows that:

Theorem 66: *Four complexes of degree n, n', n'', n''' generally have $2nn'n''n'''$ rays in common.*

(*) If one restricts oneself to real elements then – here and later – one must add the phrase "at most" to similar theorems. We will thus also give a meaning to complex line pointers (Chap. V).

(**) Three surfaces generally intersect in a finite number of points, but they can also have a curve in common; naturally, analogous outcomes are not excluded in line geometry: Three complexes can have a congruence in common.

§ 43. Equations in rod pointers.

The manifold of rods is five-fold. The rod pointers will fulfill an equation:

$$\Phi(\pi_{ik}) = 0 \tag{64}$$

that will additionally always fulfill the relation:

$$\sum_{i=1}^{3} \pi_i \pi_{i+3} = 0, \tag{16.a}$$

which will impose a condition that singles out a *four-fold* rod manifold, which we would like to call a *rod forest.* We call the three, two, and simple rod manifolds *rod complexes*, *rod congruences*, and *rod surfaces*, respectively. They will be represented by 2, 3, 4 equations, resp., in the rod pointers (i.e., "rod equations"). Since the absolute values of the rod pointers also come under consideration, these rod equations will generally *not* be homogeneous, even if we always employ homogeneous (rectangular or tetrahedral) rod pointers. Any rod of a rod structure lies in a line – viz., its carrier. The totality of these lines will define the *carrier* of the rod structure. These carriers of the rod complexes, congruences, surfaces will then generally be line complexes, congruences, surfaces, resp., while all of line space is to be considered as the carrier of a rod forest (except for questions of reality). Only when all representative equations of a rod structure are homogeneous will one enter the case in which all rods will lie on lines on which any rod of the structure at all lies that likewise belong to the structure. The carrier manifold of the rod structure is then one dimension lower than that structure. Since this case does not serve to characterize the line structures, we will exclude it from now on. Since metric properties (above all, the length of the rod) are essential to rod structures, from now on we will think in terms of rectangular, homogeneous, rod pointers and write the equations of the rod structure in the form:

$$\begin{aligned} &\Phi\,(x' - x,\ y' - y,\ z' - z,\ yz' - y'z,\ \ldots) = 0, \\ &\Psi\,(x' - x,\ y' - y,\ z' - z,\ yz' - y'z,\ \ldots) = 0, \\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \end{aligned} \tag{65}$$

In particular, when Φ, Ψ, … are entire, rational functions of their six arguments, the rod structure will be called *algebraic.*

We next consider a single equation $\Phi = 0$. If we fix the point $P \equiv (x, y, z)$ then a surface will be represented by (65), and indeed, by the locus of the endpoints $P' \equiv (x, y, z)$ of all rods of the rod forest whose carriers go through P when one puts the starting point of the rod at P itself. If the rod forest is algebraic of degree n, in particular, then it will follow that:

Theorem 67: *If one considers all rods of an algebraic rod forest of degree n whose carriers go through a point P and takes P to be the starting point of the rod then the endpoints will lie on an algebraic surface of order n.*

In general, this surface will not go through P (except when the constant term in (65) vanishes).

We now consider *two* equations $\Phi = 0$, $\Psi = 0$; a rod complex $\mathfrak{C}$ will be defined by it. Each point P will now be assigned two surfaces whose line of intersection can be considered to be the guiding line of the complex cone of $\mathfrak{C}$ that is associated with the point P. In particular, if $\mathfrak{C}$ is algebraic and n_0, n are the degrees of Φ and Ψ, resp., then it will follow from Theorem 67 that:

Theorem 68: *If one considers all rods of an algebraic rod complex whose carrier goes through a point P and takes P to be the starting point of the rod then its endpoints will lie in an algebraic space curve of order* $n_0\, n$*; the carrier of the rod complex will then also be of the same degree.*

One deduces in the same way that:

Theorem 69: *The carrier of an algebraic rod congruence that is defined by three equations of* n_0, n, n' *is a line congruence of order* $n_0 n n'$*. If one adds a fourth equation of degree* n'' *then one will obtain a rod surface whose carrier is a ruled surface of degree* $2n_0 n n' n''$.

For the last part of this theorem, confer the derivation of Theorem 65; analogously to Theorem 66, one will have:

Theorem 70: *Five rod forests of degree* n_0, n, n', n'', n''' *have* $2n_0\, n n' n'' n'''$ *rods in common.*

Let a rod complex $\mathfrak{C}$ be given by two equations of the form (65) and degree n and m, resp. We pose the problem of calculating the equation of the carrier C of $\mathfrak{C}$. In order to find the complex cone at the point $P \equiv (x, y, z)$, we have to join P rectilinearly with all points P' whose pointers x', y', z' satisfy both equations (65). The equations of such a connecting line will read:

$$\frac{\xi - x}{x' - x} = \frac{\eta - y}{y' - y} = \frac{\zeta - z}{z' - z}. \tag{69}$$

In order to obtain the equation of the complex cone, we must derive a relationship between ξ, η, ζ that is true for all values x', y', z' that fulfill (65). We thus have to eliminate x', y', z' from all four equations, and from the nature of things, the result must be capable of being written in terms of the line pointers:

$$\xi - x, \quad \eta - y, \quad \zeta - z, \quad y\zeta - z\eta, \quad z\xi - x\zeta, \quad x\eta - y\xi \tag{67}$$

alone. One asks only how one can carry out this elimination without reverting to the expressions in the point pointers when equations (65) are given in the form:

(65.a) $$\Phi\,(q_1, q_2, \ldots, q_6) = 0, \qquad \Psi\,(q_1, q_2, \ldots, q_6) = 0.$$

We can also set:

(68)
$$\begin{aligned} x' - x &= t\,(\xi - x), \\ y' - y &= t\,(\eta - y), \\ z' - z &= t\,(\zeta - z), \end{aligned}$$

instead of (66), and now eliminate the four quantities x', y', z', t' from the five equations (65) and (68). We next substitute the expressions in (65) for x', y', z' in (68); we will get, e.g.:

$$yz' - y'z = t\,(y\zeta - z\eta).$$

If we then denote the line pointers (67) by κ_i then we will still have to eliminate t from the two equations:

$$\Phi(t\kappa_1, \ldots, t\kappa_6) = 0, \qquad \Psi(t\kappa_1, \ldots, t\kappa_6) = 0.$$

We will no longer need to distinguish the κ from the q in the result, so we can state the following rule:

Theorem 71: *If a rod complex is given by the equations:*

$$\Phi(q_i) = 0, \qquad \Psi(q_i) = 0$$

then one will have to eliminate t from the equations:

(69) $$\Phi(t\,q_i) = 0, \qquad \Psi(t\,q_i) = 0$$

in order to calculate its carrier.

This rule is true for arbitrary (even transcendental) rod complexes (*). Thus, if $\mathfrak{C}$ is algebraic then we can perform the calculations even further and come to a well-known algebraic problem: Namely, if we denote the aggregate of terms of dimension ν in Φ and Ψ by φ_ν and ψ_ν, resp., then we can also write (65.a) as:

$$\begin{aligned} \varphi_n + \varphi_{n-1} + \ldots + \varphi_1 + \varphi_0 &= 0, \\ \psi_m + \psi_{m-1} + \ldots + \psi_1 + \psi_0 &= 0, \end{aligned}$$

while (69) will assume the form:

(70) $$t^n \varphi_n + \ldots + t\varphi_1 + \varphi_0 = 0,$$

(*) Transcendental line complexes will be admissible in the investigations with homogeneous line pointers when one considers them to be the carriers of a rod complex, and thus to be defined by two inhomogeneous equations in homogeneous line pointers, at least one of which is not algebraic.

$$t^m \psi_m + \ldots + t\psi_1 + \psi_0 = 0.$$

One eliminates t from equations (70) when one sets the resultant of the two entire functions of t to zero (cf., Pascal, *Determ.*, § 57):

$$D = \begin{vmatrix} \varphi_n & \varphi_{n-1} & \varphi_{n-2} & \cdots & \varphi_0 & \\ & \varphi_n & \varphi_{n-1} & \cdots & \varphi_1 & \varphi_0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \psi_m & \psi_{m-1} & \psi_{m-2} & \cdots & \psi_0 & \\ & \psi_m & \psi_{m-1} & \cdots & \psi_1 & \psi_0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = 0. \tag{71}$$

The φ appear in m rows of this determinant, while the ψ appear in n rows.

If a rod structure is defined by λ equations ($\lambda > 2$) then when we combine one equation with any of the remaining $\lambda - 1$ ones, we can find $\lambda - 1$ complexes that will define the carrier of the rod structure; their equations will replace $\lambda - 1$ of the original equations.

Theorem 72: *A rod structure can be represented in such a way that all of its equations are homogeneous, except for one of them. The carrier of the structure will be represented after one drops the inhomogeneous equation.*

We now also include the investigation of rod structures in *line geometry in the broader sense.* Non-algebraic line structures can be defined by equations in the rod pointers in which one ignores the lengths of the rods.

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§ 44. Historical remarks.

Certain line structures already appear in the theory of surfaces and curves – e.g., the normal congruence to a surface, the ruled surface of tangents, the principal normals to a space curve, etc. However, if one overlooks them then line structures had already been considered for some time before the birth of systematic line geometry – e.g., by *Binet* for a quadratic complex, as we will learn about in vol. II. However, we would now like to pursue the history of line *pointers* (i.e., line coordinates), in particular.

The essence of the general tetrahedral line and rod pointers in all of their mechanical meaning (and even more general concepts) was already completely familiar to *Grassmann* in 1844 (*Ausdehnungslehre*, Ges. W., Bd. Ia, § 117), although he did not develop the suggestions that he made in relation to that to the extent that one could actually be able to calculate with them; his book also remained unnoticed for a decade. The name "line coordinates" was first used by *Plücker* in 1846 for the quantities r, s, ρ, σ, η (cf., § 33, conclusion, here, and *Syst. der Geom. des Raumes in neuer analyt.*

Behandlung, art. 258). Cayley used six *homogeneous* line pointers explicitly in order to represent a space curve analytically by a single equation in the lines that met it ("On a new anal. repres. of curves in space," Coll. pap., vol. IV, no. 284 and 294); here, one will also find the equation of the bush of rays. He started with the definition of line pointers for the pointers (x, y, z, w), $(\alpha, \beta, \gamma, \delta)$ of two points that lie on it, which he then assumed to be pointers "of the ordinary kind." Despite the formal identity of his formulas with the ones that appeared here as equation (17) (§ 32, *b*), no tetrahedral pointers had emerged yet as of the year 1860, since the defining point pointers were made rectangular and merely superficially (*) homogeneous, in such a way that one wrote $x : w$, $y : w$, $z : w$, instead of x, y, z (**).

In his ground-breaking treatise "On a new geometry of space" (Phil. Transact., v. 155, received Dec. 1864; read Feb. 1865; Ges. math. Abh., no. 34) *Plücker* examined the line complex in Part One (pp. 725-759, in which he introduced the word "complex") and congruences, ruled surfaces of order two as the intersection of three complexes, and in connection with that, the two-fold and three-fold linear manifolds. He employed the inhomogeneous coordinates, so his calculations became asymmetric and non-intuitive. The six expressions $x' - x$, …, $yz' - y'z$, … for the homogeneous, rectangular, line pointers also appear there occasionally, but were still not regarded as line pointers and no proper symbols were being written out explicitly. In Part Two (pp. 760-774), he made applications to the refraction of light in doubly-refracting crystals, and in an "additional note" on pp. 774-788 (received Dec. 1865), he introduced homogeneous, rectangular, line pointers, when he (like *Cayley*) started with rectangular point and plane pointers that had been made homogeneous by an artifice, wrote down the equation of one and the same general complex in numerous (viz., eight) forms, and proved the general fundamental theorems on complexes and congruences of arbitrary degree (here, in § 42). He further presented line pointers, starting from twists immediately, without reverting to the point and line pointers, and remarked that the absolute values of the line pointers also have meaning, which was a suggestion (***) that was little noticed for some time. One thus already finds the essential ideas of his book *Neue Geometrie des Raumes, gegründet auf die Betractung der geraden Linie als Raumelement* (ed. by Klein, 1868) in this treatise; he was also influenced by his immediate contemporaries. For that reason, even though *Malus* (1808), *Hamilton* (1828), and *Kummer* (1860) had already published important investigations into ray congruences, 1865 must be considered to be the *year in which line geometry was born*, since that was when it was first raised to the status of a systematic development by the use of line pointers.

(*) One sees that the name "tetrahedral" pointer is more characteristic than "homogeneous."

(**) This undoubtedly emerges from the fact that he said on pp. 448 (in a loose translation): "The six line pointers cannot be divided into two groups of three that have the same character. The symmetry of the pointers is, moreover, the same as the one that one finds in the vertices (faces) of a complete four-face (tetrangle). We can divide the pointers into two groups in four ways…, where each left group corresponds to three vertices that define a triangle and each right group corresponds to the three remaining vertices that lie on a line." Had he also thought only of the tetrahedron then he would have certainly explained the grouping of the pointers by the *edges* of the tetrahedron itself and not on the basis of the more remote example of the *vertices* of a complete four-face. This is all the more striking since he also referred to the complete tetrangle (in parentheses), which can be regarded as the projection of a tetrahedron.

(***) One finds this in an extended form in art. 25 of *Neuen Geometrie*.

Tetrahedral line pointers were introduced by *Battaglini* ("Intorno ai sist. di rette do sec. grado," Atti della acc. di sc. Napoli, vol. III, 1866), *Cayley* ("On the six coord. of a line," Coll. pap., vol. VII, no. 435, read 1867, printed 1869, Transact. of the Cambr. Phil. Soc.) and *Klein* ("Über die Transf. der allg. Gl. des 2. Grades zw. Linienkoord. auf eine kanon. Form," Dissert., Bonn 1868, reprinted in Math. Ann., Bd. 23). One finds the mechanical meaning of the rectangular line pointers in *Plücker* (1865), and that of tetrahedral pointers in *Zeuthen* (1869), who made its definition his starting point ("Notes sur une syst, de coord. lin. dans l'espace," Math. Ann., Bd. I). We will come to speak of further generalizations of the concept of pointer later on (§ 49 and 81).

One finds a thorough report on the older line-geometric discoveries in Lie and Engel, *Geom. der Berührungstransf.*, chap. 7, § 2.

Practice problems.

25. The determinant of Table (14), as a skew-symmetric determinant of even order, must be a complete square; on the other hand, since it vanishes, because all of its adjoints vanish, it must be the square of the left-hand side of (16) (up to a possible constant factor). This can be confirmed immediately.

26. The quantities p and π must be independent of each other, on geometric grounds, since the planes v_i and w_i were chosen to go through g or the points y_i and z_i (§ 32, *a*) were chosen to lie on g. This can be confirmed by direct calculation.

27. In the definition of the pointers of the line g, we thought of it as being determined, once, by two planes ε_v, ε_w, and once, by two points P_y, P_z. In the first case, calculate the pointers of the connecting planes with the vertices of the tetrahedron, and in the second case, calculate the pointers of its points of intersection with the faces of the tetrahedron. Conversely, one can also calculate the points of intersection in the first case, and the connecting planes in the second case, and thus arrive at the same pointers. Do these calculations.

28. Show: The necessary and sufficient condition for two lines p, q to lie in hyperbolic position with the two opposite edges (i, k), (l, n) of the basic tetrahedron is that the two quadruples that remain after one puts primes on the pointers with the index pairs i, k; l, m must be proportional to each other.

29. In § 40, one could just as well start from the equations:

$$\rho\, u_i = \sum_{\lambda=1}^{4} A_{i\lambda} u'_\lambda \qquad (i = 1, 2, 3, 4),$$

instead of equations (55), to which they are equivalent, where the $A_{i\lambda}$ are the adjoints in $|a_{ik}|$. One would arrive at:

(72) $$\rho\,\rho'\,p_{lm} = \sum (A_{l\lambda}\,A_{m\mu} - A_{m\lambda}\,A_{l\mu})\;\,p'_{\lambda\mu}\,.$$

Derive (56) from this equation:

30. Confirm, by calculation, that (63.a) is the equation of a conic surface when the point y is fixed.

31. Should equation (71) actually represent the carrier complex, then it would have to be homogeneous of degree mn in the line pointers. Confirm this purely algebraically by a consideration of the determinant D.

32. In *Plücker's* book of 1846, in the place that was cited here in § 44, one finds the words "An equation in these four coordinates (*) does not determine a geometric locus for the straight lines, but only a rule by which the infinite space consists of straight lines." In what sense is this striking remark to be understood?

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(*) Namely, between r, s¸ ρ, σ (cf., § 28 here).

Chapter IV.

Linear rod forests, complexes, and congruences, with applications to mechanics.

§ 45. The general linear rod equation.

Since rod structures have essential metric properties, we will assume *rectangular*, homogeneous rod pointers, and thus write the general linear rod equation in the form:

$$\alpha_0 + \sum_{i=1}^{6} \alpha_i q_i = 0. \tag{1}$$

If we rotate the pointer system around the origin and choose a_1, b_1, c_1 (§ 41) in such a way that (*):

$$a_1\,\alpha_1 + b_1\,\alpha_2 + c_1\,\alpha_3 = 0,$$
$$a_1\,\alpha_4 + b_1\,\alpha_5 + c_1\,\alpha_6 = 0$$

then the coefficients of the new pointers p_1, p_4 will vanish. We can then assume that the equation (while reverting to the original symbols q and α):

$$\alpha_0 + \alpha_2\,q_2 + \alpha_3\,q_3 + \alpha_5\,q_5 + \alpha_6\,q_6 = 0 \tag{2}$$

still represents the general linear rod forest. If we once more rotate the system around the X-axis through the angle ω then, from § 41, table (58.b), we will get:

$$q_5 = \cos\omega\cdot p_5 - \sin\omega\cdot p_6\,, \qquad q_6 = \sin\omega\cdot p_5 + \cos\omega\cdot p_6\,,$$

while the linear, homogeneous function that was defined by the terms in q_2 and q_3 will go to an analogous function in p_2, p_3 . If we determine ω from:

$$\alpha_5\cos\omega + \alpha_6\sin\omega = 0 \tag{3}$$

then the coefficient of p_5 will vanish. If we now assume that the linear rod equation has the form:

$$\alpha_0 + \alpha_2\,q_2 + \alpha_3\,q_3 + \alpha_6\,q_6 = 0 \tag{4}$$

(*) This is always possible in such a way that the relations that must exist between the nine coefficients of an orthogonal substitution will be fulfilled simultaneously.

then we will have imposed no specialization of the rod forest, but only of its position relative to the pointer system. Finally, if we perform a displacement then, from § 41, equations (61), if $\mathfrak{y} = 0$ and $\mathfrak{z}$ is arbitrary then we will get:

$$q_6 = \mathfrak{x}\, p_2 + p_6 .$$

Thus, (4) will go to:

$$\alpha_0 + (\alpha_2 + \alpha_6\, \mathfrak{x})\, p_2 + \alpha_3\, p_3 + \alpha_6\, p_6 = 0.$$

We next assume:

$$\alpha_6 \neq 0,$$

and then we choose:

$$\mathfrak{x} = -\frac{\alpha_2}{\alpha_6},$$

and thus bring the general, linear rod equation into the form:

$$\alpha_0 + \alpha_3\, q_3 + \alpha_6\, q_6 = 0 \tag{5}$$

by a pointer transformation (in which we again write q, instead of p). The constant term was not affected by these calculations. The transformations will then be also valid for the special case of the linear complex ($\alpha_0 = 0$). If $\alpha_6 = 0$ then we can arrive at the form:

$$\alpha_0 + \alpha_3\, q_3 = 0 \tag{6}$$

from (4) by rotating around the X-axis, and this form will be included in (5) when we let $\alpha_6 = 0$.

Theorem 73: *The general, linear rod equation can always be brought into the form* (5) *by pointer transformations.*

We call a rod forest *general* when all three coefficients in (5) are non-zero, and *special* otherwise.

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§ 46. The linear complex.

From the result of the last paragraph, we next discuss the case of the linear complex ($\alpha_0 = 0$). When we set:

$$\frac{\alpha_3}{\alpha_6} = \mathfrak{k},$$

(5) will assume the form:

$$\mathfrak{k}\, q_3 + q_6 = 0, \tag{7}$$

or, when we introduce the pointers for the starting point $P \equiv (x, y, z)$ and the end point $Q \equiv (\xi, \eta, \zeta)$ of the rod:

$$\mathfrak{k}\,(\zeta - z) + x\eta - \xi y = 0. \tag{8}$$

However, this is precisely equation (8) of Chapter I, so it will represent a *twist*, in general; if $\alpha_3 = 0$ then $q_6 = 0$, or:

$$\frac{\eta}{\xi} = \frac{y}{x},$$

will represent all lines that cut the Z-axis; if $\alpha_6 = 0$ then $q_6 = 0$, or $\zeta = z$, will represent all lines that are parallel to the XY-plane. Thus:

Theorem 74: *The linear complex is identical to either a ray twist or a bush of rays* (*); *the axis of the latter can lie at finite points or at infinity.*

Once we know this, we can again assume general, tetrahedral pointers and write the equation of the linear complex in the forms (**):

$$\sum a_{ik}\,\pi_{ik} = 0 \tag{9}$$

or

$$\sum a_{ik}\,p_{lm} = 0; \tag{10}$$

furthermore, let:

$$a_{12}\;a_{34} + a_{13}\;a_{42} + a_{14}\;a_{23} = A. \tag{11}$$

If $A = 0$ then the a_{ik} will be pointers of a line, and indeed, we will regard them as axial pointers. (10) will then express the idea that the lines a, p intersect (§ 39, b). In this case, (9) or (10) will then be the equations of a *bush of rays*.

We now assume:

$$A \neq 0. \tag{12}$$

In this case, there will be no line of intersection for all the lines p that fulfill (10). The pointers of one of them must then be identical to the a_{ik} , whose ratios will be determined completely by five suitably-chosen sextuples p_{ik}. From Theorem 74, (9) or (10) will represent a *twist*. In order to find the association between points and planes, we introduce:

$$\pi_{ik} = y_i\,x_k - y_k\,x_i$$

into (9), and order the result in the y:

$$(a_{12}\,x_2 + a_{13}\,x_3 + a_{14}\,x_4)\,y_1 + (-\,a_{12}\,x_1 + a_{23}\,x_3 - a_{42}\,x_4)\,y_2$$

(*) We will always employ the words "twist" and "bush of rays" as we did in the Chapter I, while we will keep "linear complex" as the common term for both. We will also call a sheaf of rays a *singular complex.*

(**) When all four indices i, k, l, m appear, the sum will always refer to the three arrangements 12, 34; 13, 42; 14, 23, and the ones that arise by permutation of the pairs, and thus, to six terms.

$$+ (- a_{13}\, x_1 - a_{23}\, x_2 + a_{34}\, x_4)\, y_3 \tag{13}$$
$$+ (- a_{14}\, x_1 + a_{42}\, x_2 - a_{34}\, x_4)\, y_4 = 0.$$

If we think of the point x_k as being fixed then (13) will represent the locus of points y that yield a ray of the complex, when combined with x_k , and thus the null plane of the point x. The coefficients of the y are then the pointers u of that plane. Therefore, an association of the form:

$$\sigma u_i = \sum_{k=1}^{4} \alpha_{ik} x_k \qquad (i = 1, \ldots, 4), \tag{14}$$

with $\alpha_{ii} = 0$, $\alpha_{ik} = - \alpha_{ki}$, will be defined by the twist. If we then also set $a_{ik} = - a_{ki}$ then we can write (14) as:

$$\begin{aligned} \sigma u_1 &= && a_{12}x_2 && + a_{13}x_3 && + a_{14}x_4, \\ \sigma u_2 &= -a_{12}x_1 && && + a_{23}x_3 && + a_{24}x_4, \\ \sigma u_3 &= -a_{13}x_1 && - a_{23}x_2 && && + a_{34}x_4, \\ \sigma u_4 &= -a_{14}x_1 && - a_{24}x_2 && - a_{34}x_3 && \end{aligned} \tag{15}$$

in the present case. The determinant D of the a_{ik}, as the even-order, skew-symmetric determinant, is a complete square (cf., Pascal, *Determ.*, § 16), and in fact:

$$D = (a_{12}\, a_{34} + a_{13}\, a_{42} + a_{14}\, a_{23})^2.$$

Due to the assumption (12), equations (15) can be solved for x:

$$\tau x_i = \sum_{k=1}^{4} A_{ki} u_k ,$$

with which, the null point x to any plane u is also found, and indeed, the A_{ki} will have the common factor A [cf., § 32, a)]. After its omission, these equations will assume the form:

$$\begin{aligned} \tau x_1 &= && a_{34}u_2 && + a_{42}u_3 && + a_{23}u_4, \\ \tau x_2 &= -a_{34}u_1 && && + a_{14}u_3 && - a_{24}u_4, \\ \tau x_3 &= -a_{42}u_1 && - a_{14}u_2 && && + a_{12}u_4, \\ \tau x_4 &= -a_{23}u_1 && + a_{13}u_2 && - a_{12}u_3. && \end{aligned} \tag{16}$$

If one writes the equation of the twist in the form:

$$\sum a_i\, q_i = 0$$

then one will also have to rewrite the indices of the a according to the first two rows of the schema (25), and obtain:

$$\begin{aligned}\sigma u_1 &= \qquad\quad a_1 x_2 + a_2 x_3 + a_3 x_4, \\ \sigma u_2 &= -a_1 x_1 \qquad\quad + a_6 x_3 - a_5 x_4, \\ \sigma u_3 &= -a_2 x_1 - a_6 x_2 \qquad\quad + a_4 x_4, \\ \sigma u_4 &= -a_3 x_1 + a_5 x_2 - a_4 x_3,\end{aligned} \tag{15.a}$$

$$\begin{aligned}\tau x_1 &= \qquad\quad a_4 u_2 + a_5 u_3 + a_6 u_4, \\ \tau x_2 &= -a_4 u_1 \qquad\quad + a_2 u_3 - a_2 u_4, \\ \tau x_3 &= -a_5 u_1 - a_3 u_2 \qquad\quad + a_1 u_4, \\ \tau x_4 &= -a_6 u_1 + a_2 u_2 - a_1 u_3.\end{aligned} \tag{16.a}$$

Therefore, one must agree that the q_i mean (arbitrary tetrahedral, moreover) *ray pointers.*

Equations (14) define (also for an arbitrary choice of the a, except that the determinant cannot vanish) a spatial *correlation* (viz., a linear, reciprocal conversion). Our correlation is characterized by the fact that any point lies in the plane that is associated with it. If we ask what the most general correlation would be that has this property then one would have to fulfill:

$$\sum u_i\, x_i = 0$$

identically as a result of (14); thus:

$$\sigma \sum u_i\, x_i = \sum \alpha_{ii} x_i^2 + \sum \alpha_{ik} x_i x_k$$

must vanish for an arbitrary choice of the x. This imposes the conditions:

$$\alpha_{ii} = 0, \qquad \alpha_{ki} = -\alpha_{ik} \qquad (i \neq k)$$

on the a, which are precisely the same as the ones that are fulfilled in the present case.

Theorem 75: *The null system that is defined by a twist is the most general correlation for which each point and its corresponding plane are incident.*

For *any* correlation, a line g, as a point sequence, will correspond to a line g' (viz., its polar) as the axis of the projective pencil of planes that is associated with the point sequence. However, g is also the polar of g' for a null system (§ 5); the correlation of a null system is then *involutory*. In the event that g and g' are distinct, the projectivity would be trivial in the present case, since the point sequence g and the pencil of planes g' actually lie projectively. We then express the theorem in question for the rays of the twist (which are the lines that correspond to themselves under the correlation) as:

Theorem 76: *If a point describes a ray s of a twist then its null plane will describe a projective pencil of planes around s.*

If a line is the carrier of a point sequence and is likewise the axis of a point sequence of a projective pencil of planes then one will say that the line is *the carrier of a correlation*.

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§ 47. Further properties of a twist and ways of generating it.

We take five points, no four of which lie in a plane, link them in any sequence into a simple, spatial pentagon, whose vertices and faces might be denoted by 1, …, 5; I, …, V, respectively, in such a way that I is the face 512, etc. (Fig. 39). No four of the five planes I, …, V will then go through a point, either. Three neighboring planes – e.g., I, II, III – will then have merely the point 2 in common, which will lie in either IV or V, however. If we then associate the points 1, 2, … with the planes I, II, …, resp., then a non-degenerate correlation will be defined in any case (cf., Killing, *Analyt. Geom. II*, pp. 236), which we would like to show is a null system (v. Staudt, *Geom. d. Lage*, art. 325).

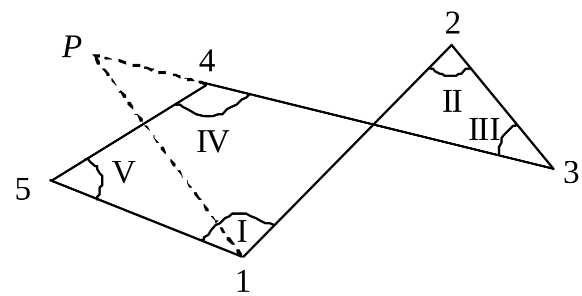

Figure 39.

Any side of the pentagon corresponds to itself, so the point of intersection $P \equiv (34, \text{I})$ will then correspond to the connecting plane (34, 1) that goes through P. In the pencil of planes (1, I), the three rays 5, P, 2 will all correspond to themselves then. For that reason, each point in I, and one of the five planes I, II, …, more generally, will correspond to a plane that goes through it. In order to show that for an arbitrary point Q of space, we draw a line g through Q; it will cut at least three of the planes I, II, … in nothing but distinct points P_i . If we regard g as a sequence of points P then g' will be the carrier of a projective pencil of planes that lies perspectively with P, since the three points P_i of P will lie in their corresponding planes. Since a twist is linked with any null system, we can say:

Theorem 77: *A twist is defined by a simple, spatial pentagon and its rays belong to the five sides of that pentagon.*

b) However, if we also take five *arbitrary* rays of space to be rays of a twist then that twist will be determined uniquely by them, in general. Then, let:

$$p_i^{(\lambda)} \qquad\qquad (\lambda = 1, \ldots, 5;\ i = 1, \ldots, 6)$$

be the pointers of the five rays, so the equation of the desired twist:

$$\sum_{i=1}^{6} \alpha_i p_i = 0$$

must be fulfilled by all five sextuples $p_i^{(\lambda)}$. The ratios of the α will be determined uniquely by the equations for:

$$\sum_{i=1}^{6} \alpha_i p_i^{(\lambda)} \qquad (\lambda = 1, \ldots, 5)$$

when the matrix of the $p_i^{(\lambda)}$ has rank five. One can then also write the equation of the twist in determinant form:

$$\begin{vmatrix} p_1 & p_2 & \cdots & p_6 \\ p_1^{(1)} & p_2^{(1)} & \cdots & p_6^{(1)} \\ \vdots & & & \vdots \\ p_1^{(5)} & p_2^{(5)} & \cdots & p_6^{(5)} \end{vmatrix} = 0.$$

Theorem 78: *A twist is determined uniquely by five rays when the matrix of their pointers has rank five.*

We can first discuss the geometric meaning of this condition later on (Theorem 102).

c) We consider two projective pencils of rays for which the connecting line a of their vertices is identical with the line of intersection of their planes, and the two pencils shall be *self-corresponding*. From § 39, c), we can then represent these projective pencils p, p' in the forms:

$$p_i = a_i + \mu\, b_i, \qquad\qquad p_i' = a_i + \mu\, b_i',$$

in which the same value of μ is assigned to corresponding rays. If we pose the condition that a ray q must cut any two corresponding rays of the pencils then we must have:

$$\sum_{i=1}^{6} (a_i + \mu\, b_i)\, q_{i+3} = 0, \qquad\qquad \sum_{i=1}^{6} (a_i + \mu\, b_i')\, q_{i+3} = 0;$$

if we eliminate μ from this then we will obtain:

$$\sum (b_i' - b_i)\, q_{i+3} = 0,$$

which is the equation of a linear complex that will be a twist when b and b' do not intersect. In fact, one will then have [cf., § 39, c)]:

$$\omega(b - b') = \omega(b) - 2\,\omega(b, b') + \omega(b') = -\,2\,\omega(b, b') \neq 0.$$

We have thus come to recognize the *Sylvester* way of generating a twist [which he first derived in Comptes R., t. 52, (1861) in a synthetic way]:

Theorem 79: *Let two projective pencils of rays with a self-corresponding line be given that nevertheless have distinct planes and vertices. The totality of lines that cut the two corresponding rays of the pencils will define a twist.*

Conversely, a twist can be generated in ∞^5 ways, since one can choose a point and a plane that goes through it to be the vertex and plane of the one pencil arbitrarily. In order to construct the null point to a plane E, one observes that E cuts the two pencils in two perspective point sequences; the center of perspectivity will then be the desired null point (dual construction?).

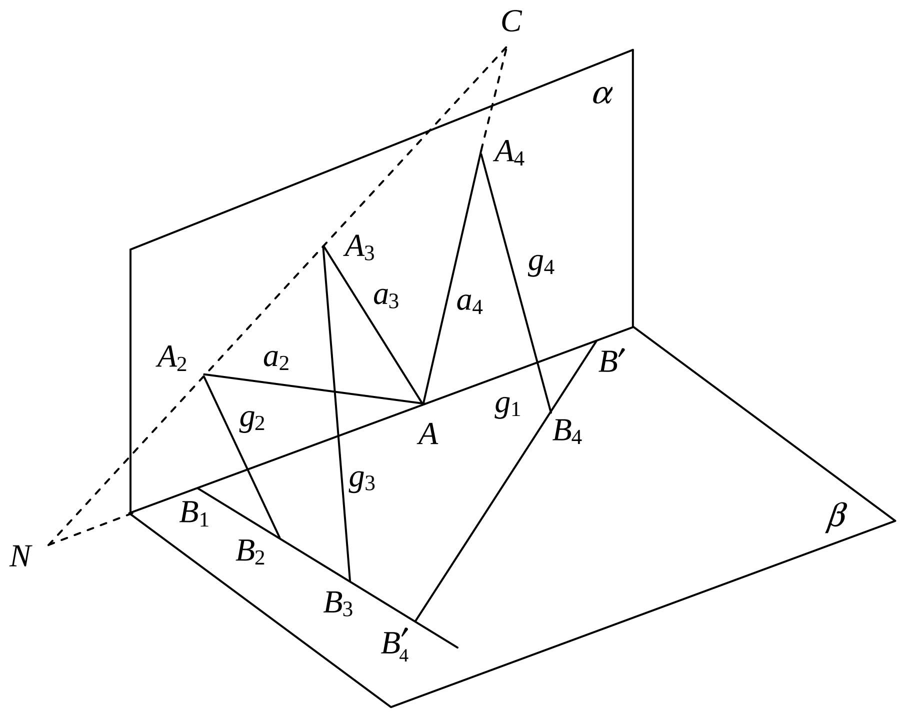

Figure 40.

d) The Sylvester generation process can thus be employed to construct a twist linearly from five rays $g_1, \ldots, g_5$ (Sturm, *Liniengeom.*, Bd. I, art. 74). If (A, a) is one ray pencil of this process of generation then, from c), we can choose A on g_1 and α on g_1 arbitrarily. Let (Fig. 40) $A_2, \ldots, A_5$ be the points of intersection of $g_2, \ldots, g_5$, resp., with a, let $B_2, \ldots, B_5$ be those with another plane β through g_1, let B be a second point on g_1, let $a_2, \ldots, a_5$ be the rays from A to $A_2, \ldots, A_5$, resp., and let $b_2, \ldots, b_5$ be the rays from B to $B_2, \ldots, B_5$, resp. We will then have to choose β, and at the same time, B, in such a way that:

$$A(g_1, a_2, a_3, a_4, a_5) \barwedge B(g_1, b_2, b_3, b_4, b_5).$$

We seek a position B' of B for which this is attained for just four pairs of rays, namely:

$$B'(g_1, b_2, b_3, b_4) \barwedge A(g_1, a_2, a_3, a_4),$$

in which we can choose the plane β arbitrarily through g_1. If B_1 is the point of intersection of (B_2, B_3, g_1) then we will determine a point B_4' on $B_2 B_3$ such that:

$$(B_1, B_2, B_3, B_4') = (g_1, a_2, a_3, a_4).$$

B_4' B_4 then will cut the point B' out of g_1. If one replaces g_4 with g_5 in this construction then one will get a point B_5' on $B_2 B_3$, instead of B_4', and a point B'' on g_1, instead of B'. The solution will then be achieved when B' and B'' coincide, with which, β can be rotated around g_1. Therefore, $B_2 B_3$ will run through the guiding family $\mathfrak{R}$ that g_1, g_2, g_3 belong to, so B_4' will itself be a ray g_4' of $\mathfrak{R}$, and likewise B_5' will be a ray g_5' of $\mathfrak{R}$. The line B_4' B_4 will then describe the guiding family of the family of rulings g_4', g_4, g_1 under a rotation of β, and will thus cut out a plane pencil β of a projective point sequence B' on g_1. B'' is likewise projective to β, and thus to B', as well. If β comes to the position α then B_4' will coincide with the point of intersection (A_2, A_3, a_4), and therefore B' (like B'') will coincide with A. A will then be the one double point of the two projective sequences B', B'' on g_1; the other double point B and the associated position of β will determine the second pencil in Sylvester's generation process.

We still have to explain how one can construct the second double point B from a double point A and two pairs P, P' and Q, Q' corresponding to points on g_1 (Fig. 41) *linearly*. One projects P, Q from an arbitrary point S onto an arbitrary line that goes through A along P_1, Q_1. The sequences $AP'Q'$ and AP_1Q_1 will then be perspective with C as their center. B will then be cut out of SC on g_1.

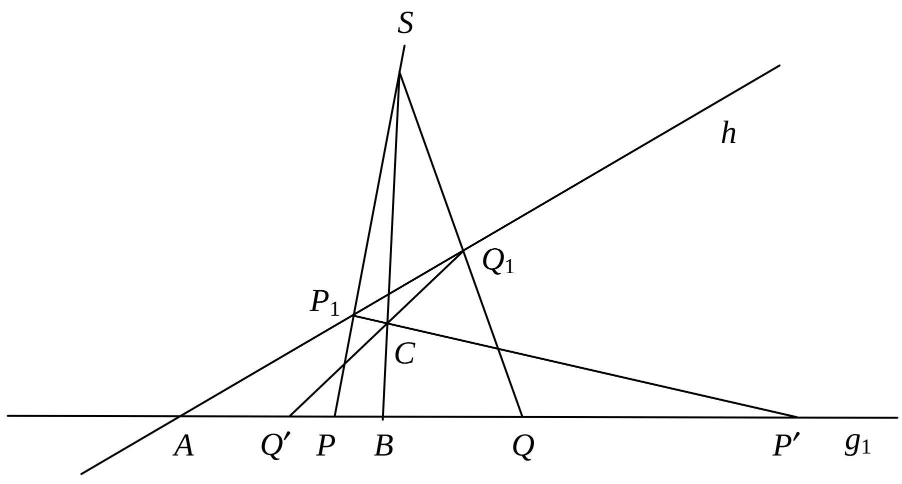

Figure 41.

e) Let:

(17) $$\sum a_{\lambda\mu}\, p_{\lambda\mu} = 0$$

be the equation of a twist. We assume that we have constructed the two polars to the opposite edges 12, 34 of the basis tetrahedron. Any line g that cuts these two edges will then be a ray of the twist. On the other hand, from § 39, one will have:

$$p_{12} = p_{34} = 0$$

for g. Equation (17) must be fulfilled for an arbitrary choice of the remaining four pointers, which are restricted only by the condition $p_{13}\, p_{42} + p_{14}\, p_{23} = 0$, and especially when we let g coincide with the edge 13 of the tetrahedron, for which only p_{42} is non-zero. Therefore, one must have $a_{42} = 0$, etc.

Theorem 80: *The equation for a twist in which the edges i, k and l, m of the basic tetrahedron are polars has the form:*

(18) $$a_{ik}\, p_{ik} + a_{lm}\, p_{lm} = 0.$$

From Theorem 47, we find that:

$$\frac{p_{ik}}{p_{lm}} = \frac{c_{ik}}{c_{lm}} \cdot \frac{M_{ik}}{M_{lm}},$$

and from equation (18), we find for the rays of the twist:

$$\frac{p_{ik}}{p_{lm}} = -\frac{a_{lm}}{a_{ik}};$$

thus, one will have:

$$\frac{M_{ik}}{M_{lm}} = -\frac{a_{lm}}{a_{ik}} \frac{c_{lm}}{c_{ik}}$$

for them; i.e.:

Theorem 81: *The ratio of the moments of an arbitrary ray of a twist relative to two fixed polars is constant.*

This shows, once more, that a twist is determined by two polars and a ray; that constant is then determined by them.

f)

Theorem 82: *If one maps all rays of a twist G by a collineation or a correlation then a twist will again arise.*

One thinks of G as being defined by a null system; it will be mapped as a correlation by a collineation or a correlation into a correlation again, and indeed, due to the conservation of the incidence relations, to a null system.

Since there are ∞^{15} collineations, but only ∞^5 twists, it will follow that a twist will go to itself under ∞^{15} collineations, or it will "admit" them; it will likewise admit ∞^{10} correlations.

§ 48. The linear rod forest.

a) From Theorem 73, the linear rod equation can be brought into the form:

$$\alpha + \beta q_3 + \gamma q_6 = 0. \tag{19}$$

We will first assume that all three coefficients are non-zero, so the rod forest $\mathfrak{S}$ will be general. If we drop the constant term then we will get the equation of a twist $\mathfrak{G}$:

$$\beta q_3 + \gamma q_6 = 0, \tag{20}$$

or, when we set $\beta / \gamma = \mathfrak{k}$:

$$\mathfrak{k}\, q_3 + q_6 = 0, \tag{21}$$

which also plays a role in the investigation of $\mathfrak{S}$, and which shall be called the twist that is *associated* with this rod forest. Its axis is also called the axis of $\mathfrak{S}$. It emerges from § 41 that only the pointers q_4, q_5 will change under a displacement of a rod along the Z-axis, and only the pointers q_1, q_2, q_4, q_5 will change under a rotation around the Z-axis, which do not, however, appear in (19). $\mathfrak{G}$ will then admit any screw around the Z-axis. Therefore, in order characterize all of the rods of $\mathfrak{S}$, it will suffice to displace a point P along the X-axis, while always carrying a plane ε that goes through it and is perpendicular to the X-axis along with it. If one has determined the length of the rods in all pencils of rays (P, ε) then any other rod of $\mathfrak{S}$ can be found from it by a screw.

From Theorem 67, any point (x, y, z) is associated with a plane by $\mathfrak{S}$ whose equation in the running pointers x', y', z' reads:

$$\alpha + \beta (z' - z) + \gamma (xy' - x'y) = 0. \tag{22}$$

In particular, the point $P \equiv (x = c, y = z = 0)$ is then associated with the plane π:

$$\alpha + \beta z' + \gamma c y' = 0. \tag{23}$$

It will cut ε along a line h that limits the lengths of the rods in the pencil (P, ε). The pitch ν' of h with respect to the XY-plane will be determined by (cf., the angle determination and the figures in § 8):

$$\tan \nu' = -\frac{\gamma c}{\beta} = -\frac{c}{\mathfrak{k}}.$$

However, from Chapter I, equation (12), the pitch of the ray s of $\mathfrak{G}$ that belongs to the pencil (P, ε) will be just as large. h will then arise from s by displacement along the Z-direction through the distance $-\alpha / \beta$, which is independent of c, and π will be parallel to the null plane of P in $\mathfrak{G}$.

Theorem 83: *The planes that are associated with the points of space by a linear rod forest arise from the null planes of the associated twist by displacement along the direction of the axis through a constant distance* [cf., also *Plücker*, "Fund. Views regarding Mechanics," (1866); Ges. math. Abh., no. 35]

If we once more consider the construction in § 15, b) (Fig. 17) then for an arbitrary choice of the g_1 in the pencil (P, ε) the end point S will always lie on the fixed line QS, which will arise from the ray of the twist σ by displacement along the vector $k' = k$. If we then choose the component k of a dyname to be equal to $-\alpha / \beta$ and determine the component m on it from $m = k\,\mathfrak{k}$ (§ 15, a) then:

$$m = -\frac{\alpha}{\beta}\cdot\frac{\beta}{\gamma} = -\frac{\alpha}{\gamma},$$

so this dyname $\mathfrak{D}$ will have the intrinsic connection with $\mathfrak{S}$ that:

Theorem 84: *The rods of a general, linear rod forest are identical with the totality of all rod crosses that are equivalent to a certain dyname.*

In order to give an expression to the connection between $\mathfrak{D}$ and $\mathfrak{S}$ even more clearly, we introduce the quantities k, m into (19), instead of the previous coefficients. When we multiply by $\alpha / \beta\gamma$, we will obtain:

(24) $$m\,q_3 + k\,q_6 - m\,k = 0.$$

The characteristic of this equation is that the constant term is equal to the negative product of the other two coefficients. This form is called the *normal form* for the equation (*).

(*) As we will soon see, it plays a role that is similar to the *Hessian* normal form for the equation of a plane. However, whereas there are two normal forms for the equation of a plane (one of them will again arise by multiplying by – 1), here, only one is present for a particular system of pointers. By contrast, one can arrive at an equivalent normal form when one reverses the direction of the Z-axis [in fact, ω was determined only up to p by equation (3)]. One must then reverse the direction of yet another axis – e.g., the Y-axis – in order for the system of pointers to remain one of the first kind. If one calls the new pointers q_i' then one will have:

$$q_i' = -q_i, \qquad \text{along with} \qquad q_4' = +q_4.$$

The q_i' will then satisfy the equation:

(24a) $$-mq_3' - k'q_6' - m'k' = 0,$$

which will again be a normal form. If one compares this with:

$$mq_3' + k'q_6' - m'k' = 0$$

then one will find that:

$$m' = -m, \qquad k' = -k,$$

If $\mathfrak{D}$ is given then one can immediately write down the equation of the rod forest, and conversely, when $\mathfrak{S}$ is given first in this special position with respect to the system of pointers, one can find $\mathfrak{D}$ when one brings the equation of $\mathfrak{S}$ into normal form. $\mathfrak{G}$ will depend upon only the ratio $m : k$, so $\mathfrak{S}$ will also depend upon the absolute values of the components of the dyname. One sees very clearly from Theorem 83 the way in which the rods of the rod cross become infinite when their carriers approach the rays of the associated twist. We call a rod forest *right-wound* or *left-wound*, analogously to the associated twist.

b) The connection in Theorem 84 no longer exists in the case of a *special* rod forest. In fact, if $\beta = 0$ ($\mathfrak{k} = 0$) then a unit force k along the Z-axis will appear in place of $\mathfrak{D}$. On the other hand, equation (23) will reduce to:

$$\alpha + \gamma c\, y' = 0. \tag{25}$$

The rod forest will then include all rods that are skew to k, while all rods that can be obtained by decomposing k into two forces will cut k itself. Furthermore, the rod forest can now be characterized geometrically by:

Theorem 85: *The equation* $\alpha + \gamma q_6 = 0$ *encompasses all rods that have the constant moment* $-\alpha / \gamma$ *with respect to a unit rod on the Z-axis.*

(25) then represents a plane that is parallel to the Z-axis. If one displaces the end point of the rod along its line of intersection with ε then the projection of the rod onto the XY-plane will not change, so its moment relative to the Z-axis, which is determined by cy', will not change either. This will immediately yield:

Theorem 86: *The equation* $\alpha + \beta\, q_6 = 0$ *encompasses all rods whose projections onto the Z-direction yield a constant vector of length* $-\alpha / \beta$.

In the case of Theorem 85, all rods of the forest will have the constant moment $-k\alpha / \gamma$ with respect to an arbitrary rod k on the Z-axis. Analogously, a mere rotational moment m will appear in place of the dyname in the case of $\gamma = 0$ ($\mathfrak{k} = \infty$), which will be represented by a field (§ 36), and all rods of the forest will determine a constant volume with the field m when one considers the field to be the base of a cylinder whose generators have the direction and length of such a rod.

We say that a special rod forest of the kind in Theorem 85 is one of the first kind and that one of the kind in Theorem 86 is one of the second kind. The former is a bush of

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which is geometrically self-explanatory. Nevertheless, only one normal form will then exist for a well-defined system of pointers, so one can always arrive at a normal form for which the coefficient of q_6 is positive, which comes from choosing the positive direction of the Z-axis to agree with the rod part of the dyname; m can then have a double sign.

rays with an axis at infinity, while the latter is associated with an axis at infinity that is "associated." One will obtain its equation when one sets the constant term equal to zero.

c) We must extend the considerations in a) to an arbitrary position of the rod forest with respect to the system of pointers S: We will obtain this when we subject equation (24) to an arbitrary pointer transformation. We first rotate S around the origin into the position S'. If the new pointers are called q_i' then, from § 41, (24) will go to:

$$m(c_1 q_1' + c_2 q_2' + c_3 q_3') + k(c_1 q_4' + c_2 q_5' + c_3 q_6') - km = 0. \tag{26}$$

If we displace S' to the final position S'' and call the new pointers p_i then (26) will go to the form:

$$\sum a_{i+3}\, p_i - a_0 = 0,$$

or when we again write q_i (*):

$$\sum a_{i+3}\, q_i - a_0 = 0. \tag{27}$$

Therefore, from § 41, equations (61), one will have:

$$\begin{cases} a_4 = mc_1 + k(c_2\mathfrak{z} - c_3\mathfrak{y}), \\ a_5 = mc_1 + k(c_3\mathfrak{x} - c_1\mathfrak{z}), \\ a_6 = mc_1 + k(c_1\mathfrak{y} - c_2\mathfrak{x}), \end{cases} \tag{28}$$

$$a_1 = k\, c_1, \qquad a_2 = k\, c_2, \qquad a_3 = k\, c_3, \tag{29}$$

$$a_0 = k\, m. \tag{30}$$

If we define:

$$\sum_{i=1}^{3} a_i a_{i+3} = A$$

then the coefficient of k^2 will be zero; thus:

$$a_1\, a_4 + a_2\, a_5 + a_3\, a_6 = a_0\,, \tag{31}$$

or:

Theorem 87: *A is non-zero for a general rod forest.*

The relationship (31) is characteristic of the quantities a when we start from a normal form for equation (24). If it is fulfilled then we would also like to call (27) a "normal form" for the equation of the rod forest here, or a *normal equation* for it. We can obtain the normal form from an arbitrary form:

(*) Why we denote the coefficients of p_i by a_{i+3}, and not by a_i, will be made clear in the next paragraph.

$$\sum \alpha_{i+3}\, q_i - \alpha_0 = 0$$

by multiplying by a suitable factor N. If:

$$\sum_{i=1}^{3} \alpha_i \alpha_{i+3} = \mathfrak{A}$$

then one can determine the "normalizing" factor uniquely from:

$$N^2\, \mathfrak{A} = N\, \alpha_0 ,$$

so:

(33) $$N = \frac{\alpha_0}{\mathfrak{A}} .$$

We return to the normal equation (27) and the geometric meaning of its coefficients: As would emerge from the schema (58), c_1, c_2, c_3 are the direction cosines of the axis a of the rod forest relative to S''; $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$ are the pointers of the origin of S'' relative to S or S'. However, if we take the standpoint that equation (27), and thus, S'', are given originally then $-\mathfrak{x}$, $-\mathfrak{y}$, $-\mathfrak{z}$ will be the pointers of a point of a relative to S''. Thus, if a dyname and its position with respect to the system of pointers is given by the quantities:

(34) $$k,\ m,\ c_1,\ c_2,\ c_3,\ -\mathfrak{x},\ -\mathfrak{y},\ -\mathfrak{z}$$

then we can write down the equation (27) of the associated rod forest by means of equations (28) to (30).

Conversely, if we would like to ascertain the geometric determining data of the associated dyname from the equation of a rod forest then we must first bring its equation into normal form and then solve equations (28) to (30) (six of which are independent) for the quantities (34). These are, in fact, eight quantities, so $\sum c^2 = 1$, and of the quantities $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$, it is the nature of things that one of them will be arbitrary, since any point of a will play the same role. We find from (29) that:

(35) $$k = \sqrt{a_1^2 + a_2^2 + a_3^2} ,$$

and we may choose the positive value of the root [cf., the remark in a)]; then, from (28) and (30):

(36) $$a_1\, a_4 + a_2\, a_5 + a_3\, a_6 = km,$$

so

(37) $$m = \frac{A}{k} ,$$

$$\mathfrak{k} = \frac{m}{k} = \frac{A}{a_1^2 + a_2^2 + a_3^2}, \tag{38}$$

and from (29):

$$c_i = \frac{a_i}{k} \qquad (i = 1, 2, 3),$$

from which (28) will assume the form:

$$\begin{aligned} a_2\,\mathfrak{z} - a_3\,\mathfrak{y} &= a_4 - \mathfrak{k}\,a_1, \\ a_3\,\mathfrak{x} - a_1\,\mathfrak{z} &= a_5 - \mathfrak{k}\,a_2, \\ a_1\,\mathfrak{y} - a_2\,\mathfrak{x} &= a_6 - \mathfrak{k}\,a_3. \end{aligned} \tag{39}$$

d) We can now choose one of the quantities $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$ arbitrarily and calculate the other two. Instead of them, we seek the pointers $\mathfrak{a}_i$ of the rod $k = QR$. If one attaches a rod k' to the origin O whose vector part is equal to that of k (Fig. 42) then, from equation (29), a_1, a_2, a_3 will be the pointers of its endpoint P, while $(-\mathfrak{x}, -\mathfrak{y}, -\mathfrak{z}) \equiv R$ will be a point of a. Let $OR'P'$ be the projection of the triangle ORP onto the XY-plane, so:

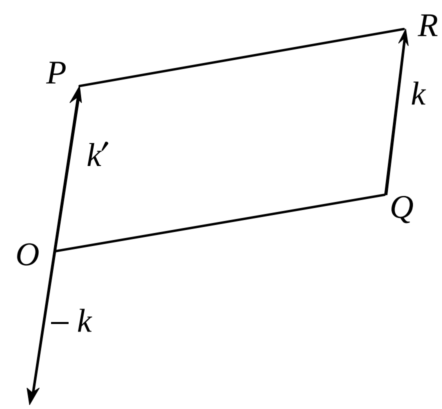

Figure 42.

$$2\,OR'P' = \begin{vmatrix} -\mathfrak{x} & -\mathfrak{y} \\ a_1 & a_2 \end{vmatrix}.$$

$2OQ'R'$ will then be just as large, which will be the pointer a_6 of k (Theorem 48); i.e., one will find both pointers of k and the components (of which we will make use) of the moment of the force pair k, $-k$ on the left-hand side of (39). The quantity A will also represent a well-defined material volume mk [equation (37)], so it will not change under a pointer transformation. For that reason, it is called an *invariant* of the rod forest; a second invariant is $\mathfrak{k}$. It is geometrically clear that there are only two independent invariants of a rod forest. From the algebraic standpoint, one can mostly simply choose:

$$A = a_1\,a_4 + a_2\,a_5 + a_3\,a_6 \qquad \text{and} \qquad a_1^2 + a_2^2 + a_3^2,$$

and from the geometric standpoint, one can choose m and k, or mk and $\mathfrak{k} = m / k$. Therefore, if a general, linear rod forest is given by its equation then one will have the following rule for the determination of the associated dyname (k, m):

Theorem 88: *One brings its equation into the normal form:*

$$\sum a_{i+3}\, q_i - A = 0.$$

One will then have (*):

(35) $$k = \left| \sqrt{a_1^2 + a_2^2 + a_3^2} \right|,$$

(38) $$\mathfrak{k} = \frac{m}{k} = \frac{A}{a_1^2 + a_2^2 + a_3^2},$$

and the pointers $\mathfrak{a}_i$ *of* k *will be:*

(40) $$\begin{aligned} \mathfrak{a}_i &= a_i, \\ \mathfrak{a}_{i+3} &= a_{i+3} - \mathfrak{k}\, a_i, \end{aligned} \qquad (i = 1, 2, 3).$$

The rod forest and the associated twist are left-wound or right-wound according to whether A is positive or negative, respectively.

The last part of the theorem follows from Theorem 14 and equation (38). Since the constant term is not affected by the pointer transformation, Theorem 88 *will also be true for a twist* ($a_0 = 0$), to the extent that it affects the determination of $\mathfrak{k}$ and $\mathfrak{a}_i$; only the ratios will come under consideration for the $\mathfrak{a}_i$.

e) If the rod forest is special then one can link the same conversions that were performed in c) with equation (24) to equation (19) (since no normal form exists now) in which (since the case of $\alpha = 0$ was just dealt with) either β or γ will be set to zero. One must correspondingly set either $m = 0$ or $k = 0$ in the equations of c) and likewise replace either k with γ again or m with β, respectively.

Theorem 89: *If the invariant A of a linear rod equation is zero then a special rod forest will be present; it will be of the first or second kind, according to whether* a_1, a_2, a_3 *are or are not all zero, respectively.*

In the first case, its geometric determining data are likewise given by equation (40) (where only the ratios of the $\mathfrak{a}_i$ come under consideration now), and from Theorem 58, by:

(41) $$M = -\frac{a_0}{\left| \sqrt{a_1^2 + a_2^2 + a_3^2} \right|}.$$

In the second case (cf., Theorem 86), the rod forest is completely characterized geometrically by a vector of length:

(*) One observes that a_1, a_2, a_3 have dimension one, a_4, a_5, a_6, m have dimension two, and A has dimension three. The connection between the pointers of a dyname and those of its axis that is expressed by equations (38) and (40), respectively, was first given by *Franke* ["Über geom. Eigensch. von Kräfte- und Rot.-Syst. in Verb. mit Linienkompl.," Wiener Ber., Bd. 84, II (1881)].

(42) $$\sigma = -\frac{a_0}{\left|\sqrt{a_4^2 + a_5^2 + a_6^2}\right|},$$

whose direction cosines are proportional to a_4, a_5, a_6 .

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§ 49. Pointers for a screw and a linear complex.

Let a linear rod forest by given by:

(43) $$\sum a_{i+3}\, q_i - a_0 = 0,$$

which will be in normal form when $A \neq 0$, so one will have $a_0 = 0$. In the previous paragraphs, we were compelled to give a mechanical interpretation for this; however, the result is just as valid for an arbitrary system of segments that are subject to the laws of geometric addition for rods, so it will be true for "screws" (§ 36) and rod crosses, which are equivalent to them. We have preferred the mechanical picture only for the sake of intuitive appeal and in order to be able to link things to § 15.

We can therefore regard (43) as an equation of a rod forest, as well as the equation for a screw that is coupled with the rod forest, and we have learned how to find the determining data for that screw from Theorem 88. The screw is determined uniquely by the six quantities $a_1, \ldots, a_6$, and conversely (except that we must count the rod and field as degeneracies of the actual screw with respect to the general concept). For that reason, we call these six quantities the *pointers of the screw* and would now like to ascertain their geometric interpretation: We first assume that $A \neq 0$, preserve the symbols k, m of the component of a dyname for the screw Σ, and we can reduce Σ at the origin O of the system of pointers (§ 14), when we parallel translate (Fig. 42) k along k' and add the force pair $\mathfrak{P} \equiv (k, -k')$, whose components might be $\mathfrak{P}_1$, $\mathfrak{P}_2$, $\mathfrak{P}_3$. We can then also write equations (28), which express the connection between the three pointers and the geometric determining data of Σ, as follows (cf., § 48, d)):

(44) $$a_{i+3} = mc_i + \mathfrak{P}_i \qquad (i = 1, 2, 3).$$

a_4, a_5, a_6 will then be the components of the total field that appears after the reduction of Σ at the origin of the system of pointers; thus:

Theorem 90: *Of the six pointers of the screw that is represented by the normal equation:*

(43) $$\sum a_{i+3}\, q_i - a_0 = 0 \qquad (A = a_0),$$

a_1, a_2, a_3 mean the pointers for the rod, and a_4, a_5, a_6 mean the pointers for the field that appears after reducing the screw at the origin of the system of pointers.

The same thing is still true, as one sees immediately from § 48, e), when the screw degenerates into a rod or a field from the outset; only the condition that (43) should be a normal equation will drop out.

If we interpret the screw as a dyname then a_1, a_2, a_3 will be the components of the force and a_4, a_5, a_6 will be those of the moment after reduction at the origin. If we interpret the screw as a twist then a_1, a_2, a_3 will be the components of the rotational velocity, and a_4, a_5, a_6 will be those of the translational velocity after reduction at the origin. a_6 will then mean two equal and opposite rotational velocities around parallel axes in the XY-plane; however, they will combine into a translation in the Z-direction (beginning of § 18). In these cases, we will also call the a_i the *pointers for the dyname* (*the twist*, resp.).

A linear complex:

$$\sum a_{i+3}\, q_i = 0 \tag{43.a}$$

belongs to a screw and a rod forest that are determined uniquely by the ratios of the a_i (and likewise conversely); we thus call the a_i (whose ratios all are that come under consideration now) the *pointers of the complex*. In particular, if $A = 0$ then the pointers of the sheaf of rays will likewise be the *ray pointers* of the carrier (*). That will allow us to distinguish *ray pointers and axis pointers*, not only for a line, but also for a linear complex and a screw. In fact, since the equation:

$$\sum a_{i+3}\, q_i = 0$$

represents a special case of a linear complex for fixed a, when the a, as well as the q mean ray pointers (axial pointers, resp.) of a line, it is logical to call the quantities a the ray pointers (axial pointers, resp.) of the twist (screw, resp.), even in the case of a twist or a screw:

$$\sum a_{i+3}\, q_i = A.$$

If b_i are the axial pointers (ray pointers, resp.) of the same structure then one will also have:

$$b_i = a_{i+3} \tag{45}$$

here. By contrast, if one writes the equation of the screw in the form:

$$\sum a_i\, q_i = A$$

then the coefficient of the i^{th} *ray* pointer of a rod q will be the same-named *axial* pointer of the screw, and conversely.

This can be correspondingly carried over to another notation for a twist equation: Thus, if the π in:

$$\sum a_{ik}\, \pi_{ik} = 0$$

(*) That would not be the case if we had written the rod equation as $\sum a_i\, q_i = 0$.

are the *ray* pointers of a *ray* twist then we would like to call the a the *axial* pointers of the twist itself. The same thing will be true for the notation:

$$\sum a_{ik}\, p_{ik} = 0$$

when the p are axial pointers. We have thus employed axial pointers for the twist in § 46; here, however, as we always do for a rectangular system, we have employed ray pointers for the screw (dyname, twist). ∞^1 metric screws belong to a complex (cf., § 36).

Theorem 91: *There are ∞^4 screws.*

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§ 50. Applications to spatial force systems.

We first calculate the moment of a dyname $\mathfrak{D}$ that is given by its pointers a_i relative to an axis a. We can replace $\mathfrak{D}$ with any equivalent force system, and therefore with one that is represented by the six pointers a_i (Theorem 90), as well.

$$M = \sum M\,(a_i,\ \alpha).$$

Here, two types of moments appear, namely, the three rod pointers a_1, a_2, a_3 and the three field pointers a_4, a_5, a_6 . If we imagine a unit rod $\mathfrak{a}$ with the pointers $\mathfrak{a}_i$ as being on α then the moment of the second kind will be expressed immediately by the pointers:

$$M(a_6,\ \mathfrak{a}) = a_6\ \mathfrak{a}_3\ ,$$

since $\mathfrak{a}_3$ can be considered to be the height of the cylinder that was constructed on page 70, (§ 36). In order to also express the other three moments most swiftly, we observe that we can exchange two rods while preserving their carriers without changing the volume of the tetrahedron, which represents their moment. $M(a_3,\ \mathfrak{a})$ will then be also equal to the moment of a rod σ of length a_3 on α relative to a unit rod ε on the Z-axis. We can now once more decompose σ into its pointers $a_3\ \mathfrak{a}_i$; therefore:

$$M(a_3,\ \mathfrak{a}) = a_3\ \mathfrak{a}_6\ ,$$

since of the six pointers of σ, only the field pointer $a_3\ \mathfrak{a}_6$ will determine a non-zero volume with ε. One generally sees from this argument that when one speaks of the moment of two arbitrary rods, one can replace any rod with an arbitrary equivalent system of rods or screws without having to consider whether the rod means a force or an axis.

Theorem 92: *If a_i are the pointers of a dyname $\mathfrak{D}$ and $\mathfrak{a}_i$ are the pointers of a unit rod $\mathfrak{a}$ on the axis a then the moment M of $\mathfrak{D}$ relative to α will be:*

$$M = \sum_{i=1}^{6} a_{i+3}\,\mathfrak{a}_i\,. \tag{46}$$

In particular, if one reduces $\mathfrak{D}$ to a rod (b_i) then $\sum b_{i+3}\,\mathfrak{a}_i$ will be six times the volume of the tetrahedron that is determined by $\mathfrak{a}$, so:

$$6 \cdot V = \sum b_{i+3}\,a_i \tag{47}$$

will be the moment of two arbitrary rods (b) and (a). The normal equation (43) of a rod forest takes on a new meaning by way of (46): We have to understand the moment of $\mathfrak{D}$ relative to a rod q_i to mean the moment of the rod relative to the carrier (§ 13), multiplied by the length of the rod. When we then multiply equation (46) by the length of the rod q_i, we will obtain $\sum a_{i+3}\,q_i$ as the expression for the moment of $\mathfrak{D}$ relative to q_i .

The moment M' of $\mathfrak{D}$ relative to the carrier of q_i is:

$$\frac{\sum a_{i+3}\,q_i}{\left|\sqrt{q_1^2+q_2^2+q_3^2}\right|},$$

since we must divide by the length of the rod. Therefore:

Theorem 93: $\sum a_{i+3}\,q_i$ *is the moment of the dyname* (a_i) *relative to the rod* (q_i). *All rods, relative to which a dyname has a constant moment, define a linear rod forest, and conversely. All axes, relative to which a dyname has a constant moment M', define a quadratic ray complex $\mathfrak{C}_2$.*

In fact, its equation is (*):

$$\left(\sum a_{i+3}q_i\right)^2 - M'^2(q_1^2+q_2^2+q_3^2) = 0. \tag{48}$$

One can, moreover, deduce the last part of Theorem 93 (and even more) in a different way: Namely, in order to find the axes of constant moment, one must look for the unit rods in the rod forest of Theorem 93, so the ones of constant length. However, since the lines that go through a point P will, from Theorem 83, be bounded by a plane, they will define a cone of rotation.

(*) This complex was already found by *Franke* in the paper that was cited in Theorem 88, and was examined more closely by *Segre* [“Sur les droites, qui ont des mom. données,…,” J. f. Math., Bd. 97 (1884)].

Theorem 94: *The complex cones of* $\mathfrak{C}_2$ *are cones of rotation whose axes are perpendicular to the null planes that are associated with their vertices in the associated twist. The complex curves are circles that have the null points of their planes E for their centers.*

The last part of the theorem follows from the fact that the moment of $\mathfrak{D}$ relative to an axis that lies in E is proportional to the distance from the axis to the null point of E (cf., practice prob. 15). If one replaces a dyname with an equivalent rod cross and considers a rod of a cross to be a rod q_i as in Theorem 93 then, from Theorem 84, one will again obtain a proof of Chasles's theorem in § 16.

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§ 51. The calculation of polar rods and lines.

We call each of two rods of a linear rod forest $\mathfrak{S}$ that are associated as in Theorem 84 the *polar rod* of the other one. If the ratios of the pointers q_i of a rod of $\mathfrak{S}$ are given then one would wish to be able to calculate the pointers q_i' of the polar rod. From Theorem, 24, this must be possible. Should the rod cross (q_i), (q_i') and the dyname (with the pointers a_i) be equivalent, then both of then would have the same moment relative to an arbitrary rod z_i in space. One would then have:

$$\sum a_{i+3}\, z_i = \sum (q_{i+3} + q_{i+3}')\, z_i\ .$$

In particular, we can let (z_i) coincide with rods in the pointer axes and with fields in the pointer planes, in succession, from which, it will follow that:

$$q_i' = a_i - q_i\,. \tag{49}$$

The problem is solved by this, since one can calculate the absolute values of the a_i from the equation of $\mathfrak{S}$:

$$\sum a_{i+3}\, q_i = A$$

or

$$\omega(a, q) = A.$$

In fact, one can first confirm that (q_i') is a rod to begin with and then secondly, that it belongs to $\mathfrak{S}$. Namely, one first has (§ 39, c):

$$\omega(q') = \omega(a - q) = \omega(a) - 2\,\omega(a, q) + \omega(q) = 2A - 2A + 0 = 0,$$

and secondly:

$$\omega(a, q') = \omega(a, a - q) = \omega(a, a) - \omega(a, q) = 2A - A = A.$$

Theorem 95: *The same-named pointers of two polar rods can be extended to the pointers of the equivalent dyname.*

This theorem will also follow immediately from the geometric interpretation of the pointers when one thinks of the rods, as well as the dyname, as having been reduced at the origin of the system of pointers.

We must solve the analogous problem for a *twist:* Its equation reads:

$$\sum_{i=1}^{6} a_{i+3}\, p_i = \sum_{i=1}^{6} a_i\, p_{i+3} = 0. \tag{50}$$

We can consider the pointers q_i of an arbitrary line in space whose polar we seek to be the carrier of a rod of an associated rod forest:

$$\sum_{i=1}^{6} a_{i+3}\, q_i = \text{const.},$$

then apply equation (49), and subsequently ignore the length of the rod (q_i') by considering only the ratios of the q_i'. However, one would wish to calculate the q_i' immediately from a formula in which the line pointers appear only homogeneously; one obtains such a formula when one multiplies (49) by:

$$\sum a_{i+3}\, q_i = A = \omega(a, q),$$

namely:

$$A\, q_i' = a_i \sum_{i=1}^{6} a_{i+3}\, q_i - A q_i .$$

When one lets q_i' denote any quantities that are proportional to q_i', instead of them, and correspondingly writes an arbitrary proportionality factor ρ on the left, instead of A, one will get (*Reye*, Journ. f. Math., Bd. 95):

$$\rho\, q_i' = a_i\, \omega(a, q) - A q_i . \tag{51}$$

One can, in fact, confirm that any ray p_i of the complex (thus, $\sum a_{i+3}\, p_i = 0$) that cuts the line (q_i) (thus $\sum p_{i+3}\, q_i = 0$) will also cut the polar (q') ($\sum p_{i+3}\, q_3' = 0$). It follows from (51) that:

$$\rho \sum p_{i+3}\, q_3' = \omega(a, q) \sum a_i\, p_{i+3} - A \sum p_{i+3}\, q_i .$$

We imagine the original rectangular line pointers. However, the proof of the latter characteristic property of polar is independent of that; (51) will also be valid for tetrahedral line pointers then.

§ 52. The moment of two screws and two complexes. The work done by a dyname on a twist.

a) Up to now, we have always thought of the q_i in the expression $\sum a_{i+3}\, q_i$ as being the pointers of a rod, so they would be subject to the relation $\omega(q) = 0$. Nothing prevents us from ignoring that and defining: One understands the moment M of two screws $\mathfrak{A}$ and $\mathfrak{A}'$, whose pointers are a_i and a_i', to mean:

$$M = \sum_{i=1}^{6} a_{i+3}\, a_i' = \sum_{i=1}^{6} a_i\, a_{i+3}' . \tag{52}$$

This extension of the concept is generally a purely formal one, but it will take on an actual content when we express the moment of two screws in terms of their determining data; moreover, it will find an important mechanical application in (b).

Let k, k' be the rod parts of the two screws and likewise the lengths of the rods, let α_i, α_i' be the pointers of the rods, let V be the volume of the tetrahedron that they determine, let m, m' be the field parts of the screws, let $\mathfrak{k}$, $\mathfrak{k}'$ be the parameters of the associated twists, let $d = NN'$ be the shortest distance, and let ω be the angle between the positive directions of the axes. It is in and of itself irrelevant which sense of rotation one chooses for ω; from § 12, b), only the sign of d will be determined by the choice of sense. We can write equations (40) as:

$$\begin{aligned} a_i &= \alpha_i & a_i' &= \alpha_i' \\ a_{i+3} &= \alpha_{i+3} + \mathfrak{k}\,\alpha_i & a_{i+3}' &= \alpha_{i+3}' + \mathfrak{k}'\alpha_i' \end{aligned} \qquad (i = 1, 2, 3),$$

so, from (52):

$$M = \sum_{i=1}^{6} \alpha_{i+3}\, \alpha_i' + (\mathfrak{k} + \mathfrak{k}') \sum_{i=1}^{3} \alpha_i\, \alpha_i' . \tag{53}$$

Now, from a basic formula of analytic geometry, one has:

$$\frac{\sum_{i=1}^{3} \alpha_i \alpha_i'}{k\,k'} = \cos \omega,$$

so, from equation (47) and Theorem 16:

$$M = 6V + k\,k'\,(\mathfrak{k} + \mathfrak{k}')\cos\omega,$$

where V is endowed with a sign, from the rule in § 12. From Chapter I, equation (15), we can write:

(54) $$M = kk'\,[(\mathfrak{k} + \mathfrak{k}')\cos\omega - d \cdot \sin\omega].$$

We have derived the moment of the carriers of two rods from the moment of the rods in § 12, c) when we replace the rods with unit rods. If we replace screws with *unit screws* (§ 36) then we will obtain an analogous expression that is characteristic of two complexes that belong to the screws $\mathfrak{A}$, $\mathfrak{A}'$. We thus call it *the moment* $\mathfrak{M}$ *of two complexes* (*).

(55) $$\mathfrak{M} = (\mathfrak{k} + \mathfrak{k}')\cos\omega - d \cdot \sin\omega.$$

From § 49, all of the formulas that were derived in this paragraph will still be true when one or both screws reduce to a rod. In the latter case, one will have $\mathfrak{k} = \mathfrak{k}' = 0$, and one will get back equation (15) of Chapter I. When $\mathfrak{A}$ reduces to a field ($\mathfrak{k} = \infty$), one will have $a_1 = a_2 = a_3 = 0$, and one will obtain immediately from (52):

$$M = \sum_{i=1}^{3} a_{i+3}\,a_i',$$

i.e., the volumes that the field determines with the rod part of $\mathfrak{A}'$ (cf., § 50). When the two screws reduce to fields, one will have:

$$M = 0,$$

in agreement with what was said in the conclusion to § 36.

If one introduces the geometric determining data k, m; k', m' of the two screws and likewise denotes the measures of the quantities in question with these symbols then one will obtain an expression for the moment whose validity will remain unchanged in all special cases:

(54.a) $$M = (km' + k'm)\cos\omega - kk'\,d \cdot \sin\omega.$$

Now, if $\mathfrak{A}$ reduces to a field then one will have $k = 0$; thus, the normal to the field will remain definitive for the determination of the angle ω.

Now, since the moment of two screws is expressed by quantities that are independent of the system of pointers, it will then follow that it is a *simultaneous invariant* of the two linear forms:

$$\sum a_{i+3}\,q_i - A, \qquad \sum a_{i+3}'\,q_i - A',$$

(*) The expression on the right-hand side of (55) first appeared in *Klein*, Math. Ann., Bd. II (1870), pp. 368.

which yields the equations of the screws in their normal form when it is set to zero. That is: If the forms go to:

$$\sum b_{i+3}\, p_i - B, \qquad \sum b'_{i+3}\, p_i - B'$$

under a pointer transformation then one must have:

$$\sum_{i=1}^{6} a_{i+3}\, a'_i = \sum_{i=1}^{6} b_{i+3}\, b'_i . \tag{56}$$

If we now denote the moment of two screws by ($\mathfrak{A}$, $\mathfrak{A}'$), and analogously for the moment of any other two geometric quantities, and further denote the aggregate of k and m, which $\mathfrak{A}$ consists of, symbolically by $k + m$ then we will get:

$$(k + m, k' + m') = (k, m') + (k', m) + (k, k') + (m, m'). \tag{54.b}$$

In fact, one recognizes that the first three terms on the right are the three terms of the right-hand side of (54.a), when $(m, m') = 0$. The distributive law is then valid. We have thus actually written down these notations in *Grassmann's* symbolism (*), and with *Grassmann's* methods, which we will not assume, equation (54.b) can define the starting point of the discussion, instead of the conclusion. We would only like to verify by an example how the *Grassmann* symbolism is especially adequate for certain problems.

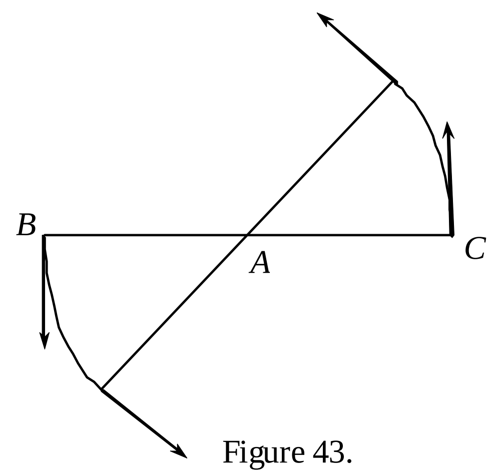

Figure 43.

b) We would now like to compute the work that a dyname with the pointers a_i performs when a body executes a twist with the pointers b_i during a certain time, but first we must preface a few remarks: A force-couple of moment m in the reference plane (Fig. 43) acts on a rigid body, which rotates only around an

(*) Except that this is even simpler when *Grassmann* would write:

$$(m + k)(m' + k') = mk' + m'k + kk' + mm'.$$

We would not like to do this, in order to admit no mixing with the usual symbolism that is applied in (54.a). In recent times, people have employed a multitude of other symbolic notations that are partly outgrowths of *Hamilton's* theory of quaternions and partly of *Grassmann's* theory of extensions. One can then, e.g., let the symbol η suggest that the neighboring symbol means a field and accordingly write the dyname as:

$$k + m\,\eta .$$

Under symbolic multiplication, one will then have to set $\eta^2 = 0$, and it becomes understandable in what way one can be allowed to introduce "complex units" for which $\eta^2 = 0$ (cf., Fortschr. d. Math., Bd. 26 (1895), pp. 804, report of *Kotjelnikoff*).

axis that is perpendicular to the plane of the couple. We can think of the forces of the couple as both acting at the same distance – e.g., at a unit distance – at *B* and *C*; each of them must then have the magnitude $m / 2$. We seek the work that the force-couple exerts under a rotation of the body through an angle ϑ. Since the force-couple moves freely in its plane, we can think of it as being carried along by the rotation. Each point of contact will then describe a path of length ϑ that comes under consideration completely in the calculation of the work, since its direction will always agree with the direction of the force. The work done by the couple will then be $m\vartheta$. By contrast, if the body rotates around any axis in the reference plane then the couple will perform no work, since the paths of the contact points are perpendicular to the forces.

The b_i mean velocity components; we imagine the velocity of the twist as being uniform, and when we multiply it by the time duration t we will get the motion of the corresponding path components of a screw $\mathfrak{S}$. We can refer to the $b_i\, t$ as the "pointers of the screw motion" (to distinguish them from the pointers of a screw in § 49). To calculate the work, we can arbitrarily replace a force with an equivalent one and take the algebraic sum of the individual works; we can likewise decompose the twist arbitrarily and do both things again: When we reduce the dyname, as well as the twist, to the origin, we will obtain the work $\mathfrak{A}$ that is done by the dyname during the screw motion, expressed in terms of the pointers of both. If we let $\mathfrak{A}(k, w)$ denote the work that is done by the force k under a motion w, which can consist of either a translation or a rotation, then we will have:

$$\mathfrak{A}(\mathfrak{D}, \mathfrak{S}) = \sum_{i=1}^{6}\sum_{k=1}^{6}\mathfrak{A}(a_i, b_k t)\,.$$

Here, one will have, e.g.:

$$\mathfrak{A}(a_3, b_6\, t) = a_3\, b_6\, t,$$

since $b_6\, t$ is a path along the *Z*-direction, and furthermore:

$$\mathfrak{A}(a_6, b_3\, t) = a_6\, b_3\, t.$$

$b_3\, t$ will then be an angle of rotation around the *Z*-axis and a_6 will be the rotational moment around it. Only six of the 36 terms in the sum will be non-zero:

$$\mathfrak{A}(\mathfrak{D}, \mathfrak{S}) = t\sum_{i=1}^{6} a_i\, b_{i+3}\,. \tag{57}$$

If we imagine the twist as being uniform in a constant force field then this equation will be valid for an arbitrary time interval t. We call:

$$\frac{d\mathfrak{A}}{dt} = \mathfrak{A}'$$

the *work velocity* (*) of the dyname under the twist and obtain:

$$\mathfrak{A}' = \sum_{i=1}^{6} a_i\, b_{i+3} \,. \tag{57.a}$$

This equation is also true as a consequence of the existence of an instantaneous axis (§ 20) for an *arbitrary* motion at any moment when the b are the pointers of the instantaneous twist and the a are the pointers of the force system that acts at any time point.

If α is the velocity of the twist (from § 18, it is dual to the intensity of the dyname) then, from equations (52), (54), and (55), we can also write:

$$\mathfrak{A}' = k\alpha\,[(\mathfrak{t} + \mathfrak{t}') \cos \omega - d \cdot \sin \omega] = k\alpha\,\mathfrak{M}; \tag{58}$$

$\mathfrak{M}$ is a purely geometric quantity, here.

Theorem 96: *If one of two screws is the carrier of a dyname and the other one is the carrier of a twist then the work velocity of the dyname on the twist will be equal to the moment of the corresponding unit screw, multiplied by the intensity of the dyname and the velocity of the twist* (**). *If the moment is zero then one is dealing with a body whose velocity is restricted to the twist that is in equilibrium under the influence of the dyname.*

The last part of the theorem, whose meaning we will later generalize (§ 85), follows from the principle of virtual displacements. In order to also encompass all of the special cases, we write down that from (54.a):

$$\mathfrak{A}' = (k\tau + m\alpha) \cos \omega - k\alpha\, d \cdot \sin \omega, \tag{58.a}$$

where τ is the translational velocity of the twist. Here, it is not excluded that arbitrarily many of the quantities that enter in will vanish.

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§ 53. The ray net.

From Theorem 64, the intersection of two linear complexes $\mathfrak{A}$ and $\mathfrak{B}$ is a *first-degree* congruence; i.e., it has order one, as well as class one. We (with Sturm) call it, more briefly, a *ray net*, or – when no misunderstanding is possible – even more briefly, a *net*. If $\mathfrak{A}$ and $\mathfrak{B}$ are given by any sort of determining data (§§ 10, 47) whatsoever then one

(*) It has the dimension of a "horse power."

(**) This was first expressed by *Klein* (in a somewhat different form), Math. Ann. IV (1871).

will find the ray of the net $\mathfrak{N}$ that goes through a point P when one constructs the two null planes of P in $\mathfrak{A}$ and $\mathfrak{B}$ and intersects them; an analogous statement will be true for the dual picture. If one considers a ray s of $\mathfrak{N}$ then the pencil of planes with the axis s will correspond to a projective point sequence in $\mathfrak{A}$, as well as in $\mathfrak{B}$. One will then have two projective point sequences on s; their possible double points D will be the points on s that correspond to the same null plane δ in both complexes. Therefore, the entire pencil (D, δ) will belong to the net. The points through which more than one ray (and therefore, an entire pencil) of the net go and the planes in which more than one ray (and therefore, an entire pencil) of the net lie are called *singular points* (*planes*, resp.) of the net.

Let:

$$\sum a_i\,p_i = 0, \qquad \sum b_i\,p_i = 0 \tag{59}$$

be the equations (*) of the two complexes $\mathfrak{A}$ and $\mathfrak{B}$ that define the net $\mathfrak{N}$ in whatever sort of homogeneous pointers. We also let $\mathfrak{A}$ and $\mathfrak{B}$ denote the linear forms on the left-hand sides of (59). The equation:

$$\lambda\,\mathfrak{A} + \mu\,\mathfrak{B} = 0 \tag{60}$$

will then represent a linear complex that includes all lines of $\mathfrak{N}$ for an arbitrary choice of the constants λ and μ, since the pointers of the two terms on the left-hand side of (60) are already individually set to zero. For different choices of $\lambda : \mu$, one will obtain ∞^1 linear complexes from (60) that define a *pencil of complexes* with the *carrier* $\mathfrak{N}$. Let:

$$\mathfrak{C}_1 = \lambda_1\,\mathfrak{A} + \mu_1\,\mathfrak{B}, \qquad \mathfrak{C}_2 = \lambda_2\,\mathfrak{A} + \mu_2\,\mathfrak{B} \tag{61}$$

be two different complexes of the pencil (thus $\lambda_1\,\mu_2 - \lambda_2\,\mu_1 \neq 0$). The equations:

$$\mathfrak{C}_1 = 0, \qquad \mathfrak{C}_2 = 0 \tag{62}$$

will then be equivalent to equations (59); i.e.:

Theorem 97: *A pencil of complexes (and its carrier) is determined by any two of its complexes, just as the original two are.*

Theorem 98: *An arbitrary line in space belongs to just one complex of the pencil, except when it belongs to its carrier.*

In fact, if one substitutes the pointers of the lines in the forms $\mathfrak{A}$ and $\mathfrak{B}$ then the ratio $\lambda : \mu$ will be determined uniquely by (60).

(*) If the geometric meaning of the coefficients does not come under consideration then the notation $\sum a_{i+3}\,p_i$ of the foregoing paragraphs will not be necessary.

We consider a line g, which might not belong to the net; from Theorems 97 and 98, we can then assume that g does not belong to either of the two complexes of the net that were defined. g will correspond to polars g_1 and g_2 in them. From Theorems (6) and (8), all rays that cut g and belong to the net will be likewise the ones that meet g, g_1, g_2. In general, they will then define a family of rulings $\mathfrak{R}$, and g, g_1, g_2 will themselves belong to its guiding family $\mathfrak{L}$; g_1 and g_2 will always be skew to g, but they can intersect each other. Two pencils of rays will appear in place of $\mathfrak{R}$, and the point of intersection of g with the plane (g_1, g_2) will be the (only) singular point on g. We return to the case in which g has no singular points; $\mathfrak{R}$ is then determined by three of its rays, and any ray of $\mathfrak{L}$ will accomplish the same thing as g, namely, it must also lead to $\mathfrak{R}$. If one moves g along $\mathfrak{L}$ then, because each of them is the null system of a correlation, g_1 and g_2 will describe two families of rulings on $\mathfrak{L}$ that are *projective* to g, and thus, to each other, as well, and which can have two double rays d_1, d_2. Let g_0 be the position of g for which g_1 and g_2 combine into d_1. g_0 and d_1 will then be polar to each other in the two complexes; therefore, when g comes to d_1, g_1 and g_2 must combine into g_0; i.e., g_0 and d_2 will be identical. In the event that two real, distinct double elements are present, there will then be only *one* pair of common polars. All of their lines of intersection will then belong to the net and will also exhaust it, as will be shown shortly. We will distinguish three main types of ray nets according to whether d_1, d_2 are distinct, combine, or are absent, and we will likewise examine them analytically. For the moment, we will deduce from this argument only the theorem:

Theorem 99: *The rays of a net $\mathfrak{N}$ that intersect a line that does not belong to $\mathfrak{N}$ define a family of rulings; if three rays of $\mathfrak{N}$ belong to one family of rulings then they will belong to all of them.*

We will employ Theorem 97 in order to replace the defining complexes of the pencil $\mathfrak{C}$ with two singular ones, where possible, and thus arrive at an intuitive picture of the rays of a net. In order for:

$$\sum (\lambda\, a_i + \mu\, b_i)\, p_i = \sum c_i\, p_i = 0 \tag{63}$$

to represent a pencil of rays, from § 46, one must have $\omega(c) = 0$, so:

$$\omega(\lambda\, a + \mu\, b) = \lambda^2 \cdot \omega(a) + 2\, \lambda\, \mu \cdot \omega(a, b) + \mu^2 \cdot \omega(b) = 0. \tag{64}$$

We now have a quadratic equation for the determination of $\lambda : \mu$. Its discriminant is:

$$\Omega\,(a, b) = [\omega(a, b)]^2 - \omega(a) \cdot \omega(b). \tag{65}$$

There are now four cases to distinguish:

a) $\Omega > 0$. One then gets two real, distinct values for $\lambda : \mu$ from (64) that will give the equations of the two pencils of rays that enter into $\mathfrak{C}$ when they are substituted in (63). $\mathfrak{N}$ will then consist of all lines of intersection of the axes of these two pencils of rays, and thus, two skew lines that are called the *focal lines* of the net; one will get the fact that they are actually skew from the discussion of case c). In particular, if crosses are *perpendicular* to the focal lines then we will call the net *rectangular*.

b) $\Omega = 0$, without all coefficients in (64) vanishing individually. This equation will then have a double root, and $\mathfrak{C}$ will contain only *one* pencil of rays Γ with the axis s, which belongs to each twist of $\mathfrak{C}$. s will then have a polar s' in a twist $\mathfrak{G}$ of the pencil, so $\mathfrak{N}$ will consist of the common rays of $\mathfrak{G}$ and Γ – i.e., the lines of intersection of s, s' - and we will be dealing with case a). $\mathfrak{N}$ will then consist of all rays of $\mathfrak{G}$ that intersect a ray s of $\mathfrak{G}$ itself. Such a net is called *special*, and s is its *focal line*. Now, from Theorem 76, s is the carrier of a correlation. One will then obtain all of the rays of $\mathfrak{N}$ when one moves a pencil of rays in such a way that its vertex S runs through the ray s and simultaneously its plane ε describes a projective pencil with the sequence S.

The type of motion can be made completely intuitive by the metric properties of such a correlation: The point at infinity U of s corresponds to a plane ε_u, and the plane ε_c that is perpendicular to ε_u (viz., the "central plane") corresponds to a point C (viz., the "central point" of the correlation). If one measures the distance t from s to C and the angle α between s and ε_c (the positive direction on s and the positive sense of rotation around s are mutually independent) then the double ratio of C, U with two other points P_1, P_2 of s will be:

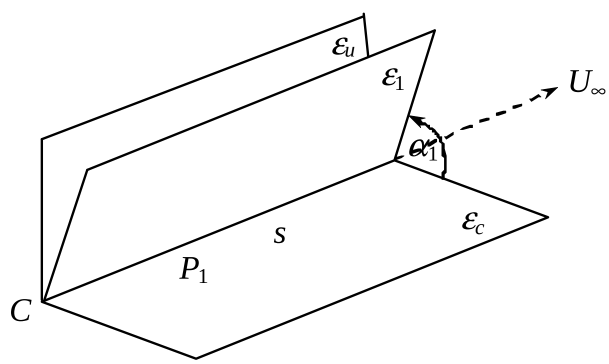

Figure 44.

$$(C\,U\,P_1\,P_2) = \frac{CP_1}{P_1U} : \frac{CP_2}{P_2U} = \frac{t_1}{t_2},$$

and that of the four corresponding planes will be:

$$(\varepsilon_c\;\varepsilon_u\;\varepsilon_1\;\varepsilon_2) = \frac{\sin(\varepsilon_c\varepsilon_1)}{\sin(\varepsilon_1\varepsilon_u)} : \frac{\sin(\varepsilon_c\varepsilon_2)}{\sin(\varepsilon_2\varepsilon_u)} = \frac{\tan\alpha_1}{\tan\alpha_2}.$$

In particular, if one sets $t_1 = 1$, $\tan \alpha_1 = K$ then one can choose K arbitrarily, because one can associate the three points $t = 0, 1, \infty$ with three arbitrary planes in order to define the correlation. The condition for P_2 and ε_2 to correspond to each other will then consist of the equality of the two double ratios, namely:

$$\frac{1}{t_2} = \frac{K}{\tan \alpha_2}.$$

If we drop the index then the corresponding points and planes will be coupled by the equation:

(66) $$\tan \alpha = K\, t.$$

K is called the *parameter* of the correlation. According to whether it is positive or negative, the plane will rotate while the points of the sequence in the positive sense run through it in the positive or negative sense, respectively. In the former case, we call the special ray net *right-wound*, while in the latter case, it is *left-wound*. If the point moves uniformly with unit velocity then the associated angular velocity φ of the plane will be given by $d\alpha / dt$; thus:

(67) $$\varphi = \frac{K}{1 + K^2 t^2}.$$

For $t = 0$, one will get $\varphi = K$, with which, we have arrived at a second geometric interpretation for the parameter.

If we then consider the position of $\mathfrak{N}$ in $\mathfrak{G}$ then ε_u must be the plane through *s* that is parallel to the axis *a* of $\mathfrak{G}$, and *C* must be the base point of the shortest distance from *s* to *a*.

c) $\omega(a) = \omega(a, b) = \omega(b) = 0$. All complexes of the pencil are singular; a_i and b_i are the pointers to two intersecting lines that determine the pencil of rays $\lambda\, a_i + \mu\, b_i$ (§ 39, c). $\mathfrak{C}$ will then consist of all pencils of rays whose carriers define a pencil of rays (S, ε). The common rays of all complexes are, firstly, all rays through *S* and secondly, all rays in ε. $\mathfrak{N}$ will then decompose into a sheaf of rays and a ray field whose carriers are incident. Such a ray net is called *singular* or *degenerate*.

We now have a complete overview of the distribution of the singular points (and planes) of the net in cases a), b), c). In a) and b), they are all points of the focal lines (planes through them, resp.), and in c), they are all points of ε (planes through *S*, resp.). We also see that the family of rulings $\mathfrak{R}$ of the Theorem 99 will decompose into two pencils of rays in cases a) and b) if and only if *g* cuts a focal line; reciprocally, rays of the net can meet only on a focal line. $\mathfrak{N}$ will always decompose in case c).

We assume that a ray net has a singular point *N*; the sheaf *N* in the common null plane ν of both complexes that were defined will then correspond to two correlative (reciprocal) fields that will thus be mutually collinear and will have at least one real double line *d*. If there is a double line *d* that does not go through *N* then it will define a common polar pair to both complexes with the lines d_1 that correspond in the sheaf, and we will be dealing with case a). If there is only one double line *d* through *N* then d_1 must lie in ν, so it must coincide with *d*, because *d*, d_1 be now also correspond to each other in the two null systems; we will have come to case b). Finally, if the two fields are identical then case c) will arise. Among all situations, the assumption then leads to just a single singular point in a case that is known already, which is why we are certain that in the case $\Omega < 0$ that we have yet to discuss, *there will be no singular points or planes.*

However, we next prove some theorems that are true for any non-singular ray net:

Theorem 100: *Two planes E_1, E_2 that go through a ray s of a ray net $\mathfrak{N}$ (without including a focal line) will be cut by $\mathfrak{N}$ in two collinear fields.*

In fact, if we fix a general line g in E_1 then, from Theorem 99, the rays of $\mathfrak{N}$ that go through it will define a family of rulings $\mathfrak{R}$ that will also belong to s. $\mathfrak{R}$ will then be cut by E_2 along a linear sequence of points that are related projectively to the sequence g by $\mathfrak{R}$. Since every ray of $\mathfrak{N}$ – and thus s, as well – will cut the possible focal lines b, b', g can also cut a focal line only in a point of s. Therefore, there are at most two pencils of rays in E_1 with positions g for which $\mathfrak{R}$ will decompose. Let γ be one such position, and let P be its point of intersection with s. All of the rays of $\mathfrak{N}$ that cut γ in a point that is different from P will then either define a pencil of rays whose vertex lies in the point of intersection of the plane (b, γ) with b', or the plane (b, g) will be associated with b, according to whether another focal line b', besides b, is or is not present, respectively. In both cases, the point sequence γ is associated with a perspective sequence in E_2. Any line g in E_1, *without exception,* is then associated with a projective sequence in E_2, which is just the characteristic property of a collinear relationship.

Theorem 101: *A ray net is determined uniquely by four of its rays that do not have hyperbolic position; if four rays of a net belong to a linear complex then all of them will belong to it.*

If a complex:

(68) $$\sum_{i=1}^{6} a_i p_i = 0$$

includes four rays $s_{\lambda i}$ ($\lambda = 1, \ldots, 4$; $i = 1, \ldots, 6$) then that will impose the four conditions:

(69) $$\sum_{i=1}^{6} a_i s_{\lambda i} = 0 \quad (\lambda = 1, \ldots, 4)$$

on the coefficients, from which, four coefficients – say, $a_3, \ldots, a_6$ – can be calculated as functions of the other two a_1, a_2, since the matrix of s has rank four (Theorem 58); a_1 and a_2 will remain arbitrary, appear in the solutions homogeneously and linearly, and will thus play the role of the quantities λ and μ in equations (60) or (63). A pencil of complexes will then be defined by four rays, and thus, a ray net, as well. One now also recognizes the geometric meaning of the conditions of Theorem 78: Should five rays determine a linear complex, then they could not belong to the same ray net. If this condition is fulfilled then four of the five rays will determine a ray net, and it will follow once more from Theorem 98 that five rays determine a linear complex. It will also follow that:

Theorem 102: *If the matrix of the pointers of five rays has rank only four then the five rays will belong to the same net, and conversely. If the determinant of the pointers of six rays has rank five then the rays will belong to the same linear complex, and conversely.*

The last part of the theorem follows from the fact that the equations:

$$\sum_{i=1}^{6} a_i s_{\lambda i} = 0 \quad (\lambda = 1, \dots, 6)$$

can be fulfilled by non-zero a_i if and only if the determinant of the $s_{\lambda i}$ is zero.

We draw two planes E_1, E_2 through a line s_0 that is one of the four defining rays of a net $\mathfrak{N}$ in such a way that the other three rays cut out an actual triangle from any plane. Thus, three pairs of corresponding points and one pair of corresponding lines (namely, when s_0 corresponds to itself), and thus, two corresponding complete quadrilaterals are known in the planes, with which, a collineation is defined (*) that must be identical to the one that is defined by $\mathfrak{N}$. Since the four rays can be chosen arbitrarily, the two planes and the collineations between them can be regarded as arbitrary, except for the situation in which s_0 corresponds to itself. The converse of Theorem 100 will then follow:

Theorem 103: *Two collinear fields that have their corresponding line of intersection in common will generate a ray net.*

In particular, we can think of a given net $\mathfrak{N}$ as being generated in such a way that we can draw two arbitrary planes through the single line of $\mathfrak{N}$ that lies in the plane at infinity, whereby the fields of Theorem 100 will become *affine* fields.

Theorem 104: *A non-singular ray net can generally be generated in ∞^2 ways by two parallel, affine fields.*

Only when a focal line of the net lies at infinity will one already have ∞^1 choices for the location of the affine fields, and correspondingly, ∞^3 generators for affine fields.

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§ 54. The ray net with no focal lines.

Theorem 104 will help us to arrive at a picture of the ray net for the still-remaining case of $\Omega < 0$ in the previous paragraph that is just as intuitive as it was in the case of real

(*) Cf., Reye, *Geom. d. Lage*, Bd. II, 3rd ed., lect. 1.

focal lines. If P, Q, …, g, … mean points or lines in a plane Σ then we will let P_1, Q_1, …, g_1, … denote the corresponding elements of a parallel affine field Σ_1, and let P', Q', …, g', … denote the perpendicular projections of the elements P_1, Q_1, …, g_1, … onto Σ, such that Σ, Σ' are in affine systems that *lie in each other.* According to Staudt, *Beitr. z. Geom. d. Lage*, art. 301 (cf., *Reye*, *loc. cit.*, Bd. II, lect. 9), the corresponding common elements of two incident collinear fields can consist of (when we exclude the case of perspective position, since it will not come under consideration here):

I) The vertices and sides of an actual triangle that are either α) all real, or β) just one vertex and the opposite side to it are real.

II) Two points, their connecting line, and a line through one of the points.

III) A point and a line through it.

However, we would now like to pursue only the case in which the ray net has no real focal lines. All other possibilities for Σ, Σ' are then excluded, except for I, β). The line at infinity then will correspond to itself; if a point on it corresponded to itself then that would give rise to parallel similar sequences in Σ and Σ_1 whose center of perspectivity would be a singular point (compare § 53). Σ and Σ' would then have a single point $M \equiv M'$ that would correspond to itself, and which would lie at infinity. MM_1 is called the *principal ray* a of the net, and M is the carrier of two projective pencils of rays in Σ and Σ', namely, the "pencils of central rays." Any two corresponding rays g, g' of it are the carriers of two similar point sequences. g, g_1 will then generate an equilateral paraboloid $\mathfrak{P}$, one of whose principal generators is a. $\mathfrak{N}$ can be decomposed into ∞^1 equilateral paraboloids.

Two affine fields Σ, Σ_1 are determined by three pairs of corresponding points, or – what amounts to the same thing – by two corresponding triangles (cf., the determination of collinear fields in the previous paragraph). We can always take M, M_1 to be one of the pairs. Since two corresponding right angles are present in the central pencil, moreover (cf., perhaps, Hankel, *Proj. Geom.*, Sect. III, § 4), we can assume that the corresponding triangles MAB, $M_1A_1B_1$ (at M and M_1) are rectangular and that one of them MAB is isosceles.

We are now dealing with the special case in which $M_1A_1B_1$ is isosceles (Fig. 45); the reference plane might coincide with Σ, and all points with the index one are thought of as being at a constant distance above the reference plane. In order to construct the point P_1 in Σ_1 that corresponds to an arbitrary point P, one draws $PQ \parallel BM$, $PR \parallel AM$; the point Q_1 must divide M_1A_1 in the same ratio as MA is divided by Q, etc. One then sees that Σ' arises from Σ by rotation around the angle ω and a similarity

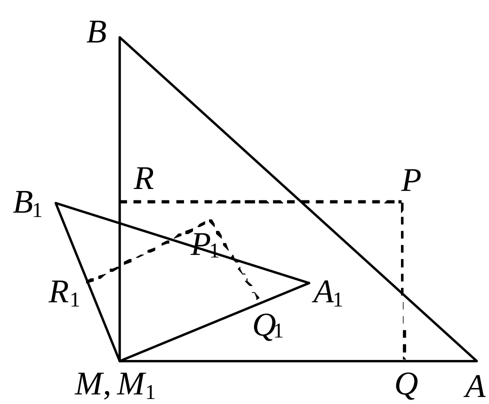

Figure 45.

transformation. The pencil of concentric circles around M corresponds to a similar pencil around M_1, so any two corresponding circles will generate a rotational family of rulings. The shortest distances from a to all rays of the net that connect corresponding points of the sequences MA and M_1A_1 will lie along a line. The shortest distances from a to all rays of a family of rulings of $\mathfrak{P}$ will then lie on the other principal generator of $\mathfrak{P}$. Therefore, all hyperboloids of rotation will have their throat circle in the same plane, viz., the *middle plane* $\mathfrak{M}$ of $\mathfrak{N}$ (*).

Theorem 105: *Among the rays nets without focal lines, there is a rotation net that is generated by rotating a family of rulings of an equilateral hyperbolic paraboloid around its principal ray and which also can be decomposed into ∞^1 rotational families of rulings with common throat planes.*

Among the ∞^2 types of generators of Theorem 104, ∞^1 of the rotation nets will be distinguished by the fact that the fields extend just as far on both sides of $\mathfrak{M}$, so they will be *congruent*, and ∞^1 of them will be distinguished by the fact that Σ_1 is rotated with respect to Σ through a right angle, whereby $\mathfrak{M}$ and the plane at infinity will be associated with each other. When Σ and Σ_1 are rotated through 45° on both sides of $\mathfrak{M}$, this way of generation will unite the two distinguished properties, so:

Theorem 106: *A rotational net can be generated in ∞^1 ways by two congruent point fields and in ∞^1 ways by two similar ones that are rotated with respect to each other by a right angle, and in a single way by two congruent fields that are rotated through a right angle.*

If we generate a rotation net by two congruent point fields then we can derive the one field from the other one by a uniform screw motion. An arbitrarily small piece of it already defines a rotation net in this way. If we pass to the limit then we will obtain (**):

Theorem 107: *If one subjects a plane E to a screw whose axis a is normal to E then the tangents to the paths at all points of E will define a rotation net with a as its principal ray.*

If one draws a plane E through the principal ray of a rotation net $\mathfrak{R}$ and changes the separation of all spatial points of E by the same ratio then a ray net $\mathfrak{N}$ will again arise from $\mathfrak{R}$; any two affine fields, by which, one can perhaps think of $\mathfrak{R}$ as being generated,

(*) We call the ray of a ray net with two focal lines that cuts the two focal lines perpendicularly the "principal ray." The middle plane is the one that that is parallel to the two focal lines and has an equal distance from them. For a special ray, we then call a ray that cuts the focal line perpendicularly at the central point a principal ray. The "middle plane" here goes through the focal line, and is perpendicular to the principal ray, and thus to the central plane, as well. This would be required by the passage to the limit of the hyperbolic net.

(**) I am grateful for an oral communication by *Study*.

will again go to affine fields under this (special) affine transformation $\mathfrak{T}$. This only raises the question of whether one can already obtain all ray nets without focal lines in this way. The number of constants would make us suspect this. In order for us to convince ourselves of this more rigorously, we think of $\mathfrak{R}$ as given by the second kind of generator in Theorem 106 by the isosceles, rectangular triangles MAB, $M_1A_1B_1$ (Fig. 46), and draw E through MB perpendicular to the reference plane Σ. A will then go to A'' through $\mathfrak{T}$, and B_1 goes to B_1'', such that:

$$\frac{MA}{MA''} = \frac{M_1B_1}{M_1B_1''};$$

therefore:

$$\Delta \equiv MA''B \sim M\,B_1''A_1 \equiv \Delta_1\,,$$

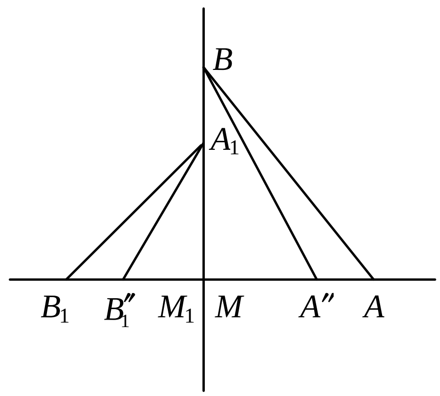

Figure 46.

therefore such that the legs of the triangle that correspond under the affinity *do not* correspond under the similarity. We will have reached our goal when we are able to show that one can generate any ray net without focal lines by an affinity that is defined by triangles Δ and Δ_1 of that kind. Here, the central pencil is involutory, since the corresponding right angles coincide. One must then, above all, be able to replace two arbitrary systems with two that have involutory central planes. Two arbitrary affine fields Σ, Σ_1' that lie in each other and have the origin M as a self-corresponding point are defined by:

(70) $$x' = \alpha x + \beta y, \qquad y' = \gamma x + \delta y.$$

The direction $\tau = y / x$ corresponds to a direction $\tau' = y' / x'$, and one finds from (70) that:

(71) $$\tau' = \frac{\gamma + \delta\tau}{\alpha + \beta\tau}$$

or

(72) $$\beta\,\tau\tau' + \alpha\,\tau' - \delta\,\tau - \gamma = 0.$$

Should the central pencil be involutory, then nothing would change in this equation under permutation of τ and τ'. One would then have:

(73) $$\alpha = -\,\delta$$

as the condition for that. Since Σ_1 arises from Σ' by a parallel displacement in the Z-direction through a segment d, the pointers to two associated points of Σ and Σ_1 will be:

$$x, y, 0; \qquad x', y', d.$$

Therefore, the equations of the connecting line in the running pointers ξ, η, ζ will be:

$$\frac{\xi - x}{x' - x} = \frac{\eta - y}{y' - y} = \frac{\zeta}{d}$$

or

(74) $$\frac{\xi - x}{(\alpha - 1)\,x + \beta y} = \frac{\eta - y}{\gamma x + (\delta - 1)\,y} = \frac{\zeta}{d}.$$

For each choice of x and y, one will obtain the equations of a ray of $\mathfrak{N}$ from (74). If one intersects $\mathfrak{N}$ with the plane $\zeta = d'$ and sets:

$$d' : d = c$$

then one will obtain a field Σ_0, and the affinity between Σ and Σ_0 will be represented by:

(75) $$\begin{aligned} \xi &= (c\alpha - c + 1)x + c\beta y, \\ \eta &= c\gamma x + (c\delta - c + 1)\,y. \end{aligned}$$

Here, the condition (73) reads:

$$c\alpha - c + 1 = -c\,\delta + c - 1,$$

so it will be fulfilled when we choose:

$$c = \frac{-2}{\alpha + \delta - 2}.$$

For any system Σ, we then obtain a system Σ_0 such that the associated central pencil is involutory; if one starts with Σ_0 then one will come back to Σ.

We can assume that Σ and Σ' already have this property and draw the axes in them with their non-corresponding legs at corresponding coincident right angles. A point $(x, 0)$ on the X-axis must then correspond to a point $(0, y')$ on the Y-axis, etc. Equations (70) will then reduce to:

(76) $$x' = \beta y, \qquad y' = \gamma x.$$

One finds that:

$$\tau\,\tau' = \frac{\gamma}{\beta},$$

so, in fact, one finds an involution of the central pencil; therefore, τ and τ' have *opposite* signs. We choose a point $A \equiv (x_0, 0)$ on the X-axis and a point $B \equiv (0, y_0)$ on the Y-axis. One will then have:

$$A' \equiv (0, \gamma x_0), \qquad B' \equiv (\beta y_0, 0).$$

Now, should we have:

(77) $$M\,A\,B \sim M\,B'\,A',$$

as would be the case for the triangles Δ and Δ_1, then we would need to have:

$$\frac{x_0}{y_0} = -\frac{\beta y_0}{\gamma x_0},$$

so:

$$\frac{y_0}{x_0} = \sqrt{-\frac{\gamma}{\beta}},$$

which must always be real. One can then always choose the right triangle $M A B$ in such a way that it fulfills condition (77) with its correspondent. Under this assumption, however, the affine system can be derived from two fields that are rotated similarly through a right angle by the transformation $\mathfrak{T}$. We have thus arrived at the most intuitive way of generating a ray net with no focal lines:

Theorem 108: *The general ray net with no focal lines can be obtained from the rotation net when one draws a plane E through the principal ray of the latter and changes all distances from E by the same ratio.*

By applying $\mathfrak{T}$ to the last part of Theorem 106, it will follow that the mutually-covering circles Σ, Σ' that correspond to each other when rotated through a right angle will go to ellipses that lie upon each other; thus:

Theorem 109: *If one assumes that there is a family of similar and similarly-lying ellipses in Σ then every point P of Σ will be associated with the closest endpoint P' of the diameter that is conjugate to the direction MP on that ellipse in a specified sense of rotation, and if this field Σ' lifts up from Σ_1 perpendicular to Σ then Σ, Σ_1 will generate a general ray net without focal lines. One can generate every such net in a single way* (*).

Moreover, one immediately recognizes the first part of the following theorem:

Theorem 110: *All rays of a net without focal lines are equally-wound with respect to the principal ray. Therefore, one can call the net itself right or left-wound* (**). *An arbitrary ray of such a net is, like the net itself, equally-wound with respect to all rays of the net in a certain neighborhood that is bounded by a family of rulings of a hyperbolic paraboloid.*

The latter follows from the fact that a ray in continuous motion can change its winding with respect to another fixed *s* only when it cuts it once or crosses it perpendicularly once. The former is excluded by the absence of singular points of the

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(*) One is then dealing with the theorems of this paragraph without having to establish the application of imaginary elements, since conversely, the theory of the latter will be founded in the next chapter on a knowledge of ray nets.

(**) We also call a *family of rulings* "right-wound" or "left-wound" according to whether each of its rays is right-wound or left-wound with respect to its neighboring rays, respectively.

net; the latter is excluded due to continuity in a certain neighborhood of *s*. This will also show one the way to bound the region for which the theorem is valid. Namely, this will happen for the rays that cross *s* perpendicularly. They cut the plane at infinity in a linear point sequence, and thus, from Theorem 99, define a family of rulings.

We now have an intuitive picture for all kinds of ray nets, and summarize it with:

Theorem 111: *Ray nets are either:*

I) "General" *and indeed, either* α) *they have two focal lines (viz., the* "hyperbolic" *case) or b*) *they have no focal lines (viz., the* "elliptic" *case).*

II) "Special," *with one focal line* (viz., "parabolic") *or:*

III) "Singular" *(sheaves of rays of a total field).*

The general ray net depends upon 8 parameters, the special one on 7, and the singular one on 5.

Thus, I, α) and II will yield just one variety when a focal line goes to infinity, III) will yield two varieties according to merely whether the point *S* (cf., § 53, c), or also the plane ε, goes to infinity, such that when we consider the element at infinity, we will have eight kinds, three of which are singular.

Six of the parameters come from the position in space. Thus, two parameters will remain for the general ray net for the form and magnitude – i.e., one of them will remain for the form. In fact, the form is determined completely, in case I, α), by the angle at which the focal lines cross, and in case I, β), by axis ratio of the ellipses of Theorem 108, which we also call the "axis ratio of the net." From § 53, b), all

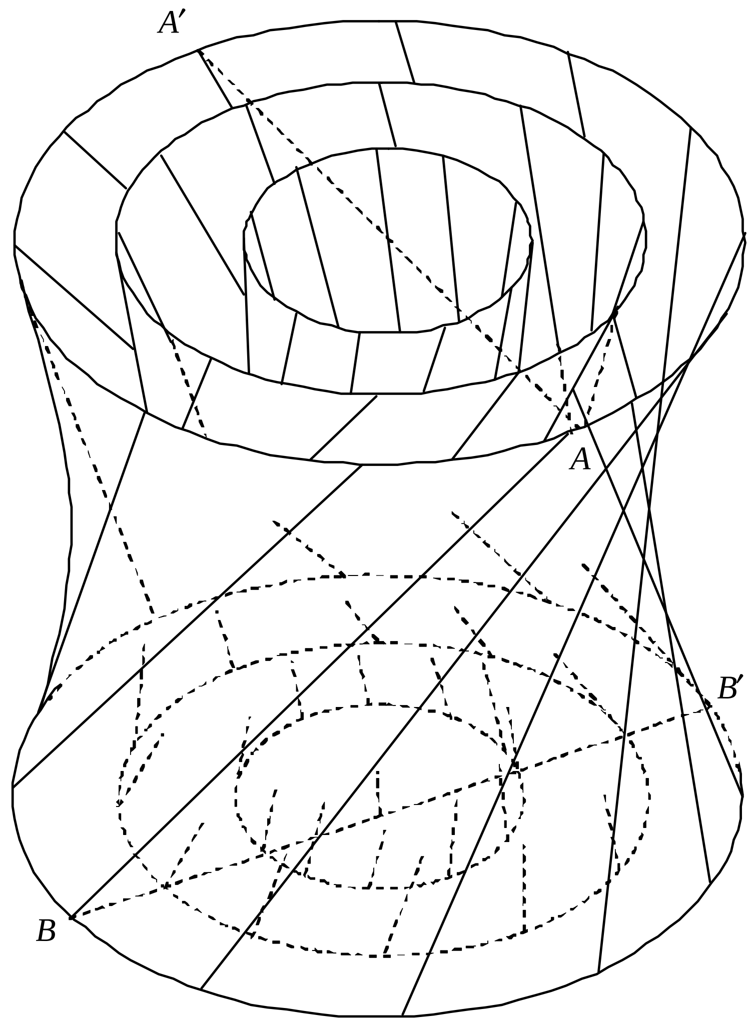

Figure 47.

special nets with the same-notated parameter are similar.

§ 55. Simplest analytical representation of the ray net.

For metric investigations, it is preferable to bring the ray net into the simplest possible position with respect to a rectangular system of pointers. We shift the XY-plane to the middle plane and the Z-axis to the principal ray of the net $\mathfrak{N}$ and then distinguish between the cases:

a) If $\mathfrak{N}$ has two focal lines b, b' that cut the Z-axis at the points N, N' then N will have the Z-pointer $-c$ for an arbitrary choice of the positive direction on the Z-axis; N' will then have the Z-pointer $+c$. We can always adjust the notation so that $c > 0$. If β, β' are the projections of b, b' onto the middle plane then we will choose the positive direction in b arbitrarily, but we will choose the positive direction in b' such that angle (β, β') is concave. Finally, we assign the positive X-axis to the bisecting line of the angle (β, β'). Now, if:

$$\angle (X, \beta') = \alpha$$

then α will always be an acute angle, and in fact, it will be less than 45° (Fig. 48.a) when the focal line pair is left-wound and greater then 45° (Fig. 48.b) when b, b' is right-wound. If one thinks of the middle plane as coincident with the (horizontal) reference plane then in both figures one must imagine that b is below the reference plane and b' is above it.

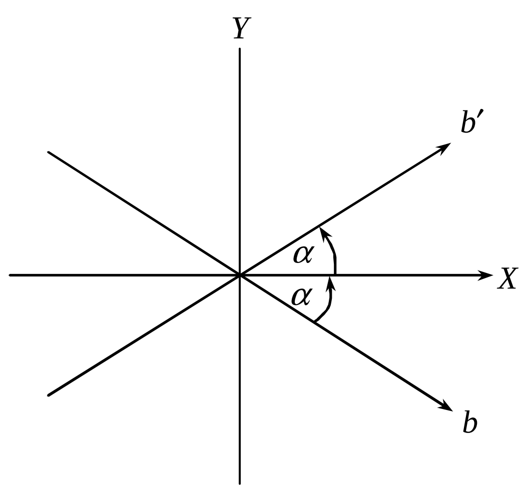

Figure 48a.

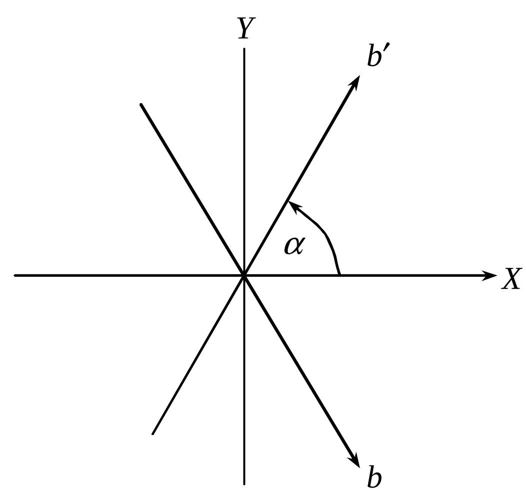

Figure 48b.

If the points of intersection S, S' of a ray s of the net with b, b' have the distances δ and δ' from N and N', resp., then the pointers of S and S' will be:

$$\begin{aligned} x &= \delta \cos \alpha, & y &= -\delta \sin \alpha, & z &= -c; \\ x' &= \delta' \cos \alpha, & y' &= \delta' \sin \alpha, & z' &= c. \end{aligned}$$

Therefore, one can summarize the line pointers q_i of s, according to § 33, equations (24), as:

$$\begin{aligned} &q_1 = (\delta' - \delta)\cos\alpha, && q_4 = (\delta' - \delta)\, c \sin\alpha, \\ &q_2 = (\delta' + \delta)\sin\alpha, && q_5 = -(\delta' + \delta)\, c\cos\alpha, \\ &q_3 = 2c, && q_6 = \delta\delta' \sin 2\alpha. \end{aligned} \tag{78}$$

We have thus arrived at a *parametric representation* of the ray net I, a); i.e., the line pointers of the doubly-infinite ray manifold of the net are expressed in terms of two independent variables δ, δ', which are the "parameters" (*). The relation:

$$\sum q_i\, q_{i+3} = 0$$

is fulfilled identically, due to (78). The q_i fulfill the equations:

$$\frac{q_4}{q_1} = c\tan\alpha, \qquad \frac{q_5}{q_2} = -c\cot\alpha, \tag{79}$$

which are free of the parameters; they are the equations of two linear complexes, and thus, of the net, in the sense of § 53.

If one introduces:

$$\delta' + \delta = u, \qquad \delta' - \delta = v,$$

in place of δ, δ', as parameters then one will get:

$$\begin{aligned} &q_1 = v\cos\alpha, && q_4 = v\, c\sin\alpha, \\ &q_2 = u\sin\alpha, && q_5 = -u\, c\cos\alpha, \\ &q_3 = 2c, && q_6 = \tfrac{1}{4}(u^2 - v^2)\sin 2\alpha \end{aligned} \tag{78.a}$$

as the representation of the net. In particular, for a *rectangular* net ($\alpha = 45^\circ$), when one introduces $u/\sqrt{2}$, $v/\sqrt{2}$, in place of u, v as the new parameters:

$$\begin{aligned} &q_1 = v, && q_4 = v\, c, \\ &q_2 = u, && q_5 = -u\, c, \\ &q_3 = 2c, && q_6 = \tfrac{1}{4}(u^2 - v^2)\,. \end{aligned} \tag{78.b}$$

One can arrive at the equations of the net in yet another way when one writes down the pointers b_i, b_i' of the focal lines: $(0, 0, c)$ and $(\cos\alpha, \sin\alpha, c)$ are two points of b', so from § 33:

$$\begin{aligned} &b_1' = \cos\alpha, && b_2' = \sin\alpha, && b_3' = 0, \\ &b_4' = -\cos\alpha, && b_5' = c\sin\alpha, && b_6' = 0. \end{aligned}$$

(*) These parameters are found in *D'Emilio*, "Le superf. rig. di una congr. lin.," Atti del Ist. Ven., Ser. VI, v. 3.b.

The equation of the sheaf of rays with the carrier b' is:

$$\sum b_i' q_{i+3} = 0,$$

so:

(80) $$c\,(-\,q_1 \sin\alpha + q_2 \cos\alpha) + q_4 \cos\alpha + q_5 \sin\alpha = 0.$$

In order to find the equation of the sheaf b, one must simultaneously replace α and c with $-\alpha$ and $-c$, resp.:

(81) $$-\,c\,(q_1 \sin\alpha + q_2 \cos\alpha) + q_4 \cos\alpha - q_5 \sin\alpha = 0.$$

Instead of equations (80) and (81), one obtains simpler ones by addition and subtraction:

$$q_4 \cos\alpha - q_1\, c \sin\alpha = 0, \qquad q_5 \sin\alpha + q_2\, c \sin\alpha = 0,$$

which agrees with (79).

b) If the one focal line b' goes to infinity then this representation will break down; We then draw the other b in the ZX-plane, where it might cut the Z-axis at the origin U at an angle ω (Fig. 49), while all rays of the net shall be parallel to the XY-plane. Is S is a point of b and $US = \delta$ then:

$$x = \delta \sin\omega, \qquad y = 0, \quad z = \delta \cos\omega$$

will be the pointers of S. One finds, by a process that is analogous to the one in a), that the pointers of the rays s, s' of the net through S that are parallel to the X and Y axes are:

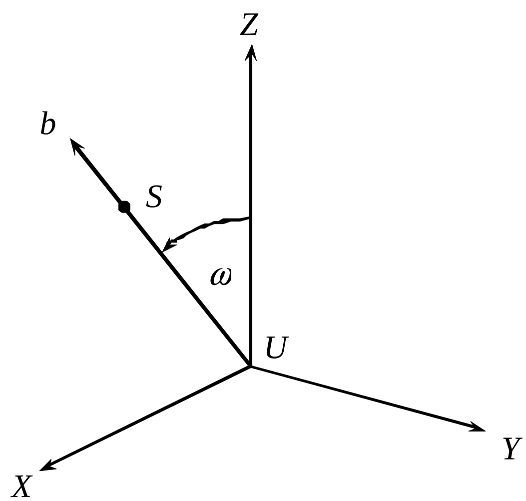

Figure 49.

$s \parallel X$	$s' \parallel Y$
1	0
0	1
0	0
0	$-\,\delta \cos\omega$
$\delta \cos\omega$	0
0	$\delta \sin\omega$

Thus, the parametric representation of the net is:

$$q_1 = \lambda, \qquad q_2 = \mu, \qquad q_3 = 0,$$

$$q_4 = -\mu\delta\cos\omega, \qquad q_5 = \lambda\delta\cos\omega, \qquad q_3 = \mu\delta\sin\omega,$$

in which δ and $\lambda : \mu$ are the parameters. As the intersection of two complexes, the net will be represented by:

$$(82) \qquad q_6 \cos\omega + q_4 \sin\omega = 0, \qquad q_3 = 0.$$

c) If $\mathfrak{N}$ is elliptic then we will start from Theorem 109: If r is the major axis of an ellipse in that theorem and m is a constant ($0 < m < 1$) then the pointers of P will be:

$$x = r\cos u, \quad y = mr\sin u, \quad z = -c,$$

in which u runs through all values from 0 to 2π. The pointers of P' are:

$$x' = -r\cos u, \quad y' = mr\sin u, \quad z' = c,$$

in which c will be positive for right-wound nets. When we again define the q_i as in § 33, we will get the parameter representation of the net I, β):

$$(83) \qquad \begin{aligned} q_1 &= -r(\cos u + \sin u), & q_4 &= cmr(\cos u + \sin u), \\ q_2 &= mr(\cos u - \sin u), & q_5 &= cr(\sin u - \cos u), \\ q_3 &= 2c, & q_6 &= m\,r^2. \end{aligned}$$

By eliminating the parameters r and u, one will obtain the same net when it is represented as the intersection of two complexes:

$$(84) \qquad q_4 + cm\,q_1 = 0, \qquad q_5 + \frac{c}{m}\,q_2 = 0.$$

If the point of intersection of a ray (u, r) with the XY-plane (viz., the middle plane) has the pointers x_0, y_0 then, since the points P, P' are at equal distances on opposite sides of the middle plane, x_0 will be the arithmetic mean of x and x', etc., as one can confirm from equations (83), moreover, when one introduces the expression in point pointers for the q on the left-hand side, then sets $z = 0$, and calculates the remaining pointers according to § 33. Thus:

$$x_0 = \tfrac{1}{2} r(\cos u - \sin u), \qquad y_0 = \tfrac{1}{2} mr(\cos u + \sin u),$$

or

$$r\cos u = \frac{y_0}{m} + x_0, \qquad r\sin u = \frac{y_0}{m} - x_0.$$

Therefore, if one introduces x_0, y_0 in place of r, u as parameters then one will obtain the following parametric representation (while dropping a factor of 2):

$$(83.a)\qquad \begin{aligned} q_1 &= -\frac{y_0}{m}, & q_4 &= c\,y_0, \\ q_2 &= m\,x_0, & q_5 &= -c\,x_0, \\ q_3 &= c, & q_6 &= m x_0^2 + \frac{1}{m} y_0^2 . \end{aligned}$$

(83) is also the *parametric representation of a family of rulings*, if one considers r to be constant. The net is decomposed into ∞^1 families of rulings by the various values of r, whose throat ellipses, which we will also call the throat ellipses of the net, all lie in the middle plane and whose axes have a common position. We also call the latter the *axes of the net.* They cut each other and the principal ray at right angles in the *center* of the net; let the two planes of Theorem 109 be called *power planes.*

As the *center* of a hyperbolic net, we refer to the middle of the segment that is cut out by the focal lines on the principal ray as the *axis* of the bisector of the lines β, β' in a).

d) For a *special* ray net, we lay the focal line along the X-axis and the middle plane (§ 54, rem.) in the XY-plane. If δ is the distance from the origin to the vertex S of a plane pencil of rays of the net whose plane defines the angle ω with the XY-plane then one will find the parametric representation of this pencil of rays in a manner that is similar to the one in b):

$$\begin{aligned} q_1 &= \lambda, & q_2 &= \mu \cos\omega, & q_3 &= \mu \sin\omega, \\ q_4 &= 0, & q_5 &= -\mu\delta \sin\omega, & q_6 &= \mu\delta \cos\omega . \end{aligned}$$

Thus, from equation (66), δ and ω will be coupled by the equation:

$$\cot\omega = K\,\delta;$$

from § 54, rem., one can, in fact, extend the α in equation (66) and the ω to $\pi/2$. When d increases, the associated plane will rotate in the negative sense, here, for a positive K. One can then choose either

$$(85)\qquad \begin{aligned} q_1 &= \lambda, & q_2 &= \mu \cos\omega, & q_3 &= \mu \sin\omega, \\ q_4 &= 0, & q_5 &= -\frac{\mu}{K} \sin\omega, & q_6 &= \frac{\mu}{K}\frac{\cos^2\omega}{\sin\omega} \end{aligned}$$

or

$$(86)\qquad \begin{aligned} q_1 &= \lambda\sqrt{1+K^2\delta^2}, & q_2 &= \mu K\delta, & q_3 &= \mu, \\ q_4 &= 0, & q_5 &= -\mu\delta, & q_6 &= \mu K\delta^2 \end{aligned}$$

as the representation of the net; in the first case, ω and $\lambda : \mu$ are the parameters, and in the second one, δ and $\lambda : \mu$ are the parameters. Moreover:

$$(87)\qquad q_4 = 0, \qquad q_2 + K\,q_5 = 0$$

are the equations of the net.

e) If the focal line of the special ray net is at infinity then we will draw it in the location of the *XY*-plane. The point sequence on the focal line – i.e., directions of the pencil of rays $\tau = y : x$ – and the pencil of planes $z = \delta$ (and therefore also an arbitrary piece of it – e.g., a point sequence on the *Z*-axis) are then projectively associated with each other. If we also project the pencil $\tau = y : x$ onto the *Z*-axis then we will have a correlation for which, from (66), the relation:

$$t = K\,\delta$$

will be true for the simplest choice of pointer system. Thus:

$$y = K\,\delta x + \nu, \qquad z = \delta$$

will be the equations of all rays of the net and:

$$\begin{array}{lll} q_1 = 1, & q_2 = K\,\delta, & q_3 = 0, \\ q_4 = K\,\delta^2, & q_5 = \quad \delta, & q_6 = -\nu \end{array}$$

will be the parametric representation, while:

(89) $$q_3 = 0, \qquad q_2 - K\,q_5 = 0$$

will be its sectional representation.

There are two distinguished directions in the pencil, namely, the one that corresponds to the plane at infinity and the one that is perpendicular to it – viz., the "principal direction"; we call its corresponding plane the "principal plane" of the net. Here, it falls on the *XY*-plane, and the principal direction, along the *X*-direction. The net admits the ∞^2 translations that take the principal plane to itself.

f) If $\mathfrak{N}$ is *singular* then we will place the point *S* [§ 53, c)] at the origin and the plane ε on the *XY*-plane. The rays through the origin are characterized by:

$$q_4 = q_5 = q_6 = 0,$$

and the ones in the *XY*-plane are characterized by:

$$q_3 = q_4 = q_5 = 0.$$

Therefore:

(90) $$q_4 = q_5 = 0$$

will be the equations of the net. In fact, conversely, when equations (90) are fulfilled, since:

$$q_1\, q_4 + q_2\, q_5 + q_3\, q_6 = 0,$$

either q_3 or q_6 must be zero.

If S goes to infinity along the X-axis then:

$$q_2 = q_3 = q_4 = 0$$

will be the equations of the ray sheaf:

$$q_3 = q_4 = q_5 = 0$$

will be those of the ray field, as before, and therefore:

(91) $$q_3 = q_4 = 0$$

will be those of the entire net.

Finally, if ε is the plane at infinity, and S represents the direction of the X-axis then, as before:

$$q_2 = q_3 = q_4 = 0$$

will be the equations of the sheaf, while the totality of the fields in space will enter in place of the ray field (§ 36):

$$q_1 = q_2 = q_3 = 0.$$

Therefore:

(92) $$q_2 = q_3 = 0$$

will be the equations of the net.

All parametric representations of this paragraph will also be true when q refers to the tetrahedral pointers (except that the possible relationships with the elements at infinity will vanish, and the parameters will no longer have such a simple geometric meaning); the elimination of the parameters will then lead to the same equations as before.

§ 56. The involution of two linear complexes.

If the simultaneous invariants (§ 52) of two linear complexes vanish:

$$\sum a_i\, p_{i+3} = 0, \qquad \sum b_i\, p_{i+3} = 0,$$

so one has:

(93) $$\omega(a, b) = \sum a_i\, b_{i+3} = \sum a_{i+3}\, b_i = 0,$$

then we will say that the complexes are *in involution* (*). We first assume that both complexes are twists. In order to understand the geometric meaning of their involutory position, we map a ray s of the second twist (b_i) by the reciprocity that is defined by the first twist; i.e., we seek the polar s_i' of s_i in (a) from equation (51):

(*) The concept of the involution of two complexes was introduced by *Klein* (Gött. Nachr., 1869).

$$\rho\, s_i' = a_i\, \omega(a, s) - A \cdot s_i\,.$$

It can be shown that s_i' also belong to the complex (b); one will then have:

$$\rho \sum b_{i+3}\, s_i' = \omega(a, s) \sum b_{i+3}\, a_i - A \sum b_{i+3}\, s_i\,.$$

Both sums on the right now vanish, so the sums on the left will, as well. Conversely, the sum on the left will vanish only when the first sum on the right also vanishes. If one or both of the complexes is special then that will yield the meaning of equation (93) immediately; thus:

Theorem 112: *If two twists are involutory then each of them will go to the other one by the null system, and conversely. If a twist and a sheaf of lines are involutory then the carrier of the latter will belong to the twist. The carriers of two involutory sheaves of rays will interest.*

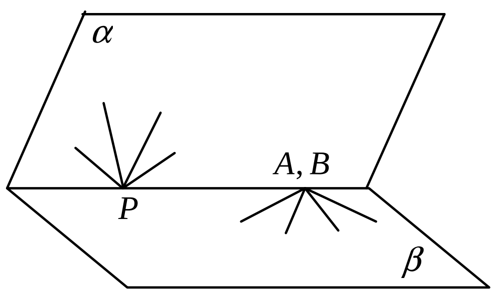

Figure 50.

One can deduce another, more intuitive, property of involutory position from this theorem: One chooses a point P on a common ray t of the complexes (a) and (b) (Fig. 50). A null plane α corresponds to it in (a) and a null plane β in (b); α has the null point A on t in (b), and β has the null point B on t in (a). Now, A and B will be identical when the complexes are in involution. The pencil (P, α) will then correspond to a pencil in β in the null system (b), because it has the vertex P, and a pencil with the vertex A, and thus, the pencil (A, β), because it lies in the plane α. On the other hand, from Theorem 112, it must once more be a pencil in (a), so since it lies in β, it must be the pencil (B, β). Therefore, A and B will be identical. If one were to start with the point $A \equiv B$, which we will call Q, then one would come back to P by this process. Each of the four elements P, Q, α, β will determine the remaining three. If one rotates α around t then the sequences P and Q on t will also be projective for two arbitrary complexes, so here, they will be involutory, as well. By combining the two reciprocities that are defined by two twists, a collineation will arise that is therefore involutory, here.

Theorem 113: *The points (planes, resp.) of space are paired in an involutory way by two involutory twists when the two null points of a plane (the two null planes of a point, resp.) that rotates around a common ray t (that moves along t, resp.) describe involutory sequences (pencils, resp.). This involution is a collineation.*

We would like to represent them analytically (*Stolz,* Math. Ann., Bd. IV, pp. 440) by calculating the pointers of g_1 from those of g. It is therefore irrelevant whether we first look for the polar g' of g in (a) and then the polar g_1 of g' in (b) or first look for the polar

g_1' of g in (b) and then the polar γ_1 of g_1' in (a); γ_1 must then coincide with g_1. We set out upon the first path: If s_i are the pointers of g, s_i' are those of g', and s_i'' are those of g_1 then, from equation (51) (cf., the beginning of this paragraph):

(94) $$\rho s_i' = a_i \cdot \omega(a, s) - A \cdot s_i,$$

(95) $$\rho s_i'' = b_i \cdot \omega(b, s') - A \cdot s_i',$$

where:

$$B = \sum b_i \, b_{i+3}.$$

If we substitute the values in (94) into (95) then we will get:

$$\rho \cdot \omega(b, s') = \sum b_{i+3} \cdot \rho s_i' = \omega(a, s) \cdot \omega(b, a) - A \cdot \omega(b, s).$$

Since $\omega(b, a) = 0$, by involution, what will remain is:

$$\rho\rho' s_i'' = -A\, b_i \cdot \omega(b, s) - B\, a_i \cdot \omega(a, s) + A\, B\, s_i,$$

or when one sets $\rho\rho' = -\sigma$:

(96) $$\sigma s_i'' = A\, b_i \cdot \omega(b, s) + B\, a_i \cdot \omega(a, s) - A\, B\, s_i.$$

Two twists $\mathfrak{A}$ and $\mathfrak{B}$ with the pointers a_i and b_i determine a pencil. Two complexes $\mathfrak{C}$, $\mathfrak{C}'$ in it might have the pointers $\lambda\, a_i + \mu\, b_i$ and $\lambda' a_i + \mu' b_i$. They will be in involution when:

(97) $$\omega(\lambda\, a_i + \mu\, b_i, \lambda' a_i + \mu' b_i)$$
$$= \lambda\lambda' \cdot \omega(a) + (\lambda\mu' + \lambda'\mu) \cdot \omega(a, b) + \mu\mu' \cdot \omega(b) = 0.$$

If one chooses $\lambda : \mu$ arbitrarily then that will yield $\lambda' : \mu'$ uniquely; i.e.:

Theorem 114: *For any twist in a pencil, there is a single twist in that pencil that is involutory with it.*

In particular, if $\mathfrak{A}$ and $\mathfrak{B}$ are already involutory then equation (97) will reduce to:

$$\lambda\lambda' \cdot \omega(a) + \mu\mu' \cdot \omega(b) = 0,$$

or

(98) $$\lambda\lambda' A + \mu\mu' B = 0.$$

If the carrier $\mathfrak{N}$ of the pencil is elliptic then, from equations (65), A and B will get the same notation (the converse is not true), and we would like to see how the helix of $\mathfrak{N}$ depends upon the signs of the invariants of the complexes in the pencil. We can assume that it has a special position with respect to the system of pointers and, from Theorem 88 and § 55, c), we can then immediately deduce the meaning of the sign of c that was given there:

Theorem 115: *All twists of a pencil that has an elliptic carrier $\mathfrak{N}$ are left-wound or right-wound according to whether $\mathfrak{N}$ is right-wound or left-wound, respectively; in the first case, the invariants of the twist will be positive.*

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§ 57. Gathered, involutory spatial systems.

If $\mathfrak{N}$ is a ray net with two focal lines b, b' then a ray of $\mathfrak{N}$ will go through any point P in space, upon which we would like to associate the point P with the point P' that is harmonically separated from P by the points of intersection with b, b'. The points of space are thus involutorily associated with each other, in such a way that the points of b, b' will correspond to themselves (dual construction?). This association is a collineation, because all points of one line g are projectively associated with the points of the line g' that is harmonically separated from g on the family of rulings (g, b, b'). The rays of $\mathfrak{N}$ correspond to themselves under the collineation. One calls a non-perspective spatial collineation (*) for which there exist ∞^2 self-corresponding rays a *gathered (gescharte) collineation*, and if it is involutory then one will call it a *gathered involution* (Staudt, *Beitr. zur. Geom. d. Lage*, art. 101, 1856).

Now, for the following chapter it will be very important to prove that a gathered involution is defined *uniquely* by *any* general ray net, independently of the existence of focal lines. The net determines a pencil of complexes; all that one needs to prove then is that the involution in Theorem 113 is independent of which of the ∞^1 involutory complex pairs in the pencil (Theorem 114) one starts with. If one determines an involution $\mathfrak{J}$ by the complexes $\mathfrak{C}$ and $\mathfrak{C}'$, instead of the complexes $\mathfrak{A}$ and $\mathfrak{B}$ (cf., the conclusion of the foregoing paragraphs) then in order to represent $\mathfrak{J}$ analytically, one must substitute:

$\lambda\, a_i + \mu\, b_i$	in place of	a_i,
$\lambda'\, a_i + \mu' b_i$	"	b_i,
$\frac{1}{2}\,\omega(\lambda\, a_i + \mu\, b_i) = \lambda^2 a_i + \mu^2\, b_i$	"	A.
$\lambda'^2 a_i + \mu'^2\, b_i$,	"	B,
$\lambda \cdot \omega(a, s) + \mu \cdot \omega(b, s)$	"	$\omega(a, s)$, or, more briefly, ω_a,
$\lambda' \cdot \omega(a, s) + \mu' \cdot \omega(b, s)$	"	$\omega(b, s)$, or, more briefly, ω_b,

in equation (96). We can now assume that $A = \pm B = \pm 1$ (and indeed, from Theorem 115, one will certainly have $A = B$ for elliptical nets; we will now examine only this case,

(*) One will get the perspective spatial involutions when one chooses a point S and a plane E, and assumes that the points on each ray s through S that are harmonically separated by A and (s, E) are associated with each other.

since the other one is completely analogous to it), because we can multiply the complex equations by an arbitrary factor. The relation (98) will then go to:

$$\lambda\lambda' + \mu\mu' = 0, \tag{99}$$

and the line S_i that corresponds to s_i in $\mathfrak{I}$ will be given by:

$$\begin{aligned}\sigma S_i = {} & (\lambda^2 + \mu^2)\,(\lambda'\, a_i + \mu' b_i)\,(\lambda'\, \omega_a + \mu'\, \omega_b) \\ & + (\lambda'^2 + \mu'^2)\,(\lambda\, a_i + \mu\, b_i)\,(\lambda\, \omega_a + \mu\, \omega_b) - (\lambda^2 + \mu^2)\,(\lambda'^2 + \mu'^2)\, s_i\,.\end{aligned} \tag{100}$$

By means of (99), one will get:

$$(\lambda^2 + \mu^2)\,(\lambda'^2 + \mu'^2) = (\lambda\mu' - \lambda'\mu)^2\,.$$

Moreover, ω_b will have the coefficients:

$$a_i\,(\lambda\mu' + \lambda'\mu)(\lambda\lambda' + \mu\mu') + b_i\,[(\lambda\mu' - \lambda'\mu)^2 + 2\,\mu\mu'(\lambda\lambda' + \mu\mu')] = b_i\,(\lambda\mu' - \lambda'\mu)^2.$$

The right-hand sides of (96) and (100) will thus differ by only the factor $(\lambda\mu' - \lambda'\mu)^2$, which is why the S_i are proportional to the s_i''.

Theorem 116: *A gathered involution is defined uniquely by any general ray net, namely, the one that determines any two involutory complexes of the pencil whose carrier is the net.*

We would like to convince ourselves that the the involution $\mathfrak{I}$ of this theorem is identical with the $\mathfrak{I}'$ that was presented in the beginning of this paragraph in the hyperbolic case: In order to find the correspondent to a point in $\mathfrak{I}$, we must look for its null point in one complex and then its null point in the other one. Now, since a true point involution is determined completely by its possible double points, the involutory sequences of $\mathfrak{I}$ and $\mathfrak{I}'$ must be identical on any ray of the net.

We would like to represent the involution analytically as a *point conversion* in the special position with respect to the system of pointers [§ 55, c)] for the case of an elliptical net: We will obtain the reciprocity of the first null system (84) when we set:

$$a_4 = 1, \qquad a_1 = cm$$

in equations (15.a) of § 46 and set all of the other a equal to zero. We will further calculate the null point x_i' to a plane in the second null system, when set:

$$a_2 = \frac{c}{m}, \qquad a_5 = 0,$$

in equations (16.a) of § 46 and set all other a equal to zero. We have to combine these two conversions:

$$(101)\qquad \begin{aligned} \sigma u_1 &= c\,m\,x_2\,, & \tau x'_1 &= u_3\,,\\ \sigma u_2 &= -\,c\,m\,x_1\,, & \tau x'_2 &= -\frac{c}{m}\,u_4\,,\\ \sigma u_3 &= x_4\,, & \tau x'_3 &= -\,u_1\,,\\ \sigma u_5 &= -\,x_3\,, & \tau x'_4 &= \frac{c}{m}\,u_2\,, \end{aligned}$$

into:

$$(102)\qquad \begin{aligned} \sigma\tau x'_1 &= x_4\,,\\ \sigma\tau x'_2 &= \frac{c}{m}x_3\,,\\ \sigma\tau x'_3 &= -\,c\,m\,x_2\,,\\ \sigma\tau x'_4 &= -\,c^2\,x_1\,, \end{aligned}$$

and finally (from the rule in § 31) replace $x_2 : x_1$, $x_3 : x_1$, $x_4 : x_1$ with x, y, z:

$$(103)\qquad \begin{aligned} x' &= \frac{c}{m}\frac{y}{z}\,,\\ y' &= -cm\frac{x}{z}\,,\\ z' &= -\frac{c^2}{z}\,. \end{aligned}$$

In fact, the solution for x, y, z:

$$x = \frac{c}{m}\frac{y'}{z'}\,, \qquad y = -cm\frac{x'}{z'}\,, \qquad z = -\frac{c^2}{z'}$$

will have the same form as the original equations, which is as it must be for an involutory relationship. Since the two complexes (84) that we started with are also involutory to each other when the q mean tetrahedral pointers, the associated involution will be represented by (102) for general pointers. The basic tetrahedron thus lies in such a way with respect to the net that the opposite edges $P_1\,P_4$ ($x_2 = x_3 = 0$) and $P_3\,P_3$ belong to the net, and the two vertices of the tetrahedron on each of these two edges are a pair of the involution; an analogous dual statement will also be true.

If we take two rays s, s' of a general net and an involution on s then any pair of points P, P_1 on them that are linked with s' as self-corresponding points will give to rise planes that correspond to each other in the gathered involution, and will thus define a point-pair of the involution around s''; i.e., if one projects the point involution onto any ray of a general net $\mathfrak{N}$ from the other rays of $\mathfrak{N}$ and cuts the plane involution around any ray of $\mathfrak{N}$

with the other rays of $\mathfrak{N}$ then one will obtain the same involutions that were defined already. In other words:

Theorem 117: *Any point involution that belongs to a ray of the net will lie perspectively to any plane involution that belongs to a ray of that net.*

In particular, the planar, affine parallel systems that can be generated by $\mathfrak{N}$ (Theorem 104) will be paired involutorily, and the involution will cut that ray out of any finite ray of the net; likewise, the directions that are located in the middle plane will be paired involutorily (we called this an involution in the central pencil in § 54) and will determine the plane involution around this ray with any finite ray. The plane at infinity will correspond to the middle plane, on which the central points of all true point involutions will then lie.

We also call a gathered involution *elliptic* or *hyperbolic* according to the character of the associated net.

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§ 58. The special ray net of a twist and the parameters of its correlations.

Any ray s of a twist $\mathfrak{G}$ with an axis α is the focal line of a special ray net. These nets are all (from § 54, conclusion) similar to two of them and differ only by the parameter K of the correlation on s, which we would like to calculate as a function of the shortest distance δ between s and α, and the pitch $\mathfrak{k}$. Let the base point of δ on s be N. From § 53, b), all that must be calculated is the angular velocity φ_0 by which the null plane rotates around s when a point on s passes through N with unit velocity. Let n be the normal to the moving null plane; the angle between two arbitrary positions of n (Fig. 51) will then have the location of a plane that goes through the X-axis. The angular velocity can then be measured by the change in the angle $(X, n) = \omega$. If t is the distance from a point P on s to N and $(Y, s) = \nu$ then the pointers of P will be:

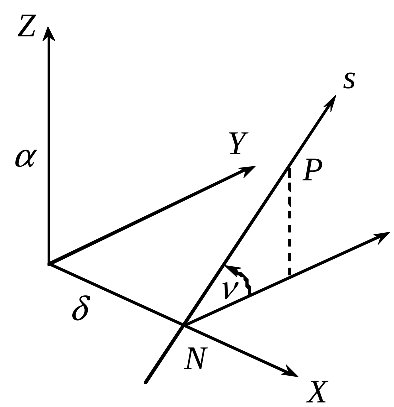

Figure 51.

$$x = d, \quad y = t \cdot \cos \nu, \quad z = t \cdot \sin \nu.$$

Now, from § 8, equation (14):

$$\tan \nu = -\frac{\delta}{\mathfrak{k}}.$$

Thus, if one sets:

$$\left| \sqrt{\mathfrak{k}^2 + \delta^2} \right| = w$$

then:

$$\sin\nu = \frac{\delta}{w}, \qquad \cos\nu = -\frac{\mathfrak{k}}{w}.$$

In fact, if one chooses the direction of *s* that subtends an acute angle with the *Z*-axis to be positive then the negative sign must stand before the cos, as Fig. 51 shows when one considers the geometric meaning of $\mathfrak{k}$ (Fig. 51 corresponds to a *right*-wound twist, so it will correspond to a negative $\mathfrak{k}$; cf., § 11). If one substitutes the pointers of *P*, namely:

$$\delta, -\frac{\mathfrak{k}\,t}{w}, \frac{\delta t}{w},$$

in the equation of the null plane (§ 7):

$$x\,\eta - y\,\xi + \mathfrak{k}\,(\zeta - z) = 0$$

of an arbitrary point then one will get:

$$w\delta\eta - \mathfrak{k}\,t\xi + \mathfrak{k}\,(w\zeta - t\,\delta) = 0$$

as the equation for all null planes of *s*. We infer from this that:

$$\cos\omega = \frac{\mathfrak{k}\,t}{\sqrt{(\mathfrak{k}^2+\delta^2)^2+\mathfrak{k}^2t^2}},$$

and must calculate $\varphi_0 = \left[\frac{d\omega}{dt}\right]_{t=0}$. When we then merely seek the absolute value and consider that when *t* is equal to zero, we will have cos ω = and sin $\omega = \pm 1$, we will immediately obtain by differentiating the equation:

$$\cos\omega\cdot\sqrt{w^4+\mathfrak{k}^2t^2} = \mathfrak{k}\,t$$

that:

$$w^2\left[\frac{d\omega}{dt}\right]_{t=0} = \mathfrak{k},$$

so

$$|\,\varphi_0\,| = \frac{\mathfrak{k}}{\mathfrak{k}^2+\delta^2}.$$

The null plane rotates in the positive sense for increasing *t* when the twist is right-wound (from Theorem 14, a left-wound screw is defined by the normal plane), so the same thing will be true (§ 11) when $\mathfrak{k}$ is negative (*). If we also wish to express the rotational sense of the null plane by the sign of the parameter K of the correlation on s then we must write [cf., § 53, b)]:

(104) $$K = -\frac{\mathfrak{k}}{\mathfrak{k}^2 + \delta^2}.$$

Thus, we can also calculate the angular velocity φ of the null plane at an arbitrary point P of s from equation (67):

$$\varphi = \frac{-\mathfrak{k}\, w^2}{w^4 + \mathfrak{k}^2 t^2}.$$

On the other hand, the distance r from the point P to α is given by:

$$\rho^2 = t^2 \cos^2 \nu + \delta^2 = \frac{t^2 \mathfrak{k}^2}{w^2} + \delta^2,$$

so:

$$\rho^2 + \mathfrak{k}^2 = \frac{t^2 \mathfrak{k}^2 + w^4}{w^2}$$

and

(105) $$\varphi = \frac{-\mathfrak{k}}{\mathfrak{k}^2 + \rho^2}.$$

This equation includes (104) as a special case and shows the result that will be employed later on:

Theorem 118: *If a point P moves on a ray of a twist with unit velocity then the angular velocity of its null plane will depend upon just the distance from the point to the axis, but not on the direction of the motion.*

---

§59. Generating a twist by translation, rotation, and screwing of a net.

A surface of rotation can be generated by not only rotating its meridian curve, but also by rotating an arbitrary curve that lies on it. In general, it will not always be obtained *completely* in that way. For example, if one rotates a sphere circle around a diameter of the sphere then only a spherical zone will be generated, in general. Analogously, since a twist $\mathfrak{G}$ admits not only a rotation, but also a screw around its axis a, any net $\mathfrak{N}$ that is contained in $\mathfrak{G}$ will generate the twist by a screwing motion around a.

(*) One sees this from Fig. 51 immediately when one considers that the normal to α at P always belongs to the null plane. In addition, it will follow that: The special ray nets of a twist are wound the same way as the twists themselves.

However, one will have to investigate whether any ray of $\mathfrak{G}$ can be obtained in that way. Since the screwing motion can be decomposed into a translation and a rotation, it suffices to consider these two cases.

a) We generate $\mathfrak{G}$ by translating $\mathfrak{N}$ along the direction *a* and ask whether the generation is complete. We seek all rays of $\mathfrak{N}$ that cut any diameter *d* of $\mathfrak{G}$; from Theorem 99, they will define a second-order family of rulings $\mathfrak{R}$, and in fact, here they will define a hyperbolic paraboloid, because they also cut the polar at infinity of *d*. If we project $\mathfrak{R}$ along the direction *d* onto the null plane of any point *P* of *d* then the projections will fill up the pencil (P, ν) completely; therefore, this pencil will also be obtained completely by the displacement when it extends from $-\infty$ to $+\infty$.

Theorem 119: *A twist will be generated completely by displacing any net that is contained in it.*

b) We generate $\mathfrak{G}$ by rotating $\mathfrak{N}$ around *a*. The line at infinity of $\mathfrak{N}$ must also be contained in $\mathfrak{G}$; i.e., the middle plane of $\mathfrak{N}$ must be parallel to *a* (this will follow immediately from Theorem 10 for the hyperbolic net). In the event that $\mathfrak{N}$ is general, we shall consider only the simplest case in which the middle plane contains *a* itself. Thus:

α) If the net $\mathfrak{N}$ is hyperbolic then the focal lines b, b' will define equal angles with *a*. We consider a plane $E \parallel a$ and ask whether all rays of the twist in it can be obtained by rotating $\mathfrak{N}$. We thus consider the starting position of $\mathfrak{N}$ to be the one in which b, b' are parallel to *E*. Under rotation around a, b, b' will describe the same hyperboloid of rotation that will be cut by *E* in a hyperbola whose asymptotes have the directions b, b', and in fact, after a half rotation of the starting position the point of intersection of *b* will describe the one branch of the hyperbola and that of b' will describe the other one. Any two points of the hyperbola that are on different branches will be associated with each other by the same position of $\mathfrak{N}$, and we known from the outset that the connecting line of associated points will move parallel to itself. Since the branches will be described completely, it must then also pass through the plane *E* completely. (The reader must complete a sketch for himself and pursue the two possibilities that *E* can or cannot be separated by b, b'.) Only when *E* goes through *b* or b' will the rays of the twist not be contained in *E*. However, since such planes *E* define only a simple manifold and their rays then define only a two-dimensional manifold, we will consider this to be no breach of completeness.

β) When $\mathfrak{N}$ is an arbitrary special net of $\mathfrak{G}$, an argument that is similar to the one in α) will show the completeness of the generation: Let *b* be the focal line of $\mathfrak{N}$ and let α be the acute angle (a, b); if we again draw $E \parallel a$ then under a half rotation of $\mathfrak{N}$ the point of intersection $S \equiv (b, E)$ will describe completely one branch of a hyperbola with the asymptotic angle 2α, whose auxiliary or principal axis will be parallel to a according to whether E is or is not separated from the plane at infinity by a, b, respectively. In the first case, the ray s of the twist will subtend an acute angle with a in E whose absolute value will be greater than α, and in the second case, it will be smaller. Thus, while S runs through the one branch of the hyperbola, the ray s that goes through S will sweep out the plane E completely in both cases.

γ) Let a, g, h be three mutually perpendicular rays that intersect at A and let a be the axis of a twist $\mathfrak{G}$, so g, h will be rays of it. The rays of $\mathfrak{G}$ whose shortest distances fall along h will all define a family of rulings $\mathfrak{R}$ of an equilateral hyperbolic paraboloid $\mathfrak{P}$, whose principal generators will be g and h (Fig. 52). If one carries the same segment on g from A in both directions and draws the second generators r, r' of $\mathfrak{P}$ through the points C, C' thus-obtained then $\mathfrak{R}$ can be regarded as the product of congruent point sequences on r, r'. If one then rotates $\mathfrak{R}$ around g then a rotation net $\mathfrak{N}$ will arise (Theorem 105) whose principal axis is g. It will belong to $\mathfrak{G}$ completely; a ray s of $\mathfrak{P}$ will then arrive at the same position twice under the rotation, where it intersects a perpendicularly and certainly belongs to $\mathfrak{G}$ then. Since this can be generated by rotating $\mathfrak{R}$, it will follow (*Zindler*, Jahresber. d. D. Math. Ver. IV) that $\mathfrak{G}$ can be generated by rotating a simple line manifold twice, namely:

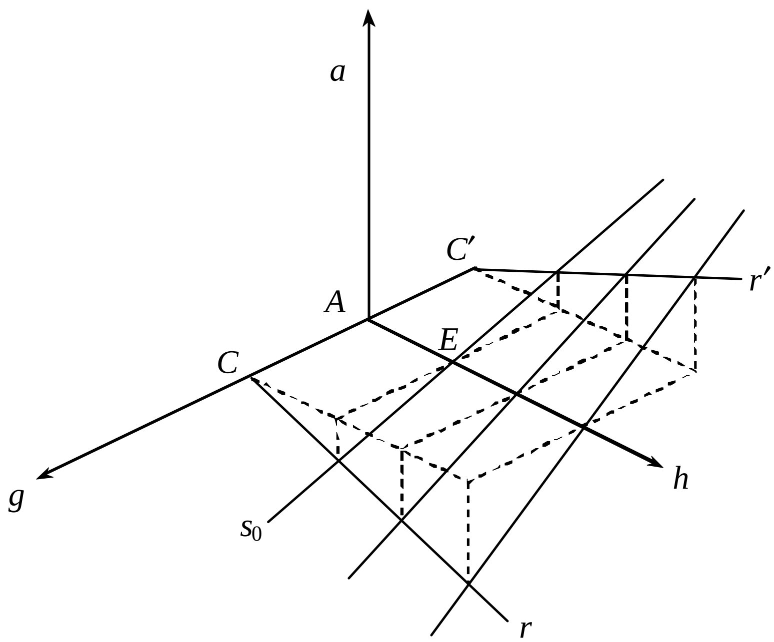

Figure 52.

Theorem 120: *If one rotates a family of rulings of an equilateral hyperbolic paraboloid around its principal generator g and the net thus obtained around a line through the vertex of the paraboloid that is perpendicular to g then a twist will arise.*

Thus, a half-rotation must be executed each time; the generation will indeed be complete then. We then ask whether all rays s can be obtained in a plane $E \parallel a$: The s_0 that intersect h (at B) already belong to $\mathfrak{N}$ in its original position. The remaining ones will then emanate from the rays of $\mathfrak{N}$ that contact the cylinder with the axis a and radius AB, so they will lie in the contact plane β of the cylinder. If $\beta \perp g$ then the ray of the net will lie at infinity in β; if one rotates β then it will come in from infinity, and indeed from the side on which it is wound like s_0 compared to g (in Fig. 52, it is left-wound, so that will be from below in the event that the initial position of β cuts the *positive* half-line g). It will move continuously into the position s_0 when β rotates through a right angle into the position E. If β rotates through another right angle then a ray of the net that lies in β (when it always defines the same angle with a) will again go from the position s_0 to infinity, but on the other side (here, from above), because it must always be wound the same compared to g; it will then sweep out the entire plane β under a half-rotation.

If one draws a plane $\varepsilon \perp a$ and changes the distance from all points of space to ε by the same ratio then this (special) affine transformation will convert the rotation net into a general elliptical one and the twist into another one; the rotation around a will remain preserved, as such. Therefore, the generation of the twist will also be *complete* when one rotates a general elliptical net.

The fact that a twist is always wound oppositely to an elliptical net that is contained in it (Theorem 115) is immediately intuitive in the case of Theorem 120: Every ray s of Π will then be wound with respect to a oppositely to the way that it is wound with respect to g.

---

§ 60. Parametric representation of a twist.

If the line pointers of a line are given as functions of three independent variables – viz., the parameters u, v, w – by:

(106) $$\sigma p_i = f_i(u, v, w) \qquad (i = 1, \ldots, 6),$$

in which the condition:

(107) $$\sum p_i \, p_{i+3} = 0$$

is fulfilled identically, then a line complex $\mathfrak{C}$ will be defined by that. If one changes just one parameter then one will get a ruled surface that is contained in $\mathfrak{C}$. A u-surface – i.e., one for which only u varies – will then be singled out by a value-pair $v = v_0$, $w = w_0$. $\mathfrak{C}$ can therefore be decomposed into ∞^2 u-surfaces, ∞^2 v-surfaces, or ∞^2 w-surfaces. Analogously, a u, v-congruence will be singled out by $w = w_0$, and a decomposition of $\mathfrak{C}$ into ∞^1 congruences will be given by the parametric representation itself, and in three ways. Any congruence will again be decomposed into ∞^1 surfaces in two ways.

For a twist, we can derive such representations immediately from the results of the previous paragraphs, since we have already learned about the parametric representation of the net. We restrict ourselves to the employment of the general net for the generation of the twist $\mathfrak{G}$.

a) For a displacement $\mathfrak{z}$ along the Z-axis, equations (61) in § 41 reduce to:

$$
\begin{aligned}
&p_i = \kappa_i \qquad (i = 1, 2, 3, 6),\\
&p_4 = -\mathfrak{z}\,\kappa_2 + \kappa_4\,,\\
&p_5 = \mathfrak{z}\,\kappa_1 + \kappa_5\,.
\end{aligned}
\tag{108}
$$

Here, the old rod pointers p are expressed in terms of the new ones κ; $\mathfrak{z}$ means the displacement of the system of pointers with respect to the structure. We would like to conversely express the new pointers in terms of the old ones and likewise introduce the displacement of the structure with respect to a fixed system of pointers in place of $\mathfrak{z}$, so we must substitute $-\mathfrak{z}$ for $\mathfrak{z}$ in (108) and then solve for κ. However, we will arrive at equations of precisely the same form in that way, which is why we will preserve equation (108) and now regard the p as the new pointers and $\mathfrak{z}$ as the displacement of the structure with respect to the system of pointers. One can also find this by mere contemplation (*).

In the representation (78.a), the principal ray of the net falls upon the Z-axis, which we would, however, ultimately like to obtain as the axis of the twist. We thus cyclically permute the axes in (78.a), which gives us:

$$
\begin{aligned}
&\kappa_1 = 2c, && \kappa_4 = \tfrac{1}{4}(u^2 - v^2)\sin 2\alpha,\\
&\kappa_2 = v\cos\alpha, && \kappa_5 = v\,c\sin\alpha,\\
&\kappa_3 = u\sin\alpha, && \kappa_6 = -u\,c\cos\alpha\,.
\end{aligned}
\tag{109}
$$

The principal ray of this hyperbolic net will now fall upon the X-axis; we obtain $\mathfrak{G}$ by displacement along the Z-axis, so equations (108) can be applied immediately:

$$
\begin{aligned}
&p_1 = 2c, && p_4 = -\mathfrak{z}\,v\cos\alpha + \tfrac{1}{4}(u^2 - v^2)\sin 2\alpha,\\
&p_2 = v\cos\alpha, && p_5 = 2c\mathfrak{z} + vc\sin\alpha,\\
&p_2 = u\sin\alpha, && p_6 = -uc\cos\alpha\,.
\end{aligned}
\tag{110}
$$

Here, u, v, w are the parameters. Relation (107) will be true, as it must be, and the p_i will fulfill the linear equation:

$$
\frac{p_6}{p_3} = -c\cot\alpha,
\tag{111}
$$

which is free of parameters, and which will then be the equation of the twist.

If we base things upon an elliptic net then we will next obtain in the same way from equations (83.a) (when we also cyclically permute the geometric meaning of the parameters x_0, y_0 and drop the index 0):

$$
\kappa_1 = c, \qquad \kappa_4 = my^2 + \frac{1}{m}\,z^2,
$$

(*) An analogous remark is true for an arbitrary pointer transformation, especially for the interpretation of equations (115).

$$(112) \qquad \kappa_2 = -\frac{z}{m}, \qquad \kappa_5 = cz,$$
$$\kappa_1 = my, \qquad \kappa_6 = -cy,$$

and then from (108):

$$p_1 = c, \qquad p_4 = my^2 + \frac{z}{m}(\mathfrak{z} + z),$$
$$(113) \qquad p_2 = -\frac{z}{m}, \qquad p_5 = c\,(\mathfrak{z} + z),$$
$$p_3 = my, \qquad p_6 = -cy$$

or

$$(114) \qquad \frac{p_6}{p_3} = -\frac{c}{m}.$$

One obtains the simplest representations when one bases (110) on a rectangular net ($\alpha = 45^\circ$) and bases (113) on a rotation net ($m = 1$); in the first case, we introduce $u/\sqrt{2}$, $v/\sqrt{2}$ as the new parameters, and in the second case, we replace $\mathfrak{z}$ with $\mathfrak{z} + z = w$, which gives:

$$(110.a) \qquad \begin{cases} p_1 = 2c, & p_4 = -\mathfrak{z}v + \tfrac{1}{2}(u^2 - v^2), \\ p_2 = v, & p_5 = 2c\mathfrak{z} + cv, \\ p_3 = u, & p_6 = -cu; \end{cases}$$

$$(113.a) \qquad \begin{cases} p_1 = c, & p_4 = y^2 + zw, \\ p_2 = -z, & p_5 = cw, \\ p_3 = y, & p_6 = cy. \end{cases}$$

b) For a rotation around the z-axis through an angle ω (and with a change of notation) equations (59) and (60) in § 41 reduce to:

$$(115) \qquad \begin{aligned} p_1 &= \kappa_1 \cos\omega - \kappa_2 \sin\omega, & p_4 &= \kappa_4 \cos\omega - \kappa_5 \sin\omega, \\ p_2 &= \kappa_1 \sin\omega + \kappa_2 \cos\omega, & p_5 &= \kappa_4 \sin\omega + \kappa_5 \cos\omega, \\ p_3 &= \kappa_3, & p_6 &= \kappa_6 . \end{aligned}$$

If we substitute the same values for κ that we prepared in equations (109) or (112) then we will get two new parametric representations (116) of the twist, whose explicit specification we will forego.

These representations will also all be true when one considers the p to be general *tetrahedral* pointers; the elimination of the parameters will then always yield a linear equation.

It is self-explanatory that a map of the rays of a twist to the spaces of *points* or *planes* will be given by any parametric representation when one interprets the parameters to be any sort of point or plane pointers. For example, if one considers them to be rectangular point pointers then the rays of a twist will be mapped completely and reciprocally by (116) onto all points of a slice of space that is bounded by two parallel planes at a distance of π, but they will be mapped to *the entire point space* by (110) and (113). One can obtain *infinitely many* such maps when one substitutes the representations that are given here for their parameter functions.

The first map of a twist onto point space was given by *Noether* (Göttinger Nachr., 1869).

Practice problems:

33. If a twist is given by two projective pencils of rays (viz., Sylvester's method of generation) then how can one construct its axis most rapidly?

34. Which special forms does the equation of a twist assume when:

α) The null plane of a vertex of the base tetrahedron falls upon a face of the latter?

β) This happens twice?

g) This happens three times?

35. The (special) affine transformations T also belong to the collineations, which consist of the ones that change the distance from all points to a fixed plane E by the same ratio q. If one draws E through the axis of a twist G then the twist must go to another one under T (Theorem 82). On the other hand, it seems as if its axial symmetry would then be lost, in the same way that an ellipsoid of rotation would become general under T. How does one resolve this paradox?

36. Investigate the arrangement of the rays of the complex $\mathfrak{C}_2$ of Theorem 94 with the help of a family of coaxial circular cylinders (similar to what we did for the twist in § 11).

37. Verify equation (56) for rectangular line pointers by direct calculation. (§ 52)

38. Which theorems will replace Theorem 106, 108, 109 for hyperbolic nets?

39. Verify that when one substitutes the pointers s_i'' in the right-hand side of equation (96), one will come back to the pointers s_i .

40. Represent the involution of the general ray net with equations (79) or (84) as a plane conversion.

41. Represent the hyperbolic net with equations (79) as a point conversion.

42. Let x, x' and y, y' be corresponding pairs of points of two projective sequence g, g'. Consider xy' and $x'y$ to be the focal lines of a ray net. The ∞^2 nets thus-defined are contained in the same twist (*Franel*, Vierteljahrsschr. d. naturf. Ges. Zürich, Bd. 40, 1895).

43. If h is a ray of a twist G that cuts the axis a perpendicularly then the rays of G whose shortest distances from a fall upon h will define a family of rulings $\mathfrak{R}$ of a hyperbolic paraboloid. Find a parametric representation for it and derive a representation of the twist from that when one first rotates $\mathfrak{R}$ around a and the translates along a.

44. How can the completeness of the ways of generation that were spoken of in § 59 be deduced from the parametric representations of the twist?

45. Show how u-surfaces, v-surfaces, u, v-congruences, etc. are given by the representations (110), (113).

46. Show that the polars of a fixed line g with respect to all complexes of a pencil define a family of rulings.

Chapter V

Imaginary elements

§ 61. The transversals to a quadruple of rays and the associated ray net.

We have already encountered a quadratic equation once before in the calculation of line pointers, namely, in the determination of the bush of rays in a pencil of complexes (§ 53). From an analytical standpoint, this is the same problem as: *Find the transversals that intersect four given lines.* Let:

$$s_{i\lambda} \qquad (i = 1, \dots, 4;\ \lambda = 1, \dots, 6)$$

be the pointers of the given rays s_i and p_λ that cut a desired common ray; the following equations must then be fulfilled:

(1) $$\sum_{\lambda=1}^{6} s_{i\lambda}\, p_{\lambda+3} = 0 \qquad (i = 1, \dots, 4),$$

(2) $$\omega(p) = \sum_{\lambda=1}^{6} p_\lambda\, p_{\lambda+3} = 0.$$

We assume that the s_i are not in hyperbolic position; the matrix $s_{i\lambda}$ will then have rank four, and we can calculate four of the p – say, $p_3, \dots, p_6$ – as linear, homogeneous functions of the remaining two:

(3) $$p_\lambda = c_\lambda\, p_1 + c'_\lambda\, p_2 \qquad (\lambda = 3, \dots, 6).$$

We can also write down $\lambda = 1, \dots, 6$ when we establish that:

$$c_1 = 1, \qquad c'_1 = 0,$$
$$c_2 = 1, \qquad c'_2 = 0.$$

We substitute the values (3) into (2) and obtain:

$$\omega(p) = \sum (c_\lambda\, p_1 + c'_\lambda\, p_2)(c_{\lambda+3}\, p_1 + c'_{\lambda+3}\, p_2) = 0,$$

or

(4) $$\omega(c) \cdot p_1^2 + 2\,\omega(c, c') \cdot p_1\, p_2 + \omega(c') \cdot p_2^2 = 0.$$

The ratio $p_1 : p_2$, and therefore all remaining pointers, will be determined by this. On the other hand, one can construct the transversals geometrically, when one intersects the hyperboloid $\mathfrak{H}$, which lies on the family of rulings $\mathfrak{R} \equiv (s_1, s_2, s_3)$, with s_4 . If S is a common point of $\mathfrak{H}$ and s_4 then the guiding ray of the family $\mathfrak{R}$ that goes through S will be a common intersector of all four rays. The reality of the solution must be independent of which triple one has selected from the four rays for the determination of $\mathfrak{H}$; thus:

Theorem 121: *The four hyperboloids that four rays determine three at a time will all be either cut or contacted or not cut simultaneously by the fourth ray.*

The reality of the roots of (4) depends upon the sign of the quantity:

$$\Omega = [\omega(c, c')]^2 - \omega(c) \cdot \omega(c'). \tag{5}$$

We know that when the matrix:

$$s_{i\lambda} \qquad (i = 1, \ldots, 5;\ \lambda = 1, \ldots, 6)$$

of pointers of five rays is reduced to rank four, the five rays will belong to the same net (Theorem 102). With this assumption, one row of the matrix can be combined with the remaining ones with constant multipliers; e.g.:

$$s_{5\lambda} = \sum_{i=1}^{4} \kappa_i s_{i\lambda} \qquad (\lambda = 1, \ldots, 6) \tag{6}$$

(cf., the argument in § 39, *e*). Conversely, the rank of the matrix $s_{i\lambda}$ will always be reduced by the existence of equations (6). Thus, if $s_{5\lambda}$ mean the pointers of any ray whatsoever then it will belong to the net that is determined by $s_1, \ldots, s_4$. If one then replaces the four original rays s_i by any other four independent ones in the same net and then seeks the common transversals then that will come down to multiplying equations (1) by four numbers four times and adding them; i.e., replacing them with an equivalent system of equations. Thus, the solutions of the equations must remain the same. I. e.:

Theorem 122: *The equation that serves to determine the pointers of the common transversals of four rays that were chosen from a net is independent of that choice.*

This is self-explanatory for a hyperbolic net; however, we have proved it in general. Thus, from Theorem 122, any general ray net will now be associated with two sextuples of numbers that fulfill the condition (2) and are real for a hyperbolic net, complex conjugates for an elliptic one, and coincide for a special one. We would like to invert the last result, namely, to show how every complex-conjugate pair of sextuples that fulfills the condition (2) is associated with an elliptic ray net (*Klein*, dissert., art. 5, Math. Ann., Bd. 23). Let:

$$\begin{aligned} p_\kappa &= a_\kappa + i b_\kappa \\ p'_\kappa &= a_\kappa - i b_\kappa \end{aligned} \qquad (\kappa = 1, \ldots, 6) \tag{7}$$

be the pair of sextuples. We seek all real lines whose pointers q_κ fulfill the incidence condition $\omega(p, q) = 0$ with the p_κ. It decomposes into two equations:

(8) $$\sum a_\kappa q_{\kappa+2} = 0, \qquad \sum b_\kappa q_{\kappa+2} = 0$$

that can be considered to be the equations of two linear complexes, and thus, a ray net. If they are fulfilled then $\omega(p', q) = 0$ will also be fulfilled. Therefore, this ray net is determined by the p'_κ, as well as by the p_κ. We assume that:

$$\omega(p) = \omega(a) + 2i \cdot \omega(a, b) + i^2 \cdot \omega(b) = 0,$$

so:

(9) $$\omega(a) = \omega(b),$$

(10) $$\omega(a, b) = 0.$$

The latter condition says that the two complexes lie in involution. Furthermore, we next assume that:

(11) $$\omega(a) \neq 0.$$

From § 53, equation (65), Ω is always negative here, so:

Theorem 123: *Any sextuple of complex numbers that fulfills the conditions* (2) *and* (11) *is uniquely associated with an elliptic net, and therefore with a gathered elliptic involution, and indeed the one that is associated with the sextuple of complex-conjugate numbers.*

If, conversely, an elliptic ray net is given by two twists, and one would like to find the associated sextuple then one will first seek the pencil that is involutorily associated with the one twist, as in § 56, with which, equation (10) will be fulfilled. Since the invariants of the two twists are denoted in the same way (Theorem 115), it will always be possible to fulfill equation (9) in such a way that one multiplies the equation of the one twist by a suitable factor. One can then compose the sextuple immediately using (7). One does not actually need to exhibit the quadratic equation (4) then (*).

Theorem 133 opens up the prospect of being able to also ascribe a geometric meaning to complex line pointers. However, since the theory of imaginary lines can be developed only in connection with the theory of imaginary points and planes, we must return to this somewhat later.

(*) The basis for this – at first glimpse, surprising – fact is geometrically transparent: One can determine the involution linearly, but not its possible double elements. Moreover, the determination of the aforementioned factor will demand the solution of a *purely* quadratic equation.

§ 62. Geometric interpretation of the imaginary elements.

We call a system of four (three, resp.) complex numbers an "imaginary point" or an "imaginary plane" according to whether it has been obtained from the solution of a problem by employing tetrahedral (ordinary, resp.) pointers as the pointer system of a desired point or plane, respectively. Analogously, we call a system of six complex numbers "an imaginary line" when it satisfies the same condition (2) that the pointers of an actual line also fulfill. In the case of tetrahedral pointers, only the ratios of the numbers shall be involved. We shall use the term "imaginary element" for the common term for imaginary points, planes, and lines (*). We call an imaginary element D' "conjugate" to another one D of the same kind when the numbers of the system that comprises it are conjugate to those of D in succession.

This definition is then purely analytical at first (**), but our next problem is to associate the imaginary elements with actual geometric structures uniquely. The simplest basic problem that appears for imaginary points already points to the path that must be pursued: If one calculates the pointers of the intersection points of a line g with a conic section (in the same plane) then they will be real or complex according to whether g does or does not cut the conic sections, respectively. On the other hand, one knows that a point involution is defined (S. S. VII, art. 93) on any line in the plane of a conic section K by that conic section, which will be hyperbolic, elliptic, or degenerate according to whether g cuts, does not cut, or contacts the conic section, respectively. In the first case, the double points of the involution are simultaneously the points of intersection with K. A hyperbolic line involution is then (also ignoring the relationship to K) associated with two real points and two pairs of plane pointers; one then asks whether an elliptic involution can be associated with a well-defined (complex) number system).

If:

$$A \equiv (a_\nu), \qquad A' \equiv (a'_\nu) \qquad (n = 1, \ldots, 4)$$

are two points of a line g then the point sequences:

$$P \equiv (\lambda a_\nu + \lambda' a'_\nu), \qquad Q \equiv (\mu a_\nu + \mu' a'_\nu)$$

will be projective when a bilinear relation:

$$c\lambda'\mu' + m\lambda\mu' + c'\lambda'\mu = 0$$

exists between the parameters $\lambda' : \lambda$, $\mu' : \mu$.

Should the projectivity be an involution, then the equation would have to be symmetric in the two parameter ratios: i.e., $m = m'$. Should A, A', in particular, be a pair of associated points of the involution then the equation:

(*) More properly, one should say "complex elements," since imaginary numbers are only a special case of the complex ones in algebra. However, the name "imaginary element" has already become sufficiently natural that one would not wish to depart from it.

(**) If one must always adhere to it then it would have no value; it is only a temporary artifice for speaking comfortably about the entire matter.

(12) $$c\lambda'\mu' + m\,(\lambda\,\mu' + \lambda'\mu) + c'\lambda'\mu = 0,$$

must be fulfilled by:

$$\lambda' = 0, \qquad \mu = 0,$$

as well as by:

$$\lambda = 0, \qquad \mu' = 0.$$

Both of these pairs will lead to the same condition, since one must have:

$$m = 0.$$

If we set:

$$\frac{c}{c'} = k$$

then it will follow that:

$$\frac{\lambda'}{\lambda}\cdot\frac{\mu'}{\mu} = -\,k.$$

One can then represent various involutions on g in the form:

(13) $$P \equiv (\lambda a_\nu + \lambda' a'_\nu)\,, \qquad P' \equiv (\lambda' a_\nu - k\lambda a'_\nu)$$

when one assigns all real values to k, in succession, and indeed one will obtain a parabolic involution for $k = 0$, an elliptic one for $k > 0$, and a hyperbolic involution for $k < 0$, since one must have:

$$\left(\frac{\lambda'}{\lambda}\right)^2 = -\,k$$

for the double elements. One can always make the absolute values of k equal to unity by a proper involution; we shall pursue this only for $k > 0$: Let:

$$\left|\sqrt{k}\right| = w,$$

so one can also write the involution (13) as:

$$P \equiv (\lambda w a_\nu + \lambda' w a'_\nu)\,, \qquad P' \equiv (\lambda' a_\nu - \lambda w \cdot w a'_\nu)$$

when one multiplies the pointers of P by w. If one now sets:

$$\lambda w = \lambda_0\,, \qquad w\,a'_\nu = a''_\nu$$

then the involution (13) will be represented by:

$$P \equiv (\lambda_0 a_\nu + \lambda' a''_\nu)\,, \qquad P' \equiv (\lambda' a_\nu - \lambda_0 a''_\nu)\,,$$

and the point A'' will be identical with A'. In order to obtain all elliptic involutions now in which A, A' is a pair, one must fix the pointers of the one point A and vary those of the other one A'. If we again revert to the original notations then we can say that:

Theorem 124: *Any proper elliptic involutions in which the pair A, A' is involved can be represented in the form:*

$$P \equiv (\lambda a_\nu + \lambda' a'_\nu), \qquad P' \equiv (\lambda' a_\nu - \lambda a'_\nu) \qquad (\nu = 1, \dots, 4), \tag{14}$$

when one chooses the absolute values of the pointers a suitably (*).

Should P and P' coincide then one would need to have:

$$\frac{\lambda'}{\lambda} = -\frac{\lambda}{\lambda'};$$

i.e.:

$$\lambda' : \lambda = \pm\, i.$$

There will then be only imaginary double elements:

$$D \equiv (a_\nu + i\,a'_\nu), \qquad D \equiv (a_\nu - i\,a'_\nu) \qquad (\nu = 1, \dots, 4), \tag{15}$$

and we will next show that they are independent of which pair A, A' one has based the representation (14) upon: Suppose that it is based upon the pair P, P', instead of A, A'; one will then obtain the double elements:

$$D_1 = (\lambda a_\nu + \lambda' a'_\nu) + i\,(\lambda' a_\nu - \lambda a'_\nu) = (\lambda + i\lambda')\,a_\nu + (\lambda' - i\lambda)\,a'_\nu,$$
$$D'_1 = (\lambda - i\lambda')\,a_\nu + (\lambda' + i\lambda)\,a'_\nu.$$

D_1 emerges from D' by multiplying by $\lambda + i\lambda'$; analogously, D and D'_1 are identical. Conversely, one can write down the associated involution (14) for which D is the one double element immediately when an imaginary element $D \equiv (a_\nu + i\,a'_\nu)$ is given. One easily convinces oneself by calculation that this involution depends upon only the *ratios* of the four numbers (D), which also emerges from the fact that only the amplitude, but not the absolute value, of the factor $\lambda + i\lambda'$ comes under consideration in the previous calculation.

Theorem 125: *Any imaginary point D is associated with a unique elliptic involution; conversely, the conjugate point D', along with D, belongs to this involution.*

(*) Any hyperbolic involution can be represented in the form:

$$P \equiv (\lambda\, a_\nu + a'_\nu), \quad P' \equiv (\lambda'\, a_\nu + a'_\nu).$$

This result can be carried over immediately to the case in which an involutory pencil of planes or lines enters in place of the proper point involution: We need only to understand a_ν and a'_ν to mean the pointers of two planes in the first case and the pointers of two incident lines in the second case (cf., § 39, *c*), while ν simultaneously goes from 1 to 6. Namely, in the latter case, a ray involution will be represented by:

$$p_\nu = \lambda\, a_\nu + \lambda'\, a'_\nu\,, \qquad p'_\nu = \lambda'\, a_\nu - \lambda\, a'_\nu \qquad (\nu = 1, \ldots, 6), \tag{14.a}$$

and one will have:

$$\omega(a) = \omega(a') = 0; \qquad \omega(a, a') = 0,$$

and this is precisely the case that we left unconsidered in the previous paragraph. In summary, we can say that:

Theorem 126: *The pairs of conjugate imaginary elements are in one-to-one correspondence with elliptic involutions, and indeed, with involutory point sequences, pencils of planes, pencils of rays, and gathered involutions.*

According to whether the sextuple:

$$a_\nu + i\, a'_\nu \qquad (\nu = 1, \ldots, 6)$$

is associated with an involutory pencil of rays or a gathered involution, one will ordinarily call it an imaginary line of the *first* or *second* kind, respectively. In the first case, one will have:

$$\omega(a) = \omega(a') = 0,$$

while in the second case one will have:

$$\omega(a) = \omega(a') \neq 0$$

[in both cases, one will have $\omega(a, a') = 0$]; we then say *special* and *general* imaginary lines, respectively, instead.

We call the p the *ray* or *axis pointers* of the imaginary lines according to whether the a and b in (7) are the ray or axis pointers of twists (of lines, in the special case), respectively. If p_ν are the ray pointers of an imaginary line and q_ν are its axial pointers then, due to § 49, equation (45), one will also have:

$$p_{\nu+3} = q_\nu$$

here.

We further understand the *carrier* of an imaginary point to mean the carrier of the associated involution; analogously, we say the *axis* of an imaginary plane.

---

§ 63. The separation of two conjugate-imaginary elements.

If the one point in a proper, elliptic, point involution traverses a line in a certain sense then the other point will move in the same sense. One can thus say that the involution itself is traversed in a certain sense. Analogously, elliptic ray and plane involutions can be traversed in two senses. Finally, for a gathered elliptic involution, all of the involutory point sequences and pencils of planes that enter into it can be traversed in two senses. However, if one fixes a certain sense on one of the point sequences *g* then, from Theorem 117, a certain sense of rotation will be fixed in all pencils of planes. They will cut out the same involution, and in fact, with the same sense of traversal, from any ray *s* of the net. The same thing will then be true on *g*. If one moves *s* from *g* continuously in the net then the sense of the intersection of *s* with a pencil of planes cannot change suddenly from one pencil to another. One can then accordingly assign two senses of traversal to the gathered involution itself. One of them is fixed (cf., the conclusion of § 57), e.g., when we fix either the sense of the involution of the planar affine system or the sense on the ray at infinity of the net. Indeed, this will yield the insight that a certain sense of traversal of a ray will be associated with the sense of rotation around it for the principal ray of a *right-wound* net, and therefore, for any ray of it, and that sense will seem positive for the fixed sense of the ray. The opposite thing will be true for the left-wound net.

All involutions in Theorem 126 can thus be traversed in two senses, and one asks whether that situation can be employed in order to separate a pair in such a way that one can associate the one element with a certain sense of the associated involution and the other element of the same involution with the opposite sense.

We next treat the imaginary points, planes, and special imaginary lines together. If:

$$A \equiv (a_\nu), \qquad A' \equiv (a'_\nu) \qquad\qquad (\nu = 1, \ldots, 4 \text{ or } 1, \ldots, 6)$$

is a pair of associated involutions then:

$$B \equiv (a_\nu + a'_\nu), \qquad B' \equiv (a_\nu - a'_\nu)$$

will be the pair in involution that is harmonically separated by A, A' (*).

The sense:

$$A\,B\,A'$$

will be then be opposite to the sense (**):

$$A\,B'\,A',$$

but will agree with the sense of:

$$B\,A'\,B' \qquad \text{or} \qquad A'\,B'\,A.$$

(*) There is just one such pair. One must then have:

$$-\frac{\lambda}{\lambda'} : \frac{\lambda'}{\lambda} = -1$$

for it; i.e., $\lambda' = \pm\lambda$.

(**) We consider the line to be closed at infinity. A sense is then also first fixed for the case of the point sequence by a succession of three of its elements.

Rule: *We associate the element:*

$$D \equiv (a_\nu + i\,a'_\nu)$$

with the sense:

$$(a_\nu), \quad (a_\nu + a'_\nu), \quad (a'_\nu)\,,$$

and the element:

$$D' \equiv (a_\nu - i\,a'_\nu)$$

with the sense:

$$(a_\nu), \quad (a_\nu - a'_\nu), \quad (a'_\nu)\,.$$

If this rule is to be useful then one must show that the association is independent of which pair one bases it on, as opposed to A, A', or – what amounts to the same thing, from the previous paragraph – that it does not change when all numbers of the imaginary element are multiplied by the same (complex) factor. However, we first remark that an arbitrary point $C \equiv (\lambda\, a_\nu + \lambda'\, a'_\nu)$ can be employed in place of $B \equiv (a_\nu + a'_\nu)$ when $\lambda' : \lambda$ is positive. If one now uses the pair P, P' as a basis, instead of A, A', whereby:

$$p_\nu = \lambda\, a_\nu + \lambda'\, a'_\nu\,, \qquad p'_\nu = \lambda\, a_\nu - \lambda'\, a'_\nu\,,$$

then

(16) $$a_\nu = \frac{\lambda p_\nu + \lambda' p'_\nu}{\lambda^2 + \lambda'^2}\,, \qquad a'_\nu = \frac{\lambda p_\nu - \lambda' p'_\nu}{\lambda^2 + \lambda'^2}\,,$$

and D will be identical with $D'_1 \equiv (p_\nu - i\,p'_\nu)$ (cf., the previous paragraph). On the other hand, from our rule, D'_1 will be associated with the sense p_ν, $p_\nu - p'_\nu$, p'_ν. Next, let $\lambda' : \lambda$ be positive. From the remark that was made above and equation (16), this sense will then be identical with the sense $P\ A'\ P'$. However, this will be the same as the sense $A\ P\ A'$, which we can replace $A\ B\ A'$ with. However, if $\lambda' : \lambda$ is negative then the sense p_ν, $p_\nu - p'_\nu$, p'_ν will agree with the sense $P\ A\ P'$ or $A\ P\ A'$. Thus, in both cases, the association of the various forms in which the imaginary elements can be given is independent of the forms of the associated involutions.

In order to derive analogues for the general imaginary lines, we must invoke a theorem on pencils of complexes:

Theorem 127: *Four complexes of a pencil determine the same double ratio δ on any plane by way of four null points (which lie in a plane), and likewise, any point will determine the same double ratio by way of four (coaxial) null planes of the point.*

Let a_{ik}, b_{ik} be the axial pointers of two complexes A, B; the pointers of the two others C, C' can then be represented in the form:

$$a_{ik} + \lambda\, b_{ik}\,, \qquad a_{ik} + \mu\, b_{ik}\,.$$

It suffices to prove one of the two dual assertions: The point x_k will be associated [§ 46, equation 14)] with the plane:

$$\sigma u_i = \sum_{k=1}^{4} a_{ik} x_k \qquad (i = 1, \ldots, 4;\ a_{ii} = 0),$$

by A, with the plane:

$$\sigma u_i = \sum_{k=1}^{4} a_{ik} x_k$$

by B, and with the plane:

$$\sigma w_i = \sum_{k=1}^{4} (a_{ik} + \lambda a_{ik}) x_k = \sigma(u_i + \lambda v_i)$$

by C, etc. Thus, the double ratio δ of the four planes that the point x is associated with by A, B, C, C' will be:

$$\delta = \frac{\mu}{\lambda},$$

and will be thus independent of x_k. One can speak of the *double ratio of the four complexes*, of the *projective association of the complexes of two pencils*, or also the same pencil, and in the last case, of the *involution of pencils of complexes, one of which lies inside the other*. The complexes of a pencil are then also paired involutorily in a new sense by the fact that they are paired involutorily [§ 56, equation (98) or the proof of Theorem 113] (*). In particular:

$$a_\kappa,\ b_\kappa,\ a_\kappa + b_\kappa,\ a_\kappa - b_\kappa$$

are four *harmonic complexes* of that pencil whose carrier is associated with the general imaginary line:

$$p_\kappa = a_\kappa + i\, b_\kappa$$

and its conjugates. We can now carry over the previous arguments to this case and extend our rule in it to:

We associate the general imaginary line $a_\kappa + i\, b_\kappa$ with the associated ray net that has the same sense and is determined on a ray s of the net and a fixed plane that goes through it by the null point of the twist:

$$a_\kappa,\ a_\kappa + b_\kappa,\ b_\kappa$$

and the line $a_\kappa - i\, b_\kappa$ of the same net with the opposite sense.

(*) One can also say that the complexes of a pencil will be mapped projectively from the null point of a fixed plane onto a straight line point sequence. If the carrier of the pencil is hyperbolic then two involutory complexes of the pencil will have two null points in that planes that are harmonically separated by the point of intersection with the focal lines (and dually); the double points of an involution then harmonically separate each pair.

The question can be posed of whether we can use the null plane of a fixed point on *s* instead of the null point of a fixed plane in this rule without changing the type of association:

For the determination of the gathered involutions, up to now, we have always thought of a fixed, involutory pair of twists and allowed, e.g., a plane to rotate around a ray *s* of the net, so that the null points of the twist-pair cut away a pair of the involution from *s* in any position. We denote this affinity between a sense of traversal on *s* and a sense of rotation around *s* by *Z*.

Instead of that, we can now leave the plane fixed and obtain the pairs of the involution as the null points of all involutory twist-pairs of the pencil. The gathered involution will then be independent of which twist-pair one resorts to (§ 57). If we now let a twist traverse the pencil continuously then the null point of a fixed plane through *s* will traverse that ray, just as the null plane of a fixed point on *s* will rotate around that line. Therefore, an affinity Z' – viz., a sense of traversal on *s* and a sense of rotation around *s* – will also be established by this, and one asks the question of whether Z' is identical with the affinity Z'' that is defined by Theorem 117 and is itself definitive of the sense of the gathered involution. In order to decide this, we compare Z' and Z'' with *Z*.

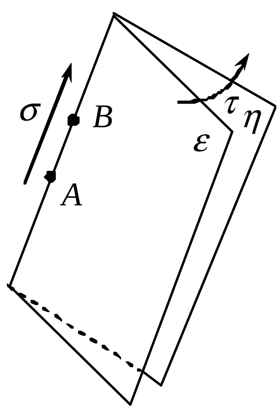

Figure 53.

Let *s* be the null plane of *A* in the twist $\mathfrak{G}$ (Fig. 53). If we change $\mathfrak{G}$ continuously into $\mathfrak{G}'$ then ε will rotate into the neighboring position η when *A* is through of as fixed, while, by contrast, *A* will be displaced into the position when *s* is through of as fixed. The sense of advance σ and the sense of rotation τ will then be associated with each other by Z'. For a fixed $\mathfrak{G}'$, the rotation of the plane from ε to η will then correspond to an advance of the null point from *B* to *A*; therefore, the associations *Z* and Z' will be opposite to each other.

In the comparison of *Z* and Z'', we assume a right-wound ray net in order to fix the presentation. Any twist in the pencil will then be left-wound (Theorem 115). Thus, the null plane will rotate around a ray of the twist in the negative sense when the direction of advance of the point is assumed to be positive (§ 58, rem.). The opposite will be true, however, for Z'' for a right-wound net (cf., the beginning of this paragraph). Thus, Z'' and *Z* will be also opposite, while *Z*, Z'' will be identical. It will then be irrelevant whether we speak of the null points of a fixed plane or the null planes of a fixed point in the rule above.

With that, the one-to-one correspondence between imaginary elements and elliptic involutions that are endowed with a given sense is concluded (*).

(*) We have placed a lot of value on the idea of completing this association and only then going into the relationships between the imaginary elements (i.e., to the incidence relations). In fact, the use of the incidence relation $\omega(p, q) = 0$ in § 61 was not essential, and served only as a heuristic tool. We can just as well say immediately that: We associate the lines:

$$p_\kappa = a_\kappa + i\, b_\kappa$$

§ 64. The incidence of imaginary elements with real ones.

The incidence between two elements, one or both of which are imaginary, will be defined by the same equations that are fulfilled by real elements. Thus, a point x and a plane u, one or both of which are imaginary, are then called *incident* when:

$$\sum_{m=1}^{4} u_m x_m = 0, \tag{17}$$

and a point x and a line p_{km}, when (*):

$$\sum_{m=1}^{4} p_{km} x_m = 0 \qquad (k = 1, \ldots, 4), \tag{18}$$

(and the analogous dual statement), and two lines p_λ, q_λ, when:

$$\sum_{m=1}^{6} p_{\lambda+3} q_\lambda = 0. \tag{19}$$

One is now dealing with the problem of ascertaining the geometric interpretation of these equations by complex pointers. We will first assume that one of the two elements is real.

If u is a real plane, and:

$$x_m = y_m + i\, z_m$$

is an imaginary point then (17) will decompose into two conditions that express the facts that the point y, as well as the point z, lie in u. Thus, the carrier of the linear point-involution that represents x will lie in u; an analogous dual statement will be true.

If p is a real line, and x is an imaginary point, as above, then (18) will decompose into two systems that express the idea that y, as well as z, lie on p, so p will be the carrier of the involution that belongs to x. Conversely, if x is a real point, and:

$$p_{km} = a_{km} + i\, b_{km}$$

is an imaginary line then (18) will decompose into the two systems:

$$\sum_m a_{km} x_m = 0, \qquad \sum_m b_{km} x_m = 0 \qquad (k = 1, \ldots, 4). \tag{20}$$

---

of the gathered involution with the involutory complexes a_κ and b_κ, without previously speaking of the fact that the ordering rays of this involution are, at the same time, the ones that fulfill the incidence relation with p.

(*) The symbol p with one and two indices shall also be connected with the same Table (25) in § 33 for imaginary lines as it was for real lines. The p are to be considered as *axial* pointers, since the system (18) is dual to the equations (38) of § 38.

Now, if the imaginary line is general then the determinants $| a_{km} |$ and $| b_{km} |$ (§ 62) (and likewise the invariants *a* and *b* of the twist) will be non-zero (§ 46). Equations (20) can only be fulfilled when all *x* are equal to zero; i.e., when there is no real point that is incident with a general imaginary line. However, if the imaginary line is special then the two determinants will be zero, and equations (20) will be precisely the same (except for notation) as the ones that served for us to find the point of intersection of two incident lines *a*, *b*. The vertex of the ray involution that belongs to *p* will be the only real point that lies in *p*; an analogous dual statement will be true.

The incidence of a real line with a general imaginary one was already discussed in § 61. For a special imaginary *p* and a real *q*, (19) will decompose into two conditions that express the ideas that *a* will be cut by two rays of the involution that belongs to *p*, and will thus be incident with the vertex or the plane of the pencil that belongs to *p*. We summarize (along with the dual results that were still not expressed up to now):

Theorem 128:

A real point lies in an imaginary plane when it lies on the axis of the associated involution.

An imaginary point lines on a real plane when the carrier of its associated involution lies in the plane.

A real line is incident with an imaginary point or an imaginary plane when it is identical with the carrier of the associated involution.

A special imaginary line is incident with a real point when it is the vertex of the representing pencil, with a real plane when it is the plane of that pencil, with a real line when it is either incident with the vertex or with the plane of the pencil, and with a general imaginary line when it belongs to the associated ray net.

A real point or a real plane cannot be incident with a general, imaginary line (*).

---

§ 65. The incidence of imaginary elements with each other.

a) (**) For the incidence of the imaginary point:

$$x_m = y_m + i\, z_m \qquad (m = 1, \dots, 4)$$

and the imaginary plane:

$$u_m = v_m + i\, w_m \qquad (m = 1, \dots, 4),$$

(17) will decompose into the two conditions:

(*) It should be mentioned that when a real line *p* and an imaginary point *x* are *not* incident, along with the general position, one can also find a special one, namely, when *p* is cut by the carrier of the involution that belongs to *x*; an analogous dual statement is true. If, moreover, a special imaginary line *q* cuts a real one *p* then the two cases that were mentioned in the text can appear simultaneously; *p* will then be a ray of the involution that belongs to *q*.

(**) From an (unpublished) lecture of *v. Dantscher* (Univ. Graz., Summer Sem. 1890).

(21) $$\sum y_m v_m - \sum z_m w_m = 0,$$

(22) $$\sum y_m v_m + \sum z_m w_m = 0 .$$

However, we can assume:

(23) $$\sum y_m v_m = 0.$$

We can then choose the starting pair of the involution to be y, z, and choose:

$$\xi_m = \lambda y_m + \lambda' z_m , \qquad \xi'_m = \lambda' y_m - \lambda z_m ,$$

to which x belongs, to be an arbitrary pair of the involution, and thus the one in which the point of intersection y of its carrier with the plane v enters. It will then follow that:

(24) $$\sum z_m w_m = 0 ;$$

i.e., the other point of that pair will lie in the plane w of the plane involution that is associated with the plane v:

$$\eta_m = \mu v_m + \mu' w_m , \qquad \eta' = \mu' v_m - \mu w_m$$

that belongs to u. We now choose one such point ξ of the one and only one such plane η of the other involution, which are incident, so, from (23) and (24):

(25) $$\lambda \mu' \sum y_m w_m + \lambda' \mu \sum z_m v_m = 0 .$$

It can now be the case that the last two sums vanish individually; y, as well as x, will then lie in v, as well as in w; i.e., the point and plane involutions will coincide. On the other hand, if one uses (22) then (25) will reduce to:

$$\lambda \mu' - \lambda' \mu = v;$$

i.e., ξ and η will be incident when $\dfrac{\mu'}{\mu} = \dfrac{\lambda'}{\lambda}$; however, one will also have $-\dfrac{\mu}{\mu'} = -\dfrac{\lambda}{\lambda'}$ then; i.e., ξ and η will also be incident, and the involutions will lie *perspectively*.

Moreover, we have associated with the element x with the sense:

$$y_m , y_m + z_m , z_m ,$$

and the element u with the sense:

$$v_m , v_m + w_m , w_m .$$

Since the elements that are arranged in a column are incident, when the point involution is traversed in the prescribed sense, the plane that is carried along by means of the perspective relation will also rotate in the prescribed sense. In such cases, we will say that the involutions are perspective *including the sense.*

Theorem 129: *An imaginary point and an imaginary plane are incident when the associated involutions are perspective including the sense (principal case), or when the carriers of the involutions are identical (special case).*

It follows immediately from this theorem (purely or algebraically) that when the point $y_m + i\, z_m$ and the plane $v_m + i\, w_m$ are incident then the same thing will be true for $y_m - i\, z_m$ and $v_m - i\, w_m$, but not perhaps for $y_m + i\, z_m$ and $v_m - i\, w_m$.

We can consider (17) to be the *equation* of an imaginary point or a plane, just as it is for real elements, according to whether the x or the u are constant, resp..

b) If the special imaginary line:

$$p_{km} = a_{km} + i\, b_{km}$$

is incident with the imaginary point:

$$x_m = y_m + i\, z_m$$

then the conditions (18) will decompose into the two systems:

$$\left.\begin{aligned} &(26) && \sum_{m=1}^{4} a_{km} y_m - \sum_{m=1}^{4} b_{km} z_m = 0, \\ &(27) && \sum_{m=1}^{4} a_{km} z_m + \sum_{m=1}^{4} b_{km} y_m = 0. \end{aligned}\right\} \quad (k = 1, \ldots, 4)$$

We knew from the outset that the two systems did not contradict each other, since the system (18), from which they were obtained, had the solutions x from the vanishing of the determinant $|\, p_{km} \,|$. On the same basis, we can assume, as we did in *a*), that:

$$\sum a_{km}\, y_m = 0.$$

It will then follow that one also has:

$$\sum b_{km}\, z_m = 0;$$

i.e., the carrier of the point involution will lie in the plane of the pencil of rays. However, it cannot coincide with a ray of the pencil, since otherwise we would choose y to be the vertex of the pencil, and it would follow that all four sums in (26) and (27) would vanish individually; i.e., z would also have to coincide with the vertex. From the representation of the involutions:

$$\pi_{km} = \lambda\, a_{km} + \lambda'\, b_{km}\,, \qquad \pi'_{km} = \lambda'\, a_{km} - \lambda\, b_{km}\,,$$

and

$$\xi = \mu\, y_m + \mu' z_m\,, \qquad \xi' = \mu' y_m - \mu\, z_m\,,$$

we can then conclude, as we did in *a*), that they lie perspectively, including the sense.

Theorem 130: *A special imaginary line is incident with an imaginary point or an imaginary plane when the two associated involutions are perspective including the sense.*

A special imaginary line p was incident with a real line q when it was (§ 64) either incident with the vertex or the plane of the involution J that was associated with p. We are now in a position to give the common plane E, in the first case, and the common point P, in the second: E is represented by the involution that projects J onto q, and P, by the involution in which q is cut by J.

c) We can now resolve the case of the incidence of two special imaginary lines p, q by reverting immediately to the discussion of equation (19). If that equation is fulfilled then the lines will have a common point, and it was previously remarked (§ 39, *c*) that this analytical fact is independent of the reality of the elements. There is then a point x that is incident with p, as well as q, and once we have already interpreted this situation geometrically, it must follow without calculation that:

Let x first be real. From Theorem 128, x must coincide with the vertices of two involutions, but their planes E, E' can be different. In order to find u, we denote the line of intersection E, E' by s (Fig. 54). It will correspond to a ray s' in the involution p and to a ray σ' in the involution q. We then think of the involution p as being given by the pair s, s', and the pair t, t' that is harmonic to it, while q is given by s, σ' and the pair τ, τ' that is harmonic to it. Now, the ray shadows (*Strahlenwürfe*) s, σ', τ, τ', and s, s', t, t' lie perspectively, and indeed in two ways, when one makes no recourse to the sense, since one has:

$$(s, \sigma', \tau, \tau') = (s, s', t, t'),$$

as well as:

$$(s, \sigma', \tau', \tau) = (s, s', t, t').$$

However, if s, t, s' is the sense that belongs to p and s, τ, σ' is the sense that belongs to q then there will be only one plane involution that lies perspectively to p, as well as to q, including the sense. One will find its axis when one, e.g., intersects the planes s, σ' and t, τ.

Dually, u can be real and x imaginary; the carrier of x will be the only line in the plane u on which the two pencils of rays p, q cut out involutions that are perspective including the sense. Naturally, x, as well as u, can be real. The two involutions will then have a common vertex and lie in the same plane.

Finally, x and u can both be imaginary (the general case); p and q will then cut out the same point involution on the line of intersection of their planes and, at the same time, the connecting line of their vertices will determine the axis of that plane involution. We spare ourselves the repeated formulation of all these possibilities and merely emphasize that:

Theorem 131: *The incidence of two special, imaginary lines subsumes four cases.*

d) When the general imaginary line:

$$p_{km} = a_{km} + i\, b_{km}$$

is incident with the imaginary point:

$$x_m = y_m + i\, z_m\,,$$

the system (18) will likewise decompose into the two systems:

$$(26)\qquad \sum_{m=1}^{4} a_{km} y_m - \sum_{m=1}^{4} b_{km} z_m = 0,$$

$$(k = 1, \ldots, 4)$$

$$(27)\qquad \sum_{m=1}^{4} a_{km} z_m + \sum_{m=1}^{4} b_{km} y_m = 0.$$

The determinants $| a_{km} |$ and $| b_{km} |$ will now be non-zero (§ 62 and § 46) and numerically equal (§ 61, equation 9). These systems do not contradict each other, on the same grounds as in *b*). However, since the *z* that is associated with the given value *y* by the system is already determined, the two systems must be equivalent.

We first write the representation of the gathered involution as a point manifold. It is defined by the two systems:

$$(28)\qquad \sigma u_k = \sum_{m=1}^{4} a_{km} z_m\,, \qquad\qquad 29)\qquad \sigma' u_k = \sum_{m=1}^{4} b_{km} y'_m \qquad (k = 1, \ldots, 4)$$

(cf., § 46). (28) expresses the connection between the pointers of a point and those if its null plane *u* in the twist *u*, (29) expresses the one from *b*, and the gathered involution indeed arises by composing these two conversions. In order to be able to immediately calculate the pointers *y* of the point that corresponds to a point *y* in that involution, we must solve (29) for the *y′* and then substitute the values of *u* from (28) in the right-hand side. However, we write the connection between the *y* and the *y′* in the form (*):

$$(30)\qquad \sum_{m=1}^{4} a_{km} y_m = \tau \sum_{m=1}^{4} b_{km} y'_m \qquad\qquad (k = 1, \ldots, 4).$$

A comparison of systems (26) and (30) shows that the point *y* that belongs to *x* in the line involution *J* and the one that belongs to *p* in the gathered involution *J′* correspond to the same point. Since we can base the representation of *J* on an arbitrary pair, instead of *y*, *z* (§ 62), it will follow that this is true for every point of *J*, so *J* will be identical with the involution that is defined on its carrier *g* by *J′*; we call it (*J′*, *g*). However, similar to what we did in the previous cases, we would also like to convince ourselves of that by

(*) If one is merely dealing with the calculation of an individual system of values *y′* for a given system *y*, one can set the proportionality factor $\tau = \sigma : \sigma'$ equal to unity. However, if one has to deal with deciding whether a system of values *y* that is calculated from (30) is equivalent to one that is given in some other way, as we are here, then this would not be allowed. By contrast, it neither necessary nor allowed to add a proportionality factor to the one term in equations (26) and (27) [and also with (21) and (22) already]. If we then fix the absolute values of the *y* then those of the *z* will also be fixed (Theorem 124), if the involution is to find its simplest representation. Likewise, when the absolute values of the *a* are fixed those of the *b* will be determined by § 61, equation (9).

calculation, in order to clarify the association of senses: Let x be represented by the involution:

$$\xi_m = \lambda\, y_m + \lambda' z_m\,, \qquad\qquad \xi'_m = \lambda' y_m - \lambda\, z_m\,.$$

We seek the point y'_m that corresponds to ξ_m in J' when we replace y_m with ξ_m in (30) and obtain, by means of (26) and (27):

$$\lambda \sum b_{km}\, z_m - \lambda' \sum b_{km}\, y_m = \tau \sum b_{km}\, y'_m\,.$$

This equation will be fulfilled identically when one sets y'_m equal to ξ'_m and $\tau = -1$. Now, there is only one solution system y'; thus, the points that correspond to ξ in J and J' will be identical. x belongs to the sense:

$$y_m\,, \qquad y_m + z_m\,, \qquad z_m\,,$$

on g; p belongs to the sense:

$$a_{km}\,, \qquad a_{km} + b_{km}\,, \qquad b_{km}$$

that the twist determines by means of the null point to a fixed plane that goes through g. We choose the fixed plane to be the null plane of y in a [and thus calculate the u from (28)], and then determine its null point y' in the second twist $a + b$ by way of:

$$\tau\, u_k = \sum_{m=1}^{4} (a_{km} + b_{km})\, y'_m \qquad\qquad (k = 1, \ldots, 4), \tag{31}$$

and in the third one b by:

$$\sigma' u_k = \sum_{m=1}^{4} b_{km} z_m\,, \tag{32}$$

and these values of z will, in fact, be identical to the ones in the representation:

$$x_m = y_m + i\, z_m\,,$$

due to the identity of the involutions J and (J', g). One can now show that the point y'_m is, in turn, identical with $y_m + z_m$. In fact, if we calculate the null plane u'_k of the latter point in the second twist then we will have to substitute $y'_m = y_m + z_m$ in (31). Of the four sums, two of them will disappear by means of (27), and we will obtain:

$$\tau\, u'_k = \sum a_{km}\, y_m + \sum b_{km}\, z_m\,, \tag{33}$$

so

$$\tau\, u'_k = (\sigma + \sigma')\, u_k\,;$$

i.e., u' will be identical with u. Thus, the involutions J and (J', g) will be identical, including the sense.

Theorem 132: *If the general, imaginary line (let its associated gathered involution be J') is incident with an imaginary point (an imaginary plane, resp.) for which g is the carrier of the associated point involution of J (for which g is the axis of the associated plane involution J, resp.) then the involution that J' determines on g (around g, resp.) will be identical with J, including the sense.*

In this case, we would also like to say that J lies perspectively with J', including the sense, when J is actually a part of J' here. A general, imaginary line thus contains ∞^2 imaginary points, namely, one on each ray of the associated net; analogously, ∞^2 imaginary planes will go through it.

e) We can now resolve the case of the incidence of a general, imaginary line p with a special, imaginary q on the same grounds as in *c*) by a simple argument: From Theorem 128, the common point x of p and q and the common plane u cannot be real. From Theorem 130, x must lie on the same ray s of the net N that belongs to p, which lies in the plane of the ray involution q, and the involution that cuts out q from s must be perspective with gathered involution J that belongs to p, including the sense. If this is the case then the following must enter into consideration by itself (and conversely): A ray t of N goes through the vertex of q, and the plane involution that projects q from t must be perspective to J', likewise including the sense. In this case, we say that the ray involution q lies perspectively to a gathered involution J', including the sense.

Theorem 133: *If a general, imaginary line is incident with a special one then the corresponding gathered and ray involutions will lie perspectively, including the sense.*

f) A general, imaginary line:

$$p_k = a_k + i\ a'_k$$

has an imaginary point x (with the carrier g), as well as an imaginary plane u (with the axis h), in common with another general line:

$$q_k = b_k + i\ b'_k\,,$$

and they must be incident with each other. One then asks whether the latter incidence is present in the principal case or the special case (Theorem 129). Above all, g and h must be common rays of the two nets N and N' that belong to p and q, resp. We thus come to the question of how many rays two (elliptic) nets can have in common. If we think of each of them as the intersection of two twists then this will come down to the determination of the common rays to four twists. We will examine the relationships between several twists more closely in the next chapter, and here we merely point out that the four equations of the twists:

$$(34) \qquad \sum_{\mu=1}^{6} a_{\lambda,\mu+3}\,\pi_\mu = 0 \qquad (\lambda = 1, \ldots, 4),$$

together with the equation:

$$\omega(\pi) = 0,$$

will generally have two systems of solutions (which is entirely similar to the situation in § 61), which naturally can also coincide, but will certainly be real, in our case (*). We first assume that g and h are different, and that g is the carrier of the common imaginary point x. If g were also the axis of the common plane u then the involution u would cut out a second common imaginary point y (Theorem 117) from the second common ray h of the net. However the gathered involutions would then be identical, since an imaginary line is determined by two of its points, just as a real one is (**). Thus, if g is the carrier of x then h will be the axis of u.

(*) Equations (34) can be independent of each other and have infinitely many solutions; however, this will be excluded by the following considerations.

(**) Namely, if the system (18) is fulfilled by two (complex) systems of values x_k and y_k then the ratios of the p will be determined completely by that. For example, one finds from the first equations of that system that:

$$p_{12} : p_{13} : p_{14} = x_3\,y_4 - x_4\,y_3 \;;\; \ldots$$

Briefly, if one denotes $x_l\,y_m - x_m\,y_l = (l, m)$ then the two systems will fulfilled identically with:

$$p_{ik} = (l, m),$$

in which the sequence of indices $i, k; l, m$ is constructed according to known rules (§ 46). However, the p also fulfill the relation $\omega(p) = 0$, which emerges from its construction [cf., the derivation of this relation in § 46] or from the fact that the factor $\omega(p)$ appears in the determinant of p (§ 32), and when it does not vanish, the system cannot be fulfilled by the values x or y. In fact, a line is then determined uniquely by two imaginary points (planes, resp.). We already know otherwise that this general or special line will be imaginary or real according to whether the carriers of the points (the axes of the planes) do not intersect, do intersect, or coincide, respectively.

The p were axial pointers in (18). If we then call the ray pointers π_{ik} then, from the conclusion of § 62, we have to set:

$$\pi_{ik} = x_i\,y_k - x_k\,y_i\,.$$

The pointers of an imaginary line thus have formally obtained the same representation by the pointers of two of their points (two of their planes, resp.) that was taken to be the starting point of the definition of line pointers for real lines. It follows immediately that the totality of the points of the connecting line x_k, y_k will also be represented by:

$$(35) \qquad \sigma z_k = \lambda\,x_k + \mu\,y_k \qquad (k = 1, \ldots, 4)$$

for imaginary elements, since all of the three-rowed determinants in the matrix:

$$x_k\,,\; y_k\,,\; z_k$$

will vanish, which is what the relations (18) imply, and this says that z will be incident with the connecting line x, y. The choice of λ and μ will now include a four-fold arbitrariness in it. However, one and the same imaginary point can be written in ∞^2 ways, due to the multiplication by a complex factor, such that we will now get ∞^2 points on the line, as it must be. If the x and y are real, but λ and μ are complex, then we will obtain the imaginary points on a real line. In fact, there will be ∞^2 elliptic involutions on one such line.

Theorem 134: *If two general, imaginary lines are incident then there will be incidence between their common point that their common plane, from the principal case or the special case of Theorem 129, respectively, according to whether the two associated ray nets have two rays or one ray in common, respectively.*

g) We have now interpreted the incidence relation between imaginary elements geometrically in all possible cases, and it is immediately obvious – analytically, as well as geometrically – that:

Theorem 135: *If two imaginary elements are incident then their conjugate elements will also be incident.*

If two involutions are perspective, including the sense, and one inverts the sense of each of them then they will once more be perspective, including the sense.

h) If one considers only the elements that lie in a fixed real plane E then the imaginary planes and general, imaginary lines will drop out (Theorem 128), and it will also follow from the second case of § 65, *e*) and *f*) in this paragraph (middle of the remark) that for the imaginary points and lines of E, any two elements of the one kind will determine one of the other kind. It is then unnecessary to develop a special "theory of the imaginary elements in the plane" beforehand. Analytically, one makes this restriction to the theory in the plane such that one thinks of E as a plane of the basic tetrahedron, so one pointer will be set to zero from the outset for all points and three pointers for all lines. Thus, imaginary points, as well as lines, in the plane will have three homogeneous pointers. In fact, it will also follow from the incidence relations:

$$\sum_k \pi_{ik} u_k = 0 \qquad (i = 1, \ldots, 4)$$

in § 38 that the three quantities π_{21}, π_{31}, π_{41} will vanish when, e.g., only u_1 is non-zero, even for imaginary elements. One can likewise speak of a theory of imaginary elements in sheaves of rays, in which only imaginary planes and special imaginary lines are present.

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§ 66. Joins and meets of imaginary elements.

The linear constructions that are single-valued for real elements will also be single-valued for imaginary ones. They will then emerge analytically from certain properties of determinants or calculations involving them that are independent of whether the numbers are real or complex. We have thus already convinced ourselves in § 65, *f*), rem., that the problem of joining two points into a line is also soluble and single-valued in the imaginary domain. Furthermore, e.g., a plane will always be determined by three independent points, since the ratios of the u can be calculated from the three equations:

$$\sum_{k=1}^{4} x_{ik} u_k = 0 \qquad (i = 1, \ldots, 3).$$

It then follows from this that a plane is also determined by a line and a point, etc. One then only has to deal with obtaining the solutions constructively. Therefore, we will often assume that an involution is given by two such pairs that are harmonically separated; we will call then a *harmonic quadruple* of the involution. From § 63 (first rem.), a harmonic quadruple of an elliptic involution is determined completely by one of its elements.

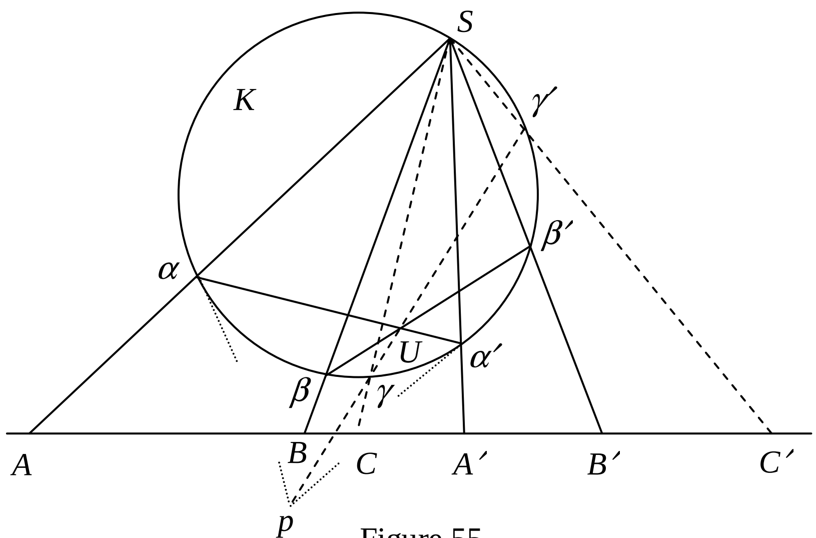

Figure 55.

If an elliptic involution is given by two arbitrary pairs A, A'; B, B' (Fig. 55) then one can find a harmonic quadruple for it as follows: One projects A, A'; B, B' from any point S of a conic section K on it to α, α'; β, β', determines the center U of the curved involution, and draws two conjugate chords through U; e.g., one joins the pole P of α, α' with U and projects the points of intersection γ, γ' with K from S to C, C'. A, A'; C, C' will then be a harmonic quadruple of the involution (cf., S. S. VII, arts. 79, 92 for this; namely, one will have $(P, U, \gamma, \gamma') = -1$; if one projects these points from α onto K then it will follow that α, α'; γ, γ' also lie harmonically).

a) *Joining two imaginary points of a real plane.*

We first think of the two points as given by the two harmonic quadruples S, S'; T, T' and S, S_1'; T_1, T_1', in which one finds the point of intersection S of their carriers g and g_1 (Fig. 56). These quadruples lie perspectively in two ways [cf., § 65, *c*)]. Thus, if the point x is represented by the involution on g with the sense $S\,T\,S'$ and the point y, by the involution on g_1 with the sense $S\,T_1\,S_1'$, and of the conjugate points are called x, y then the connecting lines will be represented as follows:

$x\,y$, by the ray involution around C with the sense σ,
$x'y'$ “ “ “ “ “ “ “ $-\sigma$,

$x\,y'$	“	“	“	“	C'	“	“	τ,
$x'y$	“	“	“	“	“	“	“	$-\tau$.

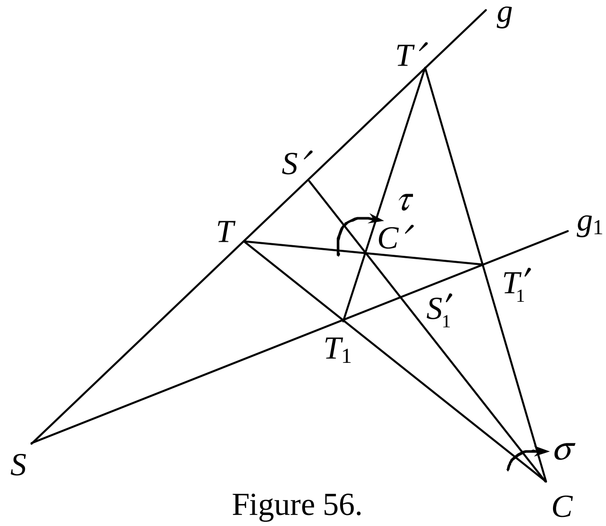

Figure 56.

Without taking recourse to a figure, the correct center the correct sense in any case – e.g., in the third case – will be determined by the two senses:

$$\begin{matrix} S & T & S' \\ S & T_1' & S_1' \end{matrix}$$

such that one joins the elements of the last two pairs that lie above and below each other. The point of intersection of the lines that arise in that way will determine the sense with one of the two rows.

If the involutions on g, g_1 are given by an arbitrary harmonic quadruple then one can revert from this case to the one that was just treated by employing a conic section in a manner that is similar to Fig. 55, or one can also solve the problem *linearly* that we, following *Grünwald* (Zeitschr. f. Math. u. Phys., Bd. 45, 1900), follow through in the dual case:

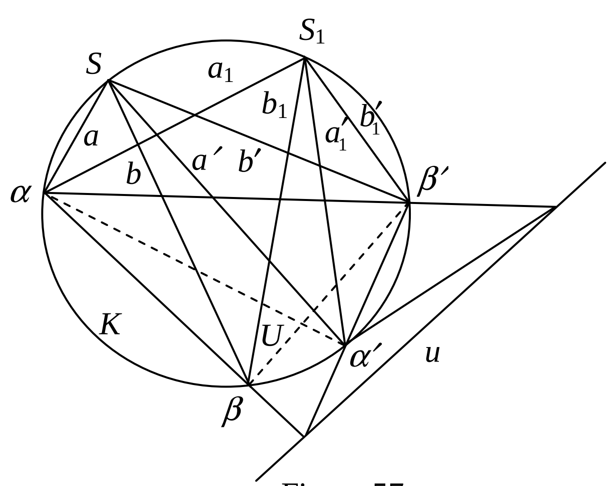

Figure 57.

b) *Intersecting two imaginary lines in a real plane.*

Let the lines g and g_1 be given by the two elliptical involutions a, a'; b, b' and a_1, a_1'; b_1, b_1', which we also call their vertices S, S_1 (Fig. 57). One is then merely dealing with the problem of finding the line in the plane on which the same involution will be cut out by S and S_1. We merely assume that $(a, a', b, b') = (a_1, a_1', b_1, b_1')$. The involutions will then be related to each other projectively, and the rays that correspond to the points of intersection α, α', β, β' will lie on the same conic section K with S and S_1, on which a curved involution J with the center U is defined by α, α', β, β'. The polar u of U has the property that the involution that is defined by K on it (S. S. VII, art. 92, 5) will also be obtained when J is projected from an arbitrary point of K (S. S. VII, art. 98, 4). u will then be the desired carrier of the imaginary point of intersection P of g and g_1. One obtains a representation of P itself when one intersects one of the involutions S or S_1 with u; u will then be found linearly from the points of intersection of the opposite edges of the tetrangle α, α', β, β'.

In particular, if the two ray quadruples are harmonic then α, α'; β, β' will also be four harmonic points on K. If one projects from one of them (e.g., from α) onto u then one will obtain a harmonic representation of P. In that way, the tangent to K at α is to be considered as the connecting line $\alpha\alpha$, and thus, the ray that is separated harmonically from $\alpha\alpha'$ by $\alpha\beta$ and $\alpha\beta'$. Its point of intersection with u will be identical with the point $(u, \beta\beta')$. One sees that in this case the imaginary point will already be determined by the simple tetrangle $\alpha\beta\alpha'\beta'$ (which is endowed with a sense of traversal) *alone*. This can also serve as a "representation" of the imaginary point then. However, the tetrangles will no longer be in one-to-one correspondence with the imaginary points, unlike the elliptic involutions.

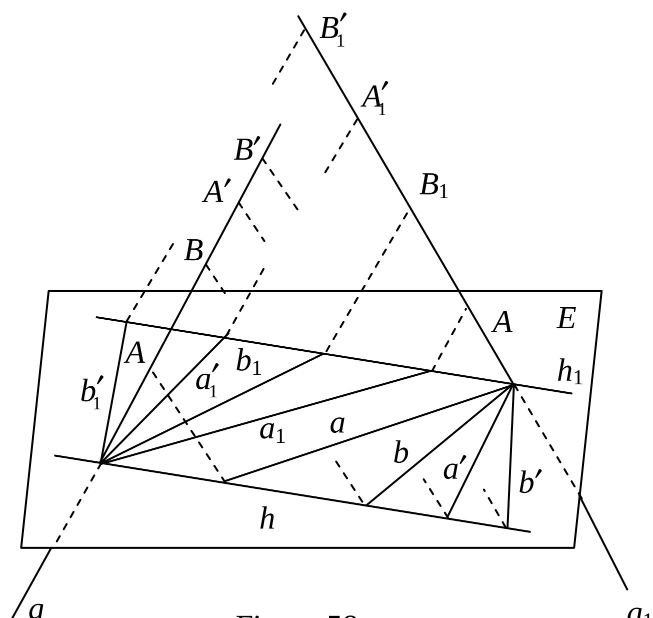

Figure 58.

c) *Intersecting a general, imaginary line* γ *with a real plane E.*

Let γ be given by two imaginary points; i.e., by the involutions A, A'; B, B' on g and A_1, A_1'; B_1, B_1' on g_1 (Fig. 58). Once again, let:

$$(A\,A'\,B\,B') = (A_1\;A_1'\;B_1\;B_1') = \mathfrak{D}.$$

One projects the involution onto g from g_1 through the planes α, α'; β, β', and conversely projects the involution onto g_1 from g through the planes α_1, α_1'; β_1, β_1' on E. The special, imaginary lines (a, a'; b, b') and (a_1, a_1'; b_1, b_1') will arise in this way, whose common point one can construct from *b*). One sees how one can then construct arbitrarily many more points of an imaginary line that is given by two points.

In particular, if $\mathfrak{D} = -1$ then the four fixed lines $\alpha\alpha_1$, $\alpha'\alpha_1'$; $\beta\beta_1$, $\beta'\beta_1'$ (which lie harmonically on the same family of rulings and determine an involution there) will cut out four points from E for an arbitrary position of E, by which the imaginary point of intersection (γE), in the sense of *b*), will be defined.

d) *Draw a plane through a general, imaginary line* γ *and a real point Q.*

Let γ be given as in *c*). If we project the involutions A, A'; B, B' and C, C'; D, D' from Q then we will obtain two special, imaginary lines that both lie in the desired plane. We thus come back to the problem that was solved in § 65, *c*).

e) *Draw a plane through three imaginary points A, B, C in general position.*

In order to convert this problem into the sub-problem that was solved already, one can make a sketch with real elements (Fig. 59). Since the reciprocal way of determining the points, planes, and lines from each other is independent of the reality of the elements, this sketch will also have a schematic meaning for imaginary elements. We draw an arbitrary real plane E through the point A, which we intersect with the general, imaginary line BC at P, as in *c*). The connecting line AP (a special, imaginary line) that was constructed as in *a*) will lie in the desired plane ε. The vertex of its involution will then be a real point of ε, and we will come back to the case *d*).

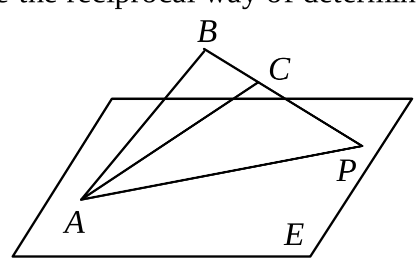

Figure 59.

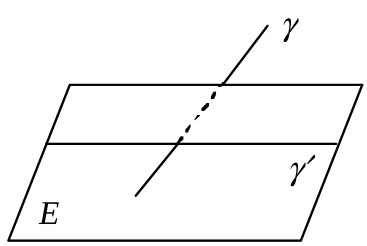

Figure 60.

f) *Intersect a general, imaginary line g with an imaginary plane.*

We think of γ as being determined by two imaginary planes and thus come to a problem that is dual to the one that was just solved.

g) *Decide whether two general, imaginary lines γ and γ' intersect.*

We make use of a real sketch (Fig, 60), as in e), and draw an arbitrary plane E through γ' (i.e., we consider any planar involution that lies perspectively to a gathered one) and intersect it with γ, as in f). If the point of intersection also belongs to γ' then γ, γ' will intersect at that point.

h) In order to be able to solve a problem in imaginary lines constructively, it is necessary to revert to the manner of determination of the associated net $\mathfrak{N}$ by two imaginary points on it by four rays (the solution of the converse problem is contained in § 66, e); i.e., to determine the associated gathered involution by two imaginary points when four rays g_1, g_2, g_3, g_4 of $\mathfrak{N}$ are given. A family of rulings $\mathfrak{R}$ is determined by g_1, g_2, g_3, and it will determine an involution J on g_4. If one projects J onto g_1 then the rays of the guiding family $\mathfrak{L}$ of $\mathfrak{R}$ will also be paired involutorily by that. $\mathfrak{L}$ will thus cut two imaginary points out of any two of the lines g_1, g_2, g_3 that determine a line γ completely; it is the desired one. $\mathfrak{L}$ also determines a gathered involution (we are anticipating Theorem 137) in which the three planes that join a point P of g_4 with g_1, g_2, g_3 correspond to three such lines that intersect at the point P that is conjugate relative to $\mathfrak{R}$. Thus, g_4 also belongs to the net of γ. The determination of the involution J is an elementary problem, and can be solved when one draws a plane through g_4, intersects it with the hyperboloid, etc.

For later purposes, we remark here that the polar g_4' of g_4 also belongs to $\mathfrak{N}$. If we then draw two planes through g_4 then we will obtain two collinear fields (Theorem 103) in which all of the intersection curves with $\mathfrak{R}$ correspond to each other, and thus also the pole of g_4 relative to it.

i) The solutions to the other elementary meet and join problems are partly dual to the foregoing ones and partly self-explanatory from what was said up to now. Whenever, e.g., a real element meets or joins with an imaginary one, that will come down to the projection or intersection of the involution in question, although when one is dealing with a general, imaginary line, it will come down to the search for the rays of the net that are incident (or coincide) with the real element.

§ 67. Involutory families of rulings and their relationship to gathered involutions.

One can pair off the rays of a family of rulings $\mathfrak{R}$ involutorily when one chooses an involution on any of its guiding rays. One then calls $\mathfrak{R}$ an *involutory family of rulings.* Such a family is said to be *included* in a gathered involution J when its rays are associated with each other in the same way that they are by J. For example, an elliptic-involutory family of rulings $\mathfrak{R}$ that is included in J is defined by three rays l, l', l'' of the net N that belongs to J when they are used as its guiding rays. Each ray p that cuts l, l', l'' will then correspond to a ray p' in J that cuts the same lines, and thus belongs to $\mathfrak{R}$. An involution in $\mathfrak{R}$ that pairs off the rays of $\mathfrak{R}$ in the same way that J does is defined by the involution on one of the lines l, l', l''.

We think of J as being determined by two imaginary pairs, namely, by the elliptic involutions x, ξ, x', ξ' on g (Fig. 61) and y, η, y', η' on h, where the sequences of symbols are defined at the same time as the senses. Let the double ratio be:

$$(x, x', \xi, \xi') = \delta.$$

We assume that y, y' is chosen to be an arbitrary pair of the involution on h, and choose η, η' to be the pair for which one also has:

$$(y, y', \eta, \eta') = \delta. \tag{36}$$

The point sequences g, h can be related to each other projectively in such a way that the quadruples x, x', ξ, ξ' and y, y', η, η' correspond to each other. Now, g and h generate an involutory family of rulings that is included in J, in which:

$$p \equiv (x, y), \qquad p' \equiv (x', y'),$$

and

$$\pi \equiv (\xi, \eta), \qquad \pi' \equiv (\xi', \eta')$$

are two pairs of corresponding rays (*).

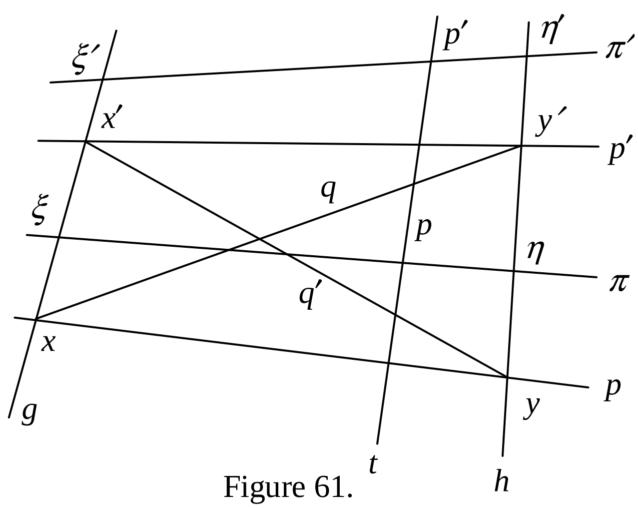

Figure 61.

(*) Since the pair y, y' (i.e., one of its points) can be chosen arbitrarily for given involutions on g and h and fixed x, x', ξ, ξ', the two involutions will give rise to ∞^1 involutory families of rulings. However, since one can base the representation of the involution on h on an arbitrary pair y, y', the following calculations will be true for all of these families of rulings.

We can set:

$$\xi_i = \lambda x_i + \mu x_i', \qquad \xi_i' = \mu x_i - \lambda x_i',$$
$$\eta_k = \lambda' y_k + \mu' y_k', \quad \eta_k' = \mu' y_k - \lambda' y_k'. \tag{37}$$

However, due to (36), we must have:

$$\frac{\mu'}{\lambda'} : -\frac{\lambda'}{\mu'} = \frac{\mu}{\lambda} : -\frac{\lambda}{\mu},$$

so we can assume that:

$$\lambda' : \mu' = \lambda : \mu,$$
$$\eta_k = \lambda y_k + \mu y_k', \qquad \eta_k' = \mu y_k - \lambda y_k'.$$

One will have:

$$\pi_{ik} = \xi_i \eta_k - \xi_k \eta_i,$$

so:

$$\pi_{ik} = \lambda^2 p_{ik} + \lambda\mu(q_{ik} + q_{ik}') + \mu^2 p_{ik}',$$
$$\pi_{ik}' = \mu^2 p_{ik} - \lambda\mu(q_{ik} + q_{ik}') + \lambda^2 p_{ik}', \tag{38}$$

if q, q' are the diagonals of the skew tetrangle x, y, y', x'. If we consider $\lambda : \mu$ to be variable in this then we will have a representation of the elliptic-involutory family of rulings.

We calculate its double rays d, d' when we set:

$$\pi_{ik} = \sigma \pi_{ik}',$$

so

$$(\lambda^2 - \sigma\mu^2)\, p_{ik} + \lambda\mu\,(1 + \sigma)\,(q_{ik} + q_{ik}') + (\mu^2 - \sigma\lambda^2) = 0.$$

We can now fulfill these six equations simultaneously when:

$$\sigma = -1, \qquad \mu = \pm i\lambda.$$

We will thus find:

$$d_{ik} = p_{ik} - p_{ik}' + i(q_{ik} + q_{ik}'),$$
$$d_{ik}' = p_{ik} - p_{ik}' - i(q_{ik} + q_{ik}'). \tag{39}$$

On the other hand, we know [§ 64, *f*), rem.] that the points of an imaginary line, any two points of which are:

$$\mathfrak{x}_i = x_i + i x_i', \quad \mathfrak{y}_k = y_k + i y_k',$$

can be allowed to coincide, just as they do for a real line. Thus, the line γ that belongs to J will have the pointers:

$$\gamma_{ik} = \mathfrak{x}_i \mathfrak{y}_k - \mathfrak{x}_k \mathfrak{y}_i = p_{ik} - p_{ik}' + i\,(q_{ik} + q_{ik}'). \tag{40}$$

If we then separate the two double rays of $\mathfrak{R}$ in a manner that is analogous to what we did with the previous structure such that we associate the ray d of the involution, whose sense is p, π, p', π', with the ray d' that has the sense p, π', p', π then the double ray d will agree precisely with the imaginary line that is associated with the involution J. Since two arbitrary rays of N can be taken to be g, h, this will be true for any involutory family of rulings that is contained in J. In fact, each of them must cut all of its guiding rays in point involutions. The carriers of the point involutions that are contained in J are, however, exhausted by the rays of N. Therefore, our construction will yield all of the involutory families of rulings that are contained in J.

For a hyperbolic-involutory family of rulings, from the remark regarding Theorem 124, merely a plus sign will enter equations (37), and therefore also (38), in place of the minus sign. Its double rays will be represented by:

$$d_{ik} = p_{ik} + p'_{ik} + q_{ik} + q'_{ik}, \qquad d'_{ik} = p_{ik} + p'_{ik} - (q_{ik} + q'_{ik}). \tag{39.a}$$

If one starts with the representation of the point-involutions on g and h by their double points s, s'; t, t', namely:

$$\xi_i = \lambda s_i + \mu s'_i, \qquad \xi'_i = \lambda s_i - \mu s'_i,$$
$$\eta_k = \lambda t_k + \mu t'_k, \qquad \eta'_k = \lambda t_k - \mu t'_k,$$

then one will get:

$$\begin{aligned} \pi &= \lambda^2 d + \lambda\mu(r + r') + \mu^2 d', \\ \pi' &= \lambda^2 d - \lambda\mu(r + r') + \mu^2 d' \end{aligned} \tag{41}$$

as the representation of the involutory family of rulings, in which r, r' mean the connecting lines $s\,t'$ and $s'\,t$.

Theorem 136: *The double rays of any involutory family of rulings $\mathfrak{R}$ that is contained in a gathered involution J are identical with the two imaginary lines that belong to J when one traverses J in both sense. All guiding rays of $\mathfrak{R}$ are rays of the net that belongs to J.*

We confirm the latter situation analytically. Since t is a line that cuts p, p', π, one will have:

$$\sum p_{ik}\, t_{lm} = 0, \qquad \sum p'_{ik}\, t_{lm} = 0,$$
$$\sum (q_{ik} + q'_{ik})\, t_{lm} = 0.$$

One will then also have:

$$\sum \gamma_{ik}\, t_{lm} = 0;$$

t will then fulfill the incidence condition with γ, so (§ 61) it will belong to the rays of N.

Since an imaginary line is determined completely by an elliptic-involutory family of rulings $\mathfrak{R}$ (all with a well-defined sense), a gathered involution must also be determined by $\mathfrak{R}$ when one adds the assumption that all of its guiding rays are ordering rays in the involution. In fact, the five independent points x, x', y, y', P (Fig. 61) will correspond to the five such points x', x, y', y, P', by which a collineation is determined (*Killing, Analyt. Geom. II*, pp. 231).

Theorem 137: *A gathered involution is determined completely by each involutory family of rulings that it contains.*

However, imaginary lines would not be defined by an involutory family of rulings, since the guiding rays of such a family would contain merely ∞^1 of the ∞^2 points of an imaginary line, so the association of general, imaginary lines and elliptic-involutory families of rulings is not, by any means, a one-to-one correspondence; moreover, ∞^3 involutory families of rulings will belong to any general, imaginary line.

We thus make note of the fact that two imaginary lines γ, γ' that belong to a gathered involution J can also be regarded as the locus of double points of that involution. Every point of γ or γ' will then be a double point of an elliptic involution that is contained precisely in J. In connection with § 62, we thus have the completely general:

Theorem 138: *The imaginary elements are always also the double elements of the involutions that they belong to.*

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§ 68. Imaginary elements in rectangular pointer systems.

From § 31, we can regard the rectangular points as a special case of the tetrahedral ones. If $x_1 = u_1 = 1$ then, from now on, we will have *three* pointers for both imaginary points and planes in a rectangular system, but *six* for an imaginary line when they are ray pointers that are connected with the point pointers by equations (24) of § 33 [cf., § 65, *f*), rem.]. All investigations of the geometric meaning of the incidence conditions, into the identity of the imaginary elements with the double elements of the elliptic involutions, etc., and more briefly, all essential results of the foregoing paragraphs will be unperturbed by this specialization. Here, one comes down to only the problem of examining the position of the imaginary elements in relation to the pointer system in which the metric properties of the involutions and their distinguished elements will also come to be valid.

a) We solve our problem for the imaginary points by giving ourselves a representation of the proper, elliptic, point involution $\mathfrak{J}$ in rectangular pointers and then looking for its double elements. $\mathfrak{J}$ will be known completely when we know its central point C, the straight line g that goes through C, on which it lies, and its power $-m^2$ (*).

(*) If δ, δ' are the distances from two associated points of an elliptic involution to the central point then (S. S. VII, art. 138):

$$\delta\,\delta' = -m^2.$$

If a, b, c are the pointers of C, α, β, γ are the direction cosines of g, and x, y, z; x', y', z' are the pointers of two associated points of the involution then one will have:

$$(x - a)\,(x' - a) = -\,m^2\,a^2$$

for the projection of the involution onto the X-axis. We will then obtain:

$$\xi = a \pm m\,\alpha\,i, \;\ldots$$

for $x = x' = \xi$. Therefore:

$$a \pm m\,\alpha\,i, \qquad b \pm m\,\beta\,i, \qquad c \pm m\,\gamma i$$

will be the pointers of the double elements of the involution $\mathfrak{J}$ and, from Theorem 138, they will likewise make up the imaginary points that belong to $\mathfrak{J}$. If we set:

$$m\,\alpha = a', \qquad m\,\beta = b', \qquad m\,\gamma = c'$$

then we will have:

$$a'^2 + b'^2 + c'^2 = m^2.$$

We point out that C and the infinite-distant point U of g are associated with each other for the separation of the two points by the sense of $\mathfrak{J}$. Thus, if $a + a'$, $b + b'$, $c + c'$ belong to a point Q as pointers then the sequence CQU will determine the sense of C along the *finite* segment on Q.

Theorem 139: *The imaginary point $a + a'i$, $b + b'i$, $c + c'i$ is associated with the involution with the central point $C \equiv (a, b, c)$, the power $-(a'^2 + b'^2 + c'^2)$, and the sense of C from $(a + a', b + b', c + c')$; the direction cosines of its carriers are proportional to a', b', c'.*

b) If an elliptic net $\mathfrak{N}$ lies in relation to the pointer system as in § 55, *c*) – i.e., if its axes coincide with the X and Y axes – then we will find the double points of the associated gathered involution from the equations [cf., § 57, equations (103)]:

(42)
$$x = \frac{c}{m} \cdot \frac{y}{z}, \qquad y = -\,cm \cdot \frac{x}{z}, \qquad z = -\,\frac{c^2}{z}.$$

The last of these can be derived from the first two; one will then find that:

(43)
$$\frac{y}{z} = \pm\,m\,i, \qquad\qquad z = \pm\,c\,i.$$

For $|\,\delta\,| = |\,\delta'\,| = m$, one obtains the two mutually-associated "power points"; the power points of all point involutions that are contained in precisely a gathered involution J will lie (as would emerge from Theorem 109) on the power planes of J; hence, the name.

Thus, the upper two and the lower two signs will belong together, while either x or y will remain arbitrary. The imaginary lines γ, γ' that belong to $\mathfrak{N}$ will be represented by these two solutions. We calculate the pointers of γ as in § 33, equations (24), from those of the two points:

$$x = 0, \qquad y = 0, \qquad z = c\,i,$$
$$x' = 1, \qquad y' = m\,i, \qquad z' = c\,i,$$

and find:

(44)
$$q_1 = 1, \qquad q_2 = m\,i, \qquad q_3 = 0,$$
$$q_4 = c\,m, \qquad q_5 = c\,i, \qquad q_6 = 0.$$

The pointers of γ' are conjugate to these. We always use γ to denote the line for which the coefficient of i in q_2 is positive – i.e., we can assume that $m > 0$, as was also done in § 55, c). The sign of c and the winding of the net is determined from q_4 in that way [§ 55, c)].

Theorem 140: *If a general imaginary line lies in such a way that the two axes of the associated ray nets coincide with the X and Y axes of the pointer system then it will be characteristic of that position that the pointers* q_3, q_6 *are zero, the pointer ratios* $q_4 : q_1$, $q_5 : q_2$ *are real, and* $q_2 : q_1$ *are pure imaginary. If one makes* q_1 *equal to unity then the absolute value of* q_2 *will be the axis ratio of the net and that of* q_5 *will be one-half the distance between the power planes.*

We now apply the formulas for the pointer transformation (§ 41) (*). If one solves equations (59) there for the new pointers p then one will next obtain:

(45)
$$\begin{aligned} p_1 &= a_1\, q_1 + b_1\, q_2 + c_1\, q_3\,, \\ p_2 &= a_2\, q_1 + b_2\, q_2 + c_2\, q_3\,, \\ p_3 &= a_3\, q_1 + b_3\, q_2 + c_3\, q_3 \end{aligned}$$

for a mere rotation around the origin. If one replaces the q in this with the expressions in (44) then one will have:

(46)
$$p_\lambda = p'_\lambda + i p''_\lambda = a_\lambda + b_\lambda\, m\, i \qquad (\lambda = 1, 2, 3).$$

One will obtain the new pointers p_4, p_5, p_6 when one raises the indices of the q by three [cf., § 41, equations (60)]:

(47)
$$p_{\lambda+3} = p'_{\lambda+3} + i\, p''_{\lambda+3} = c\,(a_\lambda\, m + b_\lambda\, i)\,.$$

It will follow from the properties of the coefficients of an orthogonal substitution that:

(48)
$$\begin{cases} \sum p'^2_\lambda - \sum p''^2_\lambda = 1 - m^2, & \sum p'_\lambda p''_\lambda = 0, \\ \sum p'^2_{\lambda+3} - \sum p''^2_{\lambda+3} = c^2(m^2 - 1), & \sum p'_{\lambda+3} p''_{\lambda+3} = 0 \end{cases}$$

(*) This is also true for imaginary elements; this will then be true for the associated real involutions whose double elements are the imaginary elements (Theorem 138).

(49) $$\sum p'_\lambda p'_{\lambda+3} = cm, \qquad \sum p''_\lambda p''_{\lambda+3} = cm,$$

(50) $$\sum p'_\lambda p''_{\lambda+3} = 0, \qquad \sum p''_\lambda p'_{\lambda+3} = 0,$$

in which $\lambda = 1, 2, 3$. Only one of the relations (49) and (50) is essential; the other one will then follow from the fact that $\sum p_\lambda\, p_{\lambda+3} = 0$. These conditions are by no means characteristic of the present position of the net, since one can multiply all p with a common factor $\alpha + \beta i$. In that way, they will then go to the quantities $P_\lambda = P'_\lambda + iP''_\lambda$, and indeed one will have:

$$P'_\lambda = \alpha p'_\lambda - \beta p''_\lambda, \qquad P''_\lambda = \alpha p''_\lambda + \beta p'_\lambda .$$

If one sets, to abbreviate:

$$\sum P'_\lambda P''_\lambda = \sigma_\lambda, \qquad \sum P'_{\lambda+3} P''_{\lambda+3} = \sigma_{\lambda+3},$$

$$\sum P'^2_\lambda = \sigma'_\lambda, \quad \sum P''^2_\lambda = \sigma''_\lambda, \quad \sum P'^2_{\lambda+3} = \sigma'_{\lambda+3},$$

$$\sum P''^2_{\lambda+3} = \sigma''_{\lambda+3}, \quad \sum P'_\lambda P'_{\lambda+3} = \tau', \quad \sum P''_\lambda P''_{\lambda+3} = \tau'',$$

$$\sum P'_\lambda P''_{\lambda+3} = \tau_{12}, \quad \sum P''_\lambda P'_{\lambda+3} = \tau_{21}$$

then one will find, with the use of equations (48) to (50), that:

(48.a) $$\begin{cases} \sigma'_\lambda - \sigma''_\lambda = (\alpha^2 - \beta^2)(1 - m^2), & \sigma_\lambda = \alpha\beta(1 - m^2), \\ \sigma'_{\lambda+3} - \sigma''_{\lambda+3} = (\alpha^2 - \beta^2)c^2(m^2 - 1), & \sigma_{\lambda+3} = \alpha\beta c^2(m^2 - 1), \end{cases}$$

(49.a) $$\tau' = (\alpha^2 + \beta^2)\, c\, m = \tau'',$$

(50.a) $$\tau_{12} = 0 = \tau_{21} .$$

There is thus a three-fold condition for a net, namely, that its midpoint must coincide with the origin. There must then be three relations between the P', P'' alone, which express that position (in which, the ones that follow from:

$$\sum P_\lambda\, P_{\lambda+3} = 0 \qquad (\lambda = 1, \ldots, 6)$$

will not be counted). One of them will be (50.a), and another one will be:

(51) $$\frac{\sigma'_{\lambda+3} - \sigma''_{\lambda+3}}{\sigma'_\lambda - \sigma''_\lambda} = \frac{\sigma_{\lambda+3}}{\sigma_\lambda}.$$

One would further have:

$$c^2 = -\frac{\sigma_{\lambda+3}}{\sigma_\lambda}.$$

We shall not pursue the derivation of the third relation and the calculation of m [cf., the conclusion of b)], but we would like to derive the main features for a rotational net. $m = 1$ for one such, so:

$$\sigma'_\lambda = \sigma''_\lambda, \qquad \sigma_\lambda = 0,$$
$$\sigma'_{\lambda+3} = \sigma''_{\lambda+3}, \qquad \sigma_{\lambda+3} = 0.$$

Conversely, the first row of these relations, or also the last one, can be fulfilled without α and β vanishing individually only for $m = 1$. The first two can be combined into:

$$\sum_{\lambda=1}^{3} P_\lambda^2 = 0, \tag{52}$$

and the last two into:

$$\sum_{\lambda=1}^{3} P_{\lambda+3}^2 = 0,$$

and indeed (52) will be true for an *arbitrary* position of the net, since the first three pointers do not change under a parallel displacement (§ 41). Thus:

Theorem 141: *If* q_λ ($\lambda = 1, \ldots, 6$) *are the pointers of a general, imaginary line then the necessary and sufficient condition for the associated ray net to be a rotational net will be:*

$$\sum_{\lambda=1}^{3} q_\lambda^2 = 0.$$

For a parallel translation ($\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$) of the pointer system [equations (61) of § 41] from the original special position, the connection between the new pointers κ and the old ones q will be:

$$\begin{aligned}
\kappa_\lambda &= q_\lambda \qquad\qquad (\lambda = 1, 2, 3)\\
\kappa_4 &= q_4 + \mathfrak{x}\, q_4 - \mathfrak{y}\, q_3 ,\\
\kappa_5 &= q_5 + \mathfrak{x}\, q_3 - \mathfrak{z}\, q_1 ,\\
\kappa_6 &= q_6 + \mathfrak{h}\, q_1 - \mathfrak{x}\, q_2 .
\end{aligned}$$

If one now replaces the q here with the expressions in (4) then one will get:

$$\begin{aligned}
\kappa_1 &= 1, & \kappa_4 &= c\, m + \mathfrak{x}\, m\, i = \kappa'_4 + i\,\kappa''_4 ,\\
\kappa_2 &= m\, i, & \kappa_5 &= -\,\mathfrak{z} + c\, i = \kappa'_5 + i\,\kappa''_5 ,\\
\kappa_3 &= 0, & \kappa_6 &= \mathfrak{y} - \mathfrak{x}\, m\, i = \kappa'_6 + i\,\kappa''_6 .
\end{aligned} \tag{53}$$

Theorem 142: *If the axes of an elliptic net are parallel to the X and Y axes of the pointer system then it will be characteristic of that position that, of the pointers of the associated imaginary line, κ_3 will be zero, and $\kappa_2 : \kappa_1$ will be pure imaginary. If only the principal ray of the net is parallel to the Z-axis then one will still have* $\kappa_3 = 0$ (*).

The last part of the theorem follows from the formulas for the rotation of the system around the *Z*-axis. If a given κ fulfills these two conditions then one will make κ_1 equal to unity; one can then deduce the quantities *m*, *c*, $\mathfrak{x}$, $\mathfrak{y}$, $\mathfrak{z}$ from equations (53).

If a general, imaginary line $q'_\lambda + i q''_\lambda$ is given, and its position with respect to the pointer system is arbitrary then one can determine a_3, b_3, c_3 from:

$$\begin{aligned} a_3\, q'_1 + b_3\, q'_2 + c_3\, q'_3 &= 0, \\ a_3\, q''_1 + b_3\, q''_2 + c_3\, q''_3 &= 0, \\ a_3^2 + b_3^2 + c_3^2 &= 1 \end{aligned}$$

(with which, the direction of the principal ray will be found), and furthermore, the remaining *a*, *b*, *c* will be obtained from:

$$\begin{aligned} a_1\, q''_1 + b_1\, q''_2 + c_1\, q''_3 &= 0, \\ a_2\, q'_1 + b_2\, q'_2 + c_2\, q'_3 &= 0, \end{aligned}$$

and the conditions for an orthogonal substitution. One will then know the pointer transformation by which the position of Theorem 142 is arrived at. By performing it, one can calculate the two constants *m* and *c* that are definitive for the form and size of the net (§ 54, conclusion).

c) For a special, imaginary line $q'_\lambda + i\, q''_\lambda$, one can calculate the pointers of the vertex and plane of the associated ray involution from § 39, *b*), after one has specialized the incidence conditions in a known way (§ 31) for rectangular pointers. Moreover, one can also write down the ray involution itself immediately:

$$p_\lambda = \mu\, q'_\lambda + \mu' q''_\lambda, \qquad p'_\lambda = \mu'\, q'_\lambda - \mu\, q''_\lambda \qquad (\lambda = 1, \ldots, 6);$$

its analytical representation does not differ at all in rectangular and tetrahedral, homogeneous, line pointers.

Things are different for the case of the imaginary plane $u'_\lambda + i\, u''_\lambda$; one has:

$$1 + \sum_{\lambda=1}^{3} u'_\lambda\, x_\lambda = 0, \qquad \sum_{\lambda=1}^{3} u''_\lambda\, x_\lambda = 0 \tag{54}$$

for the equations of the axis α of the involution, while, by contrast:

(*) The last condition is counted twice.

$$\text{(54a)} \qquad v_\lambda = u'_\lambda + \mu u''_\lambda, \qquad v'_\lambda = u'_\lambda - \frac{1}{\lambda} u''_\lambda \qquad (\lambda = 1, 2, 3)$$

are those of the involution itself. An arbitrary plane v of the pencil (54) will be represented by the left-hand equations (54.a) when the u'' themselves are also not pointers of a plane of the pencil (*); moreover, the pointers of a plane that rotates around α in such a way that it goes through the origin will become infinite in such a way that their ratios will approach those of the fixed line u''. We thus denote this plane by u''; its equation is the right-hand equation (54). Moreover, if:

$$w_\lambda = u'_\lambda + \nu u''_\lambda$$

then:

$$(u'\, u''\, v\, w) = \frac{\nu}{\mu}.$$

One must have:

$$(u'\, u''\, v\, w) = (u''\, u'\, v'\, w')$$

for the corresponding planes v', w', since u', u'' itself is a pair of the involution, so:

$$\frac{\nu}{\mu} = \frac{\mu'}{\nu'}.$$

If one thinks of the one pair w, w' as being fixed, while the other one moves, then one will find that:

$$\mu\,\mu' = \text{const.}$$

is the connection between the parameters of the associated planes of one pair. Indeed, the constant will be negative for elliptic involutions, and here it can be chosen such that the double elements of the involution will emerge from the given imaginary planes precisely, so:

$$\mu\,\mu' = -1.$$

(*) If one would like to express the pointers v of a general plane of the pencil in terms of the pointers u', u of two fixed planes in the pencil then, as is known, one will have:

$$v_\lambda = \frac{u'_\lambda + \mu u_\lambda}{1+\mu}.$$

One would then obtain:

$$v_\lambda = \frac{u'_\lambda + \mu u_\lambda}{1+\mu}, \qquad v'_\lambda = \frac{\mu u'_\lambda - k^2 u_\lambda}{\mu - k^2}$$

as the representation of an involution from this in the same manner as in the text. However, the representation (54.a) is not only simpler, but it also can be connected immediately with the pointers of the given imaginary plane.

We now ask when the ray or plane involution will be rectangular. In the former case, one must have:

$$\sum_{\lambda=1}^{3} p_\lambda \, p'_\lambda = 0,$$

and in the latter one:

$$\sum_{\lambda=1}^{3} v_\lambda \, v'_\lambda = 0$$

for every value of μ': μ or μ. This will give:

Theorem 143: *Should the involutions that belong to a special, imaginary line $q'_\lambda + iq''_\lambda$ or an imaginary plane $u'_\lambda + i\,u''_\lambda$ be rectangular then one would need to have:*

$$\sum_{\lambda=1}^{3} q'_\lambda \, q''_\lambda = 0, \qquad \sum_{\lambda=1}^{3} q'^2_\lambda = \sum_{\lambda=1}^{3} q''^2_\lambda$$

or

$$\sum_{\lambda=1}^{3} u'_\lambda \, u''_\lambda = 0, \qquad \sum_{\lambda=1}^{3} u'^2_\lambda = \sum_{\lambda=1}^{3} u''^2_\lambda \,,$$

which can be combined into one equation:

$$\sum_{\lambda=1}^{3} q^2_\lambda = 0 \qquad \text{or} \qquad \sum_{\lambda=1}^{3} u^2_\lambda = 0.$$

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§ 69. Basic tetrahedron with some imaginary elements.

From § 29, *e*), any linear transformation:

$$(55) \qquad \rho\, x_\lambda = \sum_{\mu=1}^{4} a_{\lambda\mu} x'_\mu \,, \qquad \sigma\, u_\lambda = \sum_{\mu=1}^{4} A_{\lambda\mu} u'_\mu \,,$$

$$(56) \qquad \rho' x'_\lambda = \sum_{\mu=1}^{4} A_{\lambda\mu} x_\mu \,, \qquad \sigma' u'_\lambda = \sum_{\mu=1}^{4} a_{\lambda\mu} u_\mu$$

whose determinant $|\, a_{\lambda\mu} \,|$ does not vanish can be regarded as a pointer transformation. Any of these four systems of equations will determine the other three. Up to now, we have thought of the coefficients a in the substitution as being *real*. However, even when they are complex, any (real or complex) quadruple of values x or u is in one-to-one correspondence with a quadruple x' or u', such that the values x' or u' are just as useful for the determination of a (real or imaginary) point as the x or the u. Due to the known connection between the line pointers and the point or plane pointers, the same thing will be true for the line pointers.

When we set the right-hand sides of (56) equal to zero, we will further obtain the equations of the vertices and faces of the new tetrahedron in the old system; in other words, $A_{\lambda\mu}$ ($\mu = 1, \ldots, 4$) will be the pointers of the λ^{th} plane and $a_{\lambda\mu}$ ($\mu = 1, \ldots, 4$) will be those of the λ^{th} vertices of the new tetrahedron. We also *define* these numbers to be the pointers of the vertices and faces of the new tetrahedron for complex substitutions. In fact, we are justified in saying that they exist for a real tetrahedron, as the elementary properties of determinants show.

From now on, we consider only such imaginary basic tetrahedra for which the conjugates of the elements of the tetrahedron will enter into consideration, along with each imaginary element. Only two cases can then appear:

a) Two vertices – say, P_1, P_4 – are real and the other two P_2, P_3 are complex conjugate, so their connecting line k will be real. Only two (k, P_1) and (k, P_4) of the four planes will also be real then. A pair of opposite edges will be real, while the other two pairs will be imaginary, and indeed they will cut any two conjugate edges as special, imaginary lines in a real vertex.

b) All four edges are imaginary, and perhaps P_2, P_3 are conjugate, along with P_1, P_4. Such a tetrahedron will be determined by two proper involutions on skew carriers g, g'. A pair of opposite edges g, g' will be real; each of the other two pairs will consist of two general, conjugate, imaginary lines.

a) We obtain a tetrahedron of the first kind when we choose two real columns from the a that are both complex-conjugate to each other. Since for us this only comes down to obtaining such a basic tetrahedron as real, and not in general position with respect to the original one, in order to accomplish this, we will take the simplest possible transformation of the vertices Q_1, Q_4 of the old tetrahedron, as well as the vertices P_1, P_4 of the new ones, and choose an involution on $Q_2\,Q_3$ for which:

$$Q_2 \equiv (0, 1, 0, 0), \qquad Q_3 \equiv (0, 0, 1, 0)$$

is a pair of it. Let:

$$(0, 1, i, 0), \qquad (0, 1, -i, 0)$$

be the double points of such an involution, so (§ 29, *c*):

$$\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & i & 0 \\ 0 & i & -i & 0 \\ 0 & 0 & 0 & 1 \end{matrix}$$

will be the substitution matrix. For our purposes, we can multiply its adjoint by i and thus obtain the desired transformation:

$$(57)\qquad \begin{aligned} \rho x_1 &= x_1', \\ \rho x_2 &= x_2' + x_3', \\ \rho x_3 &= i(x_2' - x_3'), \\ \rho x_4 &= x_4', \end{aligned} \qquad\qquad (58)\qquad \begin{aligned} \sigma u_1 &= u_1', \\ \sigma u_2 &= u_2' + u_3', \\ \sigma u_3 &= i(u_2' - u_3'), \\ \sigma u_4 &= u_4'. \end{aligned}$$

We combine the transformation of the line pointers, as well as the ray pointers, according to equations (56) in § 40:

$$(59)\qquad \begin{aligned} \tau\pi_{12} &= \pi_{12}' + \pi_{13}', \\ \tau\pi_{13} &= i(\pi_{12}' - \pi_{13}'), \\ \tau\pi_{14} &= \pi_{14}', \end{aligned} \qquad \begin{aligned} \tau\pi_{34} &= -i(\pi_{42}' + \pi_{34}'), \\ \tau\pi_{42} &= \pi_{42}' - \pi_{34}', \\ \tau\pi_{23} &= -2i\pi_{23}'. \end{aligned}$$

b) In order to obtain such a tetrahedron as simply as possible, we will choose that elliptic involution on the edges Q_1, Q_4 and Q_2, Q_3 of the old real tetrahedron to which the two edges belong as one of its pairs. The imaginary vertices P_1, P_4 will also emerge from the pointers of the real ones Q_1, Q_4, as P_2, P_3 already did before from Q_2, Q_3 . From *a*), we can thus write down the desired transformation for the point and plane pointers immediately, and for the line pointers from the matrix:

$$\begin{matrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & i & -i & 0 \\ 0 & 0 & 0 & -i \end{matrix}$$

analogous to before, we can summarize them as:

$$(60)\qquad \begin{aligned} \rho x_1 &= x_1' + x_4', \\ \rho x_2 &= x_2' + x_3', \\ \rho x_3 &= i(x_2' - x_3'), \\ \rho x_4 &= i(x_1' - x_4'), \end{aligned} \qquad \begin{aligned} \sigma u_1 &= u_1' + u_4', \\ \sigma u_2 &= u_2' + u_3', \\ \sigma u_3 &= i(-u_2' + u_3'), \\ \sigma u_4 &= i(-u_1' + u_4'), \end{aligned}$$

$$(62)\qquad \begin{aligned} \tau\pi_{12} &= \pi_{12}' + \pi_{13}' + \pi_{42}' - \pi_{34}', \\ \tau\pi_{13} &= i(\pi_{12}' - \pi_{13}' + \pi_{42}' + \pi_{34}'), \\ \tau\pi_{14} &= -2i\pi_{14}', \end{aligned} \qquad \begin{aligned} \tau\pi_{34} &= \pi_{12}' - \pi_{13}' - \pi_{42}' - \pi_{34}', \\ \tau\pi_{42} &= i(\pi_{12}' + \pi_{13}' - \pi_{42}' + \pi_{34}'), \\ \tau\pi_{23} &= -2i\pi_{23}'. \end{aligned}$$

c) We would like to consider the pointer system that one obtains from a rectangular one when one introduces new pointers in place of the rectangular pointers *x*, *y*, *z* by means of the equations:

$$(63)\qquad \xi = x + i\,y, \qquad \eta = x - i\,y, \qquad \zeta = z$$

or

$$(64)\qquad x = \tfrac{1}{2}(\xi + \eta), \qquad y = \tfrac{1}{2}(-\xi + \eta), \qquad z = \zeta.$$

If we recall § 31 then we can consider the new system to be a special case of *a*). In fact, the *XY*-plane and the tetrahedral plane that has been shifted to infinity will remain real, and corresponding statements will be true for the vertex of the tetrahedron at the origin and the point at infinity on the *Z*-axis. $\xi = 0$ and $\eta = 0$ are the double elements of the rectangular ray involution:

$$\lambda x + \mu y = 0, \qquad \mu x - \lambda y = 0.$$

The transformation thus comes down to introducing the double rays of this rectangular involution instead of the *X* and *Y* axes. We would like to check which transformation of the ray pointers has equations (63) as a consequence. If we think of them as once more written with primed symbols then we will find, when we recall equations (24) of § 33 and denote the new ray pointers by *p* and the old ones by *q*, that (*):

$$\begin{aligned} p_1 &= q_1 + i\,q_2, & p_4 &= -(q_5 + i\,q_4), \\ p_1 &= q_1 - i\,q_2, & p_5 &= q_5 - i\,q_4, \\ p_3 &= q_3, & p_6 &= -2i\,q_6. \end{aligned} \tag{65}$$

---

§ 70. Imaginary lines of real line structures.

a) We say that an imaginary line *g* is contained in a real line structure $\mathfrak{L}$ when its pointers $p_\lambda = p'_\lambda + i\,p''_\lambda$ fulfill the equations for $\mathfrak{L}$. If $\mathfrak{L}$ is a twist (**) then the equation:

$$\sum_{\lambda=1}^{6} a_{\lambda+3}\,p_\lambda = 0 \tag{66}$$

will decompose into two:

$$\sum a_{\lambda+3}\,p'_\lambda = 0, \qquad \sum a_{\lambda+3}\,p''_\lambda = 0. \tag{67}$$

Now, if *g* is special then equations (67) will mean that the vertex and plane of the involutory bush of rays that belongs to *g* will be associated with each other as the null point and null plane in the twist, resp. If *g* is general then these equations will mean that the given twist *a* will lie involutorily with the two twists *p′* and *p″* by which the net that *g* belongs to is determined.

(*) One can believe that equations (65) must also be obtained from equations (59) when they are solved for the π' in such a way that one replaces the symbols with two indices by symbols with one index according to the table (25) in § 33. In fact, one will obtain the same equations from this process that one does by means of the transformation:

$$\xi = \tfrac{1}{2}(x - i\,y), \qquad \eta = \tfrac{1}{2}(x + i\,y), \qquad \zeta = z,$$

which differ from (63) only inessentially.

(**) The analogous question for a real bush of rays will be answered by Theorem 128.

b) The definition of an imaginary line being contained in an arbitrary line structure is entirely analogous. Therefore, should g be contained in a ray net $\mathfrak{N}$ that is defined by the complexes a and b, then the equations:

$$\sum b_{\lambda+3} p'_\lambda = 0, \qquad \sum b_{\lambda+3} p''_\lambda = 0$$

would have to be fulfilled, in addition to (67). That is, when g is general: Each of the complexes a, b will lie involutorily to p', as well as p''. If g is special then the vertex of the ray involution will be a singular point, and its plane will be a singular plane of $\mathfrak{N}$. This condition is independent of the way that a ray complex is determined by two complexes; from § 53, it will then emerge immediately that:

Theorem 144: *If a complex c lies involutorily to two others a, b then it will also lie involutorily to the complexes of the pencil a, b.*

One can then say that c lies involutorily to the pencil a, b.

c) Up to now [§ 39, *e*)], we have represented the rays p of a family of rulings $\mathfrak{R}$ analytically by:

$$p_k = \lambda\, q_k + \mu\, q'_k + \nu\, q''_k ,$$

with the condition:

$$\omega(p) = 0,$$

in which q, q', q'' are three fixed rays of $\mathfrak{R}$. If we also assign complex values to the parameters λ, μ, ν then we will obtain imaginary rays of the real family of rulings. They will also be characterized completely by the fact that they cut any three rays s, s', s'' of the guiding family $\mathfrak{L}$, since the analytical condition in question will be fulfilled independently of the reality of the parameters. We will thus find the imaginary lines of $\mathfrak{R}$ when we seek all imaginary lines p that each have one point in common with the real lines s, s', s''. The necessary and sufficient condition for this is that s, s', s'' must belong to the net $\mathfrak{N}$ that p belongs to. We will thus obtain all imaginary lines of $\mathfrak{R}$ when we determine a net by way of s, s', s'', and an arbitrary fourth ray t, and therefore a gathered involution J. Since t can be chosen to be inside that same net in ∞^2 ways, there will be ∞^2 imaginary lines in a real family of rulings. From § 67, J will determine an involutory family of rulings on the guiding family of $\mathfrak{L}$ – i.e., on $\mathfrak{R}$ itself – to which p will belong as a double ray.

Theorem 145: *One obtains all imaginary lines of a real family of rulings $\mathfrak{R}$ as the double rays of all elliptic involutions to which the rays of $\mathfrak{R}$ can be assigned; $\mathfrak{R}$ will thus contain ∞^2 (general) imaginary lines.*

§ 71. Logical and historical remarks on the role of the “imaginary” in geometry.

We shall once more give an overview of the train of thought that we previously embarked upon in the theory of imaginary elements: In analytic geometry, complex values are frequently produced for the pointers of a desired element (i.e., point, plane, line). Every such system of complex pointers can be put in one-to-one correspondence with a well-defined real geometric structure (§ 61-63). The analytically-defined incidence conditions between the complex elements also correspond to well-defined geometric statements (§§ 64, 65). The reciprocal determination of the elements by each other is expressible analytically by equations whose validity is independent of the reality of the elements. Once the incidence relations are interpreted geometrically, the laws of meets and joins will thus also have a well-defined meaning in the imaginary domain, first of all, and secondly, they will preserve their validity. The problems that these rules gave rise to were solved constructively in § 66, once their unique solubility was already established at the beginning of that paragraph. With that, the constructions of the “geometry of position,” which rest upon merely the laws of meets and joins, are thus made independent of analytic geometry (and therefore, of the pointer system), for the imaginary domain, as well.

One can thus give a schema (cf., the following example) for a construction in the geometry of position, for which one first directs one’s attention to real elements, and then applies it, while being unconcerned about whether imaginary elements appear in the course of the construction. One is certain that each step in the schema can be translated into actual constructions in such a way that the final result will remain correct. The latter can – e.g., from the nature of things – be real and still remain useful in a schema in which imaginary elements appear. However, this “passage to the imaginary” would not be permissible if one could not endow each step with a well-defined geometric meaning. The fact that one does not always need to remember it explicitly when one is merely concerned with results is one advantage of the theory of imaginary elements that are allowed to operate with the complex structures of the involutions just as they do with real elements. On the other hand, in the analytic geometry of position one knows that each step of the calculation is meaningful without appealing to the reality of the numbers that appear in it. A completely parallelism is exhibited between the analytic operations and the geometric constructions in the designated context that makes it permissible to convert the results in the one domain into results in the other one immediately.

One has assumed three standpoints in relation to the theory of imaginary elements:

a) One converts the results that were made obvious for real elements to the case in which all of the elements that were employed as tools for proving them are now imaginary using the “principle of continuity,” without actually developing a theory of imaginary elements at all. For example, a theorem of geometry reads: If three circles in a plane have three common chords to two of them then they (viz., the three “chordals”) will intersect in the same point. This theorem can be easily recognized in the case of Fig. 62 without making any use of the property of common chords that they are power lines: If one describes three spheres with the three circles as great circles then the three points of intersection that the three spheres have in common will go through the two points P, P' that are common to all three spheres. The three planes of the circles will thus go through the connecting line PP', and their three traces on the reference plane will go through the trace point of PP'. One now concludes that the possible common chords to the three circles might also intersect in a point when the three spheres have no real points in common (e.g., Fig. 63). However, that is not correct. One cannot conclusively commit to such a method, although it is also very fruitful as a heuristic tool. One can likewise judge when the theorems of algebra on the common roots of equations, *et al.*, can be converted into mere figures of speech in geometry without affecting their actual meaning.

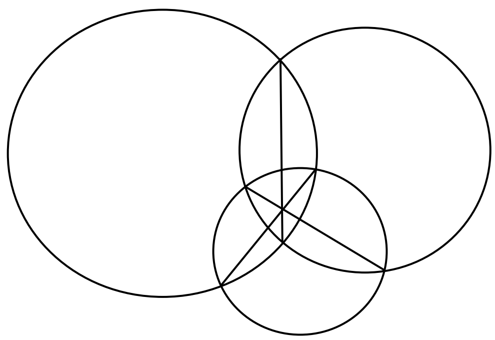

Figure 62.

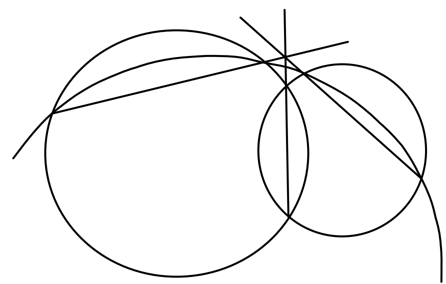

Figure 63.

b) One refuses to employ imaginary numbers in geometry or to speak of imaginary elements at all. This standpoint is logically correct, but inconvenient. One is then forced to address all case distinctions in the investigation that raise questions of reality from the outset, while one who is in possession of a theory of the imaginary elements must first separate them in the results. The advantages that the imaginary contributes to algebra are also attainable in geometry (*) when they can be assigned to actual geometric structures in that context, just as the imaginary numbers do, which emerge as the solutions to a problem that is originally posed as real. The advantage of a schema such as the one in the following example would be lost completely if one were to reject the theory of imaginary elements.

c) One develops an actual theory of imaginary elements (as we did here, at least, in a restricted context), from which, algebraic operations with complex numbers will become geometrically useful. It is, in fact, correct that all of the results that one thus obtains can also be formulated without leaving the real domain, so the theory of imaginary elements in geometry cannot add any actual content, but merely an abbreviated way of speaking. However, it is precisely in the latter situation that its greatest value exists, which also lies essentially in the "economy of thought" (cf., *Mach*, *Mechanik*, chap. IV, 4). The words "imaginary point, line," etc, are not at all trivial. They immediately remind us of the fact that the same laws are true for the corresponding structures that are true for the real ones, and make it easy for us to apply the same system of thinking or notation – indeed, schematic notations (§ 66, *e*, *f*) that lead to correct results – since the logical relationship of "complete analogy" exists between the schemas and the actual structures. Therefore, a

(*) *Staudt* emphasized this in the Foreword to *Beitr. zur Geom. d. Lage* (1856).

"mechanical implementation of the concept of an imaginary point," as well as imaginary elements, is indeed possible (this was renounced in S. S. VII, pp. 288).

One cannot reproach the use of imaginaries for a lack of logical clarity (*), as long as one applies them only within the boundaries within which the theory was founded – so, for us, as long one deals with the geometry of position. By contrast, from the previous developments, we can ascribe no meaning to the concepts of "an imaginary angle" or "the perpendicular position of imaginary lines." It would take us to far from our topic in line geometry to extend the theory of imaginary elements further in an essential way. Moreover, we have still not done everything that we would like to do, since there is also more that we could include in our presentation. For example, one finds a theory of the double ratios of imaginary elements in *Clebsch-Lindemann's Vorl. üb. Geom.*, Bd. II (pp. 115, *et seq.*). Above all, the theory of projective conversions of the basic structures was also already extended to imaginary elements by *Staudt* himself. Moreover, we will have occasion to extend the theory up to now at some points. In order to explain the statements, we consider an:

Example: Let a null system be given by five real rays $n_1, \ldots, n_6$ of the associated twist. Construct the null plane ν to a real point P.

We already encountered a similar problem, and we summarized the constructions that § 22 and § 10, *d*) implied in a table in which points were denoted by upper-case Latin characters, lines, by lower-case Latin ones, and planes, by Greek ones. The symbol "≡" should mean "incident with." If several symbols are on one or both sides of it then each element on the one side should be incident with each element on the other. One thus determines, in succession:

$g, g' \equiv n_1, n_2, n_3, n_4$	(§ 66, *h*)
$\varepsilon \equiv P, n_5$	(real)
$S \equiv g, \varepsilon; \qquad S' \equiv g', \varepsilon$	(§ 66, *c*)
$h \equiv S, S'$	(must be real)
$N \equiv h, n_5$	" " "
$t \equiv P, g, g'$	" " "
$\nu \equiv t, N$	" " "

In regard to the penultimate row of the table, we remark that t is determined for imaginary elements, in general, in that same way that it is for real ones; i.e., this row can be replaced with:

$\alpha \equiv P, g$	(dual to § 66, *c*)
$\alpha' \equiv P, g'$	" " "
$t \equiv \alpha, \alpha'$	(dual to § 66, *a*).

(*) The fact that one must remain in the realm of imaginary elements for elliptic involutions in the geometric theory, while one goes on to the double elements for the hyperbolic involutions, is a small incongruity that cannot be set aside, but will either disturb things or bring about errors.

However, this is not necessarily true here, since g, g' are (real or) conjugate-imaginary. The problem of $t \equiv P$, g, g' then comes from § 66, *d*). The axis of the plane involution is the solution. If one then knows from analytic geometry that a twist is determined by five rays then this schema will give – even when a linear construction (§ 47, *d*) is not known – the solution to the problem under all circumstances, since otherwise it would be necessary for us to force the reality of the solution by a trick, as in § 22.

The creator of the theory of imaginary elements was *Staudt* [*Beitr. z. Geom. d. Lage*, here one must consider both the first book (1856) and the second one (1857)]. He proceeded in a purely synthetic way; his main achievements were the interpretation of a general, imaginary line in terms of a gathered involution and the separation of the conjugate-imaginary elements by their sense of traversal. The synthetic theory was developed further by *Lüroth* ("Die Imaginäre in der Geometrie und das Rechnen mit Würfen," Math. Ann. IX), *Sturm* ("Über die v. Staudtschen Würfe," Math. Ann. IX), and for planes, by *Kötter* ("Grundz. einer rein geom.. Theorie der alg. eb. Kurven," Abh. d. Berl. Ak., 1887), and the connection with analytic geometry was exhibited by *Stolz* ("Die geom.. Bedeutung der kompl. Elemente in der anal. Geom.," Math. Ann. IV), *August* ("Unters. über d. Imaginäre in d. Geom.," Progr. d. Friedr.-Realsch., 1872), *Schröder* ("Über v. Staudts Rechnung mit Würfen u. verw. Prozesse," Math. Ann. X), *Segre* ("Le rappr. reali delle forme compl., etc." Math. Ann., Bd. 40). The presentation that was given here is self-contained in many points ([*]). *Ramorino* gave a thorough historical study of this topic in Giorn. di. Mat., v. 35 and 36 (1898).

The analytic-geometric theory of imaginary elements is superior to the purely-synthetic one. The latter was actually brought to a state of completion only for second-order structures by *Staudt*, and can be extended to higher structures only with great effort. With the analytic theory, however, one arrives a more general viewpoint at one stroke: For example, an n^{th}-degree algebraic equation with real coefficients in three variables (in rectangular pointers) is also satisfied by a triple of complex values. We attribute the totality of the imaginary points thus defined to an n^{th}-order surface; indeed, it can contain imaginary points exclusively. From the fundamental theorem of algebra, when the contact points are correspondingly counted multiply, every line g will have precisely n points in common with it, in which the possible imaginary ones will appear in conjugate pairs when g is real ([**]). I. e.:

([*]) E. g., the determination of the sense of a general, imaginary line was carried out in a way that was analogous to what was done for the remaining imaginary structures, by which the involutory family of rulings became unnecessary for that purpose. The same thing was treated only by way of an appendix in § 67, once the main goal of the chapter was attained in § 66.

([**]) If a multiple complex root w is present then we will say that the line g has *imaginary contact* with the surface F (except when all partial derivatives of the surface equation are also fulfilled by w). There is nothing difficult about interpreting this situation geometrically: If the line g' at a neighboring position moves into g then several involutions on g' that are defined by F must exist, that will coalesce for the position g, while their powers will also become equal (which can first happen when $n \geq 4$). It would be desirable to separate the case of imaginary contact from the case of imaginary singular points by geometric information.

Theorem 146: *The number r of the real points of intersection and the number s of the elliptic involutions that are determined by an algebraic surface of order n on a general, real line are related by:*

$$r + 2s = n.$$

There is nothing left to prove for this theorem. One can only further demand that the determination of the involutions can also be carried out geometrically (this will happen for $n = 2$ in the next paragraph). Analogously, the theorem on the number of points of intersection of three algebraic surfaces will now have a well-defined geometric meaning, etc.

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§ 72. The line-geometric representation of a second-order surface. Its imaginary elements.

a) (*) Let $a_{\lambda\mu}$ be real quantities, for which we assume that:

(68) $$a_{\mu\lambda} = a_{\lambda\mu} .$$

If one denotes:

$$\sum_{\lambda=1}^{4}\sum_{\mu=1}^{4} a_{\lambda\mu} x_\lambda x_\mu = F(x), \qquad \frac{1}{2}\frac{dF}{dx_\lambda} = \sum_{\mu=1}^{4} a_{\lambda\mu} x_\mu = F_\lambda(x)$$

then the identity will exist:

(69) $$F(x) = \sum_{\lambda=1}^{4} x_\mu F_\lambda(x),$$

and

(70) $$F(x) = 0$$

will represent a second-order surface F_2. For the sake of intuition, we would like to look for its imaginary points first under the assumption that it also possesses real points, whereby we can assume that the polar theory for second-order surfaces is known. Should:

(71) $$\rho x_\lambda = \xi_\lambda + i\,\eta_\lambda \qquad (\lambda = 1, \ldots, 4)$$

fulfill (70), then one would need to have:

$$\sum_{\lambda=1}^{4}\sum_{\mu=1}^{4} a_{\lambda\mu}(\xi_\lambda + i\eta_\lambda)(\xi_\mu + i\eta_\mu) = 0,$$

which decomposes into the two conditions:

(72) $$F(x) = F(h),$$

(*) On this, cf., *Stolz, loc. cit.*, art. 8.

(73) $$\sum \eta_\lambda F_\lambda(\eta) \quad = 0.$$

The second condition says that the points ξ and η must be conjugate with respect to F_2; i.e., that each of them must lie on the polar plane of the other one (S. S. XXV, § 2). For a fixed value of η, (73) will then represent the equation of the polar plane to the point η; its pointers will then be:

(74) $$\sigma u_\lambda = \sum_\mu a_{\lambda\mu} x_\mu ,$$

if we write x, instead of η, and switch λ, μ.

Space will be divided into two domains by F_2, and we already know from Theorem 146 that ξ and η will lie in the same domain (for positively-curved surfaces, they will lie outside of it), since otherwise their connecting line g would have to intersect it at a real point. Thus, $F(\xi)$ and $F(\eta)$ will have the same sign in any case, and (72) can be fulfilled when one multiplies the pointers of a point with a suitable constant, with which, from § 62, (71) will just now be connected with a well-defined involution:

(75) $$\rho x_\lambda = \nu\, \xi_\lambda + \nu'\, \eta_\lambda , \qquad \rho x'_\lambda = \nu'\, \xi_\lambda - \nu\, \eta_\lambda .$$

We would like to show that this involution is identical with the one that is defined on g by F_2 when one associates the points that are conjugate relative to F_2 with each other. If one substitutes x'_λ for η_λ and x_μ for ξ_μ, as in (75), in the left-hand side of (73) then one will get:

$$\sum_\lambda (\nu'\xi_\lambda - \nu\eta_\lambda) \sum_\mu a_{\lambda\mu} (\nu\xi_\mu + \nu'\eta_\mu)$$

or

$$\nu\,\nu' \,[F(\xi) - F(\eta)] - \nu^2 \sum \eta_\lambda F_\lambda(x) + \nu'^2 \sum \xi_\lambda F_\lambda(\eta),$$

and this will, in fact, vanish identically due to (72) and (73) (the two sums differ only by the ordering of the terms). One calls the reciprocal transformation that is defined by a second-order surface, which associated any point with its polar plane, a (spatial) *polar system*. What we just proved and the dual version of it (*) can then be expressed as:

Theorem 147: *The possible imaginary points of intersection of a line g (or contact plane through g) with a second-order surface F_2 are represented by the involution that the polar system of F_2 cuts out from g (is determined around g as its axis, resp.).*

This theorem also keeps its sense and validity for the case in which the form F_2 is definite, so F_2 will contain no real points. A well-defined polar system will also be linked with F_2 then, by which, the imaginary surface will be represented in a way that is similar to the way that an imaginary element is represented by an involution: Namely, we

(*) We understand a contact plane of F_2 to mean any plane whose pointers fulfill the equation for F_2 in plane pointers; we can also perform the conversion without exhibiting that equation.

start with equations (74). They associate every point x in space with a plane u, and when one solves for x:

$$(76)\qquad \rho\, x_\lambda = \sum_{\mu=1}^{4} A_{\mu\lambda} u_\mu\,,$$

one will also associate every plane u with a point x. Due to (68), we can also write:

$$(77)\qquad \rho\, x_\lambda = \sum A_{\lambda\mu}\, u_\mu\,,$$

instead of (76). Furthermore, if the point y corresponds to the plane v then:

$$(78)\qquad \sigma\, v_\lambda = \sum a_{\lambda\mu}\, y_\mu\,,$$

so we would like to assume that v and x are incident, thus:

$$(79)\qquad \sum v_\lambda\, x_\lambda = 0,$$

and show that the corresponding elements y and u will also be incident then. It follows from (78) and (79) that:

$$\sum a_{\lambda\mu}\, y_\mu\, x_\lambda = 0,$$

which we can write as:

$$\sum a_{\lambda\mu}\, x_\mu\, y_\lambda = 0,$$

according to (68), or:

$$\sum u_\lambda\, y_\lambda = 0.$$

If x then moves in the fixed plane v then u will rotate around the fixed point y. If x likewise moves around another fixed plane w then u must rotate around another fixed point z – i.e., around a line – which one can, moreover, infer immediately from the linearity of the transformation. The lines in space will also be associated with each other in this way. A plane v will then always be associated with the same point, regardless of whether one carries out this association immediately using equations (77) or first chooses three points in it and then intersects their associated planes. Due to this property, the reciprocal transformation (i.e., correlation) that is defined by equations (74) will be *involutory,* in contrast to the general correlation that is defined by the same equation when one drops the conditions (68).

One determines the points in space that lie in the associated planes (viz., the *ordering points*) from the equation:

$$\sum u_\lambda\, x_\lambda = 0,$$

which agrees with:

$$F(x) = 0,$$

due to (74) and (69). An involutory correlation is also called a *polar system*, and the locus of the ordering points of this polar system will be a second-order surface. Both of them depend upon only the $a_{\lambda\mu}$, and one can write down the equation of the surface immediately for a given polar system, or conversely. However, the polar system and the calculations that led to Theorem 147 are completely independent of the reality of the ordering surface.

We would like to provide ourselves with an intuitive picture of a polar system with no real ordering surface (*) and assume a rectangular pointer system. Any definite form in three variables can then be put into the form:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}+1$$

by a pointer transformation (except for a factor), so the equation of the imaginary surface can be put into the form:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}+1 = 0. \tag{80}$$

The associated polar field is defined by:

$$u = \frac{x}{a^2}, \qquad v = \frac{y}{b^2}, \qquad w = \frac{z}{c^2}. \tag{81}$$

Along with the above, we consider the ellipsoid:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1 = 0 \tag{82}$$

and the associated polar field:

$$u' = -\frac{x}{a^2}, \qquad v' = -\frac{y}{b^2}, \qquad w' = -\frac{z}{c^2}. \tag{83}$$

One sees that the planes u, v, w and u', v', w', which correspond to the same point in both cases, lie parallel to each other at the same distance on both sides of the origin; thus:

Theorem 148: *A spatial polar system with no real ordering surface emerges from the polar system of an ellipsoid when one leaves the points of space unchanged and transforms the plane in a manner that is centrally-symmetric with respect to the center of the ellipsoid (or conversely).*

In particular, if $a = b = c = r$ then we will call the surface:

$$x^2 + y^2 + z^2 = - r^2$$

(*) We consider only the case in which the determinant $| a_{\lambda\mu} |$ does not vanish. In the other case, the "imaginary cone" will be represented a polar system that one obtains when one projects a planar polar field with no real ordering curve from a point.

an *imaginary cone*. Its polar system is characterized completely by the fact that any point and its associated plane lie along the same line such that their distances from the origin are δ, δ', resp., and one has $\delta \cdot \delta' = - r^2$ (*).

If one intersects a (spatial) polar system with a plane E then one will obtain a polar field whose ordering curve is real or imaginary according to whether E does or does not cut the ordering surface of the polar system in real points, respectively. Since there is only one kind of second-order imaginary curve (except for pairs of imaginary special lines), as would emerge from the analytical representation, one will obtain the same polar field that would emerge from intersecting a second-order imaginary surface with a plane as the one that would result from intersecting a *real* surface of positive curvature with an unreal plane.

b) We would next like to represent the totality of tangents to a second-degree surface with an equation in line pointers. To that end, we represent the transformation of the polar system in line pointers when we set:

$$\sigma u_i = \sum_\lambda a_{i\lambda} x_\lambda , \qquad \sigma' v_k = \sum_\mu a_{k\mu} y_\mu .$$

These equations are just (74) in a different notation. The connecting line $g \equiv (x, y)$ corresponds to the line of intersection $g' \equiv (u, v)$. If we set:

$$\pi_{\lambda\mu} = x_\lambda y_\mu - x_\mu y_\lambda , \qquad p_{ik} = u_i v_k - u_k v_i$$

then we can express the axial pointers of g' in terms of the ray pointers of g by precisely the same calculation as in the beginning of § 40:

$$\tau p_{ik} = \sum_{\lambda,\mu} (a_{i\lambda} a_{k\mu} - a_{k\lambda} a_{i\mu}) \pi_{\lambda\mu} , \tag{84}$$

or when we go from symbols with two indices to ones with one index, as in § 40:

$$\tau p_n = \sum_{\nu=1}^{6} \mathfrak{p}_{n\nu} \pi_\nu \qquad (n = 1, \ldots, 6). \tag{85}$$

The tangents to the surface are characterized by the fact that their polars intersect. We must now write the incidence condition as:

$$\sum_{\mu=1}^{6} p_\mu \pi_\mu = 0. \tag{86}$$

(*) Analogously, we call the curve:

$$x^2 + y^2 = - r^2$$

an *imaginary circle*. Its polar field is likewise characterized, except that one has to say "associated line" instead of "plane." A sphere will always be cut by a plane in a (real or imaginary) circle.

From (85) and (86), we will obtain:

$$(87) \qquad \sum_{\mu=1}^{6}\sum_{\nu=1}^{6} \mathfrak{p}_{\mu\nu}\,\pi_\mu\pi_\nu = 0$$

as the condition for the lines π that cut their own polars, or when written out more thoroughly:

$$(88) \qquad \sum_{i,k}\sum_{\lambda,\mu}(a_{i\lambda}a_{k\mu} - a_{k\lambda}a_{i\mu})\,\pi_{ik}\pi_{\lambda\mu} = 0.$$

Due to the symmetry of the matrix a_{ik}, we will have:

$$(89) \qquad \mathfrak{p}_{\nu\mu} = \mathfrak{p}_{\mu\nu}$$

in (87). This is the *line-geometric representation* (*) *of the second-order surface* F_2 :

$$(70) \qquad \sum a_{ik}\, x_i\, x_k = 0.$$

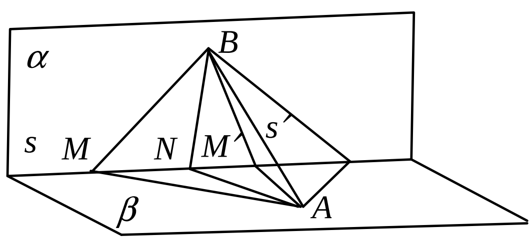

Figure 64.

We now also define the *tangents to the surface* F_2 – for imaginary pointers, as well – to be all lines whose pointers fulfill equation (87). In order to characterize this geometrically, we can go back to the origin of this equation, instead of appealing to it directly: For imaginary lines, (85) will decompose into two equations that say that the representative involutions of polar lines are also polar to each other (**). With that, we know how we have to go about looking

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(*) If a polar tetrahedron of the surface is chosen to be the basic tetrahedron then we can set $a_{ii} = a_i$, while all of the remaining a vanish (Killing, *Lehrb. d. anal. Geom.* II, § 14). Equations (74), (84), (88) will then assume the simple forms:

$$(74a) \qquad \sigma\, u_i = a_i\, x_i\,,$$

$$(84a) \qquad \tau\, p_{ik} = a_i\, a_k\, \pi_{ik}\,,$$

$$(88a) \qquad \sum a_i\, a_k\, \pi_{ik}^2 = 0.$$

(**) Namely, if π is a general, imaginary line $\pi'_\nu + i\pi''_\nu$ then it will go to $p'_n + ip''_n$, where:

$$(90) \qquad \tau\, p'_n = \sum \mathfrak{p}_{n\nu}\;\pi'_\nu\,.$$

The π'_ν are pointers of a twist $\mathfrak{G}$ with the equation:

$$(91) \qquad \sum \pi'_\nu\;u_{n+3} = 0.$$

One must show that the p' are pointers of the twist that corresponds to $\mathfrak{G}$ in the polar system. Since the polars correspond reciprocally, the inverse transformation to (85) will have the same form as (85) itself (cf., the similar state of affairs in § 57). We thus do not need (85) in order to solve for the π, but only have to for the polar to an imaginary line. Since (86) is also true for imaginary lines, we can say that in general the tangents to the surface F_2 are characterized by the fact that they intersect their own polars. The point of intersection must always belong to F_2 in its own right, since this was true for real lines initially. If one calculates the pointers of that point as functions of π, as in § 39, *b*), and substitutes them into (70) then that equation must be an identity between the π that is also true when complex numbers are substituted for real ones. Similar conclusions are true for the connecting plane E and the last part of the following theorem:

Theorem 149: *The lines g whose pointers satisfy the line-geometric equation of a second-order surface* F_2 *are the ones that cut their own polars* g'. *The point of intersection S lies in* F_2, *while the connecting plane E contacts* F_2 . *S and E correspond to each other in the polar system; the two points of intersection of g or* g' *will also coalesce in S.*

With the help of this theorem, we can now find the imaginary tangents to an F_2 by a simple argument:

I) Let S be real, and therefore E, as well. If we choose S to be a real point on F_2 and choose E to be the contact plane at S then the conditions for Theorem 149 will be fulfilled when we choose an *arbitrary* (elliptic) ray involution in the pencil of rays (S, E). Any such special, imaginary line g will thus contact F_2 . However, when F_2 is positively curved, two of these lines γ, γ' will be distinguished, namely, the ones whose involution is the one that is defined by the polar system of F_2 in the pencil (S, E). If we cut γ or γ' with

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switch π and p, although simultaneously we should worry that left axial pointers and right ray pointers will once more come about. However, if we would like to still keep the same pointers then we would have to write:

(92) $$\pi_{n+3} = \sum_{\nu} \mathfrak{p}_{n\nu}\, p_{\nu+3} .$$

If we substitute κ in place of π and q in place of p in this and then substitute the expressions in (91) then we will obtain the relation:

(93) $$\sum \pi'_n \sum \mathfrak{p}_{n\nu}\, q_{\nu+3} = 0,$$

which is satisfied by the totality of all lines q that correspond to the rays κ of the three $\mathfrak{G}$. If we write (93) in the form:

$$\sum a_\nu\, q_{\nu+3} = 0$$

then we will have:

$$a_\nu = \sum_n \pi'_n \mathfrak{p}_{n\nu} ,$$

or due to (89):

$$a_\nu = \sum_{n=1}^{6} \mathfrak{p}_{\nu n} \pi'_n .$$

However, the right-hand sides of these systems differ from (90) only be the exchange of n and ν; thus, the a will be proportional to the p', which completes the proof.

an arbitrary plane then, from Theorem 147, the point of intersection will also belong to F_2. *The lines* γ, γ' *will then lie completely on* F_2 .

II) Let S be imaginary, and therefore E, as well. One can then have:

a) g, g' are special. They must then lie as in the last case of § 65, *c*). We can then choose S to be an arbitrary, imaginary point of F_2 (which is represented by M, N, M', N' in Fig. 64), and the plane α of g can be chosen to go through the carrier s of S arbitrarily; everything else will be determined them. Namely, since S and E correspond to each other, the polar s of s' must be the axis of E. s', α intersect in a point B whose polar plane β will go through s. The point $A \equiv (s', \beta)$ will correspond to α. The two lines g, g' will then be represented by the involutions (B, S) and (A, S). If F_2 is real then at least one of the planes α, β must be real. However, the argument will also be true for an imaginary F_2 . If F_2 is positively-curved then among the positions of α there will be two distinguished ones, namely, the real contact planes to F. g will coincide with g' for them, and as in I), we will come to two lines through S that lie completely on F_2 .

Since an imaginary line that is incident with a plane is special, and the second-order curves can be obtained as plane sections of the second-order surface, the imaginary tangents to a second-order curve will also be obtained by what we did up to now.

β) g, g' can be general. The first or the second case in Theorem 134 can then be relevant – i.e., the associated nets of rays $\mathfrak{N}$, $\mathfrak{N}'$ can have either two rays a, b in common or just one a.

In the first case, let a be the carrier of S, so b will be the axis of E. From the last part of Theorem 149, a and b must correspond to each other in the polar system. If we then choose an imaginary point S on a given F_2 and then choose an arbitrary imaginary point $T \equiv (A, B, A', B')$ on the polar b of its carrier a then the elements on b that are conjugate to the elements A, B, A', B' of the involution T will determine an involution T'. The polars to the connecting line $g \equiv (S, T)$ will be determined by the two plane involutions (b, S), (a, T'), or also by its reciprocal axis intersection S, T'. Thus g, g' will, in fact, fulfill the condition of Theorem 149. If we choose T to be an imaginary point of F_2 that lies in b, in particular (this is not possible for positively-curve surfaces), then g' will be identical with g, and g must lie on F_2 entirely. In fact, this will be true for not only real lines, but also more generally for lines that are similar to the ones that were used in Theorem 149.

In the second case of Theorem 134, a is the carrier of A, as well as the axis of E, so it must correspond to itself in the polar system and lie upon F_2 ; this case can come about only for ruled surfaces. We shall investigate when the connecting line g of S with another imaginary point T of space is a tangent to F_2 . Let the carrier of T be t, and let its polar be t'; g' is determined by E and the plane involution (t', T). If g, g' are supposed to intersect then the latter involution must also cut out the involution S on a, since a, by assumption, is the only common ray of the net g, g'. If S is then chosen such that one can still choose the carrier t of T arbitrarily then T itself will be cut out of t by the plane (t', S) (*).

We direct our attention to the general, imaginary line g that lies completely on a ruled surface F_2 . If we choose an imaginary point S on a ray a of a family of rulings $\mathfrak{R}$ of F_2 then one such line will already be determined by it. S will then determine an involution in the guiding family $\mathfrak{L}$ of $\mathfrak{R}$, so by Theorem 137, it will determine an imaginary line γ. Two generators of F_2 will go though S, namely, γ and the real carrier a of S. If we choose all possible imaginary points S on a certain ray a then we will obtain ∞^2 lines γ, they will all be general, imaginary lines that lie on F_2 already. If we then determine a line γ that lies upon F_2 in the most general way such that we choose an arbitrary imaginary point S' of F_2 (whose carrier does not therefore lie upon F_2) then, from the discussion of the first case of b), we will meet up with the choice of sense on the polar t' to t. t will then determine the ray net $\mathfrak{N}$, $\mathfrak{N}'$ that t' also belongs to, from § 66, h), by means of the one, as well as the other, family of rulings on F_2, respectively. The two lines that belong to $\mathfrak{N}$ and $\mathfrak{N}'$ will then lie entirely upon F_2, since they will include not just S', but also ∞^1 imaginary points whose carriers lie upon F_2 .

We have thus not only proved the following theorem, but also made its content completely intuitive:

Theorem 150: *Two generators also go through every imaginary point P of a second-order surface F_2 . When F_2 is imaginary, they will generally be imaginary. When F_2 is real, they will be special or general imaginary, according to whether F_2 is positively or negatively curved, resp. The one will be general imaginary and the other one real in the last case only when the carrier of P lies upon F_2 .*

The splitting of the imaginary generators into two families is also easy: For the positively-curved surfaces, in order to specify the one family, one can establish a well-defined sense of rotation for the associated ray involutions everywhere – e.g., the positive one – when one considers them from the outside of the surface. For a ruled surface F_2, from the argument that we just presented, we will obtain every imaginary generator γ through an involutory family of rulings $\mathfrak{R}$ that lies upon F_2 ; γ must be counted as being in the family $\mathfrak{R}$. That agrees with the fact that the real generator of the other family is attached to an imaginary point whose carrier lies upon F_2, in this way of determining things. For positively-curved surfaces, two conjugate-imaginary generators will belong to different families, while for ruled surfaces, they will belong to the same family.

Finally, for an imaginary F_2 , let the central point of the involution of one of its imaginary points J be linked with the polar t' to the carrier t of J by a plane ε. If one chooses the one imaginary point of F_2 to be on t then a positive sense of rotation will also be fixed by its sense in ε, so the one imaginary point of F_2 will thus be distinguished on t'. If we connect an imaginary point of F_2 with a certain second point in this way then we

(*) This corresponds completely to the real behavior: By assumption, E is determined along with S; i.e., one knows a pencil (S, E) that g belongs to. A line of the pencil is determined by the condition that it cut a second line t.

will obtain the generators of the one family. Two conjugate-imaginary generators will belong to the same family here.

F*or all* F_2 , one has that one of the two generators through a point will belong to one family and the other one will belong to the other family.

Theorem 151: *When one considers the imaginary elements of a second-order surface, it will have* ∞^4 *points that can be arranged into* ∞^2 *groups of* ∞^2 *lines in two different ways.*

The decision about whether a given general, imaginary line does or does not contact an F_2 can be made as in § 66, *g*), once one has constructed its polars.

It is self-explanatory how part of the foregoing investigation can be specialized for the plane and the imaginary second-order curves – such as, e.g., the way that the polar field of one of them can be obtained from that of an ellipse.

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§ 73. The imaginary sphere-element at infinity.

We now consider the imaginary elements at infinity of a second-order surface, and in particular, a sphere. A polar field is determined by F_2, as it is in any plane, and therefore in the plane at infinity *U*, as well – i.e., any direction in space will be associated with an attitude (*Stellung*) (viz., the direction of a diameter of the attitude of the plane of the conjugate diameter). We shall pass over the paraboloids for which *U* is a contact plane here. If F_2 is a sphere *K* then any direction will be perpendicular to the associated attitude. The polar field will then be independent of the location of the sphere in this case, and will remain unchanged for all ∞^7 motions and similarity transformations of *K*. For that reason, one says that all spheres cut *U* (and each other) in *the same* imaginary sphere-circle. By abuse of terminology, it will be called "the imaginary circle," regardless of that fact that there are naturally other imaginary sphere-circles at finite points (viz., ∞^3 in each plane).

If one cuts the polar system of a sphere with a plane *E* then two points *J*, *J′* of the circle of intersection will lie on each line of *E*, and therefore also on the line *u* at infinity [cf., the last remark in § 72, *a*)]. The involution by which *J*, *J′* is represented will be cut out of *u* by *any* rectangular ray involution in *E*, and will be independent of which (real or imaginary) sphere one has intersected with *E*. For that reason, one says that all (real or imaginary) circles in a plane *E* go through the same two infinitely-distant points *J*, *J′* of *E*, namely, the *imaginary circle points* of *E*. They will likewise be the points of intersection of *E* with the imaginary sphere-circle.

Since the center *M* of a circle *k* in *E* is the pole of *u*, from § 72, *b*, II, *α*), the rectangular ray involution of the pencil (*M*, *E*) will represent two tangents to *k*; they have only *J*, *J′* in common with *k*. One calls them the two tangents at the imaginary circle points (at infinity). They will also be tangents to any cone through *k*.

The totality of all special, imaginary lines that cut the imaginary sphere-circle $\mathfrak{K}$ is represented by all rectangular ray involutions; there are ∞^5 of them. Any ∞^3 of them have the same position and will go through the same point of $\mathfrak{K}$. From Theorem 143, they will fulfill the equation:

$$\sum_{\lambda=1}^{3} q_\lambda^2 = 0.$$

Since the polar of a diameter d of a sphere lies at infinity, a rectangular plane involution with the axis d will represent a contact plane to the sphere whose contact point lies at infinity, and thus on $\mathfrak{K}$. Such a plane will itself contact $\mathfrak{K}$. It will therefore also have one such plane in common with every sphere. The planes that contact the imaginary sphere-circle will then be represented by all rectangular plane involutions. Starting from here, we can arrive at an analytic representation of $\mathfrak{K}$ immediately: Namely, from Theorem 143, the characteristic feature of a rectangular plane involution reads:

$$u^2 + v^2 + w^2 = 0 \tag{94}$$

in rectangular pointers u, v, w. This will then be the equation for $\mathfrak{K}$ in plane pointers, as well.

In order to obtain a general, imaginary tangent t to a sphere, from § 72, *b*, II, β), we can choose its imaginary contact point A arbitrarily, and connect it with an arbitrary imaginary point of the polar a' of the carrier a to A. In particular, if we choose A to be at infinity then a' will be a diameter of the sphere, and thus perpendicular to the position of A, and likewise the middle plane of the net that belongs to t. It will follow from this that a' is the principal ray of the net and that the net is a rotational net.

However, even if we do not relate to a sphere then we will see immediately that only one imaginary line of a *rotational* net has the property that it meets the imaginary sphere-circle. This then says nothing else but that its point of intersection with the plane at infinity is represented by the intersection of a rectangular ray involution. However (Theorem 141), the rotational nets are characterized by the equation:

$$\sum_{\lambda=1}^{3} q_\lambda^2 = 0.$$

Since the *special* imaginary lines of intersection of $\mathfrak{K}$ also fulfill this equation, we can say (*Lindemann*, Math. Ann., Bd. 7):

Theorem 152: *The equation of the imaginary sphere-circle reads* (*):

(*) A curve can be represented in line pointers by the totality of its lines of intersection, which defines a complex. When one considers the imaginary elements, a complex will contain ∞^6 lines. In fact, there are ∞^6 rotational nets.

(95) $$q_1^2 + q_2^2 + q_3^2 = 0$$

in (rectangular) line pointers and:

$$u^2 + v^2 + w^2 = 0$$

in plane pointers.

A curve is represented by two equations in point pointers. Since 𝔎 lies in the plane at infinity, and its points possess no actual representation in rectangular point pointers, 𝔎 can also experience no actual representation in terms of such things. However, 𝔎 can also be characterized completely as the intersection of the plane at infinity with a conic surface that 𝔎 projects from any finite point – e.g., the origin. Since the generators meet the conic surface 𝔎, their pointers must fulfill (95). We can then translate this equation into point pointers immediately if we set:

$$q_1 = x' - x, \qquad q_2 = y' - y, \qquad q_3 = z' - z,$$

as in § 33, and take x, y, z to be the fixed pointers of the vertex of the conic surface and consider x', y', z' to be the running pointers, or conversely.

Since an imaginary line 𝔖 cuts a rotational net 𝔎 and the point of intersection also belongs to any sphere K, 𝔖 will contact K at a point at infinity, if it contacts K at all; i.e., no imaginary line that contacts K at a finite point will correspond to a rotational net. This should not be surprising, since one already sees that for the special, imaginary lines *any* involution of a pencil of real tangents will represent a tangent to the sphere, so the associated involution will thus, by no means, always be a rotational structure.

From equation (88.a), we shall now exhibit the equation of the sphere:

(97) $$x^2 + y^2 + z^2 = r^2$$

in line pointers. We think of (97) as being made homogeneous by a fourth variable t and must then let the variables:

$$t, x, y, z$$

correspond to:

$$x_1, x_2, x_3, x_4,$$

respectively (§ 31), in order to remain in agreement with the chosen notations. This will give:

(98) $$q_4^2 + q_5^2 + q_6^2 - r^2(q_1^2 + q_2^2 + q_3^2) = 0.$$

For a sphere tangent 𝔖 that meets 𝔎, one will have:

$$q_1^2 + q_2^2 + q_3^2 = 0,$$

so one will also have:

$$q_4^2 + q_5^2 + q_6^2 = 0.$$

In fact, we already know that the principal net of such a tangent will go through the origin, and that from § 68, *b*), the rotational net will also fulfill the last equation for that position.

Practice problems:

47. Find the analytical way of characterizing the special position of an imaginary point with respect to a non-incident real line that was mentioned in the conclusion (rem.) to § 64.

48. Represent the gathered involution as a point transformation on the basis of equation (35) in § 65, *f*), rem.

49. Carry out the constructions that are dual to the ones in § 66.

50. Convince yourself that the double rays of an elliptic-involutory family of rulings are independent of which pair one bases the representation upon, instead of p, p' (§ 67).

51. Calculate the rectangular pair of the involution of an imaginary plane or special, imaginary line from the given pointers.

52. Exhibit the equation for a sphere with its center at (a, b, c) and a radius of r in line pointers.

53. A general, imaginary line $\mathfrak{S}$ is given in the most special position with respect to the pointer system, namely [§ 68, *b*)]:

$$q_1 = 1, \qquad q_2 = m\,i, \qquad q_3 = 0,$$
$$q_4 = n\,m, \qquad q_5 = n\,i, \qquad q_6 = 0.$$

Find the geometric locus of the centers of all spheres that contact $\mathfrak{S}$.

54. Exhibit the line equations of:

α) The rotational paraboloid $x^2 + y^2 = 2z$.

β) The equilateral hyperbolic paraboloid $xy = z$.

55. What is the relationship between second-order surfaces of rotation and imaginary sphere-circles?

56. Find the imaginary generators of a ruled surface F_2 that correspond to a rotational net.

Chapter VI

The manifold of linear complexes, with applications to mechanics and the theory of motion

§ 74. The axis manifold of the complexes of a pencil.

We have already learned several theorems on pencils of complexes (viz., Theorems 97, 98, 114, 115, 127), because they were useful in the investigation of ray nets. We now ask what would be the locus of all axes of a pencil of complexes and their associated pitches (*Steigungen*). It is irrelevant whether we base our arguments upon the abstract concepts of twist and screw or the concrete ones of dyname and winding, since the laws of composition and decomposition are the same. For the sake of intuitive appeal, we might appeal to one or the other, as the situation dictates, and when two theorems correspond to each other under duality (§ 18) in the theory of motion and mechanics, we will mostly state just one of them.

We consider two dynames $\mathfrak{A}$ and $\mathfrak{B}$ with the pointers a_n and b_n . The ratios of those numbers are also pointers of the associated complexes A and B. The dynames of the form:

$$c_n = \lambda\, a_n + \mu\, b_n \qquad (n = 1, \ldots, 6) \tag{1}$$

define a doubly-infinite manifold to which merely ∞^1 linear complexes then belong. Then, if:

$$c'_n = \lambda'\, a_n + \mu'\, b_n ,$$

and if:

$$\lambda' : \mu' = \lambda : \mu ,$$

then one will obtain the dyname $\mathfrak{C}'$ from $\mathfrak{C}$ by multiplying by a numerical factor. Thus, the pointers of the complex C will also be represented by (1), which belongs to all ∞^1 dynames for which $\mu : \lambda = \kappa$ has the same value. C will run through a pencil of complexes when one changes κ. We now first assume only a temporary orientation, such that:

I) The rod parts of the dynames $\mathfrak{A}$ and $\mathfrak{B}$ are non-zero.

Their directions (*) will then determine a common location σ in which lies the angle ϑ that the complex axes define with a direction of σ. One will find the direction of the rod part of the restultant dyname of two given ones when one combines the rod parts of the given ones as if they were vectors (§ 14). The values κ and $\tan\,\vartheta$ will then be mutually

(*) If they are parallel then the situation will be similar to the one in II).

associated with each other, and we can employ tan ϑ, instead of κ, as the parameter (*) of the pencil.

II) If the rod part of $\mathfrak{B}$ is zero then $\mathfrak{B}$ can be decomposed into two moments, one of which acts as merely a displacement of the dyname (§ 14).

Here, the axis manifold will then consist of a pencil of parallel rays, and we can employ a linear quantity as a parameter.

III) If the rod parts of both dynames are zero then we will have two moments before us, and all force systems that can be composed from them will again be only moments.

The axis manifold is then to be regarded as the pencil of rays at infinity whose vertex is given by the common direction of both fields. The carrier of the pencil is a singular ray net for which *e* (in the notation of § 54, conclusion) lies at infinity.

In absolutely all cases in which the net is singular, all complexes of the pencil will also be singular (thus, the question of the pitch will go away), and the axis manifold will be a plane pencil of rays. We can ignore this case, since the composition of the given force system will also result from elementary rules. The only cases that remain to be actually investigated are:

a) The carrier of the pencil of complexes is a hyperbolic ray net, and both focal lines lie at infinity.

b) One of them lies at infinity.

c) The carrier is an elliptic ray net.

d) The carrier is a special net whose focal line lies at infinity.

e) Its focal line lies at infinity.

We shall treat cases *a*), *c*), *d*) together and define the pointer system as simply as possible relative to the ray net, namely, as we did in § 55, *a*), *c*), *d*). We can then write equations (79), (84), (87) of § 55 in the common form:

$$\begin{aligned} q_4 + \mathfrak{k}\, q_1 &= 0 \qquad \text{(complex } A\text{)}, \\ q_5 + \mathfrak{k}' q_2 &= 0 \qquad \text{(complex } B\text{)}, \end{aligned} \tag{2}$$

where in cases *a*), *c*), *d*) one will have:

(*) From now on, in order to guard against confusion with the "parameter" of a parametric representation, we will always call the quantity $\mathfrak{k}$ that was defined in §§ 1 and 15 the "pitch" of the screw, winding, or dyname.

$$a)\quad \mathfrak{k} = -\,c \tan \alpha, \qquad \mathfrak{k}' = c \cot \alpha,$$
$$c)\quad \mathfrak{k} = cm, \qquad \mathfrak{k}' = c : m,$$
$$d)\quad \mathfrak{k} = 0, \qquad \mathfrak{k}' = K,$$

respectively. Therefore, from § 48, equation (38), $\mathfrak{k}$, $\mathfrak{k}'$ will actually be the pitches of the twist in question. It is characteristic of the cases that $\mathfrak{k}$ and $\mathfrak{k}'$ will have opposite signs in *a*) and the same signs in *c*). If the pitches are given for two twists whose axes intersect at right angles then one can calculate the constants *c* and *a* (*c* and *m*, resp.), which are characteristic of the form and magnitude of the associated net, from equations *a*) or *b*), resp.:

$$a')\quad c^2 = -\,\mathfrak{k}\,\mathfrak{k}', \qquad \tan^2 \alpha = -\,\mathfrak{k} : \mathfrak{k}',$$
$$c')\quad c^2 = \mathfrak{k}\,\mathfrak{k}', \qquad m^2 = \mathfrak{k} : \mathfrak{k}'.$$

Whenever *m* and $\tan \alpha$ are always positive, *c* will have the sign of $\mathfrak{k}'$.

A linear complex is determined by its axis and its pitch $\mathfrak{k}$: *If we carry the quantity* $\mathfrak{k}$ *on the axis, with consideration given to its sign* (by which, the sense of rotation of the associated screwing motion will determine the positive direction of the axis), then the complex will be represented by a rod, namely, its *pitch rod* (*). The rods of a pencil of complexes will then define a *rod surface* (§ 43), which we would like to find.

If we compare equations (2) with equation (7) of § 46 then we will see that the axis of *A* coincides with the *X*-axis and that of *B* coincides with the *Y*-axis. We will frequently say that a complex or a screw *lies in a plane* when the representative rod lies in a plane. The possibility of representing any pencil of complexes in the cases *a*), *c*), *d*), of § 55 in the form (2) implies the:

Theorem 153: *Any pencil of complexes whose carrier is not a singular ray net with a middle plane can be defined by the two complexes that lie in the middle plane – viz., the "principal complexes" – whose axes are perpendicular to each other and coincide with the axes of the net for a general net. In the case of a hyperbolic net, the principal complexes are oppositely wound, while in the case of an elliptic net, they will be wound the same; in the case of a special net, one of the principal complexes will be singular.*

For the pencil of complexes (2), one will have:

$$a_1 = 1, \qquad a_4 = \mathfrak{k}, \qquad \text{the remaining } a_n \text{ are zero,}$$
$$b_2 = 1, \qquad b_5 = \mathfrak{k}, \qquad \text{“ \quad “ \quad } a_n \text{ \quad “ \quad .}$$

Therefore, from (1), one will have:

(*) A pitch rod differs from a force rod in that the latter will also have meaning in the absence of a positive direction for its carrier. By contrast, a pitch rod already has an *intrinsically*-determined sign (viz., that of $\mathfrak{k}$). If one then changes the positive direction of the carrier then one must apply the rod pitch to *the same* twist in the opposite sense.

$$c_1 = \lambda, \qquad c_2 = \mu, \qquad c_3 = 0,$$
$$c_4 = \lambda\,\mathfrak{k}, \qquad c_5 = \mu\,\mathfrak{k}', \qquad c_6 = 0.$$

We call the angle that the complex axis defines with the X-direction ϑ, and we know from I) that $\tan\vartheta$ is a suitable independent variable, so we set:

$$\lambda = \cos\vartheta, \qquad \mu = \sin\vartheta,$$

and obtain the pitch $\mathfrak{k}_c$ of C and the points $\mathfrak{a}_i$ of its axis from § 48, equations (38) and (40):

(3) $$\mathfrak{k}_c = \mathfrak{k}\cos^2\vartheta + \mathfrak{k}'\sin^2\vartheta,$$

(4) $$\begin{array}{lll} \mathfrak{a}_1 = \cos\vartheta, & \mathfrak{a}_2 = \sin\vartheta, & \mathfrak{a}_3 = 0, \\ \mathfrak{a}_4 = -(\mathfrak{k}' - \mathfrak{k})\sin^2\vartheta\cos\vartheta, & \mathfrak{a}_5 = (\mathfrak{k}' - \mathfrak{k})\sin\vartheta\cos^2\vartheta, & \mathfrak{a}_6 = 0. \end{array}$$

The rod $\mathfrak{a}$ cuts the Z-axis and is parallel to the XY-plane; in order to find its distance z from that plane, we introduce the expressions for the $\mathfrak{a}$ in point pointers that are given by § 33, equations (24), whereby we immediately set: $x = y = 0$. We obtain, in double agreement:

$$z = (\mathfrak{k}' - \mathfrak{k})\sin\vartheta\cos\vartheta.$$

The equations:

(5) $$\begin{aligned} x &= r\cos\vartheta, \\ y &= r\sin\vartheta, \\ z &= (\mathfrak{k}' - \mathfrak{k})\sin\vartheta\cos\vartheta \end{aligned}$$

will then give a generator of the desired axis surface of the pencil of complexes (2) for each value of ϑ. If one eliminates r and ϑ from (5) then one will obtain the equation of *a third-order ruled surface that is called the "cylindroid"* (*):

(6) $$(x^2 + y^2)\,z = (\mathfrak{k}' - \mathfrak{k})\,xy.$$

According to whether we consider this surface as a line surface or a rod surface, we will distinguish these two constructions, when necessary, as the "line cylindroid" or the "rod cylindroid," resp.; the latter is defined by equations (5) and (3) together.

If $\mathfrak{k}' = \mathfrak{k}$ then it will follow from equation c) that:

$$m = 1.$$

(*) It has other names, among which are the "Plücker conoid" and the "Cayley line surface," that get some degree of use. It was discovered by *Hamilton* (1830), investigated more thoroughly by Plücker (*Neue Geom. des Raumes*, 1868), and most thoroughly by *Ball* in his work *A Treatise on the Theory of Screws* (Cambridge, 1900), which summarized his previous research in the realm of kinematics and mechanics (since 1870). We shall follow the latter work, in part, in § 76.

That is: If a rotation net is the carrier of a pencil then the twist axes will define a plane pencil of rays in the middle plane of the net; from (3), the pitch will then be constant. We see that the rotation net that we spoke of in § 59, *b*, γ) is the only one that belongs to a given twist.

b) In this case, the finite focal line must be a diameter of all twists of the pencil. We can then anticipate that its axes will define a pencil of parallel rays, in general. In order to correctly obtain the pitches, with their signs, then it would be best to start here with the analytical representation of the ray net in § 55, *b*), equations (82): If we first preserve the parameters λ and μ of equation (1) then that will give us, as before:

$$\mathfrak{k}_c = \frac{\mu}{\lambda} \cos \omega. \tag{7}$$

$$\begin{aligned} \mathfrak{a}_1 &= \lambda \cos \omega, & \mathfrak{a}_2 &= 0, & \mathfrak{a}_3 &= \lambda \cos \omega, \\ \mathfrak{a}_4 &= -\mu \sin \omega \cos \omega, & \mathfrak{a}_5 &= 0, & \mathfrak{a}_6 &= \mu \sin^2 \omega. \end{aligned} \tag{8}$$

The middle two equations show that all complex axes cut the *Y*-axis perpendicularly. We may then go to point pointers and immediately set $x = z = 0$ and obtain the distance y of a rod from the finite focal line:

$$y = -\frac{\mu}{\lambda} \sin \omega,$$

and from this:

$$\mathfrak{k}_c = -y \cot \omega. \tag{9}$$

Thus, for $\omega = 0$ it is immediately clear that the rod surface consists of all rods on the focal line, because the net will then consist of all pencils of rays whose vertices lie on the focal line and whose planes are perpendicular to it. With Study, we will call such a net a *normal net* ("Ein neuer Zweig der Geometrie," Jahresber. d. D. Math. Ver., Bd. 11).

e) Here, one obtains:

$$\mathfrak{k}_c = -\frac{1}{K}, \tag{10}$$

(11) $\mathfrak{a}_2 = -\mu K$, $\mathfrak{a}_6 = \lambda$, the remaining $\mathfrak{a}$ zero

from equations (89) of § 55 by the same process. All twists of the pencil will then have the same pitch, and their axes will define a pencil of parallel rays in the *XY*-plane that belongs to the *X*-axis. If one displaces a twist in a direction that cuts its axis perpendicularly then one can, in fact, see immediately that all rays will be parallel to the connecting plane of the axis and that direction will remain the same.

We have now discussed all possible cases, and summarize them as:

Theorem 154: *The axis surface of a pencil of complexes consists of:*

α) *A plane pencil of rods of constant length whose vertex coincides with the center of the net and whose plane coincides with the middle plane when the carrier is a rotation net.*

β) *A plane pencil of rods that are all parallel to the finite focal line b of a hyperbolic net with a focal line at infinity whose position does not intersect b perpendicularly, and whose lengths are proportional to their distance from b when the carrier is that hyperbolic net.*

γ) *A plane pencil of equal and parallel rods that lie in the principal plane of a special net with a focal line at infinity that is perpendicular to the principal direction when the carrier is that special net.*

δ) *A plane pencil of rays whose vertices and planes coincide with the vertices and planes of a singular net when the carrier is that singular net* (*).

ε) *The rods of the focal line of a normal net when the carrier is that normal net.*

ζ) *A cylindroid, in all (three) remaining cases.*

We also call the middle planes of the ray net from which we started the middle plane of the cylindroid [it will coincide with the *XY*-plane in the representation *b*)], and also call the axes of the principal complex the "principal generators" of the cylindroid.

---

§ 75. The cylindroid.

a) When we set $\mathfrak{k}' - \mathfrak{k} = 2h$, the rod cylindroid will be defined by the equations:

$$x = r \cos \vartheta, \quad y = r \sin \vartheta, \quad z = h \sin 2\vartheta, \tag{5}$$

$$\mathfrak{k}_\vartheta = \mathfrak{k} \cos^2 \vartheta + \mathfrak{k}' \sin^2 \vartheta, \tag{3'}$$

and the line cylindroid will be defined by (5) alone or by:

$$(x^2 + y^2)\, z = 2h\, x\, y. \tag{6'}$$

(*) Here, one cannot regard the axis surface as a rod surface, because all rods will have a length is either zero or infinite.

We first discuss the latter: A single parameter h enters into (6′) that remains unchanged when one simultaneously changes $\mathfrak{k}$ and $\mathfrak{k}'$ by the same amount; thus:

Theorem 155: *All line cylindroids are similar to each other, and* ∞^1 *pencils of complexes belong to each of them.*

One infers from (5) that all generators of the cylindroid cut the Z-axis perpendicularly. For $\vartheta = \pi / 4$ and $3\pi / 4$, that will yield the "extreme" generators:

$$z = \pm h, \qquad \frac{y}{x} = \pm 1.$$

Two generators of the cylindroid lie in each parallel plane between the "extreme" planes $z + h$ and $z = - h$; the equation $\sin 2\vartheta = z : h$ will then have two solutions ϑ. No real generators of the surface lie outside of these planes. The Z-axis is a *double line* of it and belongs to it for its entire course. In order to obtain the surface completely, it will suffice to let ϑ increase from zero to π. In order to get an intuitive picture of it, we cut it with a cylinder:

$$x^2 + y^2 = r^2.$$

The curve of intersection on it will be represented by the last of equations (5). If we develop the cylinder onto a plane then we will obtain two complete waves of a sinusoid in the extended sense of the word. If we let r increase for one and the same cylindroid then the height of that wave will remain unchanged, while its length will increase proportional to r.

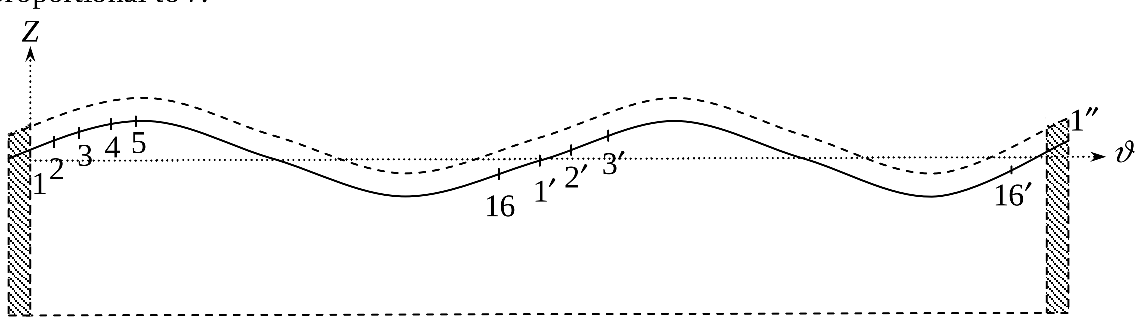

Figure 65.

One can then present a *model* of the cylindroid by the simplest means thus: One draws two waves of the curve:

(13) $$z = h \sin 2\vartheta$$

on a rigid, but still bendable, piece of cardboard in a rectangular system of pointers (ϑ, z) for a well-defined value h, and indeed, we emphasize that in order to obtain a sufficiently large neighborhood of the double line, we must take $h = 3 / 4$, as well as the six-fold scale in Figure 65. In fact, the smaller that one makes h, the larger the neighborhood of the double line that one obtains will be, but the smaller the scale of the model will be. One then divides each quadrant into – say – four equal pieces, sticks a pin through the indicated points 1, 2, 3, …, 16; 1′, 2′, …, 16′, 1″, cuts short the cardboard with the dashed line, bends it around a circular cylinder in such a way that the hatched boundaries overlap, and attaches them together. In order to assure the circular form of the cylinder base, one immediately stiffens the model by putting a circular cardboard frame around the cylinder (Fig. 66) and attaches it. One draws a string on it through 1, 1′, 2, 2′, 3, 3′, 4, etc., in sequence. It is important to choose the length of the string in such a way that the 16 generators distribute themselves on four strings. In Fig. 66, the model is depicted with a small alteration (cf., App. II).

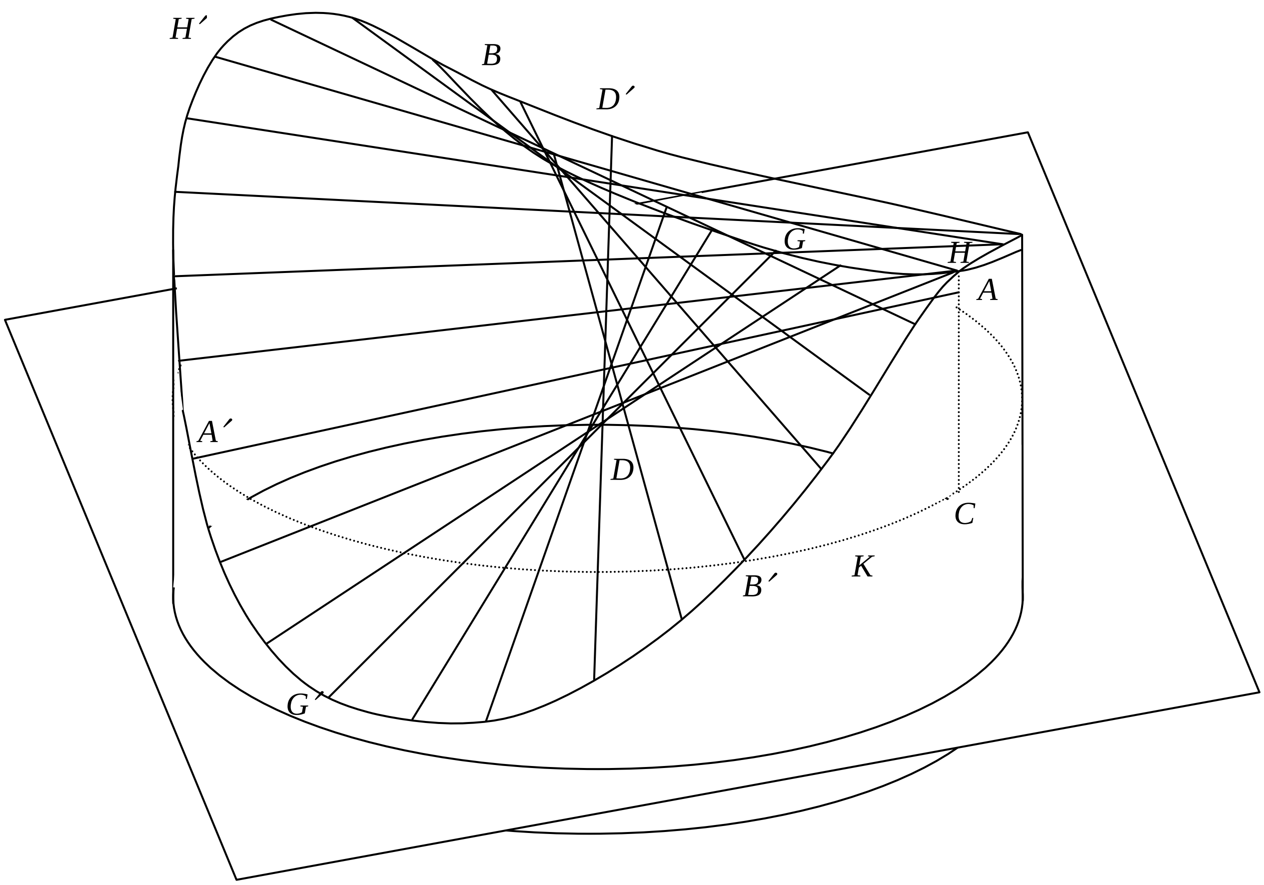

Figure 66.

If one cuts the cylindroid with a plane E that contains a generator then the rest of the intersection must be a second-order curve, and in fact, an ellipse, because it is contained completely in the strips of E between the extreme planes.

We now examine the intersection of a circular cylinder C that contains the double line d as a generator with the cylindroid $\mathfrak{C}$. Since d must be counted twice, the curve of intersection must decompose into d and a planar conic section, which we would like to confirm immediately: If ϑ_0, ρ are the polar pointers of the point of intersection of the cylinder axis with the middle plane then (Fig. 67):

$$r = 2\rho \cos (\vartheta - \vartheta_0)$$

will be the polar equation of the cylinder base. Therefore:

$$(14)\qquad \begin{aligned} x &= 2\rho\cos(\vartheta - \vartheta_0)\cos\vartheta, \\ y &= 2\rho\cos(\vartheta - \vartheta_0)\sin\vartheta, \\ z &= h\sin 2\vartheta \end{aligned}$$

will be the equation of the curve of intersection of C and $\mathfrak{C}$. If one defines:

$$\begin{aligned} &x\sin\vartheta_0 + y\cos\vartheta_0 \\ &= 2\rho\cos(\vartheta - \vartheta_0)\sin(\vartheta + \vartheta_0) \\ &= \rho(\sin 2\vartheta + \sin 2\vartheta_0) \end{aligned}$$

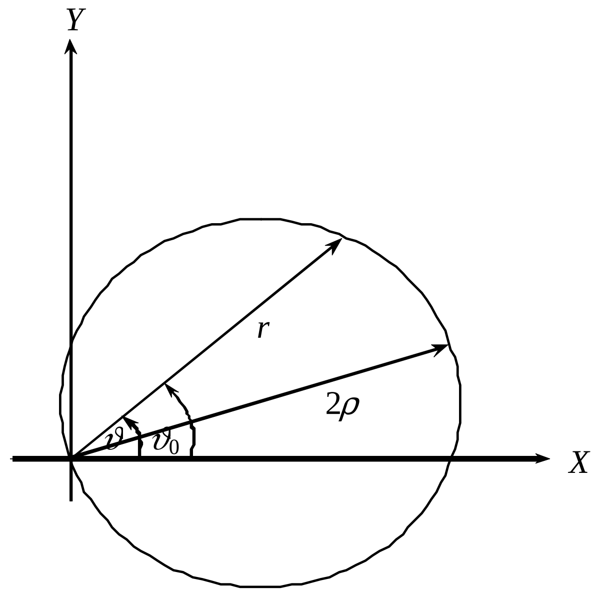

Figure 67.

then one will see that by eliminating J from equations (14) one will obtain a *linear* equation that represents the plane E:

$$(15)\qquad x\sin\vartheta_0 + y\cos\vartheta_0 = \rho\left(\frac{z}{h} + \sin 2\vartheta_0\right).$$

The intersection of C and $\mathfrak{C}$ will also be the intersection of C and E then; i.e., an ellipse. We would like to look for the generator ϑ_1 of C that lies in E. For that, one has to introduce:

$$x = r\cos\vartheta_1, \quad y = r\sin\vartheta_1$$

into (15) and impose the condition on ϑ_1 that the value of z that is calculated from (15) must agree with $h \sin 2\vartheta_1$; however, one sees immediately that the solution is:

$$(16)\qquad \vartheta_1 = -\vartheta_0.$$

If one then rotates C around the double line then the generator that E has is common with $\mathfrak{C}$ will run through the entire cylindroid in the opposite sense. Furthermore, the angle γ that the normal to E defines with the Z-axis will be:

$$(17)\qquad \cos\gamma = \frac{\rho}{\sqrt{\rho^2 + h^2}}.$$

If ρ increases from zero to infinity then the absolute value of $\cos\gamma$ will run through all values that it can possibly assume. Moreover, if one replaces ϑ_0 with:

$$\vartheta_0' = \vartheta_0 + \pi$$

then one will see from (15) that the new plane E' is the mirror image of E in the XY-plane. Thus, one sees from (16) and (17) that all possible positions of E will exhaust the ∞^2 planes that go through the generators of $\mathfrak{C}$.

Theorem 156: *Every plane through a generator of the cylinder intersects it in an ellipse whose projection onto the middle plane is a circle.*

Theorem 157: *If one chooses an ellipse and a generator d on a circular cylinder arbitrarily then the geometric locus of all lines that cut the ellipse and d, and indeed cut the latter perpendicularly, will be a cylindroid.*

This theorem also leads to a simple construction of the cylindroid that is carried out in Fig. 68, and in fact the generator is chosen to go through the endpoint of the minor axis BB' of the ellipse. In fact, one will also obtain yet another cylindroid with that special assumption, because one can set $\vartheta_0 = \vartheta_1 = 0$ in (16). The same thing will follow from Theorem 155, moreover. One can also give preference to a particular angle of inclination of the plane of the ellipse relative to the base k (one can let the latter go through BB'). Each chord $M'N'$ of the base that is parallel to BB' will belong to two equally-high points M, N of the ellipse. If one then makes $BD = M'M$ then DM and DN will be two generators of the cylinder.

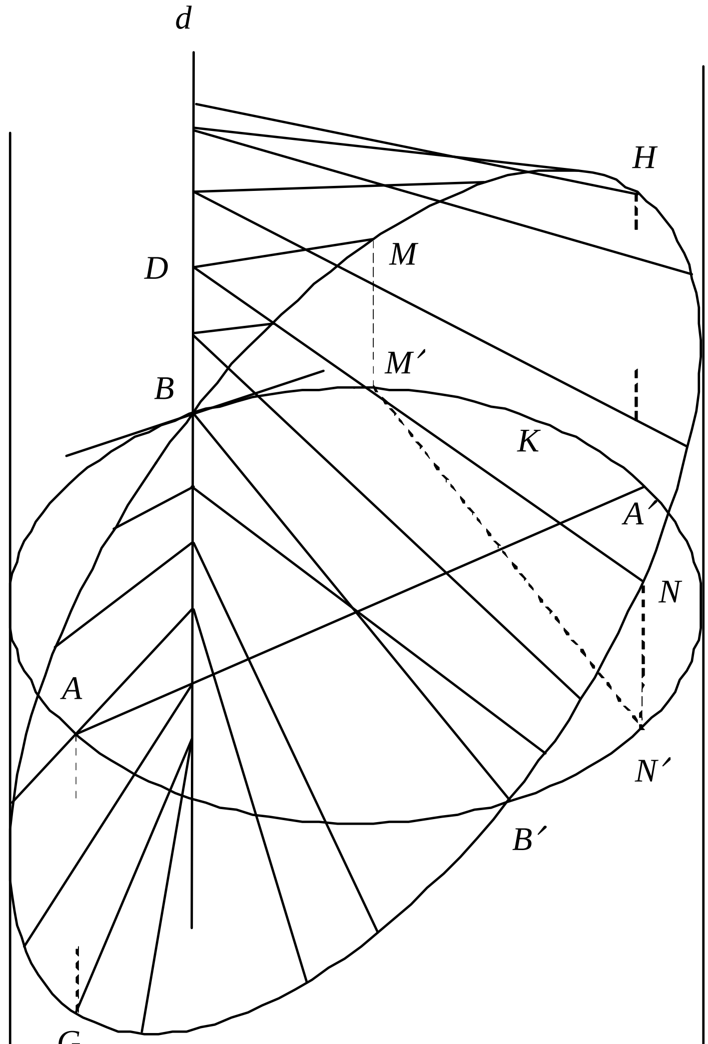

Figure 68.

γ is independent of ϑ_0. If one then rotates C around the double line then the pitch of the plane of the ellipse with respect to the base will remain unchanged, so the ellipse will also be congruent to itself. The cylindroid must then be capable of being generated by a certain motion of that ellipse, or – what amounts to the same thing – the cylinder on which it lies. The motion must be such that every point of the ellipse, and therefore also its projection onto the base – i.e., any point of the base – must describe a straight line. As is known, the motion that has this property consists of a circle K rolling inside of another one of twice the diameter (*). Thus:

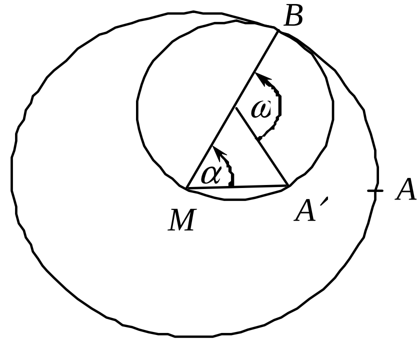

Figure 69.

Theorem 158: *If a circular cylinder C rolls on the inside of another one with twice the diameter then any point of an ellipse that remains fixed on C will describe a generator of one and the same cylindroid* (**).

One can infer some other theorems from the foregoing calculations and constructions. For example: The difference of the squares of the axes of any elliptical section of a cylindroid is a constant = h^2.

Any ellipse that lies on a cylindroid $\mathfrak{C}$ cuts the generator of its plane twice: Once on the double line, and the second time at a point S for which E is the contact plane of $\mathfrak{C}$. Two lines in E that lie completely on $\mathfrak{C}$ will then go through S.

b) We now go on to the examination of the rod cylindroid. If we carry the associated rod from the point of intersection with the double line on any generator then the endpoints of the rod will define the "characteristic curve" on the line cylindroid $\mathfrak{C}$, which is determined uniquely by its projection C. If we set:

$$\cos^2\vartheta = \tfrac{1}{2}(1 + \cos 2\vartheta), \qquad \sin^2\vartheta = \tfrac{1}{2}(1 - \cos 2\vartheta)$$

in (3′) and:

(*) If the circles originally make contact at A (Fig. 69), and if the contact point goes to B under the rolling motion then A must arrive at A'. One must have:

$$\text{arc } AB = \text{arc } A'B,$$

so

$$\omega = \angle A'B = 2\angle AB.$$

However, since one also has $\omega = 2\alpha$, it will follow that:

$$\alpha = \angle AB;$$

i.e., the point A' will be found on the connecting line MA for any position of the inner circle.

(**) If is a useful exercise to visualize this motion and its distinguished phases intuitively with the help of a string model.

$$\tfrac{1}{2}\,(\mathfrak{k} + \mathfrak{k}') = s$$

then C will have the polar equation:

$$\mathfrak{k}_\vartheta = s - h \cos 2\vartheta . \tag{18}$$

One can easily find arbitrarily many points of C with this. One draws (Fig. 70) a circle K with the origin as its center and s as its radius and a circle K' with h as its diameter that contacts the Y-axis at the origin, and whose center lies on the negative or positive side of the X-axis according to whether h is positive or negative, respectively. In order to find a value ϑ for a point of C, one makes the ray 2ϑ cut K' at A and makes $SB = OA$ (with consideration given to the sign), and analogously $S'B' = OA'$. For a certain cylindroid, h will be constant; thus:

Theorem 159: *One obtains all characteristic curves of a line cylindroid from one of them when one displaces each of their points through the same segment on the generator on which it lies, where the sense of the displacement on a generator can be chosen arbitrarily and can be determined by continuity on the remaining ones.*

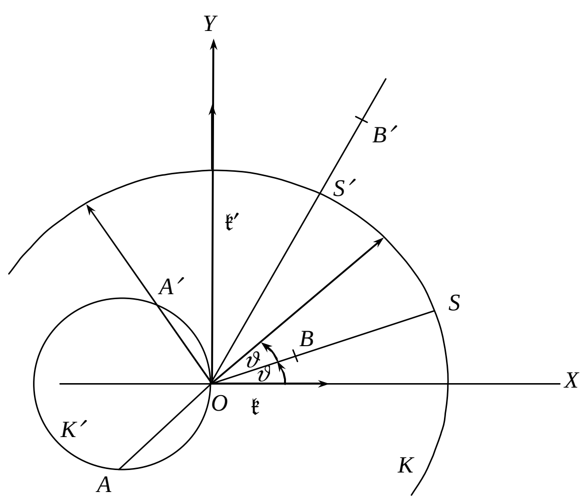

Figure 70.

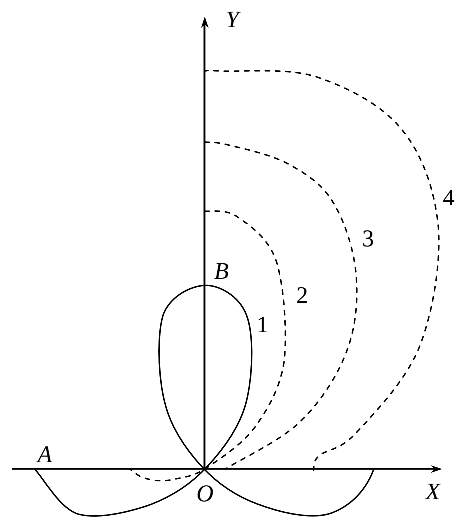

Figure 71.

We then consider the simplest case of $s = 0$. In this case, since ϑ ranges only from 0 to π, the equation:

$$\mathfrak{k}_\vartheta = -\,h \cos 2\vartheta \tag{19}$$

will represent one-half of a "four-leafed clover" (Fig. 71), in which one must observe that for a positive h, $\mathfrak{k}_\vartheta$ will be negative in the domain $0 < \vartheta < 45^\circ$, so the entire piece AOB of the curve 1 will belong to the first quadrant of ϑ. If the moving ray rotates through π then its positive direction will change; the value $\mathfrak{k}_\vartheta$ has also been carried into the opposite direction then (cf., the rem. on pp. 216). It will therefore happen that the characteristic curve is not closed, although its endpoints are associated with the same twist.

The general curve is also symmetric with respect to the Y-axis. We then need to pursue it in just one quadrant: If we carry the constant segment s in the current direction J then, from (1), we will get type 2, 3, or 4 (Fig. 71) for a positive s according to whether s is smaller than, equal to, or greater than h, respectively (*). We denote the curve C by C_1, …, C_4, according to the type that it belongs to. The equation of C_3 reads:

$$\mathfrak{k}_\vartheta = 2h \sin^2 \vartheta. \tag{20}$$

The tangent directions of the possible double point are determined by:

$$\tan^2 \vartheta = -\frac{\mathfrak{k}}{\mathfrak{k}'}. \tag{21}$$

These values of J correspond to singular complexes.

With the help of the curve C, one can, in all cases, form an intuitive picture of the distribution of pitches for the complexes of a pencil whose line cylindroid is given. The choice of system of pointers in § 55 corresponds to the assumption that one always has:

$$\mathfrak{k}' > \mathfrak{k},$$

no matter how the signs of these quantities might be arranged. h will then be positive, and the generators of $\mathfrak{C}$ will lie over the XY-plane in the first quadrant and below it in the second.

We begin with the case C_1, in which one has:

$$\mathfrak{k}' = -\mathfrak{k} = h.$$

The extreme generators a, a_1 of $\mathfrak{C}$ are simultaneously the focal lines b, b_1 of the associated net, which is therefore *rectangular*, and indeed, b will lie above the XY-plane at a distance of h, etc. The sign in the octants of Fig. 72 indicates the winding of the complexes of the pencil, such that the positive sign will correspond to a left-wound winding (§ 11, conclusion).

If we go on to the other pencils of complexes of the same cylindroid, while we let $\mathfrak{k}$ and $\mathfrak{k}'$ increase by the same amount (Fig. 73), then b and b_1 will approach the XY-plane by the same amount, until they coincide for $\mathfrak{k} = 0$, at which point, they will become a focal line of a special net. If $\mathfrak{k}$ is also positive then the net will be elliptic, but one can calculate its constants c, m from equations (c') of § 74, and thus determine its position with respect to $\mathfrak{C}$. If one sets $\mathfrak{k}' = \mathfrak{k} + h$ in them then one will get:

(*) Plücker's remark (*Neue Geom. des Raums*, art. 93) that one has to let all points either approach the double line or move away from it is therefore not true, in general.

$$c = \sqrt{\mathfrak{k}(\mathfrak{k}+h)}\,, \qquad m = \sqrt{\frac{\mathfrak{k}}{\mathfrak{k}+h}}\,; \tag{22}$$

both roots are taken to be positive. The net is left-wound, and will always approach a rotation net with increasing $\mathfrak{k}$. The major axes of the throat ellipses of the net will fall upon the X-axis. All twists of the pencil will be right-wound.

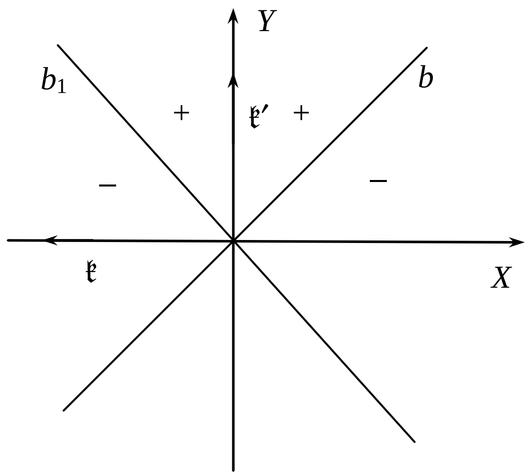

Figure 72.

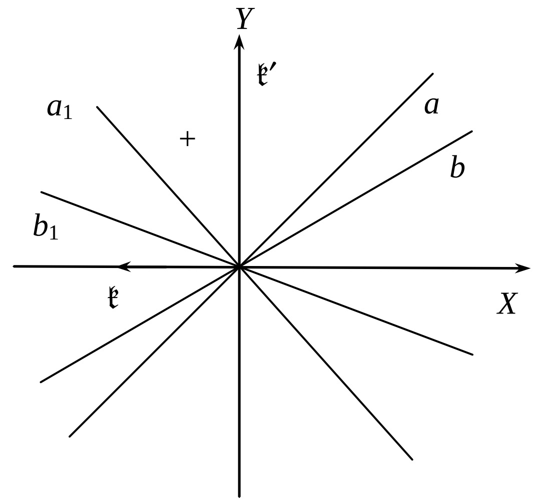

Figure 73.

It is now easy to follow through what happens when $\mathfrak{k}$ and $\mathfrak{k}'$ both decrease from their initial values by the same amount. When both of them are ultimately negative, one must take the first root in (22) to have the negative sign. One will have $m > 1$; i.e., the major axes of the throat ellipses of the net will fall upon the Y-axis, so they will always fall upon the axis that belongs to the principal complex with an absolutely smaller pitch. For all nets of the cylindroid, one will have:

$$c = h \cdot \frac{m}{1-m^2}\,. \tag{23}$$

Theorem 160: *Two special and one rectangular ray net belong to any line cylindroid, as well as* ∞^1 *hyperbolic and* ∞^1 *elliptic nets, moreover. The pair of focal lines of the former will be left-wound or right-wound according to whether the two pitches* $\mathfrak{k}$, $\mathfrak{k}'$ *of the principal complex have the positive or negative of the greater absolute value, respectively. The latter are left-wound or right-wound according to whether* $\mathfrak{k}$, $\mathfrak{k}'$ *are both positive or negative, respectively.*

c) If a ray s of a net $\mathfrak{N}$ is cut by a ray a of the associated cylindroid $\mathfrak{C}$ then s will belong to the complex C of the associated pencil whose axis falls upon a. If C is a twist then a and s will intersect perpendicularly. Since $\mathfrak{C}$ is of third order, at least point of one intersection with s will be real. It will follow from this that:

Theorem 161: *Any ray of a net $\mathfrak{N}$ (except the principal ray) cuts one and only one generator of the associated cylindroid perpendicularly. The other two points of intersection will or will not be real according to whether $\mathfrak{N}$ is hyperbolic or elliptic, resp.; in the former case, they will lie on the focal lines.*

In fact, *s* cannot be on more than one generator of $\mathfrak{C}$ without the double lines coinciding, but on the other hand, the possible focal lines of $\mathfrak{N}$, which also lie on $\mathfrak{C}$, must intersect. From the first part of Theorem 161, and from the reciprocal position of $\mathfrak{C}$ and $\mathfrak{N}$, which we studied in *b*), it will follow immediately that:

Theorem 162: *The locus of the shortest distances from the principal ray of a net to all other rays is the associated cylindroid of the net.*

Thus, ∞^1 of these shortest distances will lie on the same line. From Theorem 99, the associated rays *s* will define a family of rulings, and in fact, an equilateral, hyperbolic paraboloid.

---

§ 76. The composition of two dynames or windings.

The two equations:

(19) $$\mathfrak{k}_\vartheta = s - h \cos 2\vartheta, \qquad z = h \sin 2\vartheta$$

represent the characteristic curve of the rod cylindroid. We would now like to rotate all of its rods around the *Z*-axis in the same plane. Their endpoints will then define a plane curve whose equation we will get by eliminating ϑ from equations (19):

(20) $$(\mathfrak{k}_\vartheta - s)^2 + z^2 = h^2,$$

which is then a circle *K* with radius *h*. We draw *K* in the *XZ*-plane. Its points of intersection with the *X*-axis have *X*-pointers (Fig. 74):

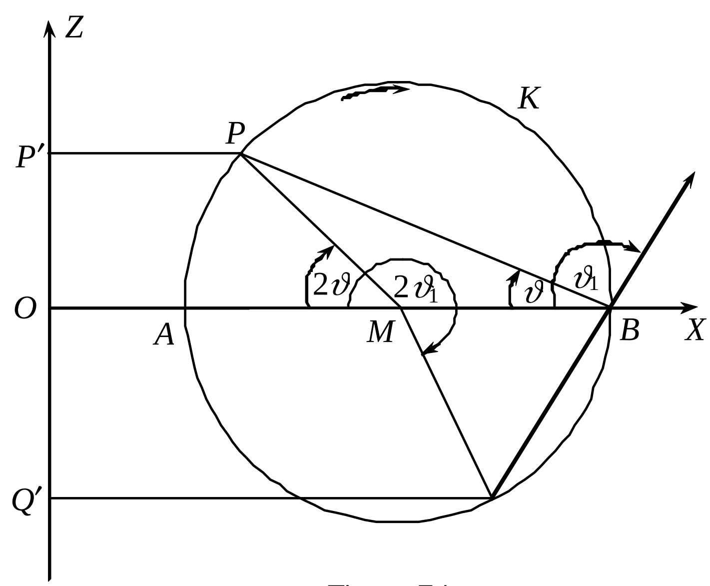

Figure 74.

$$s - h = \mathfrak{k} = OA,$$
$$s + h = \mathfrak{k}' = OB,$$

by which, it is determined completely. The angle 2ϑ of equations (19) is again found to be $\angle AMP$ in the figure. Its sense of rotation is positive, since the Y-axis points below the reference plane. In order to find the actual position of the rod $P'P$ in space, we have to rotate it around the Z-axis through the angle ϑ; i.e., through a peripheral angle ABP on the arc AP. *Lewis's theorem* then follows from this:

Theorem 163: *If a point moves in a circle K with uniform angular velocity, while K itself rotates around a Z-axis that lies in its plane with one-half that angular velocity, then the distances to the points of Z will define the rods of a cylindroid. The associated net will be hyperbolic, special, or elliptic, according to whether the circle K cuts the axis, contacts it, or does not cut it, respectively.*

One thus obtains the rod cylindroid completely when the point on K performs a complete circuit, so ϑ will run through the domain $0, \ldots, \pi$. If a rod cylindroid is given by the "principal rods" $\mathfrak{k}$, $\mathfrak{k}'$ then it will be immediately obvious how one can find the rod that is associated with each value of ϑ: One draws K, then $\angle ABP = \vartheta$, then makes the two legs intersect K at P, and then rotates $P'P$ through ϑ out of the reference plane.

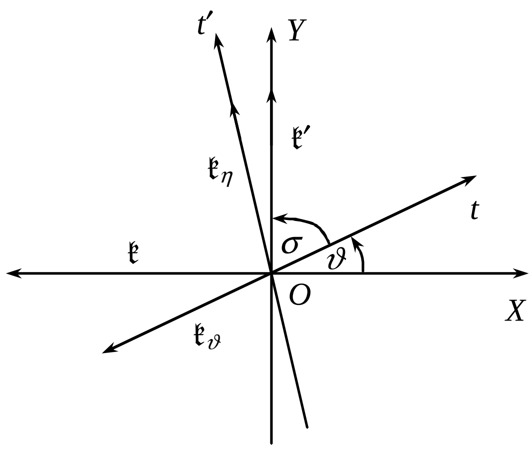

Figure 75.

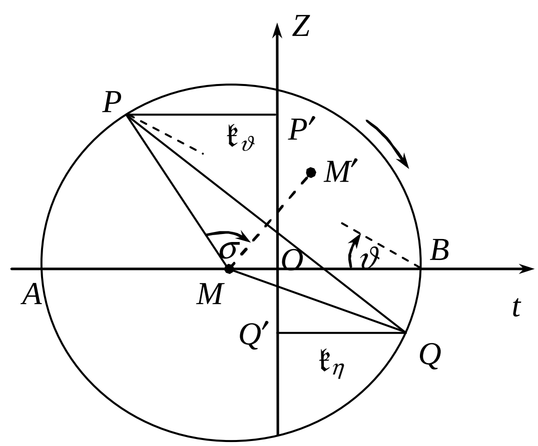

Figure 76.

However, one can also find the circle K, and therefore the principal rods $\mathfrak{k}$, $\mathfrak{k}'$, from the two *arbitrary* rods $\mathfrak{k}_\vartheta$, $\mathfrak{k}_\eta$ of the cylindroid (*): Let Z be the line of shortest distance $P'Q'$ between t and t'. We choose the direction on it to be positive when the angle $(t, t') = \sigma$ appears to be concave (Fig. 75) from it. We choose the plane Z, t to be the reference plane for K (Fig. 76) and rotate t', together with $\mathfrak{k}_\eta$, in it around Z. The circle K must then have the property that it goes through the endpoints P, Q of both rods, and σ must be its

(*) Therefore, every rod must be given a positive direction on the carriers t, t', because otherwise one could not evaluate the sign of the rod (cf., rem. 3 in § 74).

peripheral angle on the arc *PQ*. There are two such circles with centers *M*, *M′*; however, the arc $PQ = 2\sigma$ will be traversed, in some sense, for only one of them (viz., *K*), so that the positive *Z* semi-axis and *t* will be determined in that sequence. For that reason, only *K* will be the desired circle. Now, one knows the angle $ABP = \vartheta = (X, t)$, so one can draw the *X*-axis in Fig. 75, and likewise the principal rods $\mathfrak{k}$, $\mathfrak{k}'$, whose lengths and signs one infers from Fig. 76 ($\mathfrak{k} = OA$, $\mathfrak{k}' = OB$). The position of the point *O* on the shortest distance is also determined by Fig. 76, such that the problem of finding a rod cylindroid from two of its rods is solved completely in a constructive way (*). One finds a computational solution in Schell, *Theorie der Bewegung und der Kräfte*, Bd. II, pp. 220 (2nd ed.).

With that, the problem of composing two given dynames with the help of the cylindroid is also solved [*Study* has given other constructive solutions in *Geometrie der Dynamen* (I, 1901), where the theories that are discussed here are developed upon a new foundation]. Then, from the beginning of § 74, the rod pitch of the resultant dyname must belong to the cylindroid of the two given ones, and the directions and magnitudes of the forces of the given dynames must determine the direction ϑ and magnitude of the resultant force. However, there is a single rod with a given direction ϑ in a cylindroid, which we just learned how to construct; it will give us the position and pitch of the resultant dyname. We shall leave to the reader the task of carrying this out as a continuation of Figures 75 and 76. In this, one must observe: If a dyname $\mathfrak{D}$ is given by its force part *k* and its moment *m* (in the form of a rod and a rectangular field) then its rod pitch will be $\mathfrak{k}_\vartheta = m : k$. Its construction thus comes down to converting one right angle into another one, one side *k* of which is given. Thus, if the rod pitches $\mathfrak{k}_\vartheta$, $\mathfrak{k}_\eta$ of the dynames $\mathfrak{D}$, $\mathfrak{D}'$ are known then one can choose the magnitudes and senses of the forces *k*, *k′* on the carriers *t*, *t′*, of $\mathfrak{k}_\vartheta$ and $\mathfrak{k}_\eta$, resp., arbitrarily in order to give them completely.

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§ 77. The manifold of linear complexes.

The linear complexes define a five-fold infinite manifold (§ 2) in which an individual representative can be determined by the ratios of six numbers, viz., its (ray or axis) pointers (§ 49) in a tetrahedral system of pointers. We consider $k + 1$ complexes C_0, C_1, …, C_k with the pointers:

$$\begin{matrix} c_{01} & c_{02} & \dots & c_{06} \\ \dots & \dots & \dots & \dots \\ c_{k1} & c_{k2} & \dots & c_{k6} \,. \end{matrix} \qquad (k \le 5) \tag{21}$$

(*) Figures 75 and 76 are intended to represent the case in which $\mathfrak{k}_\vartheta$ is negative and $\mathfrak{k}_\eta$ is positive. The reader will convince himself that the solution is independent of the sequence of the two rods.

If this matrix has rank $k + 1$ [with the definition of rank in § 39), *c*)] then we will call the complexes *independent* of each other, and otherwise, *dependent*. We now assume that $C_0, \ldots, C_k$ are independent of each other and compose the pointers of a complex B from their pointers in the following way:

$$(22) \qquad \rho\, b_\nu = \sum_{\kappa=0}^{k} \lambda_\kappa c_{\kappa\nu} \qquad (\nu = 1, \ldots, 6),$$

where ρ is a proportionality factor. Let:

$$\rho\, b'_\nu = \sum_{\kappa=0}^{k} \lambda'_\kappa c_{\kappa\nu} \qquad (\nu = 1, \ldots, 6)$$

be the pointers of B'. B' is then identical with B only when the λ' are proportional to the λ. It will then follow from a suitable choice of $k + 1$ of the six equations:

$$\sum_{\kappa=0}^{k} (\lambda_\kappa - \sigma \lambda'_\kappa)\, c_{\kappa\nu} = 0$$

that:

$$\lambda_\kappa - \sigma\, \lambda'_\kappa = 0 \qquad (\kappa = 0, 1, \ldots, k).$$

If we let the ratios of the λ assume all possible real values then (22) will represent a k-fold infinite manifold M_k of complexes. We say that M_k has *dimension* k, or (with *Grassmann*) that it has *rank* $k + 1$ ($k + 1$-rank), because $k + 1$ complexes are necessary for its determination. We now choose $k + 1$ well-defined complexes $B_0, B_1, \ldots, B_k$ from M_k:

$$(23) \qquad \rho_\mu\, b_{\mu\nu} = \sum_{\kappa=0}^{k} \lambda_{\mu\kappa} c_{\kappa\nu} \qquad (\nu = 1, \ldots, 6;\ \mu = 1, \ldots, k).$$

It follows from the multiplication theorem for determinants that any $k + 1$-rowed determinant of the matrix of b can be obtained from the corresponding one in the matrix (21) by multiplying by:

$$\Lambda = |\, \lambda_{\mu\kappa} | \qquad (\mu, \kappa = 0, 1, \ldots, k).$$

Thus, if, as we shall now assume:

$$(24) \qquad \Lambda \neq 0$$

then $B_0, \ldots, B_k$ will also be independent of each other. We let $B, C, \ldots$ denote not only the complexes, but also the linear forms that enter into their equations, namely:

$$B_\mu = \sum_{\nu=1}^{6} b_{\mu\nu}\, p_\nu\,, \qquad C_\kappa = \sum_{\nu=1}^{6} c_{\mu\nu}\, p_\nu\,, \ldots$$

etc. In them, the p are homogeneous line pointers, and ray or axial pointers, according to whether the complex pointers are axial or ray pointers, respectively (§ 49). One then has identically:

$$\rho_\mu \, B_\nu = \sum_{\kappa=0}^{k} \lambda_{\mu\kappa} C_\kappa \, . \tag{25}$$

That is, the forms of the B are *linearly derived* from the forms of the C, which is why the manifold M is also called *linear* or a *linear complex domain*. For $k = 1$, we get the pencil of complexes that is known to us, for $k = 2$, the *net of complexes*, for $k = 3$, the *bush of complexes*, for $k = 4$, the *web of complexes* (a terminology of *Reye* and *Sturm*), and for $k = 5$, the entire *space of complexes*. If B is linearly derivable from $C_0, \ldots, C_k$ then $B, C_0, \ldots, C_k$ will be independent of each other, and conversely. Since each linear form in these variables is linearly derivable from these six forms in six variables, one will have:

Theorem 164: *More than six linear complexes are always dependent upon each other.*

Due to (24), the forms C_κ can be also represented in terms of the B_μ by way of (23). Thus, for arbitrary λ', the system of equations:

$$\sum_{\mu=0}^{k} \lambda'_{\tau\mu} B_\mu = 0 \qquad (\tau = 0, \ldots, k) \tag{26}$$

encompasses precisely the same complexes that the system of equations:

$$\sum_{\kappa=0}^{k} \lambda_{\mu\kappa} C_\mu = 0$$

does for arbitrary λ. That is:

Theorem 165: *A linear complex domain of dimension k is determined by any* $k + 1$ *of its independent complexes in the same way as it is by the original* $k + 1$.

We now let M, N, S, V always denote linear complex domains whose dimension will be indicated by the index. One can now let the first $l + 1$ ($l < k$) of the λ in equations (22) be arbitrary, and let the other ones be zero. It will then follow that:

Theorem 166: *If* $l + 1$ *independent complexes of one* M_l *of two linear complex domains is contained in the other one* M_k *then all of* M_l *will be contained in* M_k .

As a result of the last theorem, one can obtain – e.g., a net of complexes – in the following way: If one links each complex of a pencil (that is defined by C_0 and C_1) with a complex C_2 that does not belong to the pencil by another pencil of complexes then one will obtain the ∞^2 complexes that make up the net. This process will then correspond analytically to the one that couples a fixed choice of $\lambda_0 : \lambda_1$ to all values of λ_2 by defining the linear form:

$$\rho B = \lambda_0 C_0 + \lambda_1 C_1 + \lambda_2 C_2 .$$

Analogously, one can derive a bush of complexes from a net of complexes and a fourth complex, etc.

One remarks the analogy with the following theorems of elementary geometry: A line (plane, resp.) is determined by two (three, resp.) of its points. If two points of a line lie in a plane then the entire line will lie in the plane. One will obtain all points of a plane when one connects all points of a line with an external point by lines, and all points of space, when one connects all points of a plane with an external point. However, whereas this construction cannot be continued further in point space, in the manifold of linear complexes, one can go on to four-dimensional domains and the five-dimensional space of complexes. We will evaluate this remark for the geometrically fundamental concept of "multi-dimensional spaces" (§ 80). However, first we would like to look into whether the laws of meet and join for the linear structures in point space also have an analogue in the space of complexes. These laws for points, lines, and planes in space can be summarized as follows (the index will again denote the dimension, where the index zero refers to a point): In general, $\mathfrak{M}_m$ and $\mathfrak{N}_n$ have a common meet s_{m+n-3} or determine a join v_{m+n+1} according to whether $m + n \geq 3$ or $m + n < 3$, respectively; in these expressions, m, n can assume the values 0, 1, 2, 3.

In the space of complexes, we define the *join* of two domains to be the totality of all complexes of all pencils that link any complex of the one domain with a complex of the other domain. We understand the term "linear complex domains M_k and N_l *in general position*" to mean two that are not contained in any lower-dimensional linear domain as they are in all of the space of complexes when $k + l \geq 5$ and two that have no common complex when $k + l < 5$. We now prove the:

Theorem 167: *If one counts the dimension s of the meet of M_m and N_n as* – 1 *when no common complex at all is present and zero when a single one is present, and if v is the dimension of their join then:*

(27) $$s + v = m + n.$$

We think of M_m as being defined by the independent complexes $B_0, \ldots, B_m$, and N_n as being defined by the complexes $C_0, \ldots, C_n$. Any complex of M can be represented in the form:

$$\rho B = \sum_{\kappa=0}^{m} \lambda_\kappa B_\kappa$$

and any complex of N, in the form:

$$\rho' C = \sum_{\kappa=0}^{m} \lambda'_\kappa C_\kappa .$$

Now, one might be able to find a, but no more, mutually-independent C that depend upon B (a can also be zero). From Theorem 165, we can assume that these a complexes are the first a of the sequence $C_0, \ldots, C_n$. The complexes $C_0, \ldots, C_{a-1}$ will then define a meet of dimension $a - 1 = s$; moreover:

(28) $$B_0, \ldots, B_m, C_a, \ldots, C_n$$

are independent of each other. If one then had, perhaps:

$$\lambda'_a C_a + \cdots + \lambda'_n C_n = \lambda_0 B_0 + \ldots + \lambda_m B_m$$

identically then, contrary to the assumption, the linear forms on both sides of the equal sign would be additional $(a+1)^{\text{th}}$ forms that would depend upon the C, but not upon C_0, …, C_a, and would also be dependent upon B. Thus, the sequence (28) would define a the join of dimension $v = m + 1 + (n - a + 1) - 1 = m + n + 1 - a$. Therefore (cf., *Grassmann*, Ges. W. I, *b*, art. 25):

$$v + s = m + n.$$

Theorem 167 is also true when one goes to the rank numbers, because each of the four numbers will then be raised by one. For a complex domain in general position, one will have $v = 5$, in the event that $m + n \geq 5$; thus:

Theorem 168: *Two complex domains M_m, N_n in general position will have a common meet of dimension $m + n - 5$ when $m + n \geq 5$.*

When $a = 0$, all $m + n - 2$ defining complexes of both domains will be independent of each other; thus:

Theorem 169: *Two complex domains M_m, N_n in general position will have a common join of dimension $m + n + 1$ when $m + n < 5$.*

For example, one can obtain a bush of complexes from two pencils of complexes in general position as the totality of connecting pencils in precisely the same way that one obtains the entire point space as the join of two skew lines.

If one writes the quantities b_ν in (22) as the last column in the matrix (21) then one will obtain a matrix in which all $k + 2$-rowed determinants are zero. One has then eliminated the quantities λ from (22) and obtained linear, homogeneous equations between the b whose coefficients depend upon only the given fixed quantities c. We already know that these equations are satisfied by at least ∞^k systems of values $b_1 : b_2 : \ldots : b_6$, but also no more than that, because conversely the system (22) can be derived from these equations [cf., the conclusion of § 39, *c*)]. Thus, when one also considers the absolute values of the b, ∞^{k+1} systems of values b_ν will satisfy the equations; therefore, $6 - (k + 1) = 5 - k$ of the latter will be independent.

Theorem 170: *A linear complex domain of dimension k can be represented by $5 - k$ linear, homogenous equations in the complex pointers.*

For example, a web of complexes will be represented by one such equation and a pencil of complexes, by four of them. One observes the analogy with the theorems: A plane will be represented by one linear, homogeneous equation in the tetrahedral point pointers, and a line by two of them.

§ 78. Extended domains of complexes.

The condition for the involution of two complexes A and B, with pointers a_i and b_i, resp., was bilinear (§ 56):

$$\omega(a, b) = \sum a_{i+3}\, b_i = 0.$$

Now:

$$(29) \qquad \begin{aligned}(a, \lambda b + \lambda' b' + \ldots) &= \lambda \cdot \sum a_{i+3}\, b_i + \lambda' \cdot \sum a_{i+3}\, b_i' + \ldots \\ &= \lambda \cdot \omega(a, b) + \lambda' \cdot \omega(a, b') + \ldots\end{aligned}$$

If $B, B', \ldots, B^{(k)}$ are independent complexes then they will define a linear domain N_k, and it will follow that:

Theorem 171: *If a complex lies in involution with* $k + 1$ *independent complexes of a linear domain* N_k *then it will lie in involution with every complex of* N_k *.*

We now ask what all of the complexes A that lie in involution with every complex of N_k would be. They are defined by the $k + 1$ mutually-independent equations:

$$(30) \qquad \omega(a, b) = 0, \qquad \omega(a, b') = 0, \ldots,$$

so, from Theorem 170, they will define a linear domain M_{4-k} that we also call the *extended* domain of N_k ; N_k will also be the extended domain of M_{4-k} . It is then contained in it in any case, since it already has the right dimension. We also say that M and N are "extended domains of each other."

Theorem 172: *Every linear complex domain is then associated with another one in such a way that every complex of the one domain lies in involution with each of the others and the dimensions of such "mutually-extended" domains are extended to four (the rank numbers to six, resp.).*

If follows immediately from the representation (25) of a linear complex domain that:

Theorem 173: *The rays that are common to* $k + 1$ *independent complexes of a linear domain* M_k *define the totality of rays that are common to all complexes of* M_k *.*

A singular complex C is in involution with an arbitrary complex C' when (Theorem 112) C' is the carrier of C as a ray of the complex; it then follows from this that:

Theorem 174: *The carriers of the singular complexes of a linear domain are identical with the common rays of the extended domain.*

From Theorems 172 and 168, one can easily judge, in any case, how many independent complexes inside of a linear domain L_l can lie in involution with a given domain M that also belongs to L_l . If we assume, e.g., that L is a net of complexes ($l = 2$) then it will follow that there is a pencil of complexes inside L that are in involution with a complex of L. One notes the analogy with the theorem about sheaves of rays: There is a pencil of normal rays for every ray s of a sheaf. If s is normal to two rays of a pencil then it will be normal to all of them. Not only the concept of perpendicularity, but also that of angle, can find an analogue in complex domains. On this, cf., *D'Ovidio* (e.g., "Le ser. triple, etc.," Acc. dei Lincei, Ati, 1876; Ser. II, tom. 3), *Müller* ("Die Lineiengeom. nach d. Prinzipien d. Grassmannschen Ausdehnungsl.," art. 12; Monatsh. f. Math. u. Phys. II).

There is no middle ground between normal and skew position in a sheaf of rays, but things are different in complex domains: In general, there will be no complexes in an arbitrary net of complexes M_2' that lie in involution with all complexes of another net M_2. However, if M_2' and the extended domain N_2 of M_2 do not lie in general position then one, two, or three mutually-independent complexes that fulfill the stated condition can exist in M_2'. One can then call M_2' "simply," "doubly," or "triply normal" to M_2 (*D'Ovidio*, *loc. cit.*); in the last case, M_2' will be identical with N_2. We leave the examination of the other cases to the reader and remark only that the terms "simply, …, normal" should not relate to just any sort of domain, but only to the sequence of possible special positions. Thus, a net of complexes M_2' will first become simply normal to a pencil of complexes M_1 when *two* complexes can be found in M_2' that lie in involution with all complexes of M_1, because one such complex will always be present, anyway.

One calls two complex domains *completely* normal to each other when each complex of the one domain lies in involution with each of the other ones; in this, it is *not* assumed the domain is linear (*). It emerges from this that:

Theorem 175: *If G and G′ are completely normal to each other, and M is the smallest linear domain in which G is contained then G′ will be contained in an extended domain of M.*

Theorem 176: *If M and N are mutually-extended domains then any sub-domain of M will be completely normal to any sub-domain of N. The extended domain of any domain of M will contain N.*

In particular, two involutory complexes are also completely normal to each other, but the addition of the word "completely" would be superfluous here.

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§ 79. The common rays of the complexes of a linear domain M_k.

For $k = 1$, we get the ray net (§ 53); we then go on to the case:

A) $k = 2$. We imagine that the net of complexes M_2 is defined by three independent complexes A, B, C with the pointers a_i, b_i, c_i, resp. A and B determine a ray net $\mathfrak{N}$. The

(*) One imagines that nonlinear complex domains are defined by nonlinear equations in the complex pointers.

rays of $\mathfrak{N}$ that also belong to C are the common rays of M_2 . If $\mathfrak{N}$ has two focal lines then, from Theorem 9, these rays will define a family of rulings $\mathfrak{R}$. In order to also conveniently survey all special cases and the case in which $\mathfrak{R}$ is imaginary, we focus on the analytical representation of a fourth complex D of M_2 that takes the form:

$$D = \lambda A + \mu B + \nu C. \tag{31}$$

From Theorem 174, this will be the same as the problem of determining the carriers of the singular complexes of a net (namely, the extended one). We thus impose the condition that D must be singular:

$$\omega(d) = \omega(\lambda A + \mu B + \nu C) = 0,$$

or

$$\lambda^2 \cdot \omega(a) + \mu^2 \cdot \omega(b) + \nu^2 \cdot \omega(a) + 2\lambda\mu \cdot \omega(a, b) + 2\mu\nu \cdot \omega(b, c) + 2\nu\lambda \cdot \omega(c, a) = f(\lambda, \mu, \nu) = 0. \tag{32}$$

The complexes of M_2 will be mapped onto E by the null point of a fixed plane E in such a way that every pencil of complexes will correspond to a line; from Theorem 127, double ratios will also remain unchanged under this map $\mathfrak{A}$. We can then say:

Theorem 177: *A net of complexes will be mapped collinearly onto a fixed plane E by the null point of E.*

On the other hand, if one interprets $\lambda : \mu : \nu$ in (31) as homogeneous point pointers in E then one will obtain a map $\mathfrak{A}'$ with essentially the same properties (cf., the expression for the double ratio of four complexes in § 63). $\mathfrak{A}'$ is also collinear to M_2, and thus, also to $\mathfrak{A}$. One can then arrive at $\mathfrak{A}'$ and $\mathfrak{A}$ being identical by a suitable choice of the basis triangle for $\lambda : \mu : \nu$.

Now, (32) means the equation of a conic section K in E. The determinant of its equation is:

$$\Delta = \begin{vmatrix} \omega(a) & \omega(a,b) & \omega(a,c) \\ \omega(a,b) & \omega(b) & \omega(b,c) \\ \omega(a,c) & \omega(b,c) & \omega(c) \end{vmatrix}. \tag{33}$$

We first assume:

a) Let Δ be non-zero. K is then a proper conic section that can be either α) real or β) imaginary. In the first case, a line in E will correspond to a pencil of complexes with a hyperbolic, special, or elliptic carrier, according to whether it cuts, contacts, or does not cut K, respectively. The argument at the beginning of this paragraph will then be in force (since one can choose A and B to be in a pencil with a hyperbolic carrier), and the common rays of M_2 will define a real family of rulings $\mathfrak{R}$ that cuts E along K. However, in the case β), a real polar system Σ will also be always present that represents K. We express the idea that a complex $\lambda : \mu : \nu$ lies in involution with another one $\lambda' : \mu' : \nu'$, namely:

$$\omega(\lambda a + \mu b + \nu c, \lambda' a + \mu' b + \nu' c) = 0,$$

or

$$\lambda\lambda' \cdot \omega(a) + (\lambda\mu' + \lambda'\mu) \cdot \omega(a, b) + \ldots = 0. \tag{34}$$

If we fix $P \equiv (\lambda' : \mu' : \nu')$ then (34) will represent the polar of P in Σ; so:

Theorem 178: *Every complex of a net M_2 is associated with a completely normal pencil of complexes in M_2 . This association in M_2 is a polar system, and, as such, also maps onto any plane E by means of Theorem 177.*

We can show that the ∞^3 polar systems that emerge for the various positions of E are sections of *one and the same* spatial polar system Σ (*); in case α), $\mathfrak{S}$ is defined by $\mathfrak{R}$. In case β), we take A, B, C to be three complexes whose null points lie in E on the vertices A', B', C' of a polar triangle; let its sides be $A'B' \equiv c$, etc. Each of the complexes A, B, C will lie in involution with each of the other ones. If we then rotate E around c then A', B' will describe two involutory point sequences A'', B'' on c, one pair of which will be the original positions A', B', while starting with the null point C'' of the third complex C of C', it will describe the polar c' of c in C. In the various positions of E, the points A' will then be associated with the rays of the pencil (B', c'), whose vertex will always lie on the fixed line a for all possible rotations of E around A', so, along with c', it will determine a fixed plane α that corresponds to the point A' in $\mathfrak{S}$. Thus, if $\mathfrak{R}$ is not also real then the associated polar system can always be found, because the gathered involution of an elliptic net [cf., especially, § 66, *h*)] can be regarded as known. From § 72, the ordering surface of $\mathfrak{S}$ can be considered to be an imaginary second-order ruled surface, and it will follow, in connection with Theorem 174, that:

Theorem 179: *If Δ is non-zero then the carriers of the singular complexes of a net M_2 will define a family of rulings $\mathfrak{R}$, and the common rays of all complexes of M_2 will define the guiding family $\mathfrak{L}$ of $\mathfrak{R}$. Conversely, for the extended net N_2, $\mathfrak{L}$ will be the locus of the singular complexes and $\mathfrak{R}$ will be the locus of the common rays.*

When $\mathfrak{R}$ is imaginary, all complexes of M_2 will have pitches that are denoted the same, and likewise for N_2 . $\mathfrak{L}$ is also called the "basic family of rulings" of the net M_2 .

(*) This is not to be confused with the polar system that is defined by a bush of complexes:

$$\lambda A + \mu B + \nu C + \rho D$$

when one seeks the completely normal net of complexes for every complex in it and interprets $\lambda : \mu : \nu : \rho$ as homogeneous spatial point pointers.

b) Δ has rank 2. *K* then decomposes into two straight lines g, g'. The pencils B, B', which are mapped by g, g', resp., contain nothing but special complexes that are subordinate to a common *C*. We can think of *B* as being determined by the two complexes *C*, *A*, and B' as being determined by *C*, A'; the axes of *A*, *A*, A' might be called *c*, *a*, a', respectively. *c*, *a* will then intersect, as will *c*, a', but not *a*, a', since they will give rise to a third pencil of singular complexes in M_2, while *K* contains only two lines. The axes of the singular complexes will define the two pencils of rays *c*, $a \equiv (S, a)$ and *c*, $a' \equiv (S, a')$ (Fig. 77), while the common rays of M_2 will be the pencils of rays (S, α') and (S', α). We have already encountered a similar figure (Fig. 50). *c* will always be real, but g, g', and thus also *a*, a', do not need to be so; all of the complexes of M_2 will then have a single real ray *c* in common that will likewise be the carrier of the only singular complex of M_2 . We will obviously obtain this case when we connect all of the complexes of a pencil with an elliptic carrier $\mathfrak{N}$ with a singular complex whose carrier *c* also belongs to $\mathfrak{N}$. *c* will then be the only ray that is common to all complexes of M_2 . More generally, we can say:

Theorem 180: *If Δ has rank two then one can obtain the net of complexes M_2 in such a way that one connects all of the complexes in a pencil whose carrier $\mathfrak{N}$ is elliptic or hyperbolic with a singular complex whose axis belongs to $\mathfrak{N}$.*

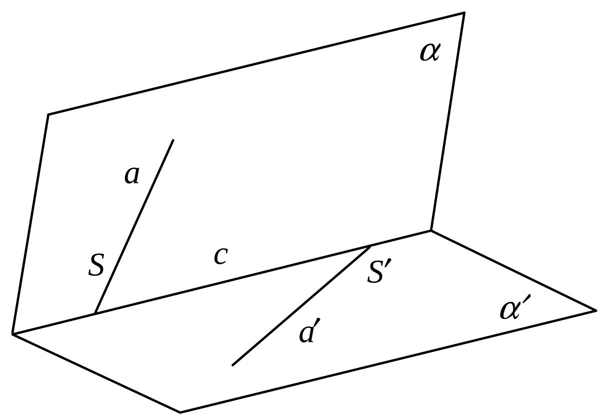

Figure 77.

c) Δ has rank one. $f(\lambda, \mu, \nu)$ is then a complete square of a linear form (cf., *Killing*, *Analyt. Geom. I*, § 17). In our mapping of M_2 onto *E*, there will then be a single line that represents the locus of the images of singular complexes. The latter define a pencil of complexes $\mathfrak{B}$. We can determine M_2 by way of two complexes *A*, *B* in $\mathfrak{B}$, as well as a third complex *C*; we will then have:

$$\omega(a) = \omega(b) = \omega(a, b) = 0.$$

Now, in order for Δ to actually have rank one, one must also have:

$$\omega(b, c) = \omega(a, c) = 0 ;$$

i.e., the carriers of *A* and *B* must be contained in *C*.

Theorem 181: *If Δ has rank one then we will obtain the net of complexes when we connect all of the complexes in a pencil of singular complexes with a twist whose carrier contains the singular complexes.*

d) Finally, Δ has rank zero; i.e., all coefficients of (32) vanish individually. The complexes of M_2 will all be singular then, and the axes of any three of them must intersect.

Theorem 182: *If Δ has rank zero then M_2 will consist of nothing but singular complexes whose axes will fill up either the same sheaf or the same field.*

We can now quickly deal with the remaining cases with the help of Theorem 174:

B) $k = 3$. The extended domain N_1 of M_3 is a pencil of complexes; thus:

Theorem 183: *The axes of the singular complexes of a pencil of complexes M_3 define a net of rays $\mathfrak{N}$. According to whether $\mathfrak{N}$ is a*) *hyperbolic, b*) *special, c*) *elliptic, d*) *singular, all complexes of M_3 will have two, one, no real rays, or a pencil of rays in common, respectively.*

Conversely, M_3 is determined by $\mathfrak{N}$; e.g., by a plane pencil of rays in the last case.

C) $k = 4$. The extended domain of M_4 is single complex; thus:

Theorem 184: *The axes of the singular complex of a web of complexes M_4 define a linear complex C. All complexes of M_4 will or will not have a ray in common depending upon whether C is* (*a*) *singular or* (*b*) *not, respectively.*

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§ 80. Logical remarks about the geometry of multi-dimensional space.

In § 77, we already remarked about certain analogies with the geometry of point space for which the manifold of linear complexes is also characterized as a linear one, independently of its analytical representation, when one first of all starts with a pencil of complexes and arrives at a domain of the next higher rank by connecting a linear domain with a complex that lies "external" to it, and secondly, verifies that the *laws of linear meets and joins* that we expressed in Theorems 165-169 are valid for the domain thus-obtained. We developed these theorems upon the concrete foundations of line geometry; however, one sees that the number five plays no role in their proof as an upper limit on the dimension of the domains in questions. Moreover, we can start with a matrix:

$$\begin{matrix} x_{01} & \dots & x_{0,\,q+1} \\ \dots & \dots & \dots \\ x_{k1} & \dots & x_{k,\,q+1} \end{matrix} \tag{35}$$

instead of (21), which is so arranged that the $q + 1$ variables $x_1, \dots, x_{q+1}$, whose ratios are all that will be considered, will produce $k + 1$ special systems of values. An *arithmetic value domain* will then be defined by:

$$\rho\, y_\nu = \sum_{\kappa=0}^{k} \lambda_\kappa x_{\kappa\nu} \qquad (\nu = 1, \ldots, q+1), \tag{36}$$

in which the y are not all simultaneously unrestricted in their variability; otherwise, their ratios could assume only ∞^k systems of values. The same domain of values can also be represented by $q - k$ linear, homogeneous equations between the y (cf., Theorem 170), and for that reason, it will be called *linear*. Now, just the number q will enter in place of five in Theorem 168, 169, and the definition of the general position. Moreover, the laws of linear meet and join will be valid for the q-dimensional manifold of the variables $x_1 : x_2 : \ldots : x_{q+1}$ and its linear sub-domains, which are completely analogous to those of our space, for which reason, one might care to refer to the entire domain of values $x_1, \ldots, x_{q+1}$ briefly as a *q-dimensional space*. The laws that underlie its sub-domains (and even the nonlinear ones) define the content of *q-dimensional geometry*. In particular, one counts those theorems that are a generalization of the investigations of analytic geometry in space or have an application to the apparent geometry of the eyes, or finally, are methods and ways of presenting things that are analogous to the ones that are accessible in synthetic geometry. We would like to give an immediate example of the latter situation. Five-dimensional geometry is, in fact, realized in the domain of complexes (*). We can then think of the space of complexes as the substrate for the linear constructions in five-dimensional space, without leaving behind the foundations of ordinary Euclidian geometry. However, since its other properties besides the laws of linear meet and join do not come under consideration, we will refer to its one-dimensional linear domains as *lines*, in order to evoke an analogy with point space, and the two-dimensional ones as *planes*, and when necessary, suggest the dimension of the domain by indices. We can also schematically map the lower-dimensional subspace to points, lines, …, and keep them in our mental picture.

Example: We choose three planes E, E', E'' (Fig. 78) in general position (§ 77) in five-dimensional space. One can then draw a single line G through a point P of E that cuts E', as well as E'', namely, the line of intersection (Theorem 168) of the join space R_3' $\equiv (P, E')$ and $R_3'' \equiv (P, E'')$. The latter actually have only one line in common, since otherwise they would be contained in an R_4, in which E' and E'' would also have to

(*) This rests upon the fact that any complex can be determined by the ratios of six numbers – viz., its pointers – that are subject to no further restrictions, while the line pointer must fulfill a quadratic equation. For that reason, the straight lines in space do not define a linear domain; it is already impossible to take the first step in the construction of a linear domain here, namely, to give a rule by which two lines would determine a simple manifold of lines in such a way that they would be determined by two other lines in it using the same rule. The linearity of the domain of complexes is naturally accessible to a synthetic examination (*Reye*, *Sturm*) that can be abbreviated by the use of a *fundamental theorem* on linear manifolds (cf., *Zindler*, "Nachweis lin. Mannigf., etc.," Wiener S. B. Math. Cl., Bd. CI, Abt. II, 1892; § 1). Therefore, the validity of the laws that define linearity needs to be proved in the domain in question only up to $q = 2$, with which, they will be true for the entire domain in its own right, regardless of what rank it has. In order to apply this theorem, one thus does not need to possess an analytical representation, or even know its dimension. One will then have to regard the possibility of determining the individual elements of a domain by numbers that are subject to no restrictions as, in fact, a sufficient characterization, but not a definition, of linearity.

intersect. Let G cut E' at Q and E'' at N; analogously, let a line G' be defined that goes through the point P' of E and cuts E' at Q' and E'' at N''. The lines PP', QQ', NN' will then also lie in the join space $R_3 \equiv (G, G')$, and will thus define a ruled surface in it that also belongs to G, G'; i.e., if one moves P along a line in E then G will move along a family of rulings. It follows from this that:

Theorem 185: *Three planes in general position in R_5 define a system of ∞^2 lines, all of which will intersect, and by means of which they will be related to each other collinearly.*

The actual meaning of theorems of this sort will emerge when one either chooses an arithmetic manifold as a substrate or a linear manifold as a geometric structure (*). Now the former is, without a doubt, logically simpler; however, the arithmetic method must always be supported by a well-defined form for the representation of the value domain (the value domain of a line in space can already be represented by two equations *in a multitude of ways or* be derived from two of its points), whereby superfluous elements will be dragged into the investigation. By contrast, the synthetic method operates with only the structures themselves, and certainly Theorem 185 could not have been proved so simply by calculating in a domain of five variables. Indeed, the synthetic methods of multi-dimensional geometry share a certain intuitiveness with ordinary geometry. Admittedly, they relate to only the subordinate depictions (cf., "Surrogatvorstellungen," Beitr. z. Th. d. math. Erkenntnis, Wiener S. B. Phil.-hist. Cl., Bd. 118, § 26) of certain other objects.

The application of multi-dimensional geometry to the geometry of point space now takes the form: If one has proved a theorem of the pure geometry of position in any n-dimensional manifold M of our space then one will first project the result into a three-dimensional domain G_3 inside of M or intersect it with G_3 ; one will then map G_3 collinearly to point space, with which, one will obtain a theorem in point space. One thus does not need to leave behind the foundations of ordinary geometry at any step. For example, Theorem 185 can be immediately interpreted as the statement that three planes in point space can be collinearly related to each other in such a way that the three corresponding projective point sequences will define *the same* family of rulings.

If one has based the methods of the "projection and intersection of higher spaces" in a logically consist way in this manner then naturally one will not need to think of their application to a substrate every time, but one can operate exactly as if the barrier had fallen that puts the constructions into our space by the limited number of its dimensions. However, one cannot assume that standpoint from the outset, or else metaphysical

(*) Since the coefficients of an m^{th}-order form in x, y are $1 + 2 + 3 + \ldots + (m + 1) = \binom{m+2}{2}$ in number, the planar algebraic curves of order m will produce linear domains up to degree $n = \binom{m+2}{2} - 1$. The interpretation of multi-dimensional geometry by systems of algebraic structures has the advantage of being valid for an arbitrary dimension at one stroke, but has the disadvantage of being complicated by the theory of imaginaries; by contrast, the real complexes define a five-dimensional domain with no exceptions in themselves (cf., also "Synth. Gewinnung geom.. lin. Mf. beliebiger Dim.," Journ. f. r. u. a. Math., Bd 111).

speculations (like the proof of the possible limitations of our perception) would come about by their introduction.

One has introduced multi-dimensional geometry from three standpoints (similar to what one does in the theory of imaginary elements):

a) One repeats the steps that lead from the plane to space, *per analogiam*, and thinks of the constructions that were true for the plane and space as being continued as if one would find points outside of our space that one could link with the points of our space. This process is logically inadequate, even if it does lead to correct results. At best, one can say hypothetically by this method: If there is any sort of higher manifold for which analogies that one starts with are verified then the further theorems that follow from them will also be true. However, this does not suffice for the application of multi-dimensional geometry to ordinary geometry. There, one must actually eliminate those manifolds that are based in either the arithmetic domain or the linear systems of geometric structures in our space.

b) One refuses to employ the manner of presentation and methods that currently comprise multi-dimensional geometry at all, and prefers to immediately establish the original real meaning that such investigations might have any time that linear systems of geometric structures appear (*Sturm*, Foreword to v. III of *Liniengeom.*). This standpoint is logically correct, but inconvenient. One will then gain the advantage of being able to apply one and the same schema to many investigations that would otherwise have to be carried out in the individual cases. The criticism that multi-dimensional geometry replaces "the intuitive with the non-intuitive" (*Sturm*, *loc. cit.*) is also inapplicable. Moreover, our simple example already shows how intuitively one can go about one's affairs with the concepts of multi-dimensional geometry; indeed, one will be forced into schematic notations, similar to what happens in the theory of imaginary elements.

c) One bases the methods of multi-dimensional geometry in one of two ways that were already discussed in the conclusion of *a*). If this is done once and for all then, consistent with that, the standpoint *a*) will be legitimate, and one will now only operate with the abstract concepts of "points, lines, … in a four or *n*-dimensional space" in order to see the advantage of those methods. These words are by no means superfluous, but contribute to an economy of thought, just as the expressions in the theory of imaginary elements do. We have also taken the standpoint *c*) and sought to develop it in the most elementary way possible on the basis of line geometry, which is eminently suited to it.

The capability of using the methods of multi-dimensional geometry – or as one sometimes says, "thinking in *n* dimensions" – has been expanded even more in the last decade by mathematicians, and is presently indispensible in most branches of mathematics. We will also occasionally use these methods from now on.

One also confers *Killing*, *Grundl. d. Geom. I*, sect. 3, for the subject of this paragraph.

§ 81. General complex pointers and line pointers. Kleinian line pointers. Co-reciprocal twists.

Up to now, we have always employed the ratios of the six numbers that were defined in § 49 (which now call $x_1, \dots, x_6$) as complex pointers. Thus, the relation:

$$\omega(x) = 2\,(x_1\,x_4 + x_2\,x_5 + x_3\,x_6) = 0 \tag{37}$$

expresses the idea that the complex x is a special one, and its pointers can then be considered to be the pointers of its carrier; the relation:

$$\omega(x, x') = \sum_{k=1}^{6} x_k x'_{k+3} = 0 \tag{38}$$

asserts the involution of the complexes x and x'. We can now put the ratios of the sextuples of values x and y into a one-to-one correspondence by a linear transformation:

$$\rho\, x_i = \sum_{k=1}^{6} a_{ik} y_k \qquad (i = 1, \dots, 6), \tag{39}$$

whose determinant $|\, a_{ik} \,| = A$ we assume to be non-zero. Along with (39), we pose:

$$\sigma\, y_k = \sum_{i=1}^{6} A_{ik} x_i \qquad (i = 1, \dots, 6), \tag{40}$$

in which the A_{ik} are sub-determinants of A or proportional to them. A complex is then determined by the ratios of the six numbers y in the same way as it is by the x, regardless of whether the transformation (39) can be interpreted as the change of pointers (§ 40) or not and regardless of whether the a_{ik} are real or complex. We call the y *general complex pointers.*

If one substitutes the form $\omega(x)$ into the expression (39) then one will obtain a quadratic form $\Omega(y)$. The equation:

$$\Omega(y) = 0 \tag{41}$$

will mean that the y are now pointers of a special complex; i.e., they are *line pointers.* We will call them *general line pointers.* Analogously, (38) will go to another bilinear condition:

$$\Omega(y, y') = 0.$$

One will likewise have:

$$2\ \omega(x, x') = \sum_{i=1}^{6} \frac{d\,\omega(x)}{dx_i}\, x_i\,.$$

One can show that one likewise has:

(42) $$2\,\Omega(y, y') = \sum_{i=1}^{6} \frac{d\Omega(x)}{dy_i} y_i\,.$$

The form on the right-hand side is called the *polar form* of the form $\Omega(y)$. Namely, the systems of values x and y correspond just as the systems x' and y' do, and due to the linearity of the transformation equations, the systems $x_i + \lambda\, x_i'$ and $y_i + \lambda\, y_i'$ will also correspond for arbitrary values of λ, such that one will have:

$$\omega(x + \lambda\, x') = \Omega(y + \lambda\, y')$$

identically, as long as one expresses the x and x' in terms of y and y' by means of (39). We develop both sides in a Taylor series for six independent variables:

$$\omega(x) + \lambda \sum \frac{d\omega(x)}{dx_i} x_i' + \lambda^2\, \omega(x') = \Omega(y) + \lambda \sum \frac{d\Omega(y)}{dy_i} y_i' + \lambda^2\, \Omega(y').$$

Now, we already know that because of (39), one will have:

$$\omega(x) = \Omega(y), \qquad \omega(x') = \Omega(y')$$

identically. The two sums must then also go to each other identically, of which, the one on the left-hand side will be equal to $2\omega(x, x')$, while we have called the one that form goes to $2\Omega(y, y')$. With that, we have proved (since the special type of form ω and the number of variables have played no role):

Theorem 186: *Under a transformation of a quadratic form by a linear substitution, the polar form of the original form will, at the same time, go to the polar form of the new one.*

If x and y are line pointers then the vanishing of an x will mean that the line cuts an edge of the basic tetrahedron [(§ 39.a)]; it will then belong to a certain special complex. Analogously, $y = 0$ now means that the line y belongs to the linear complex:

$$\sum_{i=1}^{6} A_{ik}\, x_i = 0.$$

Six arbitrary complexes – viz., the *fundamental complexes* – will now enter in place of the six tetrahedral edges.

One can now choose the a such that the form Ω assumes an especially simple form. Along with the original pointer system, one can then distinguish the *Kleinian* pointers (Math. Ann., Bd. II) (*), for which one has (up to a possible constant factor):

(*) *De Paolis* (Atti della R. Acc. dei Linc., ser. IV, Bd. I, 1885) has presented complex pointers on the basis of six co-reciprocal twists in a more synthetic way. He assumed a "unit complex" that was analogous to the unit point in the basic tetrahedron and coupled it with analogous constructions, as well. *Müller* has treated the co-reciprocal twist with the help of the *Ausdehnungslehre* (Monatsh. f. Math. u. Phys II).

(43) $$\Omega(y) = \sum_{\lambda=1}^{6} y_\lambda^2 .$$

One arrives at such pointers very simply (*Koenigs*, *Géom. réglée*, art. 75), e.g., when one recalls the identity:

$$2\ \omega(x) = (x_1 + x_4)^2 + (x_2 + x_5)^2 + (x_3 + x_6)^2 - (x_1 - x_4)^2 - (x_1 - x_4)^2 - (x_1 - x_4)^2,$$

and sets:

(40a) $$\begin{aligned} x_1 + x_4 &= y_1 , & x_1 - x_4 &= i\, y_4 , \\ x_2 + x_5 &= y_2 , & x_2 - x_5 &= i\, y_5 , \\ x_3 + x_6 &= y_3 , & x_3 - x_6 &= i\, y_6 ; \end{aligned}$$

correspondingly:

(39a) $$x_1 = \tfrac{1}{2}(y_1 + i\, y_4), \qquad x_4 = \tfrac{1}{2}(y_1 - i\, y_4), \qquad \text{etc.}$$

Naturally, the Kleinian pointers cannot be arrived at by a real transformation; however, all of the fundamental complexes can possibly be real (as they are here). One sees from their equations:

$$x_1 + x_4 = 0, \qquad x_1 - x_4 = 0,$$

etc., that none of them are special, and any two of them lie involutorily. As we would now like to show, these two properties will also be true for the most general Kleinian pointers z. One will obtain them when one applies an arbitrary, orthogonal substitution S to the y; i.e., a substitution for which one has:

$$\sum_{\lambda=1}^{6} y_\lambda^2 = \sum_{\lambda=1}^{6} z_\lambda^2 .$$

(39) and S will combine into a linear transformation T. Without also writing it down, we know that $\omega(x, x')$ will go to:

(44) $$\Omega(z, z') = \sum z_i z_i'$$

under T [Theorem 186 and equation (42)]. The equations of the fundamental complexes will now be:

(45) $$z_\lambda = 0 \qquad (\lambda = 1, \dots, 6).$$

The equation of a linear complex:

$$\sum A_\lambda\, x_\lambda = 0$$

goes to an equation:

$$\sum B_\lambda\, y_\lambda = 0$$

under T, in which the B likewise combines a substitution T' with A. Also without calculating T', one can assert that:

$$\sum A^2 = \sum B^2,$$

and for any two sequences of coefficients:

$$\sum A_i A'_{i+3} = \sum B_i B'_{i+3}.$$

For the original pointer system x, coefficients and pointers will either be identical or will become so when one changes the indices mod 3, according to the choice of notation. One can also identify whether a complex is special or lies involutorily with another one from the coefficients of its equation in the system z. One now sees from the coefficients of equations (45):

Theorem 187: *None of the fundamental complexes of a Kleinian pointer system is special, and any two of them lie involutorily.*

Ball called two involutory twists (i.e., screws) *reciprocal* and several twists, any two of which lie involutorily, *co-reciprocal*. One can obtain six co-reciprocal twists in the following way: One chooses the first one A arbitrarily, the second one B is arbitrary in the extended domain (§ 78) A_4 of A, the third one C is arbitrary in the extended domain B_3 of the pencil AB, etc.; finally, the sixth one will be determined uniquely.

Ball (*Theory of Screws*, art. 41) gave a superbly simple system of six co-reciprocal twists: Namely, from the expression for the moment of two twists in § 52, equation (55), we next infer:

Theorem 188: *If the axes of two twists cut perpendicularly, or if they cut arbitrarily (i.e., are parallel) and likewise have equal and opposite pitches then the twists will lie involutorily.*

Ball's system of "canonical" co-reciprocal twists follows from this immediately:

Theorem 189: *If one uses each axis of a rectangular pointer system as the axis of two twists with equal and opposite pitches then one will obtain six co-reciprocal twists.*

Six co-reciprocal twists determine two times 15 ray nets and three times 20 families of rulings $\mathfrak{R}$. These and the focal lines of any net have many relationships to each other. For the study of this *configuration*, we must, however, refer to the previously-cited treatises of *Klein* and *Koenigs* (*loc. cit.*, art. 78, *et seq.*). Moreover, any ray net gives rise to a gathered involution (Theorem 116), and $\mathfrak{R}$, to a polar system, although there are only ten of them, since any two $\mathfrak{R}$ will be guiding families of each other (Theorem 179). If one includes the six null systems and the identity then one will have 16 correlations and 16 collineations. This shows that the geometric conversions define a *group* with numerous subgroups (Sturm, *Liniengeom.*, I., art. 175, *et seq.*).

The ratios of the y are homogeneous pointers of a five-dimensional manifold. The lines will be cut out from inside of this by the quadratic equation:

$$\Omega(y) = 0$$

(whose special form we shall not go into further), in a way that is similar to the way that a second-order surface is cut out of point-space by a quadratic equation. One can thus also say that (at least, projectively) line geometry comes down to *developing the geometry of a four-dimensional quadratic manifold M in five-dimensional space* (*Klein*, Math. Ann. V). *Segre* has also exploited this approach to line geometry (Mem. dell' Acc. di Tor., ser. II, v. 36). For him, e.g., *M* was "contacted" by the pencils of complexes with special carriers.

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§ 82. The axis manifolds of linear complex domains of dimensions four and three.

If we represent any complex by a rod, as in § 74, then the axes of the complex will define a linear domain in a rod manifold that will represent the domain of complexes itself completely when the sign of each rod is known. Up to now, all that we have discussed precisely regarding these rod structures is the rod surface of a pencil of complexes (§ 74, 75). Our next topic is:

I) *The rod forest $\mathfrak{S}$ of a web of complexes.*

This consists of the rods of all complexes C that are reciprocal to a given one C' (Theorem 172), so, from § 52, they can be represented by the equation:

$$(\mathfrak{k} + \mathfrak{k}') \cos \omega - d \sin \omega = 0, \tag{46}$$

in which $\mathfrak{k}'$ is constant, and $\mathfrak{k}$, d, ω are variable. Now, if α_i are the pointers of the rod of C, and α_i' are those of the rod of C' then we can deduce the corresponding rod equation immediately from § 52, equation (53):

$$(\mathfrak{k} + \mathfrak{k}') \sum_{i=1}^{3} \alpha_i \alpha_i' + \sum_{i=1}^{6} \alpha_{i+3} \alpha_i' = 0. \tag{47}$$

Therefore, in order for the rods to have the associated lengths, from Theorem 48, one must have:

$$\mathfrak{k} = \sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}\,, \quad \mathfrak{k}' = \sqrt{\alpha_1'^2 + \alpha_2'^2 + \alpha_3'^2}\,. \tag{48}$$

If one gets rid of the root signs on the *variable* quantities in (47) then one will come to an equation of degree four in the α_i .

Theorem 190: *The rod forest $\mathfrak{S}$ of the web of complexes is of degree four.*

In order to examine it more closely, we define the pointer system such that the rod α_i' falls on the Z-axis, so all α will be zero, except for α_3'. The equation of $\mathfrak{S}$ will then read:

$$(\mathfrak{k} + \mathfrak{k}')\, \alpha_3 + \alpha_6 = 0. \tag{49}$$

If we select a rod in $\mathfrak{S}$ for which $\mathfrak{k}$ has a well-defined value then we will see:

Theorem 191: *The rod forest $\mathfrak{S}$ can be decomposed into ∞^1 twists with common axes by the lengths of its rods* (*).

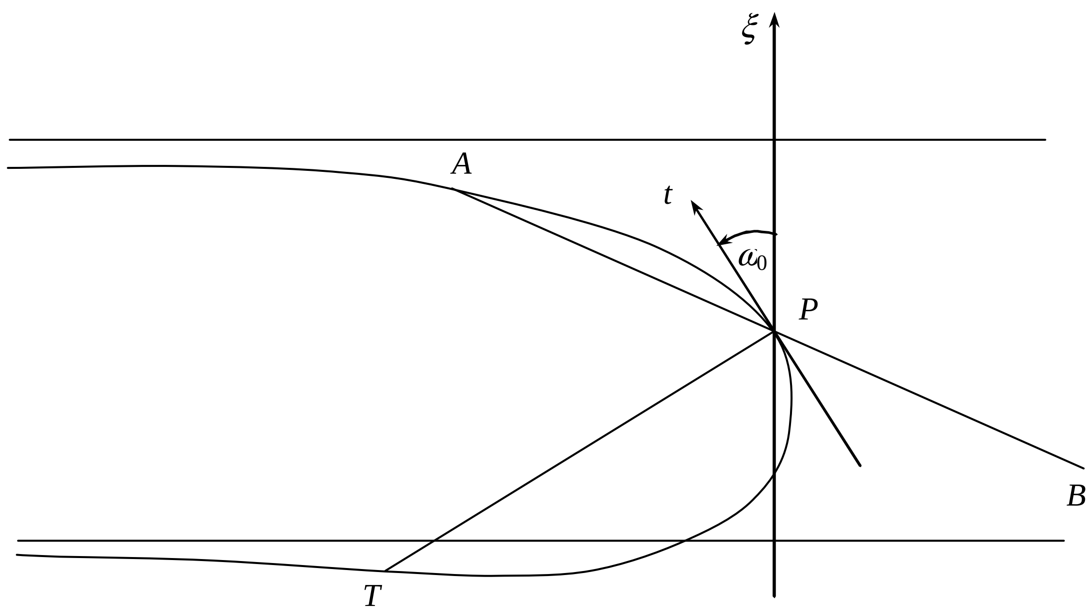

Figure 79.

This axis is the Z-axis here. If we drop the normal n with the foot N onto it from a point P then it will follow that:

Theorem 192: *The fourth-order surface F_4 that is associated with the point P through $\mathfrak{S}$ according to Theorem 67 can be generated by rotating a circle with the variable radius r whose center is always at P around n.*

We infer the dependency of r upon the position of the circle plane from (46):

$$\tan \omega = \frac{r + \mathfrak{k}'}{d}. \tag{51}$$

(*) One can find this theorem and the analogous ones for other axis manifolds (in another form) in *D'Emilio*, Gli Assoidi… [Atti del Ist. Ven. (6), 3, *b*; 1885].

Thus (from § 12, *b*), d can always be chosen to be positive, but then the sense of rotation of:

$$\omega = (\mathfrak{k}, \mathfrak{k}')$$

will be determined by *NP* as the positive semi-normal of the position $(\mathfrak{k}, \mathfrak{k}')$. Since negative values of $\mathfrak{k}$ are also meaningful, r must run through the entire interval $-\infty, \ldots, +\infty$. We get an intuitive picture of the *pectenoid* F_4 when we draw a plane E' through P perpendicular to PN and intersect it with F_4. (51) or:

$$r = d \tan \omega - \mathfrak{k}' \tag{52}$$

will then be the polar equation of the curve of intersection. The polar axis ζ will be parallel to the Z-axis. The latter will lie *below* the reference plane E' at a distance of d. From the limiting value that $\zeta = r \cos \omega = d \sin \omega - \mathfrak{k}' \cos \omega$ assumes for $\omega = \pm \pi / 2$, one sees that the curve will have the two lines $\zeta = \pm d$ for asymptotes. The figure is indicated for positive $\mathfrak{k}'$ (and indeed for $\mathfrak{k}' = 2d / 3$). If T is the point that is furthest from the asymptote then TP will be perpendicular to the tangent t at the origin P; t will be determined by $\tan \omega_0 = \mathfrak{k}' : d$. The complete F_4 will arise when one makes $PB = AP$ for every point A of the curve and describes a circle over AB as its diameter whose plane is perpendicular to the reference plane E'. However, by an argument that is similar to the one that was made for the rod cylindroid (where one took only half of the characteristic curve), ω can run through only the interval $0, \ldots, \pi$, so it can move on just the semi-circle whose projection covers the segment AP twice. The interval $\omega_0 < \omega < \pi / 2$ will correspond to rods with positive signs, and thus left-wound complexes, while the other two intervals will correspond to right-wound ones. One finds a drawing of the pectenoid in Ball's *Theory of Screws*, art. 236. One sees immediately (cf., Theorem 188) that F_4 decomposes into a sphere and a (doubly-counted) plane for $d = 0$.

This investigation will take on a *mechanical* interpretation when we represent $\mathfrak{k}'$ as the image of a winding W. From Theorem 96, the rod forest of all $\mathfrak{k}$ will then be a representation of all dynames, under whose influence a body that is only free to perform the winding W will remain in equilibrium. The investigation is also true for $\mathfrak{k}' = 0$, when W reduces to a pure rotation. By contrast, for $\mathfrak{k}' = \infty$ (i.e., W is a translation), we must revert to equation (54.a) in § 52. If we set $k' = 0$ there then we will obtain $k \cos \omega = 0$ as the condition for a dyname (k, m) to perform no work under the translation. $\mathfrak{S}$ will now consist of all rods that are perpendicular to the Z-axis and all rods at infinity – i.e., rotational moments – which is a result that is mechanically self-explanatory.

Naturally, the dual interpretation is also possible: One can understand $\mathfrak{k}'$ to mean the image of a dyname D. The rod forest will then represent all windings that a body can perform without D doing any work. In particular, the rods t of length zero whose direction is restricted to the contact plane of the pectenoid will represent the rotational axes (which were already known to *Möbius*) for which D does not perturb the equilibrium.

II) *The rod complex $\mathfrak{G}$ of a bush of complexes.*

A bush of complexes can then be defined in such a way that it is the extended domain to a pencil of complexes $\mathfrak{B}$. When we carry over the term "reciprocal" from the complexes and screws to the rods that represent them, we can say: We have to look for all rods that are reciprocal to the rods of the rod surface $\mathfrak{F}$ of $\mathfrak{B}$; for this, it will suffice for them to be reciprocal to two rods $\mathfrak{k}'$, $\mathfrak{k}''$ of $\mathfrak{F}$.

a) For the main case, in which $\mathfrak{F}$ is an actual cylindroid, we take the rods of the principal complexes to be $\mathfrak{k}'$ and $\mathfrak{k}''$, and assume, as in § 74, that $\mathfrak{k}'$ lies on the X-axis, and $\mathfrak{k}''$, on the Y-axis. We thus subsume the case α) in Theorem 154 (i.e., $\mathfrak{k}' = \mathfrak{k}''$), as well. If we let $\mathfrak{k}$ denote the variable rod that is reciprocal to $\mathfrak{k}'$ and $\mathfrak{k}''$ then its pointers α_i must fulfill the equations:

(53) $$(\mathfrak{k} + \mathfrak{k}')\,\alpha_1 + \alpha_4 = 0, \qquad (\mathfrak{k} + \mathfrak{k}'')\,\alpha_2 + \alpha_5 = 0,$$

which emerge from (49) by cyclic permutation. One thus has, once more:

$$\mathfrak{k} = \sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}\,.$$

In order to find the carrier complex $\mathfrak{T}$ of $\mathfrak{G}$, from Theorem 71, we have to eliminate t from the equations:

$$(t\,\mathfrak{k} + \mathfrak{k}')\,\alpha_1 + \alpha_4 = 0, \qquad (t\,\mathfrak{k} + \mathfrak{k}'')\,\alpha_2 + \alpha_5 = 0.$$

This will give:

(54) $$(\mathfrak{k}' - \mathfrak{k}'')\,\alpha_1\alpha_2 + \alpha_2\alpha_4 - \alpha_1\alpha_5 = 0$$

as the equation of $\mathfrak{T}$.

Theorem 193: *The axes of the complexes of a bush of complexes define a quadratic complex $\mathfrak{T}$.*

(54) will always be fulfilled by $\alpha_1 = \alpha_2 = 0$; i.e., the complex cone of any point will include the parallel p to the principal ray of the ray net of $\mathfrak{B}$ (as the double line of the cylindroid $\mathfrak{F}$). Equations (53) will represent $\mathfrak{G}$ as the intersection of two rod forests; it will then follow from Theorem 191 that:

Theorem 194: *$\mathfrak{G}$ can be decomposed into ∞^1 ray nets according to the length (endowed with a sign) of its rods.*

The equations of one such net that has a length of $\mathfrak{k} = r$ are:

$$(55) \qquad (r + \mathfrak{k}')\alpha_1 + \alpha_4 = 0, \qquad (r + \mathfrak{k}'')\alpha_2 + \alpha_5 = 0.$$

We seek the rod of the net that goes through the point $P' \equiv (x', y', z')$. If we let λ, μ, ν denote its direction cosines then we will have:

$$(56) \qquad \alpha_1 = x - x' = \pm \lambda r, \qquad \alpha_4 = zy' - z'y = \pm r(\nu y' - z'\mu), \qquad \text{etc.}$$

Equations (55) will go to:

$$(r + \mathfrak{k}')\lambda - z'\mu + y'\nu = 0,$$
$$z'\lambda - (r + \mathfrak{k}'')\mu - x'\nu = 0,$$

so

$$(57) \qquad \lambda : \mu : \nu = [x'z' - y'(r + \mathfrak{k}'')] : [y'z' + x'(r + \mathfrak{k}')] : [(r + \mathfrak{k}')(r + \mathfrak{k}'') + z'^2].$$

If we establish P' then we will have represented the complex cone $\mathfrak{K}$ as the rod surface when we know the direction of the rod as a function of its length. For $r = \infty$, one will have $\lambda : \nu = \mu : \nu = 0$. p will then be the asymptote of the "characteristic curve" on which the endpoints of the rod lie; r will have to run through the entire interval $+\infty, \ldots, -\infty$. For $r = 0$, the curve will go through the vertex of the cone, whereby the associated screws will change their sense of winding.

By means of (56), one will deduce from (54) that:

$$(58) \qquad (\lambda^2 + \mu^2)z' - \lambda\mu(\mathfrak{k}' - \mathfrak{k}'') + \lambda\nu x' - \mu\nu y' = 0.$$

One can regard this as the equation of the curve of intersection $\mathfrak{U}$ of $\mathfrak{K}$ with the plane at infinity in the homogeneous pointers λ, μ, ν. Should $\mathfrak{U}$ decompose into two lines then one would need to have:

$$\begin{vmatrix} 2z' & (\mathfrak{k}'' - \mathfrak{k}') & x' \\ (\mathfrak{k}'' - \mathfrak{k}') & 2z' & -y' \\ x' & -y' & 0 \end{vmatrix} = 0$$

or

$$(x'^2 + y'^2)z' = (\mathfrak{k}' - \mathfrak{k}'')x'y'.$$

Theorem 195: *The complex cone of $\mathfrak{G}$ decomposes into two planes for the points of the cylindroid that belongs to the reciprocal pencil of complexes.*

There is nothing difficult about representing the pointers x, y, z of a point of the characteristic curve as functions of r with the help of (56) and (57).

For the analogous investigations into the remaining cases, we can briefly assume:

b) In the cases *β*) and *γ*) of Theorem 154, $\mathfrak{F}$ consists of nothing but parallel rods. We decompose one of them $\mathfrak{k}'$ along the *Z*-axis; another $\mathfrak{k}''$ shall cut the *Y*-axis at the point $y = y_0$. The non-zero pointers of $\mathfrak{k}''$ will then be:

$$\alpha_3'' = \mathfrak{k}'', \qquad \alpha_4'' = y_0\, \mathfrak{k}'',$$

and the equations of $\mathfrak{G}$ will be:

$$(59) \qquad (\mathfrak{k} + \mathfrak{k}')\alpha_3 + \alpha_6 = 0, \qquad (\mathfrak{k}'' - \mathfrak{k}')\alpha_3 + y_0\, \alpha_1 = 0,$$

and the latter are, at the same time, the equations of the carrier complex $\mathfrak{T}$, which is *linear* and singular with an axis at infinity, here.

c) In the case *ε*) of Theorem 154, we decompose *b* into the *Z*-axis and obtain:

$$(\mathfrak{k} + \mathfrak{k}')\alpha_3 + \alpha_6 = 0, \qquad (\mathfrak{k} + \mathfrak{k}'')\alpha_3 + \alpha_6 = 0.$$

Both equations can be fulfilled only by $\alpha_3 = \alpha_6 = 0$, which is mechanically self-explanatory, from Theorem 46. The complex $\mathfrak{G}$ will consist of all rods that lie on the rays of a normal net, here.

d) Finally, if the carrier of the pencil of complexes $\mathfrak{B}$ is singular then all complexes will be singular, and their axes will define a planar pencil. If its plane *E*, as well as its vertex *S*, lie at infinity then one can regard this as a limiting case of case *a*) when one lets $\mathfrak{k}' = \mathfrak{k}'' = 0$ in the results there. Finally, if *S*, as well as *E*, lie at infinity then one can make sense of the pencil of complexes in terms of all translations that are perpendicular to a certain direction *ρ*. Any translation then belongs to a singular complex with an axis at infinity. One can now immediately look for either the dynames that perform no work relative to these motions or infer from the discussion of the case $\mathfrak{k}' = \infty$ in I) that $\mathfrak{S}$ consists of all rods that go through the point at infinity on *ρ*.

§ 83. The rod congruence $C_3^{(2)}$ of a net of complexes.

A) In the main case, we can determine the rod surface of a pencil of complexes from two rods $\mathfrak{k}$, $\mathfrak{k}'$ in it that intersect perpendicularly (§ 74). If we define the pointer system as we did there and add a third rod $\mathfrak{k}''$ on the *Z*-axis then the three rods that intersect each other perpendicularly will, in any event, define a net of complexes $\mathfrak{N}$. One asks only whether it is general; the constant count would suggest that (*). In order to decide whether that is true, we calculate the family of rulings $\mathfrak{R}$ of singular complexes in $\mathfrak{N}$ that is determined by the three complexes:

(60) $$p_4 + \mathfrak{k}' p_1 = 0, \qquad p_5 + \mathfrak{k}'' p_2 = 0, \qquad p_6 + \mathfrak{k}''' p_3 = 0.$$

In order to find the point equation of $\mathfrak{R}$ for an *arbitrary* net of complexes, we have to express the idea that a point $P \equiv (x, y, z)$ should lie in a common line of the three defining complexes:

(61) $$\sum a_i\, p_{i+3} = 0, \qquad \sum b_i\, p_{i+3} = 0, \qquad \sum c_i\, p_{i+3} = 0.$$

The condition for the incidence of P and p was expressed by the four equations (40) in § 38, although only two of them were independent. Therefore, two of these equations, the three equations (61), and:

(62) $$\omega(p) = \sum p_i\, p_{i+3} = 0,$$

must all be fulfilled by the values of p for a point P of $\mathfrak{R}$. However, we can also choose the other two of the aforementioned equations (40), instead of (62). The compatibility of equations (40) will then bring the fulfillment of (62) along with it, since the factor $\omega(p)$ is contained in the determinant of (40). We then have seven linear, homogeneous equations in the p. The seven sixth-order determinants from the matrix of their coefficients must vanish. Of the equations that we thus obtain, the four that contain the three sequences a, b, c will be equivalent, since equations (40) can go to each other by changing the notation, while the other three will be fulfilled identically, since otherwise that would give a condition for a point to lie on the common ray of two complexes. A single condition for P will then remain that will reduce to:

(63) $$\begin{vmatrix} & -x_4 & x_3 & -x_1 & & \\ x_4 & & -x_2 & & -x_1 & \\ -x_3 & x_2 & & & & -x_1 \\ \mathfrak{k}' & & & 1 & & \\ & \mathfrak{k}'' & & & 1 & \\ & & \mathfrak{k}''' & & & 1 \end{vmatrix} = 0$$

in our case (if we ignore the first row of the matrix). By developing the last three rows, one will get:

(63a) $$\mathfrak{k}'\,\mathfrak{k}''\,\mathfrak{k}''' + \mathfrak{k}'\,x_2^2 + \mathfrak{k}''\,x_3^2 + \mathfrak{k}'''\,x_4^2 = 0.$$

(*) A net of complexes, as a plane in a five-dimensional space, depends upon $3 \cdot 5 - 3 \cdot 2 = 9$ constants; this is the same as the number of triples of rods in it that define a rectangular trigon.

The calculations up to this point are true for an arbitrary pointer system; for our rectangular one, x_1, x_2, x_3, x_4 are replaced by 1, x, y¸ z (§ 31). The equation of $\mathfrak{R}$ will then be:

$$\frac{x^2}{\mathfrak{k}''\mathfrak{k}'''}+\frac{y^2}{\mathfrak{k}'''\mathfrak{k}'}+\frac{z^2}{\mathfrak{k}'\mathfrak{k}''}+1=0. \tag{64}$$

This will be an imaginary surface or a hyperboloid, according to whether α) all three quantities $\mathfrak{k}$ have the same symbol, or β) they do not.

Theorem 196: *If the singular surface F_2 of a net of complexes $\mathfrak{N}$ is an actual second-degree midpoint surface then $\mathfrak{N}$ can be defined by three complexes whose axes fall upon the principal axes of F_2 .*

With our assumption, we have thus, in fact, met up with the main case – viz., the "general" net of complexes. If we change the sign on all three $\mathfrak{k}$ then, from Theorem 189, we will obtain the extended domain of $\mathfrak{N}$, of which, we already know (Theorem 179) that it possesses the same singular surface, which we confirm here. From the pointers:

$$\begin{array}{llllll} a_1 = 1 & & & a_4 = \mathfrak{k}' & & \\ & b_2 = 1 & & & b_5 = \mathfrak{k}'' & \\ & & c_3 = 1 & & & c_6 = \mathfrak{k}''' \end{array}$$

of the three "principal complexes" of the net, we can compose those of the one arbitrary fourth complex d_i from the formula:

$$d_i = \lambda\, a_i + \mu\, b_i + \nu\, c_i \qquad (i = 1, \dots, 6)$$

namely:

$$\begin{array}{lll} d_1 = \lambda & d_2 = \mu & d_3 = \nu \\ d_4 = \lambda\,\mathfrak{k}' & d_5 = \mu\,\mathfrak{k}'' & d_6 = \nu\,\mathfrak{k}''', \end{array} \tag{65}$$

and calculate the pitch $\mathfrak{k}$ of d from Theorem 88 and the line pointers $\mathfrak{a}_i$ of its axis. We can then assume that $\lambda^2 + \mu^2 + \nu^2 = 1$; i.e., these quantities can be interpreted as the direction cosines of the axis.

$$\mathfrak{k} = \sum d_i\, d_{i+3} = \lambda^2\,\mathfrak{k}' + \mu^2\,\mathfrak{k}'' + \nu^2\,\mathfrak{k}'''. \tag{66}$$

$$\left\{\begin{array}{lll} \mathfrak{a}_1 = \lambda & \mathfrak{a}_2 = \mu & \mathfrak{a}_3 = \nu \\ \mathfrak{a}_4 = \lambda(\mathfrak{k}' - \mathfrak{k}) & \mathfrak{a}_5 = \mu(\mathfrak{k}'' - \mathfrak{k}) & \mathfrak{a}_6 = \nu(\mathfrak{k}''' - \mathfrak{k}). \end{array}\right. \tag{67}$$

Equations (67) exhibit a *unit rod* on the complex axis, while (66) give the length of the representative rod of d. (67) then represents the axis congruence C of $\mathfrak{N}$ without having to recall the pitch of the complex – i.e., as a *line structure* – and, by contrast, (67) and (66) together represent it as a rod congruence. If we express the idea that the rod (67) is incident with the point (x, y, z) then when we set, to abbreviate:

$$\mathfrak{k}' - \mathfrak{k} = \delta', \qquad \text{etc.},$$

we will get:

(68) $$\lambda\,\delta' x + \mu\,\delta'' y + \nu\,\delta''' z = 0,$$

(69) $$\begin{cases} \lambda\delta' + \mu z - \nu y = 0, \\ -\lambda z + \mu\delta'' + \nu x = 0, \\ \lambda y - \mu x + \nu\delta''' = 0 \end{cases}$$

from equations (40) of § 38. If will follow from (69) that:

(70) $$\begin{vmatrix} \mathfrak{k}' - \mathfrak{k} & z & -y \\ -z & \mathfrak{k}'' - \mathfrak{k} & x \\ y & -x & \mathfrak{k}''' - \mathfrak{k} \end{vmatrix} = 0$$

will determine $\mathfrak{k}$, from which each root will be associated with a unique direction $\lambda : \mu : \nu$. However, if we express the idea that the rod (67) is incident with the plane (u, v, w) then it will follow analogously from equations (39) of § 38 that:

(70a) $$\begin{vmatrix} 1 & \delta'' w & -\delta''' v \\ -\delta' w & 1 & \delta''' u \\ \delta' v & -\delta'' u & 1 \end{vmatrix} = 0.$$

This equation is only of degree two in $\mathfrak{k}$, so:

Theorem 197: *The axis congruence of a general net of complexes is of order three and class two.*

However, only one of the rays that go through the points at infinity will be real, since, from (67), the pointers are given uniquely as functions of the direction. We derive three equations from (67) that are linear for a constant $\mathfrak{k}$:

(71) $$\mathfrak{a}_4 = \mathfrak{a}_1\,\delta', \qquad \mathfrak{a}_5 = \mathfrak{a}_2\,\delta'', \qquad \mathfrak{a}_6 = \mathfrak{a}_3\,\delta'''.$$

However, the structure that is defined by the intersection of three linear complexes is a family of rulings, so:

Theorem 198: *The axis congruence of a general net of complexes can be decomposed into* ∞^1 *families of rulings according to the lengths of their rods.*

We can find the equations of this family of rulings in the same way that we previously found the equations of F_2. However, we see from a glance at the formulas that we now need to set $\mathfrak{k} - \mathfrak{k}'$, etc., in place of $\mathfrak{k}'$, etc., in the results. One will then have:

$$\frac{x^2}{(\mathfrak{k}-\mathfrak{k}')(\mathfrak{k}-\mathfrak{k}''')}+\frac{y^2}{(\mathfrak{k}-\mathfrak{k}''')(\mathfrak{k}-\mathfrak{k}')}+\frac{y^2}{(\mathfrak{k}-\mathfrak{k}''')(\mathfrak{k}-\mathfrak{k}')}+1=0 \tag{72}$$

for the equation of this ruled surface (naturally, F_2 is itself included in it for $\mathfrak{k} = 0$). For all values of $\mathfrak{k}$, it represents a coaxial system of surfaces that, in fact, contains only hyperboloids, along with imaginary surfaces. We separate the cases:

α) For example, let all three principal rods be positive. We can always assume that the middle lengths lie on the Y-axis, but in order to ensure that the pointer system is one of the first kind, we must admit two cases:

$$\mathfrak{k}' < \mathfrak{k}'' < \mathfrak{k}''' \qquad \text{or} \qquad \mathfrak{k}' > \mathfrak{k}'' > \mathfrak{k}'''.$$

It suffices to discuss one of them – say, the first one: We obtain real surfaces for the intervals $\mathfrak{k}' < \mathfrak{k} < \mathfrak{k}''$ and $\mathfrak{k}'' < \mathfrak{k} < \mathfrak{k}'''$, and in the first interval, we set:

$$\begin{aligned}(\mathfrak{k}-\mathfrak{k}'')(\mathfrak{k}-\mathfrak{k}''') &= a^2,\\ (\mathfrak{k}-\mathfrak{k}''')(\mathfrak{k}-\mathfrak{k}') &= -b^2,\\ (\mathfrak{k}-\mathfrak{k}')(\mathfrak{k}-\mathfrak{k}'') &= -c^2.\end{aligned} \tag{73}$$

ai, b, c are the three principal axes of the hyperboloid H, respectively. If we let $\mathfrak{k}$ decrease to $\mathfrak{k}'$ then H will enclose the X-axis ever more closely. However, if we let it increase to $\mathfrak{k}''$ then the asymptotic cone of H will approach the YZ-plane, while, at the same time, the throat ellipse in this plane will approach a certain finite segment on the Y-axis. If one lets $\mathfrak{k}'$ decrease monotonically through the interval $\mathfrak{k}''' \ldots \mathfrak{k}''$ then this second part of the family of hyperboloids will begin with ones that closely encircle the Z-axis and conclude with ones whose throat ellipse is the same segment as before, but this time it will approach the XY-plane. One can construct an intuitive picture of the distribution of rods in this congruence in that way.

In the case β), nothing in the foregoing discussion will change, except that one of the hyperboloids will become the singular surface F_2. Moreover, one sees from (72) that when one changes all three quantities $\mathfrak{k}'$, $\mathfrak{k}''$, $\mathfrak{k}'''$ by the same amount the family of surfaces will remain the same, except that the singular surface will be displaced inside the family, and, above all, the individual surfaces will be associated with the values of $\mathfrak{k}$ in a different way. An analogous situation already appears for the cylindroid and is true for the axis manifold of a linear manifold of a linear complex domain of *arbitrary* dimension. Since only the sums of the pitches appear in the condition for the reciprocity of two complexes:

$$(\mathfrak{k} + \mathfrak{k}') \cos \omega - d \sin \omega = 0$$

it will then follow that (Ball): If one increases the pitches of all complexes of a linear domain by the same amount (while keeping the same axes) and simultaneously decreases the pitches of all complexes of the extended domain by the same amount then, as before, each complex of the one domain will be reciprocal to each complex of the other one. However, since its dimensions must be preserved, the domain must remain linear (cf., Theorem 175).

Theorem 199: *Any axis manifold of a linear complex domain belongs to* ∞^1 *other domains of the same dimension. The associated rod structures emerge from one of them when one changes the lengths of all rods by the same amount.*

When $\mathfrak{k}' = \mathfrak{k}''$, only the second part of the family *H* will remain, which will now consists of hyperboloids of rotation. The case of $\mathfrak{k}' = \mathfrak{k}'' = \mathfrak{k}'''$ is especially noteworthy. It follows from (66) here that all rods will have the same length and from (67) that they will go through the origin. In fact, one can also infer this synthetically: From Theorem 154, $\mathfrak{k}'$, $\mathfrak{k}''$ will then determine a pencil of rods of constant length. If we couple a varying rod of it with $\mathfrak{k}'''$ then we will obtain ∞^1 such pencils that make up the congruence.

Theorem 200: *The totality of complexes of constant pitches whose axes intersect in the same point or lie in the same plane also belong to a net of complexes.*

The last part of the theorem is, in fact, illuminated by an entirely analogous construction. In these cases, the extended nets consist of all complexes of opposite pitch and the same axes.

We return to the general case and direct our remarks to the case in which $\mathfrak{k}$ was one of the principal rods; e.g., $\mathfrak{k} = \mathfrak{k}'$. Here, equations (71) will become:

$$\mathfrak{a}_4 = 0, \qquad \mathfrak{a}_4 = \mathfrak{a}_2 (\mathfrak{k}'' - \mathfrak{k}'), \qquad \mathfrak{a}_6 = \mathfrak{a}_3 (\mathfrak{k}''' - \mathfrak{k}').$$

The last two equations define a ray net. One must look for those of them that cut the *X*-axis, but themselves belong to the rays of the net. Here, the ruled surface then decomposes into two (real or imaginary) ray pencils, as we have already frequently encountered.

One obtains the equations of its planes from (72) when one first clears the denominators and then specializes $\mathfrak{k}$; e.g., for $\mathfrak{k} = \mathfrak{k}''$:

$$\frac{z}{x} = \sqrt{\frac{\mathfrak{k}'' - \mathfrak{k}'}{\mathfrak{k}''' - \mathfrak{k}''}}.$$

One sees that: Of the three pairs of pencils of rays that are present in the family of rulings, *two of them are imaginary and one of them is real*. The distances from the vertices to the origins are equal to the limiting values that the principal axes of the ruled surface that are laid along the same pointer axes assume; thus, for, e.g., $\mathfrak{k} = \mathfrak{k}''$:

$$e_2 = \sqrt{(\mathfrak{k}''' - \mathfrak{k}'')(\mathfrak{k}'' - \mathfrak{k}')}.$$

We confirm this when we seek all rays of the congruence that go through the point 0, e_2, 0 from (69) and (70). Here, (70) will yield:

$$\delta''(\delta' \delta''' + e_2^2) = 0.$$

In fact, μ becomes indeterminate and $\nu : \lambda$ becomes real for $\delta'' = 0$ or $\mathfrak{k} = \mathfrak{k}''$. We will then get the pencil of rays that is already known to us. The equation:

$$(\mathfrak{k}' - \mathfrak{k})(\mathfrak{k}'' - \mathfrak{k}) + e_2^2 = 0$$

will once more have the root $\mathfrak{k} = \mathfrak{k}''$, but in addition, it will also have:

$$\mathfrak{k} = \mathfrak{k}' - \mathfrak{k}'' + \mathfrak{k}''',$$

for which, $\mu = 0$ and $\nu : \lambda$ will be real. Thus, a ray of the congruence that is not contained in the pencil will then go through the vertex of the pencil. Thus:

Theorem 201: *The general axis congruence $C_3^{(2)}$ can also be determined by a pencil of singular complexes and a twist whose axis goes through the vertex of the pencil.*

Now, if one of the three principal complexes of the net is singular (e.g., $\mathfrak{k}' = 0$) then the axis congruence and the family H will show absolutely no peculiarity (Theorem 199), in and of themselves, except that the surface F_2 of the axes of the singular complexes will be inserted into a decomposable surface of the family H (three of which will be present in the family, in general), and we will have the case *A*, *b*) of § 79 before us.

For $\mathfrak{k}' = \mathfrak{k}''$, H will decompose into two pencils of rays only when $\mathfrak{k} = \mathfrak{k}'''$. By contrast, one likewise recognizes from (71) that for $\mathfrak{k} = \mathfrak{k}' = \mathfrak{k}''$ it will reduce to a pencil of rays in the *XY*-plane, and the case *A*, *c*) of § 79 will lie before us. Finally, for $\mathfrak{k}' = \mathfrak{k}'' = \mathfrak{k}''' = 0$, we will have the case *A*, *d*) of § 79.

We thus have only to examine the cases in which the basic family of rulings of the net (or its degenerate forms) exhibits a special behavior with respect to the plane at infinity, but before that we will consider the relationship of two congruences *C*, *C′* to each other that are defined by a net of complexes $\mathfrak{N}$ and the extended one $\mathfrak{N}'$, resp. The analytic representation of C' will emerge from that of C when one simultaneously changes the signs of $\mathfrak{k}'$, $\mathfrak{k}''$, $\mathfrak{k}'''$ in the formulas. The family of surfaces (72) will not change as a result of this; moreover, in each surface of the family, the one family of rulings will go through C, while the other one will go through C', as Theorem 188 will confirm. If a line g cuts two rays t_1, t_2 of C perpendicularly then it will belong to C'; one then draws a ray t of C through an arbitrary point P of g. It is the axis of a twist G of $\mathfrak{N}$, whose pitch we assume is $\mathfrak{k}$. If now lets g be the axis of a twist G' with an pitch of $-\mathfrak{k}$ then, from Theorem 188, G' will be reciprocal to G, as well as to the two twists of $\mathfrak{N}$ whose axes lie upon t_1, t_2. Thus, G' will belong to $\mathfrak{N}'$.

Theorem 202: *Of the axis congruences that belong to two mutually-extending general nets of complexes, each of them will be the system of shortest transversals between any two rays of the other one.*

If one fixes a hyperboloid in the family (72) and a family of rulings R of C on it then that will already define ∞^2 shortest transversals that therefore belong to C'. Conversely, if a ray s' of C' cuts two rays t_1, t_2 of R at real points then, from Theorem 188, this will happen perpendicularly. The pitches that belong to s' and t_1 cannot be equal and opposite then, since s' and t_1 will lie on different surfaces of the family. Thus:

Theorem 203: *The system of ∞^2 shortest distances* (*) *between any two rays of a family of rulings of second order is (apart from reality questions) identical with the axis congruence of a net of complexes.*

B) We now turn to the cases in which the axis congruence of a net of complexes N does not contain three mutually-perpendicular, intersecting rays, which are just the cases that we have still not encountered up to now (cf., § 79):

a) When the basic family of rulings R of N lies on a hyperbolic paraboloid P.

b) When N can be obtained in the manner of Theorem 180, and indeed in such a way that α) one or β) both vertices of the pencils of rays into which R decomposes lie at infinity. In the last case, γ) one of the pencils can itself lie at infinity.

c) When N arises in the manner of Theorem 181, such that the vertex of the pencils of singular complexes lies at infinity.

(*) *Waelsch* ("Über eine Strahlenkongruenz beim Hyperboloid," Wiener Sitzgber., Bd. 95, II; 1887) examined this congruence from this starting point, whose definition is suitable for any surface of the family H, even ones that decompose into two pencils of rays. Cf., also *Demoulin, Applic. d'une meth. vect.*, etc., Bruxelles, 1894.

d) When the axes of the complexes of N fill up α) a sheaf with a vertex at infinity or β) the field at infinity (the case of the field at finite points is dealt with in Theorem 200).

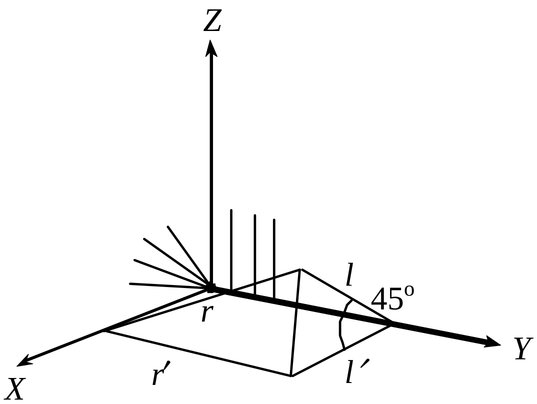

Figure 80.

a) We saw in *A*) that decomposable surfaces could appear there among the surfaces H that could serve as the starting point of the examination, but that was no advantage there. Here, one might likewise expect that the cases *a*) and *b*, α) are identical, which is why we will first examine *b*, α). Only in the case for which P is *equilateral* do we require its special feature:

Let the X and Y axes of a rectangular system be the principal generators of P, and indeed, the latter are the ones that are included in R. The guiding family L of R will then be the locus of the axes of the singular complexes of N and can be defined by the X-axis, the line at infinity u in the YZ-plane, and a line l that cuts the Y-axis perpendicularly and defines the same angles with the other axes. We can choose the non-zero pointers a_i, b_i, c_i of these three lines to be:

$$a_1 = 1, \qquad b_4 = 1,$$
$$c_1 = c_3 = 1, \qquad c_4 = -c_6 = 1,$$

resp. The three singular complexes a, b, c define N completely, but so do the three complexes a_i, b_i, $c_i - b_i - a_i$; i.e., of the last row, we need only to keep:

$$c_3 = 1, \qquad c_6 = -1,$$

and we now summarize the pointers of an arbitrary complex of N using the formula $d_i = \lambda a_i + \mu b_i + \nu c_i$:

$$(74) \qquad \begin{aligned} d_1 &= \lambda, & d_2 &= 0, & d_3 &= \nu, \\ d_4 &= \mu, & d_5 &= 0, & d_6 &= -\nu . \end{aligned}$$

By Theorem 88, the axis of d is determined by:

$$(75) \qquad \mathfrak{k} = \frac{\lambda\mu - \nu^2}{\lambda^2 + \nu^2} .$$

$$(76) \qquad \begin{aligned} \mathfrak{a}_1 &= \lambda, & \mathfrak{a}_2 &= 0, & \mathfrak{a}_3 &= \nu, \\ \mathfrak{a}_4 &= \mu - \lambda\,\mathfrak{k}, & \mathfrak{a}_5 &= 0, & \mathfrak{a}_6 &= -\nu(1 + \mathfrak{k}). \end{aligned}$$

The axes thus always satisfy the three equations:

$$(77) \qquad \mathfrak{a}_2 = 0, \qquad \mathfrak{a}_5 = 0, \qquad \mathfrak{a}_6 + \mathfrak{a}_3 (1 + k) = 0,$$

the first two of which define the normal net Σ of the Y-axis; for constant $\mathfrak{k}$, the third one represents a twist whose axis lies on the Z-axis, so a family of rulings of an equilateral paraboloid will then be singled out of Σ.

Theorem 204: *The axis congruence of a net of complexes whose basic family of rulings R lies on an equilateral paraboloid P consists of the normal net of the principal generators of R. The congruence decomposes into* ∞^1 *families of rulings of equilateral paraboloids that all have the principal generator in common with P according to the lengths of their rods.*

One of these families of rulings – viz., $\mathfrak{k} = -1$ – decomposes into two pencils of rays, one of which has its vertex at infinity (Fig. 80).

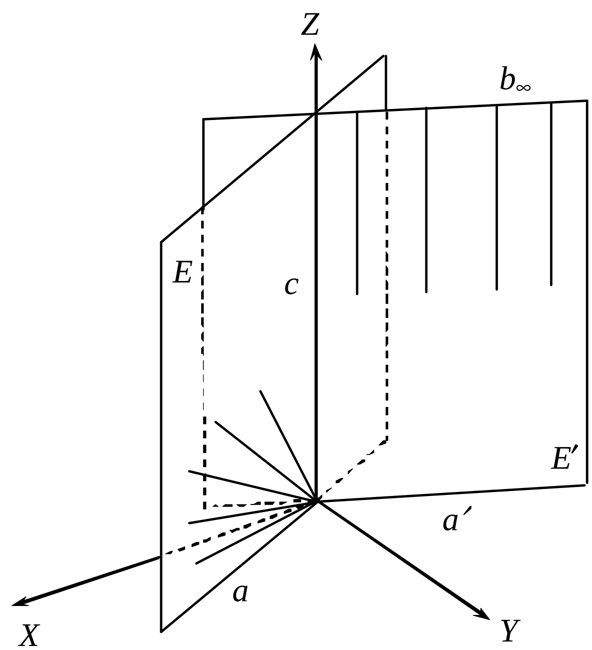

Figure 81.

b, α) Let two pencils of rays with a common ray be defined by two lines *a*, *b*, *c*, of which, *a*, *c* intersect, as well as *b*, *c*, but not *a*, *b*. Let *E*, *E′* be the planes of the pencils, and let *b* be at infinity (Fig. 81). We place the origin of a rectangular system at the vertex of the pencil (*a*, *c*), the *Z*-axis along *c*, and the *X* and *Y* axes in the bisecting planes of the two wedges (*E*, *E′*). If we finally assume that $a \perp c$ then we can assume that the non-zero pointers of *a*, *b*, *c* are:

$$a_1 = 1, \qquad a_2 = 1, \qquad b_4 = 1, \qquad b_5 = 1,$$

$$c_3 = 1.$$

We can leave aside the case $m = 1$, in which $E \perp E'$, which was just dealt with, and assume that $m < 1$, which comes from locating the X-axis in the acute wedge. As the axes of singular complexes, *a*, *b*, *c* will define a net of complexes *N*, so an arbitrary complex of *N* will have the pointers:

$$(78) \qquad \begin{array}{lll} d_1 = \lambda, & d_2 = \lambda\mu, & d_3 = \nu, \\ d_4 = \mu\, m, & d_5 = \mu, & d_6 = 0, \end{array}$$

and the axis (Theorem 88):

$$(79) \qquad \begin{cases} \mathfrak{a}_1 = \lambda, & \mathfrak{a}_2 = \lambda m, & \mathfrak{a}_3 = \nu, \\ \mathfrak{a}_4 = \mu m - \lambda\,\mathfrak{k}, & \mathfrak{a}_5 = \mu - \lambda m\,\mathfrak{k}, & \mathfrak{a}_6 = -\nu\,\mathfrak{k}, \end{cases}$$

where:

$$(80) \qquad \mathfrak{k} = \frac{2\lambda\mu m}{\lambda^2(1+m^2)+\nu^2}.$$

The $\mathfrak{a}$ will satisfy the equations:

$$(81) \qquad m\,\mathfrak{a}_1 - \mathfrak{a}_2 = 0, \qquad \mathfrak{k}\,\mathfrak{a}_3 + \mathfrak{a}_6 = 0, \qquad \mathfrak{k}\,(1 - m^2)\,\mathfrak{a}_1 + \mathfrak{a}_4 - m\,\mathfrak{a}_5 = 0,$$

which represent a family of rulings as the intersection of three complexes for a constant $\mathfrak{k}$. One will obtain its equation in point pointers by a calculation that is similar to the one in *A*), namely:

$$(82) \qquad y^2 - m^2\,x^2 - 2m\,\mathfrak{k}\,z + \mathfrak{k}^2\,(1 - m^2) = 0.$$

For $\mathfrak{k} = 0$, we will obtain the plane pair that we started with, but for all other values of $\mathfrak{k}$ we will get paraboloids with common principal planes E, E'. They will all have the same form; their magnitude will be proportional to $\sqrt{\mathfrak{k}}$. With that, we will have an intuitive picture of the distribution of rods in space. Equation (82) is only special to the extent that the rod $\mathfrak{k} = 0$ falls on the decomposable surface. We need only to write $\mathfrak{k} - \mathfrak{k}'$ instead of $\mathfrak{k}$ (Theorem 199) in order to make the axes of the singular complexes shift over to a general surface of the family:

$$(83) \qquad \frac{y^2 - m^2x^2}{(\mathfrak{k}-\mathfrak{k}')^2} - \frac{2mz}{\mathfrak{k}-\mathfrak{k}'} + 1 - m^2 = 0.$$

Since the paraboloids that appear are general, and the entire family is determined by each of them, we will subsequently see that the case *a*) is also resolved with that. There is nothing difficult about writing down the equations that correspond to (70) and (70.a) here, and to then see from them that the axis congruence is now of order and class two. If one takes the family of rulings that is parallel to a certain principal plane E from all paraboloids then the shortest transversals to any two rays will all be perpendicular to E. In fact, this will separate a sheaf of rays with a vertex at infinity from the general $C_3^{(2)}$ of case *A*).

Theorem 205: *If the basic family of rulings of a net of complexes lies on a skew paraboloid P then the axis congruence will be of order and class two and will decompose, according to the lengths of the rods, into* ∞^1 *families of rulings (amongst which, there is a decomposable one) on paraboloids that have their principal planes (but no longer their vertices) in common with P.*

b, β) We obtain this case when we couple a pencil of complexes *B* with a singular complex *C* whose axis falls on the ray at infinity of the ray net *N* of *B*. Let the rod surface of *B* be a cylindroid. A rod of it defines a pencil of complexes with *C* whose axis surface consists of nothing but equal parallel rods (Theorem 154, γ).

Theorem 206: *If the basic family of rulings of a net of complexes decomposes into two pencils of rays with vertices at infinity then the axis congruence will decompose according to the lengths of its rods into* ∞^1 *pencils of parallel rays and can be obtained from a rod cylindroid in which one displaces any of its rods along a direction that is perpendicular to the double line (but does not coincide with it).*

b, γ) The net complex *N* can be defined here by two lines at infinity in two (mutually perpendicular) planes *E*, *E′*, and a line *g* in *E* that is the axis of three singular complexes. The line of intersection *s* of *E*, *E′* is the axis of a pencil of planes whose lines at infinity *u* all define singular complexes that belong to *N*. *N* can then also be defined as the totality of the pencils of complexes that link *g* with all bushes of rays *u*. These are nothing but pencils of complexes whose rods make up a pencil of parallels that *g* belongs to, as in Theorem 154, β). The rods of *N* thus define a sheaf of parallels, in general, that must be defined by three of its rods. In fact, one sees immediately that: If one defines a net of complexes by three parallel rods in the direction ρ and links their starting and ending points with a plane then, from Theorem 154, β) and γ), all rods in the direction ρ that are bounded by the two planes will belong to the rod congruence of the net.

b, δ) However, if *g* is perpendicular to *s* then the rod congruence (cf., § 74, *b*) will consist of all rods of a pencil of parallel rays. One obtains this case when one chooses the three parallel rods to be in the same plane (but not bounded by two lines).

c) Let *E* be the plane of the pencil *B* of singular complexes *C* with parallel axes α, and let *a* be the axis of twist *G* that includes all axes α; one will then have $a \parallel E$. If we define a net of complexes *N* by *G* and two of the *C* then all twists *G* that arise from *G* by a displacement in the direction α will also belong to *N*. They will then define a pencil of complexes whose carrier is a special net of rays with a focal line at infinity (Theorem 154, γ), that is, on the other hand, defined by *G* and the line at infinity of *E*. Any *G′* and any *C* will define a cylindroid whose rod surface belongs to the axis congruence $\mathfrak{G}$ of *N*. We will obtain ∞^2 such cylindroids, which, however, contain only ∞^2 rays. Each of them can be obtained from every other one by a translation of the position of *E*. We will once more come down to the case *b*, β). $\mathfrak{G}$ will be of order two and class one, here. A ray of $\mathfrak{G}$ in an arbitrary plane $\mathfrak{E}$ can only have the direction (*E*, $\mathfrak{E}$) that enters a cylindroid only once.

However, if the axes of the singular complex *C* define a pencil at infinity then any *C* will define a special ray net with focal lines at infinity, along with *G*. From Theorem 154, γ), one will thus come to the case in which the rod congruence consists of nothing but parallel rods of equal length, which is included in *b*, γ).

d, α) is treated as in *b*, γ); we shall say nothing further about *d*, β).

We now summarize the various degeneracies of the axis congruences of a net of complexes, without once more specifying the cases in which they appear:

Theorem 207: *The rod congruence that represents the axes and pitches of the complexes of a net is, in general, of order three and class two. In special cases:*

α) *It is of order and class two,*

β) *It is of order two and class one,*

γ) *It is a normal net,*

δ) *Its rods belong to the same sheaf (they are of equal length or can be unequal, according to whether the vertex of the sheaf does or does not lie at infinity, respectively),*

ε) *Its rods belong to the same field (and are of equal length when the field lies at finite points),*

ζ) *Its rods fill up a pencil of parallel rays.*

____

§ 84. The degrees of freedom for motion.

The position of a freely-moving, rigid body *K* depends upon six constants c_i (e.g., one needs three of them to fix a point *P* in it, two more to fix an axis *a* that goes through *P*, and the last one, to establish the azimuth). If the motion of the body is subject to conditions from the outset then that will reduce the number of constants. For example, the restriction that a point *Q* of the body is constrained to remain on a surface is a "simple condition": i.e., it will result in *one* equation between the constants c_i . This will be the equation of the surface itself when one selects the pointers of *Q* from the c_i; the body will, moreover, assume only ∞^5 positions. One now says that the motion of a body has *k* degrees of freedom when it can assume ∞^k positions, or, what amounts to the same thing, when the choice of the values of *k* parameters, in addition to the given conditions, is necessary for the determination of its position. For example, when $6 - k$ points of a body are constrained to move on a surface, its motion will have *k* degrees of freedom ($k = 1, \ldots, 6$), and when one point of a body is fixed, its motion will have three degrees of freedom.

If *K* exhibits any motion from a certain initial position *L* then the beginning of the motion will correspond to a certain instantaneous twist (§ 20) that is determined (except for its velocity and its sense) by the pitch rod (endowed with a sign) of the associated instantaneous screw. We now address the question of finding the connection between the degree of freedom in the motion of all instantaneous screws that correspond to the position *L*. If two windings W_1 and W_2 are compatible with the conditions then one can combine them with arbitrary velocity ratios, and in that way obtain a winding *W* that is also compatible with the given conditions. Then, instead of immediately subjecting *K* to the given conditions, one can think of this body as being coupled with another one K' in such a way that only the instantaneous winding W_1 can enter into it, while K' should be subject to the same conditions as *K* was before. In that way, *K* will keep the same degree of freedom, since the mobility of K with respect to K' is restricted to a winding that K' can already enter into with an arbitrary velocity. However, the fact that the winding velocities of that screw can be added only algebraically is clear. The replacement above is then actually equivalent to the original conditions. If we now let K belong to the winding W_1 relative to the K' and, at the same time, let K' itself belong to the winding W_2, both of which have arbitrary velocities, then we will obtain (§ 17) an instantaneous winding of K that is composed of W_1 and W_2 in the same way that a dyname is composed of two given ones by means of duality (§ 18). We have, however, discussed the last case quite thoroughly (§ 74 – 76), and can therefore carry over the results to the present discussion: If one represents W_1 and W_2 by a rod then any winding of the rod surface F that is defined by this (which is, in general, a rod cylindroid; however, cf., Theorem 154) will be compatible with the conditions.

If, along with W_1 and W_2, yet a third winding W_3 of K that is independent of them (i.e., it does not belong to F) is compatible with the conditions then every rod surface that is defined by W_3 and a rod of F will represent nothing but windings that are compatible with the conditions, and we will thus arrive at the rod congruences that were studied precisely in § 83, etc.

Theorem 208: *The instantaneous windings that a body with k degrees of freedom in its motion can enter into from a certain position define a linear domain of rank k.*

We call this linear domain the *associated winding domain* to any position. We can carry over all theorems that we learned about linear complex domains and their representations by rod structures to the present context. Here, any rod represents a winding. If the rod lengths were zero, while the carrier preserved a certain position, then the winding would go over to a pure rotation. If one shifts its carrier to infinity then it will represent the translation that is perpendicular to the position of its representative plane.

We first consider *two* degrees of freedom more closely: The entire winding domain G that is associated with a position will be defined by two windings W_1, W_2 here. Two linear complexes C_1, C_2 that have a ray net N in common are given, along with W_1, W_2. Let W be an arbitrary winding of the domain, and let C be the associated linear complex. A point P will then be associated with a direction of advance by W that lies in the normal to the null plane of P relative to C. If we let W assume all possible positions on the rod surface then C will describe the pencil of complexes whose axis surface is F; v will then rotate around the ray of N that goes through P. However, if an entire pencil of such rays of N goes through P then v will constantly remain in the normals of that pencil. Finally, if an entire sheaf of rays of the net goes through P (which is possible for only singular nets) then P will be fixed. Conversely, since a pencil of complexes is also established uniquely by a ray net, we can say (*):

Theorem 209: *A motion with two degrees of freedom is determined completely by a ray net N. The velocities that a point can assume under all allowable motions are, in general, restricted to a plane pencil of rays whose normal is the ray of N that goes*

(*) This theorem essentially originated with *Schönemann* ("Über die Konstr. der Normalen, etc.," Berliner Akad., 1855, reprinted in Journ. f. Math., Bd. 90).

through P. It is only when ∞^1 *or* ∞^2 *rays of N go through P that the velocity of P will be restricted to a single direction (with both senses) or zero, respectively.*

If N is hyperbolic then for any point of one focal line b the normal to its connecting line with the other focal line b' will be the only possible direction of advance. This agrees with the fact that any winding of G can be composed of two rotations around b, b'; in fact, b, b' are the common polar pairs for all ray twists of the pencil (cf., § 53 and Theorem 31). One can compose any winding of G from two whose axes intersect perpendicularly (one takes the principal rod of the cylindroid) in other ways that are distinctive and independent of the reality of focal lines; if N is parabolic then the one winding will go to a rotation (*).

If one poses the argument that leads to this theorem in a completely analogous way for the higher degrees of freedom then that will yield:

Theorem 210: *If the motion of a rigid body possesses k degrees of freedom* ($k \geq 3$) *then one of its points P can move in all directions in space, in general. It is only when one,* ∞^1, *or* ∞^2 *common rays of the associated winding domain go through P that the directions of motion will be restricted to a plane pencil of rays or a line, or cease to exist, respectively.*

For example, if the three degrees of freedom of the common rays of the associated net of complexes define a real family of rulings then the points of that hyperboloid will have only two degrees of freedom, so to speak. We shall pass over the remaining cases, since we almost have to do nothing but repeat the discussion in § 79. Many theorems on the decomposition and composition of windings for higher degrees of freedom are also derived from the present standpoint in itself; e.g. (cf., the determination of the rod congruence in § 83 by the three principal rods):

Theorem 211: *In the general case of motion with three degrees of freedom, any possible winding can be composed of three windings whose axes define a rectangular trigon.*

If we prescribe a surface that cannot be ignored for four points of a rigid body then we will have two degrees of motion, and the four normals to the surface at the four points will determine the associated ray net for any position of the body. Since we can determine all ray nets by four rays, we can, conversely, obtain two degrees of freedom. The choice of the four points of the body is restricted only by the fact that the four normals must actually determine a net; nothing will prevent, e.g., three of them from lying in a plane.

We can no longer obtain three degrees of freedom in an analogous way. If we then prescribe three points of a surface then the three normals will be the common rays of all complexes of the associated winding domain. The basic family of rulings $\mathfrak{G}$ of this domain will thus always be real. This poses the question of whether the case in which $\mathfrak{G}$ is not real and all of the remaining (special) cases can be realized by real motions at all.

(*) *Schönemann*'s belief that the motion in this case will reduce to a rotation around the single focal line (*loc. cit.*, art. 3) is therefore incorrect.

If we define an arbitrary given winding domain of rank k by k independent windings W_1, …, W_k then we can couple the body K with another one W_1 in such a way that it can belong to W_1 relative to K_1, then couple K_1 with K_2 in such a way that K_1 can belong to only W_2 relative to K_2, etc. Finally, K will have k degrees of freedom compared to K_k with the given winding domain.

Theorem 212: *Any linear winding domain can appear as the associated domain of a real motion.*

Theorem 213: *k degrees of freedom can be obtained in such a way that one forces* $6 - k$ *suitable points of the body to remain on a surface only for* $k = 1$ *and* 2.

Thus, only the contact planes of the surfaces will come under consideration for a particular position. Naturally, the process for realizing an arbitrary degree of freedom that was just suggested is not the simplest one. For example, five degrees of freedom can be attained in the most general way by a simple mechanical device; on this, one might look at *Thomson and Tait*, *Theor. Phys. I*, art. 201.

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§ 85. The equilibrium of a rigid body.

We first begin with a theorem from the theory of motion that we can evaluate for force systems in terms of duality. If one defines a new winding W linearly from k independent windings $W_1, \ldots, W_k$, which therefore define a linear domain of rank k – i.e., we compose k windings with arbitrary velocity ratios – then W will also belong to the domain G (cf., the proof of Theorem 208), and conversely: Any winding that belongs to G must be linearly derivable from $W_1, \ldots, W_k$ (cf., § 77). *Any* winding at all must be derivable from six independent windings. If we replace the resulting winding W by the opposite one W', which also belongs to G, then the $k + 1$ windings $W', W_1, \ldots, W_k$ will give a zero resultant.

Theorem 214: *If n windings have a zero resultant then they will belong to the same winding domain of rank* $n - 1$; *its representative rods will thus belong to the same* $n - 2$-*dimensional rod structure, namely, for* $n = 3, 4, 5, 6$, *in general, they will belong to a rod cylindroid, a rod congruence of order three and class two, a quadratic rod complex, and a rod forest of degree four, respectively* (§ 82). *Seven windings can always be composed with velocities in such a way that they will yield a zero resultant.*

In particular, if we demand that all windings possess the same pitch then it will follow from the decomposition of this rod structure according to the lengths of its rods (Theorem 191, 194, 198, 204, 205) that:

Theorem 215: *If n windings of equal pitch have a zero resultant then for* $n = 4, 5, 6$, *their axes will belong to the same family of rulings, the same ray net, and the same linear complex, respectively.*

The theorem is also true for the special case of rotations (i.e., zero pitch); we express it in the dual domain as:

Theorem 216: *If n forces are in equilibrium then for* $n = 4, 5, 6$ *their lines of action will belong to the same family of rulings, the same ray net, and the same linear complex, respectively. One can always find forces on seven lines that are in equilibrium.*

Naturally, the special case of forces in equilibrium admits an elementary treatment: For example, if one draws a line g that cuts the lines of action of three of four forces then the moment of the fourth force with respect to g must also be zero (*Möbius*), etc. For five forces, one can conclude the corresponding theorem in this way only when the associated ray net has real, separate focal lines.

If n lines $g_1, \ldots, g_n$ fulfill the conditions of Theorem 216 then that will pose the question of how one can construct n forces on them that are in equilibrium.

a) $n = 4$. The associated force polygon must be closed. If one then chooses a force on g_1 arbitrarily and draws parallels g_2' and g_4' to g_2 and g_4 at the ends of their rods then one will find a single ray g_3' that cuts g_2', g_4', and is parallel to g_3. The sides of this closed, skew tetrangle will represent the magnitudes of the forces.

Theorem 217: *Two families of rulings of the same ray net or one family of rulings and one ray net of the same linear complex have two common rays.*

In the first case, one can, in fact, represent the ray net by two linear equations between line pointers and each of the families of rulings by yet a third equation; if one recalls the relation between the line pointers then that will yield the first part of the theorem, and the second one analogously.

b) $n = 5$. We treat this case as *Sturm* did ("Sulle forze in equil.," Ann. di Mat., ser. 2, v. 7, 1875), where the remaining cases are also addressed, along with references to the older literature (*). g_1, g_2, g_3 and g_1, g_4, g_5 determine two families of rulings R and R', which, from Theorem 216, belong to the same ray N, and therefore, from Theorem 217, they will have, in addition to g_1, yet a second ray g' in common. It can be found when one draws two planes E, E' through g_1 and considers N to be the generator of two

(*) Among them are (in addition to the previously-cited work of *Möbius*), the works of *Sylvester,* and especially *Chasles* (Comptes R., t. 16 and 52) are to be stressed. We find a supremely simple argument in the latter: If six forces with the lines of action $g_1, \ldots, g_6$ are in equilibrium in a rigid body K then the sum of the works that they do during an arbitrary motion of K must be zero. If one now determines a twist G by $g_1, \ldots, g_6$ and assigns the instantaneous winding that is thus defined to K then the corresponding forces will be perpendicular to the paths of their points of application (§ 22), so they will perform no work. Therefore, the sixth force can also do no work, and will act perpendicular to the path of its point of application; i.e., its line of action will belong to G.

collinear fields (Theorem 100). In that way, R, R' will be mapped to E by lines r, r'. The point of intersection of r, r' will determine g'. If one now chooses two equal and opposite forces k, $-k$ on g', then constructs a system S that is in equilibrium with g_1, g_2, g_3, g' and k belongs to, as in a), and furthermore, a system S' that is in equilibrium with g_1, g_4, g_5, g' and $-k$ belongs to then the system $S + S'$ will also be in equilibrium. However, the forces on g' cancel each other in such a way that only a system of five forces on the given lines of action will remain. If the point of intersection of r, r' falls on g_1 then g' will coincide with g_1, and R, R' contact along g_1. One can then let one of the other five lines play the role of g_1.

The theorems up to now on forces in equilibrium (and the dual version of Theorem 214) can be regarded as theorems about the equilibrium of a free, rigid body. We now turn to the most general theorem that can be posed about the equilibrium of a rigid body ([*]): The windings that K can perform at a given moment are defined by a linear winding domain G (§ 84). Should K remain in equilibrium for a dyname D then D must be reciprocal to any winding of G (cf., Theorem 96). Conversely, this condition will be sufficient. If K belonged to a winding W for D then D would thus do work, so it would not be reciprocal to W. Now, the screws on which all of the dynames that are reciprocal to all windings of G will also define a linear domain, namely, the extended domain (§ 78); thus:

Theorem 218: *A rigid body K with m degrees of freedom in its motion remains in equilibrium for all dynames in equilibrium whose screws fill up the* $6 - m$*-rank extended domain of the winding domain of K.*

Since we have discussed linear screw domains, their relationship to the extended domains, and their representation by rod structures precisely, despite its great generality, we can also endow this theorem with an intuitive content and deduce numerous consequences from it: If one determines a family of rulings by way of the three axes of three windings of a body with three degrees of freedom then the axes of the dynames for which K remains in equilibrium will define the system of shortest transversals of any two rays of R (cf., Theorem 202 and 203).

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§ 86. Pointers for linear complex domains.

We can repeat the steps that led us from the point pointers to the line pointers in space of complexes: Let:

(84) $$x_{i1}, x_{i2}, \ldots, x_{i6} \qquad i = (1, \ldots, \mu;\ \mu < 6)$$

([*]) Ball, *Theory of Screws*, art. 73. We once more emphatically draw attention to this comprehensive and original work.

be the pointers of μ linear complexes. We assume that the matrix (84) has rank μ, so a rank μ linear complex domain G will be defined by it. We call the $\binom{6}{\mu}$ determinants of order μ that can be defined by it the *homogeneous pointers of G*. Only their ratios will enter into this, which are independent of the choice of the complex inside of G. If one then sets:

$$\sum_{i=1}^{\mu} \alpha_{i\kappa} x_{i\nu} \qquad (\nu = 1, \ldots, 6;\ \kappa = 1, \ldots, \mu),$$

in place of $x_{\kappa\nu}$ then the pointers will all be multiplied by the determinant of the substitution $|\alpha_{i\kappa}|$ (cf., prob. 26). Since, by assumption, one and the same G can be determined by μ complexes in $\infty^{\mu(\mu-1)}$ ways, it will depend upon:

$$5\,\mu - (\mu - 1)\,\mu = \mu\,(6 - \mu)$$

constants. There must then be:

$$\binom{6}{\mu} - 1 - m\,(6 - m)$$

relations between the ratios of the pointers and just as many *homogeneous* relations between the pointers themselves. The argument does not change when we base it upon an arbitrary n-dimensional space, instead of the five-dimensional complex space, and accordingly substitute $n + 1$ for 6, in the formulas:

Theorem 219: *Of the* $\binom{n+1}{\mu}$ *homogeneous pointers of an* $m - 1$*-dimensional domain in an n-dimensional space, all of the remaining ones can be determined by* $1 + \mu\,(n + 1 - \mu)$ *suitable ones that are mutually independent.*

For a pencil of complexes, the matrix will be:

$$\begin{matrix} x_{11} & \cdots & x_{16} \\ x_{21} & \cdots & x_{26} \end{matrix} \tag{85}$$

We set $x_{1i}\,x_{2k} - x_{1k}\,x_{2i} = u_{ik}$. In order to find the six relations between the u, we once more think of writing down the same matrix as in (85), such that we can now define fifteen four-rowed determinants from the new matrix that are all zero and produce one relation in precisely the same way as we did for line pointers (§ 32). Of these relations, however, nine suitably-chosen ones must be consequences of the six remaining ones. In particular, we shall consider the six relations that we obtain when we couple two fixed columns – e.g., the last two – with two others in all ways in the definition of the determinants. These six relations are certainly independent of each other; each u_{ik} ($i, k \leq 4$) is then present in only of them. For that reason, these relations will also be soluble rationally in terms of the aforementioned six u_{ik}.

Theorem 220: *Six independent quadratic relations exist between the fifteen homogeneous pointers of a pencil of complexes (ray net); the remaining six pointers can be expressed rationally in terms of nine suitably-chosen ones of them.*

The qualification "suitably-chosen" is necessary if one is to exclude certain special choices; e.g., it is clear that when one has nine pointers that involve only five columns of the matrix, the remaining pointers cannot be determined in terms of them, since the ray net itself will not be determined.

In order to find the ten relations for $\mu = 3$, we must apply the process of *Vahlens* ("Über die Relationen zw. den Determ. einer Matrix," Journ. f. Math., Bd. 112): We let u_{klm} denote the determinant of the k^{th}, l^{th}, and m^{th} columns of the matrix:

$$\begin{matrix} x_{11} \dots x_{16} \\ x_{21} \dots x_{26} \\ x_{31} \dots x_{36} \end{matrix} \tag{86}$$

and denote the adjoint of $x_{\lambda\mu}$ in u_{123} by $\xi_{\lambda\mu}$, in which we can assume that u_{123} is non-zero. From the multiplication theorem, when we couple columns with columns, we will get:

$$\begin{vmatrix} x_{1k} & x_{1l} & x_{1m} \\ x_{2k} & x_{2l} & x_{2m} \\ x_{3k} & x_{3l} & x_{3m} \end{vmatrix} \begin{vmatrix} \xi_{11} & \xi_{12} & \xi_{13} \\ \xi_{21} & \xi_{22} & \xi_{23} \\ \xi_{31} & \xi_{32} & \xi_{33} \end{vmatrix} = \left| \sum_{i=1}^{3} x_{i\lambda} \xi_{i\mu} \right| \qquad (\lambda = k, l, m;\ \mu = 1, 2, 3). \tag{87}$$

Any element of the determinant on the right will itself be a determinant that arises when one deletes either the first, second, or third column from the matrix of u_{123}, according to the value of μ, and substitutes the k^{th}, l^{th}, or m^{th} column of (86) for them, according to the value of λ, respectively; thus:

$$u_{klm} \cdot u_{123}^2 = \begin{vmatrix} u_{k23} & u_{1k3} & u_{12k} \\ u_{l23} & u_{1l3} & u_{12l} \\ u_{m23} & u_{1m3} & u_{12m} \end{vmatrix}. \tag{88}$$

If we set k, l, m (in any order, whose choice will only have possible changes of sign on both sides as a result) equal to 1, 2, 3 then we will obtain the identity:

$$u_{123}^3 = u_{123}^3 .$$

If we choose only one of the three numbers k, l, m to be greater than three then we will also get an identity. By contrast, if we choose two of the numbers to be greater than three – e.g., $k = 1$, $l > 3$, $m > 4$ – then we will get:

$$u_{1lm} \cdot u_{123} = u_{1l3} \cdot u_{12m} - u_{1m3} \cdot u_{12l} .$$

There will be nine of these relations. If we finally choose k, l, m = 4, 5, 6 then we will get a relation of degree three. These ten relations will be independent of each other, since one u – namely, the u_{klm} that is on the left-hand side – will be present in each of them, which cannot appear in any of the remaining ones. In fact, only one index will be greater than three on the right, but at least two of them in u_{klm} will be greater then three on the left. The relations will be likewise solved for these ten u.

Theorem 221: *Ten independent relations exist between the twenty pointers of a net of complexes (a family of rulings), one of which is of degree three, while the rest are of degree two. The remaining ones can be expressed rationally in terms of ten suitably-chosen pointers.*

For μ = 4, we will again get six independent quadratic relations from the same process, as the duality between the linear complex domains and their extended domains would suggest from the outset. For μ = 5, we will obtain no further relations.

We have regarded the complex domains of intermediate dimensions (μ = 2, 3, 4) as domains that are bounded by the definition of their pointers. One can also regard them as intersection structures in the domain of highest dimension (μ = 5) and thus obtain two kinds of pointers for the three kinds of domains in a manner that is similar to how we distinguished ray pointers and axial pointers for the lines in space. The connection between these two kinds of pointers was found by *Pasch* ("Zur Th. d. lin. Kompl.," § 1, 2, Journ. f. Math., Bd. 75); thus, we can also distinguish *tetrahedral* and *rectangular, homogeneous pointers.* We would not like to pursue this further, but only remark that by linking the results of this theorem and the theorems of § 84, like the following one, it will become obvious that:

Theorem 222: *The conditions to which a rigid body with three degrees of freedom in its motion can be subjected to can be characterized completely at each moment by twenty homogeneous pointers, ten of which are independent (i.e., nine essential parameters).*

Practice problems:

57. When does an ellipse that lies on a cylindroid contact the generator of its plane E?

58. Derive the foregoing theorems on the cylindroid from Theorem 163, to the greatest extent that is possible.

59. In Fig. 76, $\mathfrak{k}_\vartheta$ is negative and $\mathfrak{k}_\eta$ is positive. Draw some other cases.

60. If one replaces a, b, c with three other complexes of the net then the determinant (33) will only be multiplied by a factor.

61. Specify the (six) cases in which a linear complex domain will have a complex in common with its extended domain.

62. If a complex of a pencil B can be found that lies involutorily to another pencil B' then conversely a complex B' can be found that lies involutorily to all complexes of B (Ball, *Theory of Screws*, art. 118).

63. Which case of a net of complexes does one obtain when one couples a pencil of singular complexes whose axes define a plane pencil B with a twist whose axis belongs to B?

64. If a line g with three points A, B, C moves on three surfaces then any point of g will move on a surface whose normal lies hyperbolically with the three surface normals at A, B, C (*Schönemann*, Journ. f. Math., Bd. 90).

65. Show that the order in which one draws the lines of the construction in § 85, *a*) has no effect on the result.

66. Treat the cases of $n = 6, 7$, as well, by analogy to the problem § 85, *b*).

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Mixed practice problems:

67. If a line is tangent to one of its points under a motion then so will its polar with respect to the twist that belongs to instantaneous screw will be so, as well.

68. If a line is characteristic of a plane E then its polar will be characteristic of another plane that is perpendicular to E.

69. When a line g moves, the tangents to the paths of its points will define a paraboloid.

70. The volume of the parallelepiped that is determined by any three generators of the same family of rulings is constant.

71. If one orders the vertices of two tetrahedra in any manner then the four connecting rays will define a ray net, just as the lines of intersection of the pairs of given planes will. These two nets will be of the same kind.

72. If the parameters δ, δ' of a ray net satisfy the equation:

$$a\ \delta\delta' + b\ \delta + c\ \delta' + \delta = 0$$

in the representation § 55, *a*) then a family of rulings will be distinguished in the net that will be a paraboloid for $a = 0$.

73. Show that the line:

$$x = \lambda z + m a, \qquad y = \mu z + \lambda b$$

traces out a ray net for varying λ and μ.

74. If two transversals to four lines coincide in a single one *t* then the double ratio of their four points of intersection will be equal to the double ratio of their four connecting planes with *t*.

75. Derive the parameter representation of a twist for a parabolic net in a manner that is similar to the one that was employed in § 60 for elliptic and hyperbolic nets.

76. Three mutually-involutory complexes group the points of space into tetrahedra (cf., the grouping of four elements P, Q, α, β in § 56 into two involutory complexes). Four mutually-involutory complexes give rise to two groups of eight points and eight planes that can be regarded as two *Möbius* tetrahedra in four ways.

77. The complexes of a pencil are pair-wise reciprocal with respect to one of them.

78. A ray net can be generated by ∞^1 families of rulings, two rays of which, s, s', will be fixed, while the third one t will move on a family of rulings R that go through one of the fixed lines s.

79. A twist can be generated by ∞^2 families of rulings, two rays of which, s, s', are fixed, while the third one t moves in a ray net that also belongs to one of the fixed lines s. It can also be generated by ∞^2 families of rulings, one ray of which is fixed, while the other two move freely in a ray net. It can also be generated by ∞^1 families of rulings, three rays of which s, s', s'' are fixed, while the fourth one t moves in a family of rulings, to which two rays of the family of rulings (s, s', s'') belong.

80. The polars of a fixed line relative to all complexes of α) a net of complexes, β) of a bush, γ) of a web, define α) a net of rays, β) a linear complex, or γ) fill up all of space, respectively.

81. The geometric locus of all lines on which two projective pencils of planes determine an involution is a linear complex.

82. Calculate the double ratio ($a\ b\ c\ d$) of four hyperbolic lines from their pointers a_i, b_i, c_i, d_i .

Appendix I

Hints for solving the practice problems

Some of the easier problems will be tacitly passed over here. As for the other ones, we shall, according to the nature of the problem, suggest how to begin the problem, the main steps in the process of solving it, or the final result, or even refer to a treatise in which the solution can be found. Wherever an author was already cited in the statement of the problem, the solution that will be given here will be a different one.

Chapter I

1. If a ray m rotates in a plane E that is parallel to the axis then for every position of m the rays of the twist that are parallel to m will fill up a plane. This plane will define a planar pencil of parallels that also belongs to E.

2. Its pointers are:

$$x = \frac{\mathfrak{t}B}{C}, \qquad y = -\frac{\mathfrak{t}A}{C}, \qquad z = -\frac{D}{C}.$$

3. One simultaneously introduces:

$$\begin{aligned} x &= x_1 \cos \alpha - y_1 \sin \alpha, \\ y &= x_1 \sin \alpha + y_1 \cos \alpha, \\ \xi &= \xi_1 \cos \alpha - \eta_1 \sin \alpha, \\ \eta &= \xi_1 \sin \alpha + \eta_1 \cos \alpha \end{aligned}$$

into equation (8), etc.

5. Cf., the construction in Fig. 17.

6. If one chooses the direction of the axis to be perpendicular to g then g' will be cut by the axis; if one chooses it to be parallel to g then one will not get an actual twist, but a bush of rays.

9. Construct the null plane of the point at infinity on s. The direction in it that is, at the same, the shortest distance from g to g' will be the direction of the axis. With that, one comes to the manner of determination b). However, when this solution fails (in what way?), one provides a second polar pair. The simplest way to do that is to look for the polar to the connecting line of a point of g and one of s.

10. Construct the polar to a transversal of g and h, etc.

11. If l_1' and d_1' are the perpendicular projections of l_1 and d_1 onto the horizontal plane through P then one will have:

$$d_1 = d_1' = d \cdot \sin \alpha.$$

All that one will then have to show is that:

$$\sin \alpha \cdot \cot \nu_1 = \cot \nu,$$

which one does with the help of a spherical triangle whose vertices lie on l_1, l_1', d.

13. The height of the screw path increases in a quadratic proportion to the radius of the cylinder, since the tangent to the pitch angle will, in fact, increase in a simple proportion with the circumference.

Chapter II

14. *a*) Relative to the normals to the null plane n of P.
b) and *c*) The sphere contacts n at P (Möbius, *Statik*, § 89).

15. *b*) If M_1 and M_2 are the moments relative to a_1 and a_2, respectively, then the null point must lie on a line whose points have the distance ratio $M_1 : M_2$ from a_1 and a_2, etc. (Möbius, *Statik*, § 96).

16. *c*) If k, m is the given dyname, K is the magnitude of the rod of the cross, and d is its separation from the axis then Theorem 25 will yield:

$$K = \frac{k}{2}\sqrt{2}, \qquad d = m : k = \mathfrak{k}.$$

18. *b*) From 17, b), the limiting position will be the same as if g had rotated around g', so it will be the foot of the shortest distance from g', and thus the axis, as well.

19. The normal of E that intersects both polars is a ray of the associated twist; its point of intersection with E will then have a velocity vector that is in E itself.

21. The attainable zone will be bounded by two circles with radii ω and $\omega + 2\gamma$ (Fig. 25) and their centers at A''.

22. *a*) The altitude is also an altitude in the reciprocal tetrahedron, but the vertex and foot are reversed.

b, α) The octahedron corresponds to a body that is bounded by two squares and four broken tetrangles (Fig. 82).

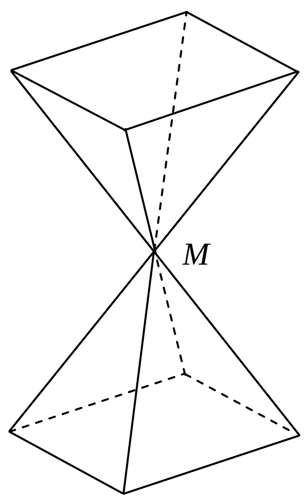

Figure 82.

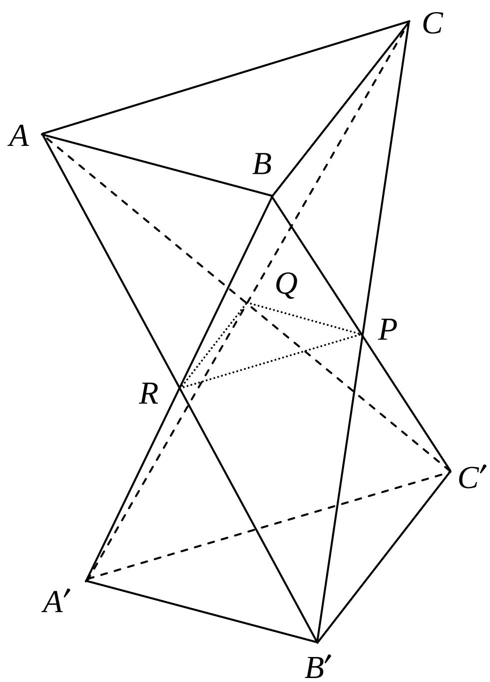

Figure 83.

M should not be considered to be a vertex of a polyhedron.

β) The cube (*Würfel*) is a body that is bounded by two triangles *ABC* and *A′B′ C′* that go to each other by parallel translation and six triangles *A′ B C*, *B′ C A*, *C′ A B*, *A B′ C*, *B C′ A*, *C A′ B* (Fig. 83). The sides of the triangle *PQR* are not to be considered as edges of a polyhedron, nor should the overlapping tetrangle *A B A′ B′*, etc., be considered to be faces of a polyhedron.

23. Half of the solution is indicated in Fig. 84. One obtains the other half by a reflection along *MN*. One would start by drawing $PQ = a = 5c / 2$ parallel to *a*, then drawing 1, 2 parallel to the rods with the same names in Fig. 31. One would then make $PS = c$, etc.

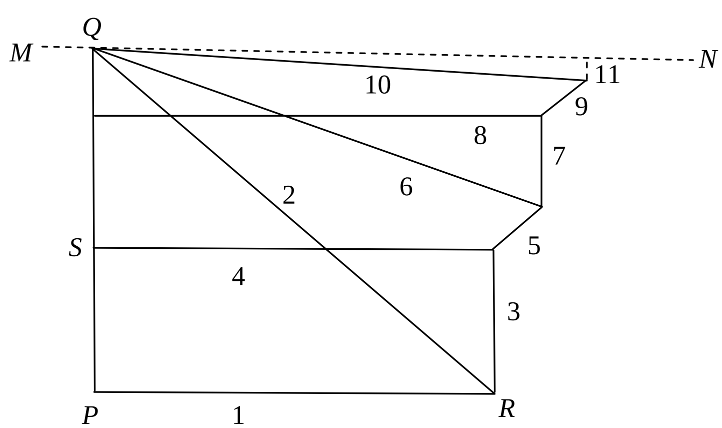

Figure 84.

24. Confer the paper of *Hauck* that was cited in § 27. One will find even more general theorems in *Hauck*, Journ. f. d. r. u. a. Math., Bde. 100 and 120. *Schmidt*, Monatsh. f. Math. u. Phys., Bd. VIII.

Chapter III

26. If one introduces linear, homogeneous forms in the pointers y, z:

$$y_i' = \lambda y_i + \mu z_i , \qquad z_i' = \nu y_i + \rho z_i ,$$

in place of those pointers, then the line pointers will be multiplied by factors that are independent of the indices.

29. One has:

$$\begin{vmatrix} A_{l\lambda} & A_{l\mu} \\ A_{m\lambda} & A_{m\mu} \end{vmatrix} = \pm A \begin{vmatrix} a_{i\iota} & a_{i\kappa} \\ a_{k\iota} & a_{k\kappa} \end{vmatrix},$$

in which the upper or lower sign is valid, according to whether the permutations l, m, i, k and λ, μ, ι, κ belong to the same or different classes (cf., Pascal, *Determ.*, § 8). However, since the permutation l, m, i, k is fixed in equation (72), and only the ratios of the coefficients of p' enter into it, we can let the sign depend upon the permutation λ, μ, ι, κ alone, etc.

31. Let a determinant of order ν that has the form:

$$\Delta = \begin{vmatrix} \varphi_\lambda & \varphi_{\lambda-1} & \cdots & \varphi_{\lambda-\nu+1} \\ \varphi_{\lambda+1} & \varphi_\lambda & \cdots & \varphi_{\lambda-\nu+2} \\ \vdots & & & \vdots \\ \varphi_{\lambda+\nu-1} & \varphi_{\lambda+\nu-2} & \cdots & \varphi_\lambda \end{vmatrix}$$

be given, in which each element is a homogeneous function of any other quantities p_i whose degree is indicated by the index. Δ will then be a homogeneous function of degree $\lambda\nu$ in the p_i , as one convinces oneself by going from ν to $\nu + 1$. (If one changes all of the indices in a row or column in determinant of this sort by the same amount then it will remain homogeneous in the p_i if it was before.) Nothing will change in this when one replaces arbitrarily many elements with zeroes.

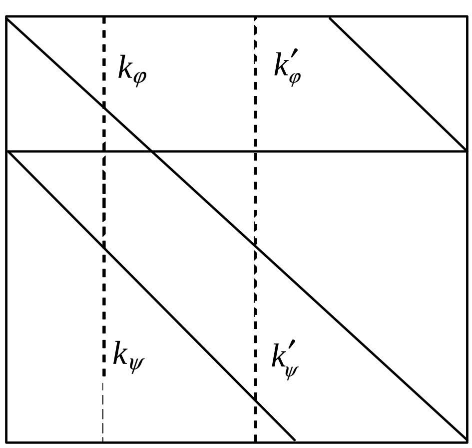

Figure 85.

If we assume that $m \leq n$ then the construction of D can be made tangible by Fig. 85, in which the two strongly-bounded parallelograms are filled with elements φ and ψ (including the slanted boundaries), and the rest of space with zeroes. If one develops D in the sub-determinants of the first m rows then one will get nothing but products of the determinants $\Delta_\varphi \cdot \Delta_\psi$, whose principal terms will be:

$$\Delta_\varphi = \varphi_n^m , \qquad \Delta_\psi = \psi_0^n .$$

The principal term in Δ_φ will then have dimension mn, while that of Δ_ψ will have dimension zero. One will obtain all other terms from the principal term when one replaces the sub-columns k_φ ($k \leq m$) of Δ_φ by other k'_φ ($k' > m$) and simultaneously switches the extended sub-columns k'_ψ and k_ψ in Δ_ψ. The dimensions of Δ_φ and Δ_ψ will then change by the same number in the opposite senses.

32. Here, the word "locus" can be understood only in the sense of point geometry. However, two equations in line pointers will not determine a "locus" then.

Chapter IV

33. If projective pencils are given by (S; a¸b, c) and (S', a, b', c') (Fig. 86) then the direction α of the axis will be constructed as the null point of the plane at infinity when one draws $\beta \parallel b'$, $\gamma \parallel c'$ through S and intersects the planes $b\beta$ and $c\gamma$. Even though the spatial structure is, for the most part, still not fixed by the elements that are suggested in the drawing, one can still construct a point T of α when one employs an auxiliary plane $E_0 \parallel a$. In order to find the axis itself, one must look for the null point of a plane that is perpendicular to α (which is no longer done, and would first become possible if one eliminated the aforementioned multi-valuedness in the figure).

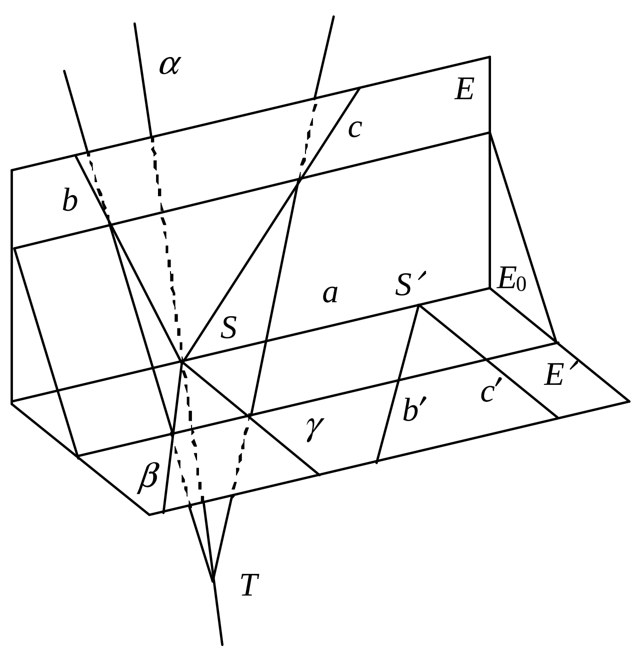

Figure 86.

35. In fact, the twist does not lose its axial symmetry, but its parameter will change in proportion to q, as one can easily confirm in the null planes of the point in E, as well as in the ones whose shortest distance from the axis is perpendicular to E.

36. If one makes the Z-axis the axis of the twist then the equation of the associated $\mathfrak{C}_2$ will assume the simpler form:

$$(a_6\, q_3 + a_3\, q_6)^2 - M'^2\ (q_1^2 + q_2^2 + q_3^2) = 0.$$

One can restrict oneself to the examination of the complex cone whose vertex (with the abscissa x) lies on the X-axis, and here again, to the search for two rays that cut the X-axis perpendicularly. If one then sets:

$$\frac{a_6}{a_3} = \mathfrak{k}, \qquad \frac{M'^2}{a_3^2} = m,$$

in which $\mathfrak{k}$ is then the parameter of the associated twist $\mathfrak{G}$, then one will get:

$$(\mathfrak{k}^2 - m)\, z'^2 + 2\mathfrak{k}\, x\, y'z' + (x^2 - m)\, y'^2 = 0. \tag{1}$$

The direction coefficient $\lambda = z' : y'$ can be calculated from this, and likewise the vertex angle of the cone.

$\mathfrak{C}_2$ can thus generate the ruled surface (1) by displacement and a screwing motion – i.e., a fourth-order ruled surface – in the same way that the twist will generate the hyperbolic paraboloid:

$$\lambda = -\frac{x}{\mathfrak{k}}$$

by displacement and a screwing motion.

If one next has:

I) $m = \mathfrak{k}^2$ then it will follow from (1) that all diameters of $\mathfrak{G}$ belong to $\mathfrak{C}_2$.

In fact, it is mechanically obvious from the outset that for $M'^2 = a_6^2$ – i.e., when the given moment is equal to the field of the equivalent dyname – any diameter of $\mathfrak{G}$ will solve the problem. The rod part of the dyname will then have a zero moment with respect to it, and the field part will have the same moment with respect to all diameters. Here, the degree of ruled surface will reduce to three.

II) $m < \mathfrak{k}^2$ then it will be even more mechanically obvious that real rays of $\mathfrak{C}_2$ must go through each point, since the absolute values of the attainable moments for the axes through a point will have a maximum, but no non-zero minimum (problem 14).

III) $m > \mathfrak{k}^2$ then no rays of $\mathfrak{C}_2$ will enter into the interior of the cylinder of radius $\sqrt{m-\mathfrak{k}^2}$. It will be contacted by rays whose pitch is:

$$\frac{\mathfrak{k}}{\sqrt{m-\mathfrak{k}^2}}.$$

All of the rays of $\mathfrak{C}_2$ can be associated with tangents to helices, two families of ∞^1 of which will lie on each coaxial cylinder. One should note the special case of $\mathfrak{k} = 0$ (cf., § 48, *b*).

38. Each hyperbolic net can be obtained from a rectangular net $\mathfrak{N}$ by a transformation $\mathfrak{T}$ as in Theorem 108. If one cuts $\mathfrak{N}$ with two planes parallel to the middle plane that harmonically separate the focal lines then one will obtain two such affine systems Σ, Σ_1 whose projections Σ, Σ' onto each other will have mutually-involutory central pencils. If one makes their double rays into axes then the affinity between Σ and Σ' can be represented by the equations:

$$x' = \mu\, x, \qquad y' = -\mu\, y. \tag{1}$$

It will emerge from:

$$\frac{y'}{x'} = -\frac{y}{x}, \qquad x'\,y' = -\mu^2\, x\,y$$

that the system Σ' can be derived from Σ by reflection along a line and a similar enlargement with a ratio μ. Therefore, $\mathfrak{N}$ can be generated by two such affine systems in ∞^1 different ways, which are derivable from each other by reflection, similarity transformations, and translations.

If one cuts $\mathfrak{N}$ with two planes that have the same distance from the middle plane on both sides then the equations:

$$x' = \kappa x, \qquad y' = \frac{1}{\kappa}\, y \tag{2}$$

will enter in place of (1), in which κ means a positive or negative constant, according to whether the two planes lie between the focal lines or outside of them, respectively.

The further equations:

$$\kappa\,\frac{y'}{x'} = \frac{1}{\kappa}\frac{y}{x}, \qquad x'\,y' = x\,y$$

will no longer admit such a simple interpretation.

41. $$x' = c\cot\alpha\cdot\frac{y}{z}, \qquad y' = c\tan\alpha\cdot\frac{x}{z}, \qquad z' = \frac{c^2}{z}.$$

42. If one considers y, y' to be fixed and x, x' to be moving then one will come to Sylvester's method of generation. The point x is associated with the plane (g, x') as its null plane, which will always be the same then, no matter which pair one considers to be fixed, instead of y, y'. Since g, g' are also rays of all twists, they will be identical. This method of determination by two projective point sequences then comes down to giving a special ray net and one ray from the twist.

44. For example, one can deduce from (113.a): If one chooses any quantities p_i that satisfy the equation:

(1) $$c\, p_3 + p_6 = 0$$

and the equation:

(2) $$\sum p_i\, p_{i+3} = 0$$

then one can calculate the four *real* quantities σ, y, z, w from the four equations:

$$\sigma p_1 = c, \qquad \sigma p_2 = -z, \qquad \sigma p_3 = y, \qquad \sigma p_5 = c\, w.$$

The two extra equations:

$$\sigma p_6 = -c\, y, \quad \sigma p_4 = y^2 + z\, w$$

will then be fulfilled by means of (1) and (2) by themselves.

45. In (110), the u-surfaces and the v-surfaces are hyperbolic paraboloids, the u, $\mathfrak{z}$ and v, $\mathfrak{z}$ congruences are special ray nets, and indeed in one case the focal lines will be at infinity; analogous statements will be true for (113).

46. One chooses two points on g; its null planes relative to the complex of the pencil will describe two projective planar pencils, etc.

Chapter V

48. If one sets:

$$x_k = a_k + i\; a'_k, \qquad y_k = b_k + i\; b'_k,$$

and writes:

$$\begin{array}{ll} \lambda + i\,\lambda', & \text{instead of } \lambda, \\ \mu + i\,\mu', & \quad\text{“} \quad \mu, \\ z_k + i\; z'_k, & \quad\text{“} \quad z_k, \end{array}$$

then one will have:

$$\sigma z_k = a_k\,(\nu\,\lambda + \nu'\lambda') + a'_k\,(\nu'\lambda - \nu\,\lambda') + b_k\,(\nu\,\mu + \nu'\mu') + b'_k\,(\nu'\mu - \nu\,\mu')$$
$$\sigma\, z'_k = a_k\,(\nu'\lambda - \nu\,\lambda') - a'_k\,(\nu\,\lambda - \nu'\lambda') + b_k\,(\nu'\mu - \nu\,\mu') - b'_k\,(\nu\,\mu + \nu'\mu').$$

If one solves each of the two systems in parentheses and sets the solutions proportional to each other then one will obtain a representation of the gathered involutory collineation that is free of parameters. When one denotes the adjoint of every element in the determinant:

$$|\,a_k \;\; a'_k \;\; b_k \;\; b'_k\,| \qquad (k = 1, \ldots, 4)$$

with the corresponding Greek symbol, it will read:

$$\tau \sum \alpha_k\, z'_k = \sum \alpha'_k\, z_k\,, \qquad \tau \sum \beta_k\, z'_k = \sum \beta'_k\, z_k\,,$$
$$\tau \sum \alpha'_k\, z'_k = -\sum \alpha_k\, z_k\,, \qquad \tau \sum \beta'_k\, z'_k = -\sum \beta_k\, z_k\,.$$

One sees from the equations that the collineation is involutory and that no real double points are present. It will then follow that $\tau^2 = -1$ for the double points.

(52) $$(b^2 + c^2 - r^2)\; q_1^2 + (c^2 + a^2 - r^2)\; q_2^2 + (a^2 + b^2 - r^2)\; q_3^2$$
$$q_4^2 + q_5^2 + q_6^2 - 2\,[a\,b\,q_1\,q_2 + b\,c\,q_2\,q_3 + c\,a\,q_3\,q_1]$$
$$+\, 2a\,(q_3\,q_5 - q_2\,a_3) + 2b\,(q_1\,q_6 - q_3\,a_4) + 2c\,(q_2\,q_4 - q_1\,a_5) = 0.$$

53. The locus is the equilateral paraboloid:

$$\frac{ab}{c} = -\,\frac{n}{m}(1 - m^2).$$

If one chooses its center to be on this surface then the radius of the sphere is determined from:

$$b^2 - a^2\, m^2 + (c^2 - r^2 - n^2)\,(1 - m^2) = 0.$$

In fact, it is a two-fold condition for an imaginary line to contact a sphere, since it will be contacted by its conjugate, as well.

(54) $$q_3^2 - q_6^2 + 2(q_1 q_5 - q_2 q_4) = 0,$$
$$q_3^2 + q_6^2 + 2(q_2 q_5 - q_1 q_4) = 0.$$

Cf., also *Zeuthen*, Math. Ann., Bd. I.

55. In double contact. They will then have only two common points with it, and the number of points of intersection can only reduce in an imaginary domain, since some of them will coincide. In fact, one can make this intuitive when one passes to the limit for a general surface for which two umbilic points coalesce into a vertex, so their tangent involutions will cut out the same involution from the lines at infinity of all parallel circle planes.

56. They will be the ones that meet the imaginary sphere-circle $\mathfrak{K}$. This can happen only at the common points of $\mathfrak{K}$ and the infinitely-distant curve C of F_2 . C will also be the intersection of the asymptotic cone A of F_2 with the plane at infinity. The ray involutions in the cyclic planes of A define imaginary rays of A that also meet $\mathfrak{K}$. They thus cut the four points of intersection of C and $\mathfrak{K}$ from the plane at infinity, since they are taken in both senses. One thus has the following theorem: An involution is defined by a hyperboloid H in a pencil of parallel planes that contains a cyclic plane of H. This also cuts out an involution from any generator e of H, which then also defines a gathered spatial involution (Theorem 137). *The ordering rays of the latter define a rotation net.* We have formulated the result in the real case; however, it was not so easily derived without the theory of imaginary elements.

Chapter VI

57. If E goes through one of the external generators of the cylindroid.

61. From the classification of § 79, these will be the cases *A*) *b*, *c*, *d*; *B*) *b*, *d*; *C a*. The common complexes can only be singular.

62. Let C be the complex of B that lies completely normal to B' (§ 78). B' will then lie in the extended domain C_4 to C (Theorem 175). However, the extended domain B_3 of B will also lie in C_4, so here it will cut B' in a complex C' that is completely normal to B.

63. The case in which the basic family of rulings is an equilateral paraboloid (§ 83, *B*, *a*).

64. Since one might also consider g to be the component of a body with three degrees of freedom, the surface normals at A, B, C will define the basic family of rulings of the associated linear winding domain.

66. Cf., the paper of *Sturm* that was cited in § 85.

Mixed practice problems

69. *Chasles*, C. R. **16** (1843). Refer to the two books by Schönflies *Geometrie der Bewegung in synth. Darst.* (1886) and Mannheim *Géométrie cinématique* (1884) on this subject.

70. *Franel*, Vierteljahrsschr. der naturf. Ges. Zürich, Bd. 40, 1895.

71. *Franel*, *ibidem*; for the case in which the nets are identical, cf., *Kohn*, Wiener Sitzsber., Bd. 107, II, 1898.

72. A projectivity will be defined on the focal lines by the equation (*D'Emilio*, the paper that was cited in § 55).

73. It will always cut the two lines:

$$x\sqrt{b} - y\sqrt{a} = 0, \qquad z = \sqrt{ab}$$

and

$$x\sqrt{b} + y\sqrt{a} = 0, \qquad z = -\sqrt{ab}\,.$$

(*Hermes*, "Über Strahlensyst. 1. Ord. u. Kl.," Journ. f. Math., Bd. 67, 1867). The net is hyperbolic or elliptic, according to whether a, b do or do not have the same symbols, respectively.

74. t is then a focal line of a parabolic ray net that can be defined by a correlation with the carrier t.

76. *Klein*, Math. Ann., Bd. II. *Caporali* and *del Pezzo*, "Introd. alla teoria dello sp. rig." (in the Mem. di. Geom. of Caporali).

77. Every ray of the associated net will then be mapped to itself by a complex of the pencil (*Caporali* and *del Pezzo*, *loc. cit.*, § 8).

78. Let R be defined by s, t', t''. s, s', t', t'' will then determine a ray net N that all of the ∞^1 families of rulings will belong to (Theorem 99), but it will be exhausted, as one knows, when one draws two planes through s and generates N by two collinear fields (Theorem 100).

79. Let N be defined by s, t_1, t_2, t_3 . A twist G will then be determined by s, s', t_1, t_2, t_3 that all of the ∞^1 families of rulings will belong to (Theorems 101 and 11), but it will also be exhausted. If one then chooses an arbitrary ray p of G then it will determine a family of rulings with s, s' that, from Theorem 217, will cut N a second ray besides s that is taken to be t if one would like to come to p. Argue analogously in the other cases.

80. Sturm, *Liniengeometrie I*, arts. 130, 142, 154.

81. *Caporali* and *del Pezzo*, *loc. cit.*, § 4.

82. We cut a, b, c, d with two rays t, t' of the guiding family. One will then have (Fig. 87):

$$(a\ b\ c\ d) = (A\ B\ C\ D) = (\alpha\,\beta\,\gamma\,\delta).$$

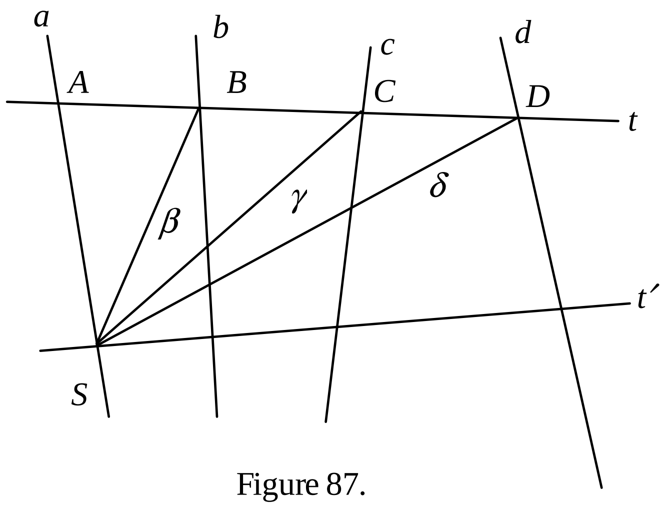

Figure 87.

Now, from § 39, *c*), the pointers of *g* and *d* can be written in the form:

$$\gamma_i = a_i + \lambda\, b_i\,,$$
$$d_i = a_i + \mu\, b_i\,,$$

in which λ and μ are to be determined from the condition that γ, c cut, just as δ, d. If we then set:

$$\omega(a, c) = \sum a_{i+3}\, c_i = (a\, c),$$

etc., then we will have:

$$(a\, c) + \lambda\,(\beta\, c) = 0, \qquad (a\, d) + \mu\,(\beta\, d) = 0,$$

and from Theorem 55:

$$(a\; b\; c\; d) = \frac{\lambda}{\mu} = \frac{(ac)}{(\beta c)} : \frac{(ad)}{(\beta d)}\,.$$

We now switch a, b, and likewise t, t', and let B play the role of S. We will then get:

$$(b\; a\; c\; d) = \frac{(bc)}{(\beta c)} : \frac{(bd)}{(\beta d)}\,,$$

and by eliminating β:

$$(a\; b\; c\; d) = \sqrt{\frac{(ac)}{(cb)} : \frac{(ad)}{(db)}}\,.$$

We still have to determine the sign of the root, whose positive value we will call u: If we calculate:

$$(a\; c\; b\; d) = \pm\, v$$

using the same rule then we will get, on the one hand:

$$(a\; b\; c\; d) + (a\; c\; b\; d) = 1,$$

while, on the other hand, only one of the three equations:

$$u + v = 1, \qquad u - v = 1, \qquad -u + v = 1$$

can be fulfilled by positive values of u, v. The double ratio can, from the nature of things, also be expressed *rationally* in terms of the pointers (*Voss*, Math. Ann., Bd. 8, pp. 61).

Appendix II

Producing the figures

While the following explanations are not indispensible for an understanding of the book, they can still give a geometer some idea of how spatial structures are drawn correctly. Knowledge of the elements of descriptive geometry is assumed for some of the following explanations. All figures in the book are, where not expressly stated to the contrary, drawn in axonometric projection, to the extent that they, above all, represent *spatial* structures.

Chapter I

Fig. 1. The circle is divided into twelve equal parts and the points of the parts are projected onto the ellipse by parallels to the axis. When one carries the segment 11′ once on 1, twice on 2, etc., in the direction a, one will get the points of the helix.

Fig. 10. Since the inclination of the plane of the ellipses that represent circles is determined completely by the ratio of the axes of the ellipses, and therefore the position of the entire spatial figure relative to the image plane, the inclination ν of the cylinder tangent l with respect to the circle plane – i.e., with respect to the circle tangent t at B – will also be determined. The arbitrary choice of ray l of the twist through B in the image will then determine the parameter k of the twist. Its ray l_1, with the pitch v_1 (and analogously for l_2) must be constructed from the proportion [Equation (14)]:

$$\tan v_1 : \tan v = c_1 : c.$$

This happens in the accompanying figure (on a larger scale). In order to find the direction of t, one draws $PT \parallel DA$. One will then have $t \parallel TQ$, since PTQ is a right angle in space. One has $AN \parallel t$ (arbitrary length), $t_2 \parallel t_1 \parallel t$, $n_2 \parallel n_1 \parallel n \parallel a$. S is determined as the point of intersection of l and n. NS then cuts points of l_1 and l_1 out of n_1 and n_2, resp., since n, n_1, n_2 are proportional to the tangents of the angles ν, ν_1, ν_2, respectively.

Fig. 13. The lines a, l_1, d, d_1 are chosen arbitrarily in the image, and one then draws $n \parallel a$ through Q, $d_1' \parallel d_1$ through N, with which l_1' is determined. One now sees likewise that one can, in fact, choose those four lines arbitrarily. One can then consider n, l_1', d_1' to be the axis cross, etc. Finally, one must draw $l \parallel NQ$.

Chapter II

Fig. 20. The hyperboloids are merely sketched and are thought to be transparent.

Chapter III

Fig. 33. The trigon P_1 is cut by E in STU and by $\mathfrak{E}$ in $\mathfrak{STU}$. The vertices of these two triangles are chosen arbitrarily from the edges, and at the same time, $\mathfrak{E}$ is thought of as being horizontal. The trace E_h of E on $\mathfrak{E}$ is then constructed from this, etc. Let $P_1 P_1' = \mathfrak{e}_1$ be the perpendicular of P_1 on E, and let N be the connecting line ($\mathfrak{e}_1$, δ_1). $\mathfrak{e}_1$ and N_h can then be chosen arbitrarily, E_h, N_h, $\mathfrak{e}_1$ can now be considered to be the directions of the axes. However, the trace N_e of N on E must be constructed (e.g., with the help of P_1 s). Finding δ_1 is just a matter of constructing the perpendicular from P_1 to N_e . Thus, if $AC \perp P_1B$ then ACB will also be a right angle in space, so AC will be the altitude of the triangle ABP_1 that lies in N. One will find a second altitude, and thus the point of intersection ω of the altitude, when one draws $B\,\omega \parallel N_h$. P_1D through ω is the desired segment δ_1. One draws δ_2 parallel to it through P_2 and bounds it with DS. One finds $\mathfrak{e}_2$ analogously.

Fig. 34. E_1 and E_2 are first assumed to be bounded by right angles in space, and then P_1 is chosen in E_2 and P_2 in E_1. If one draws α, x parallel to the boundaries of the planes then the perpendicular projection P_2'' of P_2 onto E_2 can still be chosen arbitrarily on x. However, three mutually-perpendicular directions x, y, $P_2''P_2$ are now known that can be considered to be the directions of the axes, and the projection P_1' of P_1 onto E_1 must be constructed, since the position of the figure with respect to the image plane is well-defined (up to its distance from it). If we then lay P_2'' in the image plane then we will obtain two points of the image trace β of E_1 by way of $\sigma \perp y$ and $\tau \perp P_2''P_2$; P_1P_1' must then be drawn perpendicular to β and bounded by parallels to the boundaries. One now knows the projections k' and k'' of k onto E_1 and E_2, resp., so one can draw the segments d' and s' when S' and $\mathfrak{S}'$ are chosen on k.

Chapter IV

Fig. 40. In order to get a correct sketch as simply as possible, one makes $A_3C = A_2A_3 = NA_2$, and likewise $B_3 B_4' = B_2 B_3 = B_1 B_2$.

Fig. 47. The pairs of congruent ellipses of the two families in Theorem 109 determine hyperboloids for which the principal ray of the net is an axis. Three of these hyperboloids were indicated and three pairs of conjugate directions in the ellipses (e.g., AA', BB'). The contours of the hyperboloids are not drawn everywhere; however, their contact points with the rays of the net everywhere separate a completely extended and a dotted part of the ray. The rays of the net can also be associated with equilateral, hyperbolic paraboloids when one combines all of them that cut a diameter of a family of

ellipses (e.g., AA', and thus also BB'). The same figure can be regarded as the image of a rotation net, as well as a general elliptical one.

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Chapter V

Fig. 58. Since the positions of the lines g, g_1 are still not determined by their axonometric projections alone, one can choose the directions of the traces h, h_1 of the pair of parallel planes through g and g_1, resp., arbitrarily in E. If one then projects the point shadow (*Punktwurf*) onto g in the direction g_1, and conversely, then one will obtain points on the rays of the quadruples a, a'; b, b' and a_1, a_1'; b_1, b_1'. One can now immediately connect this with the further construction of the common points with b). This construction will then have a projective character; it will therefore make no difference that here it is merely the image of a construction that takes place in another plane, in reality.

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Chapter VI

Fig. 66. A circle K of the circular cylinder is divided into 32 equal parts (using the same method as in Fig. 1). Perpendiculars to the plane of the circle are drawn through the points of the parts and the ordinates of the curve $z = \frac{1}{2} \sin 2\vartheta$ are laid on them above or below the circle. Thus, the semi-major axis of the ellipse K will be the unit of true length, so the eccentricity of K will be the shortening of its length in the Z direction. If one then takes one-half the eccentricity CH to be the unit, then defines a scale, and considers $\frac{1}{32}$ of the circle to be equivalent to $22\frac{1}{2}{}^{\circ}$ then one can immediately read off the value of z that is laid down from a table of sines, and then remove the scale.

AA' and BB' are the axes of the principal complex of the pencil and lie in the middle plane. GG' is the lowest generator, and HH' is the highest one. Both of them are the "outermost" ones. Their directions bisect the (right) angle between AA' and BB'. DD' is the double line, insofar as two real generators will intersect on it. The figure of the model is, for the sake of clarity, bounded by the sinusoid, instead of the dashed line in Figure 65, which cannot be done for an actual model. The cylinder is thought of as being opaque, but the cylindroid is transparent, as would indeed be the case for a string model.

Fig. 68. An ellipse K is drawn arbitrarily and considered to be the projection of a circle. Furthermore, d is chosen to go through a point B of K parallel to the minor axis of the ellipse. The diameter AA' of K that is conjugate to BB' is perpendicular to BB' in space. If one makes $A'H = -AG \parallel d$ then GH will be the major axis of the ellipse in space that lies on the circular cylinder through K, which can then be drawn from the conjugate diameters BB' and GH (the curvature circle at the endpoints of this diameter, according to Rohn and Papperitz, *Darst. Geom.*, art. 210). One now finds the generators of the cylindroid as is suggested in the text. The ones that go through G and H will be the

lowest and the highest ones, respectively. BB' and the tangent to K at B will be the principal generators.

Fig. 81. One should note that X, a, Y, a' define a harmonic pencil. One arranges this most simply when one lays two equal segments in succession in a parallel to a', through whose endpoints X, a, Y must go

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